GRAVITATIONAL MEMORY OF CASIMIR EFFECT

(accepted for publication in PRD)

F. SORGE

I.N.F.N. PADOVA

QGSKY – GENOVA, 06-07 OCTOBER, 2023

MOTIVATION

- Study of influence of gravity on vacuum energy
- Search for permanent effects («memory» effects)
- Relationship with the cosmological constant problem
- Use of Schwinger's approach to clearly disentangle particle creation effects from polarization effects
- Handling of divergences by means of analytic continuation

OUTLINE

- *GRAVITATIONAL BACKGROUND AND CASIMIR CAVITY*
- *PROPER-TIME SCHWINGER'S APPROACH*
- *STATIC CASIMIR EFFECT*
  - *Flat background (a consistence check)*
  - *Bianchi Type-I background*
- HANDLING THE DIVERGENCES
- THE GRAVITATIONAL WAVE CASE
- CONCLUSIONS

GAVITATIONAL BACKGROUND

Starting point: Bianchi-I spacetime
(the simplest generalization of FLRW universe)

$$ds^2 = dt^2 - \sum_{i=1}^{3} a_i^2(t) dx_i^2$$

directional scale factors

Assume *small* anisotropies,
satisfying the constraints:

$$ds^2 = dt^2 - \sum_{i=1}^{3} [1 + h_i(t)](dx^i)^2$$

anisotropies

$$\max|h_i(t)| \ll 1$$

$$\sum_{i=1}^{3} h_i(t) = 0,$$

$$h_3(t) \equiv h_z(t) = 0,$$

$$\lim_{t \to \pm\infty} h_i(t) = 0$$

→ asymptotic Minkowskian spacetime regions

↓

unambigous definition of in- and out- vacua

Recall that for Bianchi-I type coordinates can be chosen so that spatial metric is **diagonal** and **traceless**

(see, e.g., the *holy* text by Birrell & Davies)

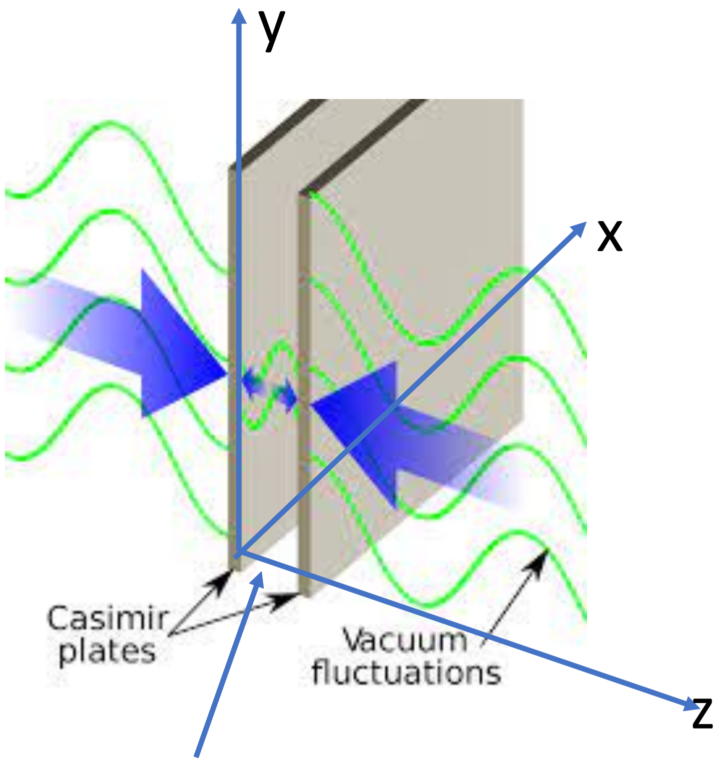
CASIMIR CAVITY

The requirement $h_3(t) \equiv h_z(t) = 0$
guarantees that proper and coordinate
distance between the plates coincide at any time.

This allows us to avoid possible complications
arising from *tidal* effects.

