Adibatic Ground States in Non-smooth Spacetimes

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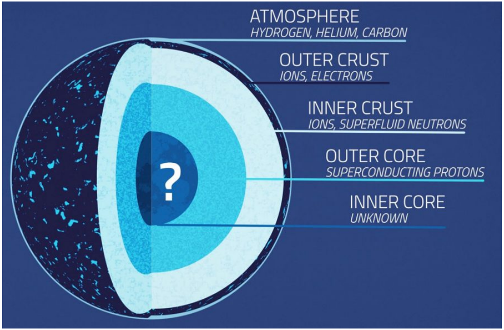
Why **non-smooth spacetimes?**



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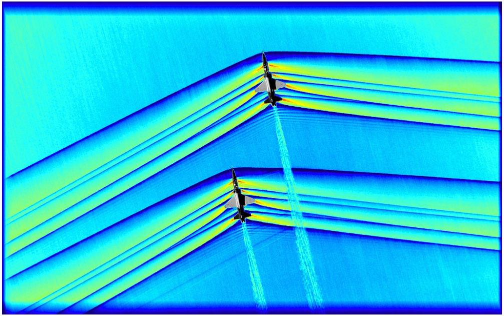
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[1] NASA's Goddard Space Flight Center / Conceptual Image Lab
[2] NASA

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THÉORÈME D'EXISTENCE POUR CERTAINS SYSTÈMES D'ÉQUATIONS AUX DÉRIVÉES PARTIELLES NON LINÉAIRES.

Par

Y. FOURÈS-BRUHAT.

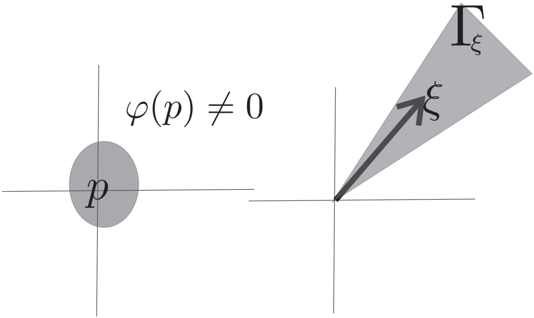
[3] George M. Bergman, Oberwolfach Photo Collection

$$G_{\mu\nu} = \frac{8\pi G}{c^4} \langle \hat{T}_{\mu\nu} \rangle_{\psi}$$

Sobolev Wavefront Set and Non-Smooth ΨDOs.

$\mathbf{H^s}$ – wavefront set :

$$(p,\xi) \notin WF^{s}(u) \iff \int_{\Gamma_\xi} (1+|\chi|^2)^{s} |\widehat{\varphi u}(\chi)|^2 d^n\chi < \infty$$



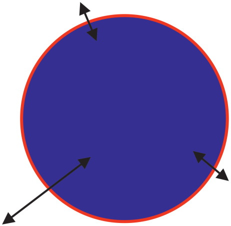
$$(p,\xi) \notin WF^{s}(u) \iff u \in H^{s}_{mcl}(p,\xi)$$

Properties:

- $WF^s(u) \subset T^*(M)\backslash\{0\}$
- $WF^s(u) = \emptyset \iff u \in H^s_{loc}$

Example: $WF^s(\chi_{D^1}) \subseteq N^*S^1$

$$\chi_{D^1} = \begin{cases} 1 & (x,y) \in D^1 := \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\} \\ 0 & \text{else} \end{cases}$$



Non-smooth symbols

A **symbol** $p(x,\xi) \in C^\tau S^m_{1,\delta}(\mathbb{R}^n \times \mathbb{R}^n)$ if and only if,

$$||D^\alpha_\xi p(\cdot,\xi)||_{C^\tau} \le C_\alpha \langle \xi \rangle^{m-|\alpha|+\tau\delta} \text{ for } |\alpha| \ge 0.$$

$$|f|_{C^{0,\beta}} = \sup_{x \neq y} \frac{|f(x)-f(y)|}{\|x-y\|^\beta}, \beta \in (0,1)$$

$$\|f\|_{C^{\tau=k+\beta}} = \|f\|_{C^k} + \max_{|\rho|=k} |D^\rho f|_{C^{0,\beta}}$$

Non-smooth ΨDO

The Ψ**DO op**(**p**) associated to $p(x,\xi) \in C^\tau S^m_{1,\delta}$ is

$$(op(p)u)(x) = \int_{\mathbb{R}^n} e^{ix\cdot\xi} p(x,\xi)\hat{u}(\xi) d\xi$$

Mapping properties:

$$op(p) : H^{s+m} \to H^s$$

for $-\tau(1-\delta) < s < \tau$.

