GRAVITATIONAL MEMORY OF CASIMIR EFFECT

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MOTIVATION

 \triangleright Study of influence of gravity on vacuum energy

ØSearch for permanent effects («memory» effects)

 \triangleright Relationship with the cosmological constant problem

ØUse of Schwinger's approach to clearly disentangle particle creation effects from polarization effects

 \triangleright Handling of divergences by means of analytic continuation

OUTLINE

Ø*GRAVITATIONAL BACKGROUND AND CASIMIR CAVITY*

Ø*PROPER-TIME SCHWINGER'S APPROACH*

Ø*STATIC CASIMIR EFFECT*

v*Flat background (a consistence check)*

v*Bianchi Type-I background*

ØHANDLING THE DIVERGENCES

ØTHE GRAVITATIONAL WAVE CASE

EXAMPLE SCONCLUSIONS

GAVITATIONAL BACKGROUND sent the directional scale factors along the axes *{x, y, z}* $T_{\rm eff}$ we solve the Klein-Gordon equation for a mass-form \sim namely the simplest generalization of the simplest generalization of the homo--------------------------------geneous spatially flat Friedmann-Lema_če spatially flat Friedmann-Lema_če spatially flat Friedmann-Lema
Lemački

3

Starting point: Bianchi-I spacetime (the simplest generalization of FLRW universe) our ting point. Dianom is space the simplest generalization T_{max} solve the Klein-Gordon equation for a massanization of ferry aniverse_f small (proper) distance *L*. In section III we follow the (the simplest generalization of FLRV hiverse) the conditions of the conditions \mathbf{r}_i lim
Lim *t*!*±*1 *h*_i *h*_i *b*_{*i*} *h*_i *b*_{*i*} *i s*_i *x*_i *s*_i *x*_i *s*^{*n*} *a*^{*n*} *s*^{*n*} *b*^{*n*} *s*^{*n*} *a*^{*n*} *s*^{*n*} *b*^{*n*} *s*^{*n*} *a*^{*n*} *s*^{*n*} *a*^{*n*} *a*^{*n*} *a*^{*n*} *a*^{*n*} *a*^{*n*} *a*^{*n*} \blacksquare Starting point: Bianchi-Lspacetime ing supplest generalization of Ferry Gilly

 $ds^2 = dt^2 - \sum$ 3 *i*=1 $a_i^2(t)dx_i^2$ *ⁱ ,* (1) $\begin{array}{ccc} \text{inchi-I} & \text{onacetime} \end{array}$ on of FLRW universe) $ds^2 = dt^2 - \sum a_i^2(t) dx_i^2$ *i*=1 $\frac{1}{2}$ **h**
directional scale factors

i=1

the plates, each of (proper) area *A*, are orthogonal to

i=1

Assume *small* anisotropies, a permanent change in the directional scale factors directional scale factors ρ entries the constraints: satisfying the constraints: $as = ai$ less scalar field, minimally coupled to the gravitational field and confidential and confidential and confident continuously concerned to a Casimir continuously represents a Casimir care and continuously represent to a Casimir care and continuously represents a continuously repre ds^2 check our computations, finding the Casimir energy den-Assume *small* anisotropies, $\frac{1}{d}$ $\frac{2}{d}$ Assume *small* anisotropies,

[1 + *hi*(*t*)](*dxⁱ*

$$
\text{directional scale factors}
$$

Assume **shiqu antisotropies,**

\n
$$
ds^{2} = dt^{2} - \sum_{i=1}^{n} [1 + h_{i}(t)] (dx^{i})^{2}
$$
\nantisfying the constraints:

\n
$$
\sum_{i=1}^{3} h_{i}(t) = 0,
$$

\n
$$
\max |h_{i}(t)| \ll 1
$$

$$
h_3(t) \equiv h_z(t) = 0,
$$

\n
$$
\frac{\lim_{t \to \pm \infty} h_i(t) = 0}{\frac{\lim_{t \to \pm \infty} h_i(t) = 0}{\frac{\lim_{t \to \pm \infty} h_i(t)}{\frac{\text{asymptotic Minkowskian}}{\text{spacetime regions}}}}
$$

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$$
h_3(t) \equiv h_z(t) = 0,
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h_3(t) \equiv h_z(t) = 0,
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$$
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$$

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$$
h_3(t) \equiv 0,
$$

(see, e.g., the *holy* text by Birrell & Davies) σ and the results of connection with the Westerlands with the Weak Energy \sim \sim 100 σ $\sqrt{(sec, e.g., the *nory* text by builter α Davies) \qquad \qquad \text{Of } \mathbf{i}$ (see, e.g., the *holy* text by Birrell & Davies) are of in**the** *z* axis, placed at \overline{a} \overline{b} and \overline{c} \overline{c} and \overline{c} \overline{c} and \overline{c} \overline{c} and \overline

3

 $\frac{3}{2}$ $\frac{1}{2}$

This appears as a sort of a *gravitational memory* of the

small (proper) distance *L*. In section III we follow the

p**n** so dia \overline{c} that spatial metric is
pronal and **traceless** *,* (2) of in- and out- vacua and the California cavity is oriented in the Case of the Casimir cavity is oriented in the Casimir **haated bead areas for Bianchi-I type** $\left($ Recall that for Bianchi-I type unambigous definition $\begin{vmatrix} 3 & 38 \\ 3 & 1 \end{vmatrix}$ diagonal and traceless

CASIMIR CAVITY \overline{X} *i*=1 *hi*(*t*)=0*,* (5) the spatial metric is diagonal metric in $\mathsf{CASIMIR}$ CAVITY. sume that the Casimir cavity is oriented in space so that

The requirement $h_3(t) \equiv h_z(t) = 0$ badiances that proper dud socialistic distance between the plates confered at any time. guarantees that proper and coordinate distance between the plates coincide at any time. constraint (6) guarantees that the proper and the coor-

sume that the Casimir cavity is oriented in space so that this anows us to avoid possible complications at *ising* from *tiddi* effects. This allows us to avoid possible complications arising from *tidal* effects. This allows us to avoid possible complications time. This will allows us to avoid possible complications dusing from *tidal* enects.