For sake of simplicity, assume a **massless scalar field**,
minimally coupled to the gravitational background

$$\frac{1}{\sqrt{-g}}\partial_\mu\left[\sqrt{-g}g^{\mu\nu}\partial_\nu\phi(x)\right] + \xi R(x)\phi(x) = 0$$



L = proper plate separation

A = proper area of the plates

CASIMIR CAVITY – FIELD MODES/1

Assume Dirichlet b.c.

To the lowest order, KG equation reads

$$(\Box + \hat{V})\phi = 0,$$

$$\hat{V} = h_x(t)\partial_x^2 + h_y(x)\partial_y^2 = h(t)(\partial_x^2 - \partial_y^2)$$ and

$$h(t) \equiv h_x(t) = -h_y(t)$$ ← ***metric perturbation***

Spatial translation invariance of Bianchi-I spacetime is broken by field confinement. However, it is still assured along x and y directions.

We guess:

$$\phi(x) = N e^{i\vec{p}_\perp \cdot \vec{x}_\perp} \sin\left(\frac{n\pi z}{L}\right)\eta(t)$$

$$N = \left(\frac{2}{(2\pi)^3 L}\right)^{1/2}$$

$$\vec{p}_\perp = (p_x, p_y), \ \vec{x}_\perp = (x, y)$$

requiring

$$\eta(t) \to e^{-i\omega t}, \qquad h(t) \to 0.$$

CASIMIR CAVITY – FIELD MODES/2

From KG equation we obtain

$$\eta(t) = e^{-i\omega t} + \int_{-\infty}^{t} dt' \frac{\sin(\omega(t-t'))}{\omega} h(t') p_\perp^2 \cos 2\theta e^{-i\omega t'} = \alpha_p(t) e^{-i\omega t} + \beta_p(t) e^{i\omega t},$$

$$\alpha_p(t) = 1 + \frac{i}{2\omega} \int_{-\infty}^{t} dt' \, h(t') p_\perp^2 \cos 2\theta,$$

$$\beta_p(t) = -\frac{i}{2\omega} \int_{-\infty}^{t} dt' \, h(t') p_\perp^2 \cos 2\theta e^{-2i\omega t'},$$

Bogolubov coefficients in the limit $t \to +\infty$

$$|\alpha_p|^2 - |\beta_p|^2 = 1.$$

$\tan\theta = p_y/p_x$, and $\omega^2 = p_\perp^2 + (n\pi/L)^2$.

$$\phi(x) = N \left(\alpha_p(t) e^{-i\omega t} + \beta_p(t) e^{i\omega t} \right) e^{i\vec{p}_\perp \cdot \vec{x}_\perp} \sin\left(\frac{n\pi z}{L}\right)$$

PROPER-TIME SCHWINGER'S APPROACH

According to Schwinger, the **effective action W** reads

total Trace

Proper-time Hamiltonian

$$W = \lim_{\nu\to 0} W(\nu)$$

$$W(\nu) = -\frac{i}{2}\int_0^\infty ds\, s^{\nu-1}\mathrm{Tr}\, e^{-is\hat{H}} + \text{c.t.}$$

In presence of a **time-dependent** spacetime background, the effective action W can become ***complex***, being related to the vacuum **persistence amplitude** in the so-called *in-out* formalism

particle creation

$$\langle 0\,\mathrm{out}|0\,\mathrm{in}\rangle = e^{iW}\,.$$

NB: the additional *counterterm* is introduced to subtract divergent terms, hence recovering the required physical normalization.

TRACE EVALUATION

The **total Trace** $\mathrm{Tr}\, e^{-is\hat{H}} = \sum\!\!\!\!\!\!\int d^4x \langle x|e^{-is\hat{H}}|x\rangle$

has to be evaluated all over the **continuous** as well the **discrete** degrees of freedom, including those of spacetime.