$m > 0$ loses regularity; $m < 0$ gains regularity.

Symbol decomposition: $p(x,\xi) \in C^{\tau} S^{m}_{1,0}$

$$p(x,\xi) = \underbrace{p^{\#}(x,\xi)}_{\text{smooth}} \quad + \quad \underbrace{p^{b}(x,\xi)}_{\text{non-smooth, but lower order}}$$

$$p^{\#}(x,\xi) \in S^{m}_{1,\delta}$$

$$p^{b}(x,\xi) \in C^{\tau} S^{m-\tau\delta}_{1,\delta};\, \delta \in (0,1)$$

Example: The Klein-Gordon Operator

$p_{KG}(t,x,\xi_0,\xi) = (-{\xi_0}^2 + h^{ij}(x)\xi_i\xi_j) + i\frac{1}{\sqrt{h}}\partial_{x^i}(h^{ij}\sqrt{h}(x))\xi_j + m^2$

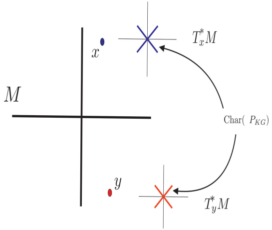
$h^{ij} \in C^\tau, p(t,x,\xi_0,\xi) \in C^{\tau-1}S^2_{1,0}$

$op(p_{KG}) = \partial_{tt} - \Delta_h + m^2 := P_{KG}$

Example: The Klein-Gordon Operator

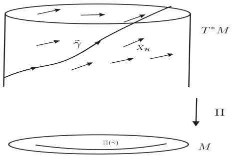
$$p_{KG}(t,x,\xi_0,\xi) = \underbrace{(-{\xi_0}^2 + h^{ij}(x)\xi_i\xi_j)}_{\mathcal{H}(t,x,\xi_0,\xi)} + i\frac{1}{\sqrt{h}}\partial_{x^i}(h^{ij}\sqrt{h}(x))\xi_j + m^2$$

$$\mathrm{Char}(P_{KG}) := \{(t,x,\xi_0,\xi) \in T^*M\backslash\{0\} : \mathcal{H}(t,x,\xi_0,\xi) = 0\}$$



$$X_{\mathcal{H}} = (\partial_{\xi_0}\mathcal{H}, \partial_{\xi}\mathcal{H}, -\partial_t\mathcal{H}, -\partial_x\mathcal{H})$$

A single flow line in $\mathrm{Char}(P_{KG})$ is called a **null bicharacteristic strip**.



The **bicharacteristic relation** C is defined as:

$$C = \Big\{(\tilde{x}, \tilde{\xi}, \tilde{y}, \tilde{\eta}) \in \mathrm{Char}(P_{KG}) \times \mathrm{Char}(P_{KG}),$$
$$(\tilde{x}, \tilde{\xi}) \text{ and } (\tilde{y}, \tilde{\eta}) \text{ lie on the same null bicharacteristic strip}\Big\}$$

where $\tilde{x} = (t, x), \tilde{\xi} = (\xi_0, \xi), \tilde{y} = (s, y), \tilde{\eta} = (\eta_0, \eta)$.

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$$(\tilde{x}, \tilde{\xi}, \tilde{y}, \tilde{\eta}) \in C \iff$$

$$\exists \text{ null geodesic } \gamma \text{ such that}$$

$$\tilde{\gamma}(a) = \tilde{x}, g(\cdot, \dot{\tilde{\gamma}})|_{T_{\tilde{x}}M} = \tilde{\xi}; \tilde{\gamma}(b) = \tilde{y}, g(\cdot, \dot{\tilde{\gamma}})|_{T_{\tilde{y}}M} = \tilde{\eta}$$

.

Propagation of singularities. (Taylor)
Let $P_{KG}(x,\xi) \in C^1 S^2_{1,0}$, γ a null bicharacteristic strip, V a neighbourhood of γ.
If $v \in D'(X)$ solves

$$P_{KG} v = f \iff P^{\#}_{KG} v = g := f - P^{b}_{KG} v$$

If $g \in H^{\sigma}_{mcl}(V)$, $\gamma(0) \in WF^{\sigma+1}(v)$ then $\gamma \in WF^{\sigma+1}(v)$.