For sake of simplicity, assume a **massless scalar field**, minimally coupled to the gravitational background solutions For sake of simplicity assume a massless scalar fiel cussed in section of the constant of the massless scalar mondern constant of filliming coupied to the gravitational back

$$
\frac{1}{\sqrt{-g}} \partial_{\mu} \left[\sqrt{-g} g^{\mu \nu} \partial_{\nu} \phi(x) \right] + \xi R(x) \phi(x) = 0
$$
 L = propel
A = propel

L = proper plate separation

A = proper area of the plates

CASIMIR CAVITY – FIELD MODES/1 We are interested in a time-dependent background CASIMIR CAVITY – FIELD MODES responding quantity has to be regarded as an *operator*) with *h*(*t*) ⌘ *hx*(*t*) = *hy*(*t*), [see (5) and (6)]. with *h*(*t*) ⌘ *hx*(*t*) = *hy*(*t*), [see (5) and (6)]. CASIMIK CAVITY – FIELD MODES/I $\overline{\mathbf{10}}$

Assume Dirichlet b.c.

(1) is broken by the field confinement as an *i*s direction and *is a* direction and Assume Di

To the lowest order, KG equation reads To the low **h** a find the lowest order, KG equation reads and (\sim To the lowest order, KG equation reads \Box $(\Box + V) \phi$

$$
(\Box + \hat{V})\phi = 0,
$$

$$
\hat{V} = h_x(t)\partial_x^2 + h_y(x)\partial_y^2 = h(t)(\partial_x^2 - \partial_y^2)
$$
 and

$$
h(t) \equiv h_x(t) = -h_y(t)
$$
 metric perturbation

= *h*(*t*)(@² (1) opatial translation invariance of bianchier spacetime is broken tuininenielit. Tiowever, it is still assured along **x** and y dired
We guess: \cdot field Spatial translation invariance of Bianchi-I spacetime is broken by field confinement. However, it is still assured along x and y directions. We guess: ⌘(*t*)*,* (10) ⇣ ² ⌘¹*/*² (2⇡)3*L* is an overall normalization con-Using (10) in (8) yields, to the lowest order in *h* \therefore *X* \therefore \therefore

 $h(t)$

 $\overline{}$ and $\overline{}$

$$
\begin{aligned}\n\text{We guess.} \\
\boxed{\phi(x) = N e^{i\vec{p}_{\perp} \cdot \vec{x}_{\perp}} \sin\left(\frac{n\pi z}{L}\right) \eta(t)} \\
\text{requiring} \quad \boxed{\eta(t) \to e^{-i\omega t}, \qquad h(t) \to 0.} \\
\end{aligned}
$$

⌘¹*/*²

$$
\eta(t) \qquad N = \left(\frac{2}{(2\pi)^3 L}\right)^{1/2}
$$

$$
\vec{p}_{\perp} = (p_x, p_y), \ \vec{x}_{\perp} = (x, y)
$$

tional perturbation, the spacetime is Minkowskian and

(10) reduces to the usual mode solution inside the cavity,

where *N* = g $\begin{bmatrix} \eta \end{bmatrix}$ requiring

CASIMIR CAVITY – FIELD MODES/2 where tan ✓ = *py/px*, and !² = *p*² CASIMIR CAVITY – FIELD MOD 2 CAVITY - FIE *h*(*t* 0)*p*² ? cos 2✓*ei*!*^t* 0 ⌘(*t*) = *ei*!*^t* + = ↵*p*(*t*)*ei*!*^t* + *p*(*t*)*eⁱ*!*^t* CAVITY - FI \underline{F} **CASIMIR CAVITY — E** (10) reduces to the usual model with C

From KG equation we obtain riverent in de la distribution de
La distribution de la distribution
 namely

p(*t*) = *ⁱ*

$$
\eta(t) = e^{-i\omega t} + \int_{-\infty}^{t} dt' \frac{\sin(\omega(t - t'))}{\omega} h(t') p_{\perp}^{2} \cos 2\theta e^{-i\omega t'} = \alpha_{p}(t) e^{-i\omega t} + \beta_{p}(t) e^{i\omega t},
$$

$$
\alpha_{p}(t) = 1 + \frac{i}{2\omega} \int_{-\infty}^{t} dt' h(t') p_{\perp}^{2} \cos 2\theta,
$$

 Bogolubov coefficients

$$
\beta_{p}(t) = -\frac{i}{2\omega} \int_{-\infty}^{t} dt' h(t') p_{\perp}^{2} \cos 2\theta e^{-2i\omega t'},
$$

$$
\tan \theta = p_{y}/p_{x}, \text{ and } \omega^{2} = p_{\perp}^{2} + (n\pi/L)^{2}.
$$

$$
\phi(x) = N\left(\alpha_p(t)e^{-i\omega t} + \beta_p(t)e^{i\omega t}\right)e^{i\vec{p}_\perp \cdot \vec{x}_\perp} \sin\left(\frac{n\pi z}{L}\right)
$$