The **p-t Hamiltonian** reads

$$\hat{H} = \hat{H}_0 + \hat{V} = -\hat{p}_0^2 + \hat{p}_\perp^2 + \left(\frac{n\pi}{L}\right)^2 - h(t)p_\perp^2 \cos 2\theta, \qquad \hat{p}_0 = i\partial_t,\, \hat{p}_\perp = -i\vec{\nabla}_\perp$$

We get

NB: the rapidly oscillating term can be removed: **R.W.A.**

$$\mathrm{Tr}\, e^{-is\hat{H}} = N^2 \int d^4x \int d^2p_\perp d\omega \sum_n \left(|\alpha_p(t)|^2 + |\beta_p(t)|^2 + 2\Re\mathrm{e}\left(\alpha_p(t)\beta_p(t)^* e^{-2i\omega t}\right) \right)$$
$$\times \sin^2\left(\frac{n\pi z}{L}\right) e^{-isp_\perp^2} e^{-is(n\pi/L)^2} e^{is\omega^2}.$$

CASIMIR EFFECT

Following Schwinger, let us write the **vacuum Casimir energy density** as

$$\langle \epsilon_{Cas} \rangle = -\frac{1}{AL} \lim_{\nu \to 0} \left[\lim_{t \to +\infty} \frac{\partial}{\partial t} \Re\mathrm{e}\, W(\nu) \right].$$

Using $W(\nu) = -\frac{i}{2} \int_0^\infty ds\, s^{\nu-1} \mathrm{Tr}\, e^{-is\hat{H}}$

we find

$$\langle \epsilon_{Cas} \rangle = \frac{1}{2(2\pi)^3 L} \lim_{\nu \to 0} \Re\mathrm{e} \left\{ i \int_0^{+\infty} ds\, s^{\nu-1} \left[\int_0^{2\pi} d\theta \int_0^{+\infty} p_\perp\, dp_\perp \int_{-\infty}^{+\infty} d\omega \sum_n \left(1 + 2|\beta_p|^2\right) e^{-isp_\perp^2} e^{-is(n\pi/L)^2} e^{is\omega^2} \right] \right\}$$

CASIMIR EFFECT– flat background (a check)

- As a consistence **check**, consider the flat spacetime case, $\beta_p = 0.$
- Integrations in square brackets are readily performed and we find

$$\langle \epsilon_{Cas} \rangle_0 = \lim_{\nu \to 0} \Re\mathrm{e} \left\{ \frac{\sqrt{i}}{16\pi^{3/2} L} \sum_n \int_0^{+\infty} ds\, s^{\nu - \frac{3}{2} - 1} e^{-is\left(\frac{n\pi}{L}\right)^2} \right\}.$$

- The remaining integral can be converted into a **Gamma function** and the infinite sum yields a **Riemann zeta-function**

$$\langle \epsilon_{Cas} \rangle_0 = \lim_{\nu \to 0} \Re\mathrm{e} \left\{ \frac{-(i)^{-\nu}}{16\pi^{3/2} L} \left(\frac{\pi}{L} \right)^{2\nu - 3} \zeta(2\nu - 3) \Gamma(\nu - 3/2) \right\}$$

- Performing **analytic continuation** $(\nu \to 0)$ we find the well-known results:

$$\langle \epsilon_{Cas} \rangle_0 = -\frac{\pi^2}{1440 L^4}$$

Casimir energy density

$$f_{Cas}^{(0)} =: -\frac{1}{A} \frac{\partial A L \langle \epsilon_{Cas} \rangle_0}{\partial L} = -\frac{\pi^2}{480 L^4}.$$

Attractive force per unit surface

CASIMIR EFFECT– Bianchi-I s-t background/1

The **correction** to the flat Casimir result now reads:

$$\langle\delta\epsilon_{Cas}\rangle = \frac{1}{2(2\pi)^3 L}\lim_{\nu\to 0}\Re\mathrm{e}\left\{ i\int_0^{+\infty} ds\, s^{\nu-1}\left[\int_0^{2\pi} d\theta \int_0^{+\infty} p_\perp\, dp_\perp \int_{-\infty}^{+\infty} d\omega \sum_n 2|\beta_p|^2 e^{-isp_\perp^2} e^{-is(n\pi/L)^2} e^{is\omega^2}\right]\right\}.$$

Suppose a **metric perturbation** $h(t) = He^{-\sigma^2 t^2}$,

(gaussian profile)