Adiabatic Ground States

A quasifree state ω_N is an **adiabatic state** of order N if its two-point function Λ_{2N} satisfies for all $s \leq N + \frac{3}{2}$

$$WF'^s(\Lambda_{2N}) \subset C^+$$

$$C^+ = \left\{(\tilde{x}, \tilde{\xi}, \tilde{y}, \tilde{\eta}) \in C; \tilde{\xi}^0 \geq 0, \tilde{\eta}^0 \geq 0\right\},$$

where
$WF'(\Lambda_{2N}) := \{(\tilde{x}, \tilde{\xi}; \tilde{y}, -\tilde{\eta}) \in T^*(M \times M); (\tilde{x}, \tilde{\xi}; \tilde{y}, \tilde{\eta}) \in WF(\Lambda_{2N})\}$

Ground State

Ultrastatic setting: $M = \mathbb{R} \times \Sigma$, Σ compact

$ds^2 = dt^2 - h_{ij}(x)dx^i dx^j$

Klein-Gordon operator: $\partial_{tt} - \Delta_h + m^2$

The **ground state**, ω_g, is completely determined by its two-point function

$$\omega_g^{(2)}(t, x; s, y) = \sum_{l \in \mathbb{N}} \frac{e^{i\lambda_l(t-s)}\phi_l(x)\phi_l(y)}{\lambda_l}$$

The eigenvalues of $-\Delta_h \phi + m^2$ are $\{\lambda_j^2\}_{j\in\mathbb{N}}$ and the set of eigenvectors $\{\phi_l\}$.

Theorem

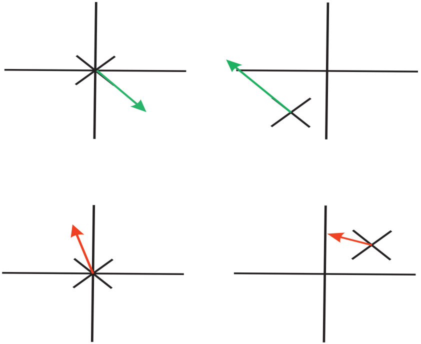
(Schrohe,- '22) Let (M, g) be a C^τ ultrastatic spacetime with $\tau > 2$, dim $M = 4$ and $\omega_g^{(2)}$ the two-point function of the ground state. Then $WF'^{-\frac{3}{2}+\tau-\tilde{\epsilon}}(\omega_g^{(2)}) \subset C^+$ for every $\tilde{\epsilon} > 0$

If the spacetime is smooth, then the ground states are adiabatic states of infinite order (Hadamard states).

Lemma

For any $\tilde{\epsilon} > 0$

$$WF^{-\frac{1}{2}-\tilde{\epsilon}+\tau}(\omega_G^{(2)}) \subset \mathrm{Char}(P_{KG}) \times \mathrm{Char}(P_{KG}).$$

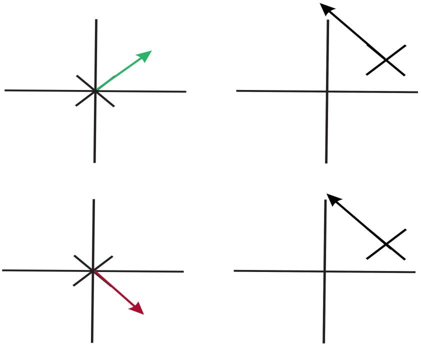


Idea: Consider $P_{KG}^{(t,x)}\omega^{(2)} = 0$, $op(\frac{1}{p_{KG}^{\#}})$, $\mathrm{Char}(p_{KG}^{\#}) = \mathrm{Char}(P_{KG})$

and mapping properties of p_{KG}^{b}

Lemma

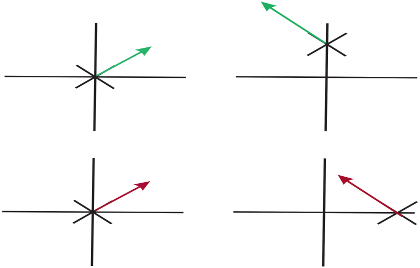
(Positivity) For all $s \in \mathbb{R}$,
$WF^s(\omega_g^{(2)}) \subset \{(\tilde{x}, \tilde{\xi}, \tilde{y}, \tilde{\eta}) \in T^*(M \times M); \tilde{\xi}^0 > 0\}$