PROPER-TIME SCHWINGER'S APPROACH \overline{O} $\overline{O$ *p***8000 PROPER-TIME SCHWINGER'S APPROATLE** *^eip*~?*·*~*x*? sin ⇣*n*⇡*^z L* Ì, \bigcap **COLO TIME SCHMUNICED'S ADDPOAC** (17) Let us start writing the e↵ective action *W*

According to Schwinger, the *effective action W* reads

$$
W = \lim_{\nu \to 0} W(\nu)
$$
total Trace
Proper-time Hamiltonian

$$
W(\nu) = -\frac{i}{2} \int_0^\infty ds \, s^{\nu - 1} \text{Tr } e^{-is\hat{H}} + \text{c.t.}
$$

.
.

particle creation

In presence of a **time-dependent** spacetime background, the effective action W can become *complex,* being related to the vacuum **persistence amplitude** in the so-called *in-out* formalism

and the end of called *in-out* formalism ecients, connecting the *in-* and *out-* vacua, satisfying In presence of a time-dependen mproduction. A anti- appointent operation. Being carried the south the so-called the so-called **in-out formalism** of a *time-dependent* spacetime background, the effecti W can become *complex*, being related to the vacuum persistence tions. The additions is interesting to the vocuum parcictance of original terms in the recovering terms of

$$
\langle 0 \text{ out} | 0 \text{ in} \rangle = e^{iW}.
$$

dditional counterterm is **EU priyonder normalization.** *L* NB: the additional *counterterm* is introduced to subtract divergent terms, hence recovering the required physical normalization. ? cos 2✓*e*2*i*!*^t* $\langle 0.001 | 0.11 \rangle = e$
Counterterm is introduced t *h*^{*p*2}, (24)^{*p*}_{*p*2} (24)^{*p*}_{*p*2} (24)^{*p*}_{*p*}² (24)^{*p*}

TRACE EVALUATION TRACE EVALUAT 4

The total Trace
$$
\text{Tr}e^{-is\hat{H}} = \sum f d^4x \langle x|e^{-is\hat{H}}|x\rangle
$$

In this section we will follow Schwinger's proper time approach $\overline{3}$ o be evaluated all over the continuous as well the uiscrete edom, including those of spacetime. has to be evaluated all over the **continuous** as well the **discrete** degrees of freedom, including those of spacetime. and the evaluated all over the continuous as well the tids to be evaluated all over the continuous as well the ater .,
. divergent terms, hence recovering the required physical phys has to be evaluated all over the **continuous** as well the **discrete** degrees of freedom including those of spacetime of freedom, including those of spacetime. *^d*↵ ^h*x|*↵ih↵*|eis*(*H*ˆ0+*V*^ˆ) *|x*i*,* **MOUS** US WEN THE UISTINE UP IN EXT crete set, and the Dirac delta function $\mathcal{O}(\mathcal{A})$ s to be evaluated all over the **continuous** as well (↵*,* ↵⁰ α is the Kronecker symbol in the Committee of the Kronecker symbol is the Kronecker symbol is a discrepation of α rreedom, including those of spacetime.

The **p-t Hamiltonian** reads Thon + Hamiltonian **The p-t F** \mathbf{H} **de** *p***-t Hamiltonian** reads
 reads |↵0 continuous). Replacing (29) in (26) and using *|*h*x|*↵i*|*

$$
\hat{H} = \hat{H}_0 + \hat{V} = -\hat{p}_0^2 + \hat{p}_\perp^2 + \left(\frac{n\pi}{L}\right)^2 - h(t)p_\perp^2 \cos 2\theta, \qquad \hat{p}_0 = i\partial_t, \ \hat{p}_\perp = -i\vec{\nabla}_\perp
$$
\nWe get

\n
$$
\boxed{\text{NB: the rapidly oscillating term can be removed: R.W.A.}}
$$
\n
$$
\boxed{\text{Tr } e^{-is\hat{H}} = N^2 \int d^4x \int d^2p_\perp d\omega \sum_n \left(|\alpha_p(t)|^2 + |\beta_p(t)|^2 + 2\Re\left(\alpha_p(t)\right)p(t)\right)^* e^{-2i\omega t}\right)}
$$
\n
$$
\times \sin^2\left(\frac{n\pi z}{L}\right) e^{-isp_\perp^2} e^{-is(n\pi/L)^2} e^{is\omega^2},
$$

CASIMIR EFFECT vacuum energy. As we will see the will see, however, the third see, \sim C ACIMID EEEECT

Following Schwinger, let us write the **vacuum Casimir energy density** as case. owing Schwinger, let us write the <mark>vacuum Casim</mark>i *W* = lim *W*(⌫)*,* (22) **possimir energy density as an interaction**. \mathcal{L} namely reasoning might lead to expect no shift in the sh

$$
\langle \epsilon_{Cas} \rangle = -\frac{1}{AL} \lim_{\nu \to 0} \left[\lim_{t \to +\infty} \frac{\partial}{\partial t} \Re \epsilon W(\nu) \right].
$$

Using $W(\nu) = -\frac{i}{2} \int_0^\infty ds \, s^{\nu - 1} \text{Tr} \, e^{-is\hat{H}}$

Z ⁺¹ $\mathop{\text{find}}$ we find

$$
\langle \epsilon_{Cas} \rangle = \frac{1}{2(2\pi)^3 L} \lim_{\nu \to 0} \Re e \bigg\{ i \int_0^{+\infty} ds \, s^{\nu-1} \bigg[\int_0^{2\pi} d\theta \int_0^{+\infty} p_\perp \, dp_\perp \int_{-\infty}^{+\infty} d\omega \sum_n \left(1 + 2|\beta_p|^2\right) e^{-isp_\perp^2} e^{-is(n\pi/L)^2} e^{is\omega^2} \bigg] \bigg\}
$$