NB:
$1/\sigma$ ~ time duration of the gravitational perturbation

then

$$\beta_p = -\frac{iH\sqrt{\pi}}{2\omega\sigma} p_\perp^2 e^{-(\omega/\sigma)^2}\cos 2\theta.$$

Integrations over variables *s*, θ and $p_\perp$ can be carried on, using also the recipe

$$\int_0^{+\infty}\frac{q^\mu dq}{(q^2+C^2)^\nu} = \frac{\Gamma\left(\frac{1+\mu}{2}\right)\Gamma\left(\nu-\frac{1+\mu}{2}\right)}{2\Gamma(\nu)\, C^{2\nu-1-\mu}}.$$

CASIMIR EFFECT– Bianchi-I s-t background/2

- **Correction** to Casimir energy density is

$$\langle \delta\epsilon_{Cas} \rangle = \frac{H^2}{16\pi\sigma^2 L} \lim_{\nu\to 0} \Re\mathrm{e}\left\{ i^{1-\nu}\Gamma(\nu-3)\left(\frac{L}{\pi}\right)^{2\nu-6} I(\nu) \right\}$$

DIVERGENCES!

$$I(\nu) = \int_0^{+\infty} \frac{d\omega}{\omega^2} e^{-2\omega^2/\sigma^2} \sum_n \frac{1}{\left(n^2 - \frac{\omega^2 L^2}{\pi^2}\right)^{\nu-3}}.$$

Performing a change of variable ($\omega = -i\,u$)
and a **Wick rotation**, we convert $I(\nu)$ into

$$\tilde{I}(\nu) = i\int_0^{+\infty} \frac{du}{(u+\epsilon)^2} e^{-\frac{2(u+\epsilon)^2}{\sigma^2}} \sum_n \frac{1}{\left(n^2 + \frac{(u+\epsilon)^2 L^2}{\pi^2}\right)^{\nu-3}},$$

Epstein-Hurwitz zeta-function!

$$\zeta_{EH}(s, q^2) = \sum_{n=1}^{\infty} (n^2 + q^2)^{-s}$$

NB: $\epsilon \to 0$ at the end of calculations

CASIMIR EFFECT– Bianchi-I s-t background/3

- Epstein-Hurwitz zeta-function can be **analytically continued** (E. Elizalde, *J. Math. Phys.* **31**, 170 (1990))

$$\zeta_{EH}(s,q^2)=\sum_{n=1}^{\infty}(n^2+q^2)^{-s} = -\frac{q^{2s}}{2}+\frac{\sqrt{\pi}}{2\Gamma(s)}\Gamma(s-1/2)q^{-2s+1} + \frac{2\pi^s}{\Gamma(s)} \quad q^{1/2-s}\sum_{n=1}^{\infty}n^{s-1/2}K_{s-1/2}(2\pi n q)$$

Modified Bessel function

- Hence, Casimir energy can be written as the sum of **three** contributions

$$\langle\delta\epsilon_{Cas}\rangle=\langle\delta\epsilon_{Cas}\rangle_1+\langle\delta\epsilon_{Cas}\rangle_2+\langle\delta\epsilon_{Cas}\rangle_3$$

CASIMIR EFFECT– Bianchi-I s-t background/3

- Three contributions to Casimir energy density:

$$\langle \delta\epsilon_{Cas}\rangle_1 = \frac{H^2}{32\pi\sigma^2 L}\lim_{\nu\to 0}\Re\mathrm{e}\left\{ i^{-\nu}\Gamma(\nu-3)\frac{\sigma^{-2\nu+5}}{2^{-\nu+7/2}}\Gamma\left(-\nu+\frac{5}{2}\right)\right\},$$

$$\langle \delta\epsilon_{Cas}\rangle_2 = -\frac{H^2}{16\pi\sigma^2}\lim_{\nu\to 0}\Re\mathrm{e}\left\{ i^{-\nu}\frac{\sqrt{\pi}}{2}\Gamma\left(\nu-\frac{7}{2}\right)\frac{\sigma^{-2\nu+6}}{2^{-\nu+4}}\Gamma(-\nu+3)\right\},$$