We define $F : \mathbb{R} + i\,]0, \epsilon_1[\, \subset \mathbb{C} \to \mathcal{D}'(\Sigma \times M)$ for $\epsilon_1 > 0$ by
$F(z) := F(t, \epsilon) = \sum_j e^{i(t+i\epsilon)\lambda_j} e^{-is\lambda_j} \phi_j(x)\phi_j(y)$.
Then, F is holomorphic and $\lim_{\epsilon \to 0} F = \omega_g^{(2)}$.

Lemma

Let $(\tilde{x}, \tilde{y}) \in M \times M$ be such that $\tilde{x}$ and $\tilde{y}$ are not causally related, i.e. $\tilde{x} \notin J(\tilde{y})$. Then $(\tilde{x}, \tilde{\xi}, \tilde{y}, \tilde{\eta}) \notin WF^{-\frac{1}{2}-\epsilon+\tau}(\omega_g^{(2)})$.



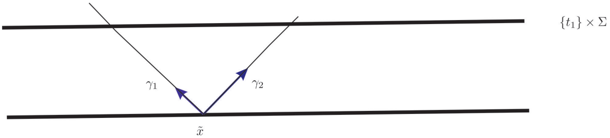
Idea: $\omega_G^{(2)}|_{\mathcal{Q}}$, $\mathcal{Q}$ the set of pairs of causally separated points, "flip" map $\rho(\tilde{x}, \tilde{y}) = (\tilde{y}, \tilde{x})$, $\rho^* \omega_G^{(2)}|_{\mathcal{Q}} = \omega_G^{(2)}|_{\mathcal{Q}}$, wavefront set of the pullback. $N_\pm := \{(t, x, \xi_0, \xi) \in \mathrm{Char}(P_{KG}); \pm\xi_0 > 0\}$

$$N_+ \times N_- \supset WF^{-\frac{1}{2}-\epsilon+\tau}(\omega_G^{(2)}|_{\mathcal{Q}}) = WF^{-\frac{1}{2}-\epsilon+\tau}(\rho^* \omega_G^{(2)}|_{\mathcal{Q}})$$
$$\subset \rho^* WF^{-\frac{1}{2}-\epsilon+\tau}(\omega_G^{(2)}|_{\mathcal{Q}}) \subset \rho^*(N_+ \times N_-) = N_- \times N_+$$

Lemma

If $(\tilde{x}, \tilde{\xi}, \tilde{x}, \tilde{\eta}) \in WF^{-\frac{3}{2}+\tau-\tilde{\epsilon}}(\omega_g^{(2)})$ *for* $\tilde{\epsilon} > 0$*. Then* $\tilde{\eta} = -\tilde{\xi}$*.*

Let $(\tilde{x}, \tilde{\xi}, \tilde{x}, \tilde{\eta}) \in WF^{-\frac{3}{2}-\epsilon+\tau}(\omega_G^{(2)})$ with $\tilde{\eta} \neq \lambda\tilde{\xi}$ then $(\gamma(\tilde{x}, \tilde{\xi}), \gamma(\tilde{x}, \tilde{\eta})) \in WF^{-\frac{3}{2}-\epsilon+\tau}(\omega_G^{(2)})$ $(P^b\omega_G^{(2)} \in H^{-\frac{5}{2}-\epsilon+\tau=\sigma})$.

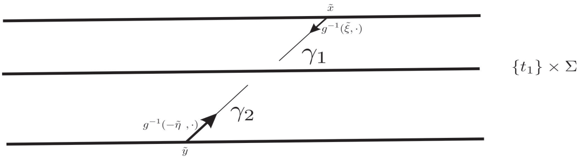


Exists $\{t_1\} \times \Sigma$ where $\Pi\gamma(\tilde{x}, \tilde{\xi}) = (t_1, w_1), \Pi\gamma(\tilde{x}, \tilde{\eta}) = (t_1, w_2)$ are causally separated !

Then $\tilde{\eta} = \lambda\tilde{\xi}, \lambda \in \mathbb{R}$.

From $(\partial_t + \partial_s)\omega_G^{(2)} = 0$ (stationary) we have $\lambda = -1$.