⌘2

⇣*n*⇡

CASIMIR EFFECT- flat background (a check) 1 *n*

- \triangleright As a consistence check, consider the flat spacetime case, $\beta_p = 0$. \overline{v} a consistence cheek, consider the natispacement
- \triangleright Integrations in square brackets are readily performed and we find where use has also been made of the relation *|*↵*p|* ² = 1, and *^p* = lim*^t*!+¹ *p*(*t*) is given by (20). where use has also been made of the relation *|*↵*p|* ² = 1, and *^p* = lim*^t*!+¹ *p*(*t*) is given by (20). **brackets are readily performed a**

$$
\langle \epsilon_{Cas} \rangle_0 = \lim_{\nu \to 0} \Re \{ \frac{\sqrt{i}}{16\pi^{3/2}L} \sum_n \int_0^{+\infty} ds \, s^{\nu - \frac{3}{2} - 1} e^{-is \left(\frac{n\pi}{L} \right)^2} \}.
$$

 \triangleright The remaining integral can be converted into a **Gamma function** and the infinite sum yields a Riemann zeta-function function and $\int (1)^{-\nu} (1)^{2\nu-3}$

$$
\langle \epsilon_{Cas} \rangle_0 = \lim_{\nu \to 0} \Re e \left\{ \frac{-(i)^{-\nu}}{16\pi^{3/2}L} \left(\frac{\pi}{L} \right)^{2\nu - 3} \zeta(2\nu - 3) \Gamma(\nu - 3/2) \right\}
$$

 \angle Pe ming **analyti** ly ul $\overline{}$ <u>15 Inuati</u> *L* \triangleright Performing analytic continuation $(\nu \to 0)$ we find the well-known results: \triangleright Perf $v \to 0$ ($10\pi^{0/2}L \setminus L$)
Performing analytic continuation $(v \to 0)$ we find the well-known rest \mathcal{L} (per unit surface), obtaining \mathcal{L} F₍₁₎ C₁_C WC₁^E_N $\overline{}$

$$
\langle \epsilon_{Cas} \rangle_0 = -\frac{\pi^2}{1440L^4}
$$

Casimir energy density

$$
f_{Cas}^{(0)} = -\frac{1}{A} \frac{\partial A L \langle \epsilon_{Cas} \rangle_0}{\partial L} = -\frac{\pi^2}{480L^4}.
$$

Attactive force per unit surface

.
t รเ erra
Attra *A* @*AL*h✏*Cas*i⁰ @*L* extending to the casimir energy density and the casimir energy density and the casimir energy density and the casimir surface $\frac{1}{2}$ **Face**

CASIMIR EFFECT– Bianchi-I s-t background/1 recalled - we do not a flat control of the creation in a flat creation in a flat control of the creation in a f spacetime background. Let us now consider the case *^p* 6= 0, corresponding to $\text{h}-\text{k}$ and k **SIMIR EFFECT – BIANCHI-I S-T DACKGrOUNG/ 1** *h*(*t*) = *He* ²*t*² nchi-l s-t hackg *I*(⌫) is potentially plagued by singularities. However, in $\mathsf{und}/\mathbf 1$ λ ICHI-I S-L DACK BIOUTIC

The **correction** to the flat Casimir result now reads: *independent* degrees of freedom. Before we go on, we need an explicit expression for the quantity *p*. For computa-The correction to the flat Casimir result now reads: The correction to the flat Casimir result now reads: *p* \mathbf{a} = \mathbf{b}

$$
\langle \delta \epsilon_{Cas} \rangle = \frac{1}{2(2\pi)^3 L} \lim_{\nu \to 0} \Re \left\{ i \int_0^{+\infty} ds \, s^{\nu-1} \left[\int_0^{2\pi} d\theta \int_0^{+\infty} p_\perp \, dp_\perp \int_{-\infty}^{+\infty} 2 \left[\beta_p \right] e^{-is p_\perp^2} e^{-is(n\pi/L)^2} e^{is\omega^2} \right] \right\}.
$$
\nSuppose a metric perturbation

\n
$$
h(t) = He^{-\sigma^2 t^2},
$$
\nwhere θ is the equation of the gravitational potential of the gravitational perturbation

\nIntegrations over variables s , θ and p_\perp

\n
$$
h(t) = \frac{H \sqrt{\pi}}{2\omega \sigma} p_\perp^2 e^{-(\omega/\sigma)^2} \cos 2\theta.
$$
\nIntegrations over variables s , θ and p_\perp

\n
$$
\int_0^{+\infty} \frac{q^\mu dq}{(q^2 + C^2)^\nu} = \frac{\Gamma(\frac{1+\mu}{2}) \Gamma(\nu - \frac{1+\mu}{2})}{2 \Gamma(\nu) C^{2\nu-1-\mu}}.
$$

CASIMIR EFFECT- Bianchi-I s-t background/2 $d/2$ a/2

Ø **Correction** to Casimir energy density is \geq Corr 2(2⇡)³*L* lim
Limba a prre rtior L *ds s*⌫¹ \overline{a} $\overline{}$ \mathbf{S} *p*? *dp*? $\overline{ }$

$$
\langle \delta \epsilon_{Cas} \rangle = \frac{H^2}{16\pi \sigma^2 L} \lim_{\nu \to 0} \Re e \left\{ i^{1-\nu} \Gamma(\nu - 3) \left(\frac{L}{\pi} \right)^{2\nu - 6} \left(I(\nu) \right) \right\}
$$