$$\langle \delta\epsilon_{Cas}\rangle_3 = -\frac{H^2}{16\pi\sigma^2 L}\lim_{\nu\to 0}\Re\mathrm{e}\left\{ i^{-\nu}2\pi^{\nu-3}\left(\frac{L}{\pi}\right)^{\nu-5/2}\sum_n n^{\nu-7/2}J(\nu)\right\},$$

$$J(\nu) = \int_0^{+\infty} du\, e^{-\frac{2u^2}{\sigma^2}} u^{-\nu+3/2} K_{\nu-\frac{7}{2}}(2nLu).$$

INTERLUDE – A few remarks

When working in a **flat** spacetime background, **analytic continuation** often allows to get rid of the **divergences** usually appearing in the evaluation of the vacuum energy, thus straightforwardly leading to the **physical** result one is looking for.

This is just what happened when computing the Casimir energy density in **flat** spacetime.

However, in presence of a **time-dependent** background, such a mathematical tool is generally **not enough**, and **further physical considerations** are required in order to remove the emerging infinities.

CASIMIR EFFECT– Handling the divergences/1

$$\langle \delta\epsilon_{Cas} \rangle_1 = \frac{H^2}{32\pi\sigma^2 L} \lim_{\nu\to 0} \Re\mathrm{e}\left\{ i^{-\nu}\Gamma(\nu-3)\frac{\sigma^{-2\nu+5}}{2^{-\nu+7/2}}\Gamma\left(-\nu+\frac{5}{2}\right)\right\}$$

- Manifestly divergent, due to the $\Gamma(\nu - 3)$ pole.
- Such a term gives an **infinite contribution** to the Casimir energy,

$$E_{Cas} = AL\langle \delta\varepsilon_{Cas} \rangle$$

which is proportional to *A*, **without any reference to the plate separation** *L*.

- Following Schwinger's argument, such energy has to be **normalized to zero**.

[J. Schwinger, *Lett. Math. Phys.* **24**, 59 (1992)]

CASIMIR EFFECT– Handling the divergences/2

$$\langle \delta\epsilon_{Cas}\rangle_2 = -\frac{H^2}{16\pi\sigma^2}\lim_{\nu\to 0}\Re\mathrm{e}\left\{i^{-\nu}\frac{\sqrt{\pi}}{2}\Gamma\left(\nu-\frac{7}{2}\right)\frac{\sigma^{-2\nu+6}}{2^{-\nu+4}}\Gamma(-\nu+3)\right\}$$

- Uniform spatial density of vacuum energy, **independent of *L***.
- Being interested in vacuum energy **dependence on the plate separation**, we can **discard** this term, again absorbing it in the *W* counterterms.

CASIMIR EFFECT– Handling the divergences/3

$$\langle \delta\epsilon_{Cas}\rangle_3 = -\frac{H^2}{16\pi\sigma^2 L}\lim_{\nu\to 0}\Re\mathrm{e}\left\{ i^{-\nu} 2\pi^{\nu-3}\left(\frac{L}{\pi}\right)^{\nu-5/2}\sum_n n^{\nu-7/2} J(\nu)\right\},$$

$$J(\nu) = \int_0^{+\infty} du\, e^{-\frac{2u^2}{\sigma^2}} u^{-\nu+3/2} K_{\nu-\frac{7}{2}}(2nLu).$$

- By means of **analytic continuation** (again!) we perform the last integral in terms of Whittaker functions:

➢ Main result:

$$\langle \delta\epsilon_{Cas}\rangle \equiv \langle \delta\epsilon_{Cas}\rangle_3 = \frac{H^2}{2^{15/4}\pi\sigma^{1/2}L^{9/2}}\sum_{n=1}^{\infty} n^{-9/2} e^{\frac{(\sigma L n)^2}{4}} W_{-\frac{3}{4},-\frac{7}{4}}\left(\frac{(\sigma L n)^2}{2}\right).$$

CASIMIR EFFECT– The gravitational wave case/1

- As an **example**, consider the following spacetime metric

$$ds^2 = dt^2 - (1 + h_+(u))\, dx^2 - (1 - h_+(u))\, dy^2 - 2h_\times(u)dxdy - dz^2,$$

$$h_+(u) \equiv h(t-z) = He^{-\sigma^2(t-z)^2}$$

representing a **gravitational plane wave pulse**.