Proof of the Theorem: Let $(\tilde{x}, \tilde{\xi}, \tilde{y}, -\tilde{\eta}) \in WF^{-\frac{3}{2}-\epsilon+\tau}(\omega_G^{(2)})$ then $(\gamma(\tilde{x}, \tilde{\xi}), \gamma(\tilde{y}, -\tilde{\eta})) \in WF^{-\frac{3}{2}-\epsilon+\tau}(\omega_G^{(2)})$ and $(t_1, w_1, \chi, t_1, w_1, -\chi) \in WF^{-\frac{3}{2}-\epsilon+\tau}(\omega_G^{(2)})$



Define

$$\tilde{\gamma}(\lambda) = \begin{cases} \gamma_1(\lambda) := \Pi(\gamma(\tilde{x}, \tilde{\xi}))(\lambda) & \lambda \in (-\infty, t_1] \\ -\gamma_2(\lambda) := -\Pi(\gamma(\tilde{y}, -\tilde{\eta})(\lambda)) = \Pi(\gamma(\tilde{y}, \eta)(\lambda)) & \lambda \in (t_1, \infty) \end{cases}$$

where $-\gamma_2$ denotes the curve with opposite orientation.
This implies $(\tilde{x}, \tilde{\xi}, \tilde{y}, \tilde{\eta}) \in C$. Hence, $WF'^{-\frac{3}{2}-\epsilon+\tau}(\omega_G^{(2)}) \subset C$.
Combining this with positivity and $(\partial_t + \partial_s)\omega_G^{(2)} = 0$ we obtain

$$WF'^{-\frac{3}{2}-\epsilon+\tau}(\omega_G^{(2)}) \subset C^+. \blacksquare$$

Thank you.

Let $\{\psi_j; j = 0, 1, \ldots\}$ be a Littlewood-Paley partition of unity on $\mathbb{R}^n$, i.e., a partition of unity $1 = \sum_{j=0}^{\infty} \psi_j$, where $\psi_0 \equiv 1$ for $|\xi| \leq 1$ and $\psi_0 \equiv 0$ for $|\xi| \geq 2$ and $\psi_j(\xi) = \psi_0(2^j \xi) - \psi_0(2^{1-j}\xi)$. The support of ψ_j, $j \geq 1$, then lies in an annulus around the origin of interior radius 2^j and exterior radius 2^{1+j}.
Given $p(x, \xi) \in C^\tau S^m_{1,0}$ and $\gamma \in (0, 1)$ let

$$p^{\#}(x, \xi) = \sum_{j=0}^{\infty} J_{\epsilon_j} p(x, \xi) \psi_j(\xi). \tag{1}$$

Here J_ϵ is the smoothing operator given by $(J_\epsilon f)(x) = (\phi(\epsilon D) f)(x)$ with $\phi \in C_0^\infty(\mathbb{R}^n)$, $\phi(\xi) = 1$ for $|\xi| \leq 1$, and we take $\epsilon_j = 2^{-j\gamma}$.

Lemma

$WF^{-\frac{1}{2}-\epsilon}(\omega_g^{(2)}) = \emptyset$ *for every* $\epsilon > 0$.

For

$u(t,s,x,y) = \sum_{j,k} u_{jk}(t,s)\phi_j(x)\phi_k(y)$ with $u_{jk} = \langle u, \phi_j \otimes \phi_k \rangle$

$$u \in H^s(\mathbb{R}^2 \times \Sigma^2)$$
$$\iff \sum_{j,k} \int_{\mathbb{R}^2} (\xi_0^2 + \eta_0^2 + \lambda_j^2 + \lambda_k^2)^s |(\mathcal{F} u_{jk})(\xi_0, \eta_0)|^2 d\xi_0 d\eta_0 < \infty\}.$$

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$$\omega_g^{(2)}(t,x;s,y) = \sum_{l \in \mathbb{N}} \frac{e^{i\lambda_l(t-s)}\phi_l(x)\phi_l(y)}{\lambda_l}$$

$$\|\omega_g^{(2)}\|^2_{H_{loc}^{-\frac{1}{2}-\epsilon}(M\times M)} \leq \sum_l \frac{C}{\lambda_l^{3+\frac{2}{3}\epsilon}} \underbrace{\leq}_{\text{Weyl's law}} \sum_l \frac{C'}{l^{1+\epsilon}} < \infty$$