\n
$$
I(\nu) = \int_0^{+\infty} \frac{d\omega}{\omega^2} e^{-2\omega^2/\sigma^2} \sum_n \frac{1}{\left(n^2 - \frac{\omega^2 L^2}{\pi^2} \right)^{\nu - 3}}.
$$

\nPerforming a change of variable $(\omega = -i \ u)$

*h***(***t***) =** *Her***torming a change of variable** Performing a change of variable ($\omega = -i u$) and a **Wick rotation**, we convert $I(\nu)$ into

$$
\tilde{I}(\nu) = i \int_0^{+\infty} \frac{du}{(u+\epsilon)^2} e^{-\frac{2(u+\epsilon)^2}{\sigma^2}} \left[\sum_n \frac{1}{\left(n^2 + \frac{(u+\epsilon)^2 L^2}{\pi^2}\right)^{\nu-3}}, \right]
$$
\n
$$
NR: \epsilon \to 0 \text{ at the end of calculations}
$$

 $NB: \epsilon \rightarrow 0$ at the end of calculations where we do not the end of calculations

 $\frac{1}{\sqrt{2}}$ 2⇡*^s* 1 **Epstein-Hurwitz zeta-function!**

$$
\zeta_{EH}(s,q^2) = \sum_{n=1}^{\infty} (n^2 + q^2)^{-s}
$$

CASIMIR EFFECT- Bianchi-I s-t background/3 *ries and Products*, (Elsevier Academic Press, Burlington, which can continued to give form to find that *I*(⌫) is made of three contributions, stemming FFECT – Bianchi-I s-t background/3 \mathbf{t} the Casimir energy density (43) can be \mathbf{t} F FECT-Bianch 1 *ⁿ^s*1*/*²*K^s*1*/*2(2⇡*nq*)*,* (49)

> Epstein-Hurwitz zeta-function can be analytically continued (E. Elizalde, *J. Math. Phys.* **31**, 170 (1990))

$$
\zeta_{EH}(s,q^2) = \sum_{n=1}^{\infty} (n^2 + q^2)^{-s} = -\frac{q^{2s}}{2} + \frac{\sqrt{\pi}}{2\Gamma(s)} \Gamma(s-1/2) q^{-2s+1} + \frac{2\pi^s}{\Gamma(s)} q^{1/2-s} \sum_{n=1}^{\infty} n^{s-1/2} K_{s-1/2}(2\pi n q)
$$

Modified Bessel function

> Hence, Casimir energy can be written as the sum of three contributions $^{\circ}$ + (*u*₂ → $^{\circ}$ (48) $\ddot{}$ where, casifility **behavior ≻** Hence, Casimir energy can be written as the sum of three contributions of (49) allows to explicitly perform the *u*integration in

 \overline{a}

$$
\left(\langle \delta \epsilon_{Cas} \rangle = \langle \delta \epsilon_{Cas} \rangle_1 + \langle \delta \epsilon_{Cas} \rangle_2 + \langle \delta \epsilon_{Cas} \rangle_3 \right)
$$

also recognize that the infinite sum in (48) represents a

CASIMIR EFFECT- Bianchi-I s-t background/3 Λ SIMIK EFFECT – BIANCIII-I S-L DACK<mark>g</mark> h✏*Cas*i = h✏*Cas*i¹ + h✏*Cas*i² + h✏*Cas*i3*,* (50) **h** 32⇡²*L* lim ال مايلياني.
مايل $\frac{1}{2}$ ⇢ i und/3

n=1 \triangleright Three contributions to Casimir energy density:

$$
\langle \delta \epsilon_{Cas} \rangle_1 = \frac{H^2}{32\pi\sigma^2 L} \lim_{\nu \to 0} \Re \{ i^{-\nu} \Gamma(\nu - 3) \frac{\sigma^{-2\nu + 5}}{2^{-\nu + 7/2}} \Gamma \left(-\nu + \frac{5}{2} \right) \},
$$

$$
\langle \delta \epsilon_{Cas} \rangle_2 = -\frac{H^2}{16\pi\sigma^2} \lim_{\nu \to 0} \Re \{ i^{-\nu} \frac{\sqrt{\pi}}{2} \Gamma \left(\nu - \frac{7}{2} \right) \frac{\sigma^{-2\nu + 6}}{2^{-\nu + 4}} \Gamma(-\nu + 3) \},
$$

$$
\langle \delta \epsilon_{Cas} \rangle_3 = -\frac{H^2}{16\pi\sigma^2 L} \lim_{\nu \to 0} \Re \{ i^{-\nu} 2\pi^{\nu - 3} \left(\frac{L}{\pi} \right)^{\nu - 5/2} \sum_n n^{\nu - 7/2} J(\nu) \},
$$

$$
J(\nu) = \int_0^{+\infty} du \, e^{-\frac{2u^2}{\sigma^2}} u^{-\nu + 3/2} K_{\nu - \frac{7}{2}}(2nLu).
$$

INTERLUDE – A few remarks

When working in a **flat** spacetime background, **analytic continuation** often allows to get rid of the **divergences** usually appearing in the evaluation of the vacuum energy, thus straightforwardly leading to the **physical** result one is looking for.

This is just what happened when computing the Casimir energy density in **flat** spacetime.

However, in presence of a **time-dependent** background, such a mathematical tool is generally **not enough**, and **further physical considerations** are required in order to remove the emerging infinities.