If $\sigma L \ll 1$, we may expand $h(t-z)$ around *z* = 0 (one of the plate locations). So:

$$h(t-z) \simeq h(t) = He^{-\sigma^2 t^2}$$

…just like in our previous Bianchi-I spacetime model!

CASIMIR EFFECT– The gravitational wave case/2

- Thanks to the rapid convergence of the sum, we expand in $\sigma L \ll 1$,

$$\langle \delta\epsilon_{Cas} \rangle = \frac{H^2}{2^{15/4}\pi\sigma^{1/2}L^{9/2}} \sum_{n=1}^{\infty} n^{-9/2} e^{\frac{(\sigma L n)^2}{4}} W_{-\frac{3}{4},-\frac{7}{4}}\left(\frac{(\sigma L n)^2}{2}\right).$$

- To the leading order in $\sigma L n$ we find

$$\langle \delta\epsilon_{Cas} \rangle \simeq \frac{15H^2}{64\sqrt{2\pi}\sigma^3 L^7}.$$

- Once the gravitational wave pulse is over ($t \to +\infty$), the total Casimir energy in the cavity is (in SI units):

$$\langle E_{Cas} \rangle = -\frac{A\hbar c\pi^2}{1440L^3}\left(1 - \frac{675c^3H^2}{2\sqrt{2}\pi^{5/2}\sigma^3 L^3}\right)$$

$$\sigma \simeq \frac{1}{\Delta t_{\rm pert}},$$

SOME REMARKS

$$\langle E_{Cas} \rangle = -\frac{A\hbar c\pi^2}{1440L^3}\left(1 - \frac{675c^3H^2}{2\sqrt{2}\pi^{5/2}\sigma^3L^3}\right)$$

- It might seem that a sufficiently **long** gravitational pulse could cause the **complete vanishing** of the Casimir energy, or even a **change** in its sign, turning the Casimir force in a **repulsive** one (!).
- Such an occurrence cannot be considered too seriously, since our calculations have been carried on following a **perturbative approach**.
- This requires that: $\frac{675c^3H^2}{2\sqrt{2}\pi^{5/2}\sigma^3L^3} \ll 1$ ⇒ $\Delta t_{\text{pert}} \ll H^{-\frac{2}{3}}L \;\; \text{ns.}$
- For example, with $L = 10^{-6}\ \text{m}$, and $H = 10^{-21}$ ⇒ $\Delta t_{\text{pert}} \ll 10^{-1}\ \text{s.}$

CONCLUSIONS/1

- A gravitational perturbation, leaving Minkowskian the s-t in the far future causes a **permanent shift** in the vacuum Casimir energy («memory» effect)
- Such a shift acts in order to **reduce** the absolute value of the (negative) Casimir energy
- Total vanishing or even sign change in the Casimir energy (and force) are probably **ruled out**, due to the followed perturbative approach
- **Reduction** of the absolute value of the Casimir energy could recall (or represent) a manifestation of the so-called **Quantum Energy Inequalities** (first pionereed by Ford) [see, e.g., L. H. Ford, M. J. Pfenning and T. A. Roman, Phys. Rev. D 57, 4839 (1998)]

CONCLUSIONS/2

- QEIs dictate bounds on the **duration** of negative energy states, hence almost preserving the **Weak Energy Conditions**, violated by Casimir effect.
- QEIs require that WEC violations are small or (as in our case) **short-lived**.
- The present approach can be straightforwardly extended to electromagnetic field, giving – as expected –an **extra factor of two**.
- Also the analysis can be carried on considering gravitational waves of **arbitrary direction** w.r.t. the Casimir cavity.
- The present technique applies also to **more general** spacetimes as, e.g., Bianchi-Type IX.

THANK YOU