CASIMIR EFFECT– Handling the divergences/1 ⇣*EH*(*s, q*²) = ^X *n*=1 (*n*² + *q*²) where, in the limit ✏ ! 0

$$
\left\langle \delta \epsilon_{Cas} \rangle_1 = \frac{H^2}{32\pi\sigma^2 L} \lim_{\nu \to 0} \Re \left\{ i^{-\nu} \Gamma(\nu - 3) \frac{\sigma^{-2\nu + 5}}{2^{-\nu + 7/2}} \Gamma\left(-\nu + \frac{5}{2}\right) \right\} \right\}
$$

[25] L. Parker, *Phys. Rev. Lett.* 21, 562 (1968)

har ily divergent, due to the T
אמים rm gives an **infinite contr** v
L \rightharpoonup Manifestly divergent, due to the Γ(ν - 3) pole.

Curved Space (Cambridge University Press, Cambridge,

infinite c antri \mathbf{r} pole.
n to the → Manifestly divergent, due to the $\Gamma(\nu-3)$ pole.

→ Such a term gives an **infinite contribution** to the Casimir energy,

$$
E_{\text{Cas}} = AL \langle \delta \varepsilon_{\text{Cas}} \rangle
$$

16⇡²*L* which is proportional to *A*, without any reference to the plate separation *L*. [5] H. Casimir, *Proc. K. Ned. Akad. Wet.* 51 793 (1948) (201) , without any reference to the plate separation *L*.

 $\sum_{k=1}^{n}$ term in the additional constant appearing in $\sum_{k=1}^{n}$ $\sum_{k=1$ \mathcal{C}^1 , *D* \mathcal{C}^i . i ▶ Following Schwinger's argument, such energy has to be **normalized to zero**. [J. Schwinger, *Lett. Math. Phys.* **24**, 59 (1992)]

CASIMIR EFFECT- Handling the divergences/2 *H*² 32⇡²*L* lim \blacksquare ⇢ ndling the d <u>the al</u> \overline{z} arge 5 2 $\frac{\text{ccs}}{2}$

$$
\langle \delta \epsilon_{Cas} \rangle_2 = -\frac{H^2}{16\pi\sigma^2} \lim_{\nu \to 0} \Re \{ i^{-\nu} \frac{\sqrt{\pi}}{2} \Gamma \left(\nu - \frac{7}{2} \right) \frac{\sigma^{-2\nu+6}}{2^{-\nu+4}} \Gamma(-\nu+3) \}
$$

- **n**
Therm spatial den <u>l</u> densit Ω \overline{O} acuum energ $\frac{1}{2}$ *<u><i><u>dent</u>* of *l*</u> Ø Uniform spatial density of vacuum energy, **independent of** *L*.
- a term in the additional constant appearing in (23) that appearing in (23) that appearing in (23) that appearing in (23) that appearing in (23) Ø Being interested in vacuum energy **dependence on the plate separation**, we can **discard** this term, again absorbing it in the *W* counterterms.

CASIMIR EFFECT– Handling the divergences/3 ^h✏*Cas*i² ⁼ *^H*² $ITCCT$ \mathbb{R}^2 2 a
Alian 1975
Alian 1975 2 ²⌫+4 (⌫ + 3) ^h✏*Cas*i³ ⁼ *^H*² 1₁₆ $\overline{}$

$$
\langle \delta \epsilon_{Cas} \rangle_3 = -\frac{H^2}{16\pi\sigma^2 L} \lim_{\nu \to 0} \Re \{ i^{-\nu} 2\pi^{\nu-3} \left(\frac{L}{\pi} \right)^{\nu-5/2} \sum_n n^{\nu-7/2} J(\nu) \},
$$

$$
J(\nu) = \int_0^{+\infty} du \, e^{-\frac{2u^2}{\sigma^2}} u^{-\nu+3/2} K_{\nu-\frac{7}{2}}(2nLu).
$$

 $\sum_{k=1}^{\infty} p_k$ means of analytic continuation (again) we next any the last integral i n ter erms of Whittaker functions **6** agam:, w The second contribution (52) represents a uniform space \mathcal{L}_c tial density of vacuum energy, independent of *L*. Since we in terms of Whittaker functions: which are not all the manner \sim **≻** By means of **analytic continuation** (again!) we perform the last integral $rac{1}{\sqrt{2}}$

V. HANDLING THE DIVERGENCES are interested in vacuum energy dependence on the plate on
The plate on the pl separation, we can discard the can discard this term, and the can discard this term, and the can discard it is N when working in a flat spacetime background, and N **P** of the divergence of the divergences of the divergences of the divergences of the divergences of the divergence of the divergence of the divergence of the divergences of the divergences of the divergence of the diverge ^h✏*Cas*i⌘h✏*Cas*i³ ⁼ *^H*² lim *{*⌫*,*⌫0*}*!0 sult:

$$
\left\langle \delta \epsilon_{Cas} \right\rangle \equiv \left\langle \delta \epsilon_{Cas} \right\rangle_3 = \frac{H^2}{2^{15/4} \pi \sigma^{1/2} L^{9/2}} \sum_{n=1}^{\infty} n^{-9/2} e^{\frac{(\sigma L n)^2}{4}} W_{-\frac{3}{4}, -\frac{7}{4}} \left(\frac{(\sigma L n)^2}{2} \right).
$$

ever, in presence of a time-dependent background, such a time-dependent background, such a time-dependent back
Contract background, such a time-dependent background, such a time-dependent background, such a time-dependent

*H*²

(53)

(*Ln*)2

✓(*Ln*)²

◆⌫5*/*²

◆

X 1
1910 - Johann Barnett, amerikansk mange
1910 - Johann Barnett, amerikansk mange

ing slightly modified form of the integral appearing in

CASIMIR EFFECT- The gravitational wave case/1 communication as a short perturbation wave, and the short perturbation as a short perturbation as a short per $\cos \theta / 1$ casc/ I coming as a short period as a short pe bation propagating along the *z* direction. In such a case **MIR EFFECT- The gravitational wave case** polarization of the wave and *u* = *t z*. Let us assume, with *h*+(*u*) and *h*⇥(*u*) being the two physical states of CASIMIR EFFECT— The gravitational wave case/1 polarization of the wave and *u* = *t z*. Let us assume, f_{max} (ASIMIR EFFECT – The gra 675*c*³*H*² 2 ^p2⇡⁵*/*²³*L*³ ⌧ ¹*,* (63)

Ø As an **example**, consider the following spacetime metric bation propagating along the *z* direction. In such a case *≽* As an example, consider the following spacetime metric for the sake of simplicity, that the sake of simplicity, that the wave has the wave has the wave has the form
The form of the form of th \triangleright As an **example**, consider the follow general case of a gravitational pulse, propagating at an anti-pulse, propagating at an anti-pulse, propagating
The case of an anti-pulse, propagating at an anti-pulse, propagating at an anti-pulse, propagating at an antipaceume metric \triangleright As an exam e, f_0 *J* α *¹/2²*

$$
ds^{2} = dt^{2} - (1 + h_{+}(u)) dx^{2} - (1 - h_{+}(u)) dy^{2} - 2h_{\times}(u) dx dy - dz^{2},
$$

$$
h_{+}(u) \equiv h(t-z) = He^{-\sigma^{2}(t-z)^{2}}
$$

with $\frac{(\mathbf{u} \cdot \mathbf{v})^2}{\|\mathbf{u}\| \|\mathbf{v}\|}$ and $\frac{(\mathbf{u} \cdot \mathbf{v})^2}{\|\mathbf{u}\| \|\mathbf{v}\|}$ being the two physical states of \mathbf{u} representing a gravitational plane wave pulse. for the same of simplicity, that the same of simplicity, that the wave \mathbf{r} of a *linearly* polarized, short gaussian pulse (the more *strain*, while gives a rough estimate of the time duse paration is *L* \sim 106 m, and a gravitation is \sim 106 m \sim representing a gravitational plane wave paise. separation is *L* = 10⁶ m, and a gravitational wave pulse having a strain *H* \sim 1021, the above constraint would wo

be discussed in Appendix B), so that *h*⇥(*u*) = 0 and

 $f(t)$ that same of simplicity, the same has the formulation $f(t)$ \overline{u} \overline{u} \overline{u} \overline{u} and \overline{u} \overline{u} and \over $\mathsf{gcd}(\mathsf{OUT} | \mathsf{S}) \mathsf{.} \mathsf{SO}(2) \quad \mathsf{S} \quad \mathsf{O} \mathsf{.}$ ound $z = 0$ (one of the plate *strain*, while gives a rough estimate of the time du $r_{\rm L},$ we may expand $r_{\rm R}(t-z)$ around z = 0 (one of the plat If $\sigma L \ll 1$, we may expand $h(t-z)$ around $z = 0$ (one of the plate locations). So: having a strain *H* = 10²¹, the above constraint would

$$
h(t-z) \simeq h(t) = He^{-\sigma^2 t^2}
$$

$$
\bullet
$$

$$
\text{Bianchi-I spacetime model!}
$$

... just like in our previous
Bianchi-I spacetime model!

CASIMIR EFFECT- The gravitational wave case/2 *strain*, while gives a rough estimate of the time ducacinaid effect time background is **positively** The gravitational wave case/ $\sqrt{2}$ \mathcal{Q} is the Bianchi-I space-I spa Inspection of (57) shows that h✏*Cas*i, induced by the casimi time background is *positive*, while the Casimir energy *inside* the Casimir cavity. This implies that we can study $\tt{t-tcl - I}$ ine gravitational wave \tt{c} MIR FFFFCT-The gravitational w tains trace of the gravitational perturbation at *t* ! +1, \mathbf{E} e case/2

Example 3 × Thanks to the rapid of 1**pid co** $\frac{1}{2}$ *{*⌫*,*⌫0*}*!0 ce of the sun W \overline{d} *in* σ *l* \neq 1 \triangleright Thanks to the rapid convergence of the sum, we expand in $\sigma L \ll 1$, t is a section we extend in $\sigma I \ll 1$ \mathbf{E} the rapid convergence of the sum appearing conv \rightarrow ∣nank the ranid convergence of the sum We a Exploiting the rapid convergence of the sum appearing when the perturbation has left the perturbation has left the cavity. Furthermore, \mathcal{L}_{max} anks to the rapid convergence of the sum, we expa

$$
\langle \delta \epsilon_{Cas} \rangle \ = {H^2 \over 2^{15/4} \pi \sigma^{1/2} L^{9/2}} \sum_{n=1}^\infty n^{-9/2} e^{{(\sigma L n)^2 \over 4}} \ W_{-{3 \over 4},-{7 \over 4}} \left({(\sigma L n)^2 \over 2}\right).
$$

 \overline{C} e fi \triangleright To the leading order in σLn we find $\sum_{k=1}^{n}$ To the leading ards \sim TV the leading order \triangleright To the Ln we find $\frac{1}{\sqrt{\delta t}}$ the leading order in σLn we find

the leading order in
$$
\sigma Ln
$$
 we find
\n
$$
\langle \delta \epsilon_{Cas} \rangle \simeq \frac{15H^2}{64\sqrt{2\pi}\sigma^3L^7}.
$$

 $\frac{1}{\Delta t_{\rm pert}},$

of the gravitational pulse. Looking at (62) it might seem

Casimir energy in the cavity is (in SI units): around *z* = 0 (one of the plate locations), considering *real* part of the e↵ective action *W*, h✏*Cas*i represents a \blacktriangleright Once the gravitational wave pu $\frac{1}{2}$ $\frac{1}{2}$ *^h*(*^t ^z*) ' *^h*(*t*) = *He* \triangleright Once the gravitational wave pulse is over $(t \to +\infty)$, the total \mathcal{L}

 $\overline{}$. (57) can find application in the interesting case of interesting cases $\overline{}$

$$
\langle E_{Cas} \rangle = -\frac{A\hbar c\pi^2}{1440L^3} \left(1 - \frac{675c^3H^2}{2\sqrt{2}\pi^{5/2}\sigma^3L^3}\right) \quad \boxed{\sigma \simeq \frac{1}{\Delta t_{\rm pert}}},
$$

SOME REMARKS
SOME REMARKS pulse is over (at *t* ! +1), the total Casimir energy the complete vanishing of the Casimir energy, or even a *gµ*⌫ = ⌘*µ*⌫ + *hµ*⌫, and once the transverse traceless gauge has been employed, the spacetime line element reads of the spacetime line element reads of the spacetime line e carried on following a *perturbative* approach [see, e.g., the the smallness of the perturbation $\boldsymbol{\delta}$ in the perturbation $\boldsymbol{\delta}$ SONAE DENAADKS spectrum in the results can be respected to the results can be results can be results can be results can be re *,* (58)

$$
\langle E_{Cas} \rangle = -\frac{A\hbar c \pi^2}{1440L^3} \left(1 - \frac{675c^3H^2}{2\sqrt{2}\pi^{5/2}\sigma^3L^3}\right)
$$

- VI. THE GRAVITATIONAL WAVE CASE where we have considered to the construction of the constructi **te vanishing** of the Casimir energy, or even a chang
the time duration of the tim its sign, turning the Casimir force in a repulsive one (!). > It might seem that a sufficiently long gravitational pulse could cause the **complete vanishing** of the Casimir energy, or even a **change** in *gµ*⌫ = ⌘*µ*⌫ + *hµ*⌫, and once the transverse traceless gauge with *h* is in the two physical states of the two physical s For the same that a sufficiently long gravitational pulse could cause ^p2⇡⁵*/*²³*L*³ ⌧ ¹*,* (63) \rightarrow for the sake of simplicity, that the wave has the form iclently long gravitational puise could cat
of the Casimir energy, or even a change i polarization of the wave and *u* = *t z*. Let us assume, **definition** a *line* complete va enceral case of a gravitation of a gravitation of a gravitation of a gravitation of an anti-> It might seem that a sufficiently long gravitational pulse could cause Casimir force in a repul remight seem that a samolently tong gravitation and complete valuating of the easing criere even a change in
- > Such an occurrence cannot be considered too seriously, since our calculations have been carried on following a **perturbative approach**. change in its sign, turning the Casimir force in a *repulsive ,* (58) calculations have been carried on follow arbitrary direction and with both polarization states will be and will be and will be and will be a state of t ب
n following a perturb $\overline{}$ arbitrary direction and with both polarization states will rus sign, turning the casinin force in a reparation of the state of the state of the state of the state of the s
Such an occurrence cannot be considere be discussed in the so that in the set of the *calculations t*wing a perturba calculations have been carried on following a **perturbative approach**.

► This requires that:
$$
\frac{675c^3H^2}{2\sqrt{2}\pi^{5/2}\sigma^3L^3} \ll 1
$$
 $\Delta t_{\text{pert}} \ll H^{-\frac{2}{3}}L$ ns.
▶ For example, with $L = 10^{-6}$ m, and $H = 10^{-21}$ $\Delta t_{\text{pert}} \ll 10^{-1}$ s.

CONCLUSIONS/1

- \triangleright A gravitational perturbation, leaving Minkowskian the s-t in the far future causes a **permanent shift** in the vacuum Casimir energy («memory» effect)
- ØSuch a shift acts in order to **reduce** the absolute value of the (negative) Casimir energy
- ØTotal vanishing or even sign change in the Casimir energy (and force) are probably **ruled out**, due to the followed perturbative approach

Example 2 Reduction of the absolute value of the Casimir energy could recall (or represent) a manifestation of the so-called **Quantum Energy Inequalities** (first pionereed by Ford) [see, e.g., L. H. Ford, M. J. Pfenning and T. A. Roman, Phys. Rev. D 57, 4839 (1998)]

CONCLUSIONS/2

- ØQEIs dictate bounds on the **duration** of negative energy states, hence almost preserving the **Weak Energy Conditions**, violated by Casimir effect.
- Ø QEIs require that WEC violations are small or (as in our case) **shortlived**.
- ØThe present approach can be straightforwardly extended to electromagnetic field, giving – as expected –an **extra factor of two**.
- \triangleright Also the analysis can be carried on considering gravitational waves of **arbitrary direction** w.r.t. the Casimir cavity.
- ØThe present technique applies also to **more general** spacetimes as, e.g., Bianchi-Type IX.

THANK YOU