University of Genoa

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DIRAC HYDRODYNAMICS

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↓

Jakobi and Lochak (1955) find the way: polar form of the spinor in covariant form

Every (regular) spinor can always be written as

$$\psi=\phi e^{-\frac{i}{2}\beta\boldsymbol{\pi}}\boldsymbol{L}^{-1}\left(\begin{array}{c}1\\0\\1\\0\end{array}\right)$$

where phi is the module and beta is the chiral angle
while L is a spinorial transformaion: DEFORMATION

Covariant derivatives of spinors $\boldsymbol{\nabla}_\mu \psi = \partial_\mu \psi + \boldsymbol{C}_\mu \psi$

where

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In general $\boldsymbol{L}^{-1} \partial_\mu \boldsymbol{L} = \frac{1}{2} \partial_\mu \xi^{ab} \boldsymbol{\sigma}_{ab} + iq\partial_\mu \xi \mathbb{I}$

so we define

$$\partial_\mu \xi_{ij} - C_{ij\mu} \equiv R_{ij\mu}$$

$$q(\partial_\mu \xi - A_\mu) \equiv P_\mu$$

The parameters combine with spin connection and gauge potential to become longitudinal parts of R and P: real tensors

Then

$$\boldsymbol{\nabla}_\mu \psi = (-\tfrac{i}{2} \nabla_\mu \beta \boldsymbol{\pi} + \nabla_\mu \ln \phi \mathbb{I} - iP_\mu \mathbb{I} - \tfrac{1}{2} R_{ij\mu} \boldsymbol{\sigma}^{ij})\psi$$

Dirac equations (in general with torsion) $i\boldsymbol{\gamma}^{\mu}\boldsymbol{\nabla}_{\mu}\psi-XW_{\mu}\boldsymbol{\gamma}^{\mu}\boldsymbol{\pi}\psi-m\psi=0$

The last two after diagonalization become

$$\nabla_{\mu}\ln\phi^{2}+R_{\mu a}{}^{a}-2P^{\rho}u^{\nu}s^{\alpha}\varepsilon_{\mu\rho\nu\alpha}+2ms_{\mu}\sin\beta=0$$
$$\nabla_{\mu}\beta-2XW_{\mu}+\tfrac{1}{2}\varepsilon_{\mu\alpha\nu\iota}R^{\alpha\nu\iota}-2P^{\iota}u_{[\iota}s_{\mu]}+2ms_{\mu}\cos\beta=0$$

The 8 Dirac equations are converted into 2 vectorial equations specifying all space-time derivatives for both degrees of freedom

Relativistic spinning quantum mechanics with chiral angle, spin, density and velocity

Eur. Phys. J. C78, 783 (2018)

Advantages: 1 --- quantum mechanics as an incompressible fluid

We have the momentum

$$
\begin{aligned}
P^{\alpha} = & \, m \cos\beta u^{\alpha} + \\
& +(\nabla_{\mu}\beta/2 - XW_{\mu} + \tfrac{1}{4}\varepsilon_{\mu\rho\sigma\nu}R^{\rho\sigma\nu})u^{[\mu}s^{\alpha]} + \\
& +(\nabla_{\mu}\ln\phi + \tfrac{1}{2}R_{\mu\rho\sigma}g^{\rho\sigma})u_{j}s_{k}\varepsilon^{jk\mu\alpha}
\end{aligned}
$$

de Broglie-Bohm interpretation extended to the relativistic case: visual QM

Found. Phys. 53, 54 (2023)

Advantages: 2 --- Lie derivative

Vanishing of Lie derivative on the spinor or its observables?

$$\frac{1}{4}(\partial\xi)_{\mu\nu}s_{\tau}u_{\sigma}\varepsilon^{\mu\nu\tau\sigma}=2\xi^{\mu}(P_{\mu}-V_{\mu})$$

$$V_{\mu}=\frac{1}{4}R_{ij\mu}\varepsilon^{ijcd}u_{c}s_{d}$$

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General proof of no spherical symmetry for spinors

Advantages: 3 --- QFT spin-sums

$$\sum_{\mathrm{spin}} \psi\overline{\psi} \equiv (m\mathbb{I} + P_a \boldsymbol{\gamma}^a)$$

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and even more into $\psi\overline{\psi} = \frac{1}{2}\phi^2 e^{-i\beta\boldsymbol{\pi}}(e^{i\beta\boldsymbol{\pi}} + u_a \boldsymbol{\gamma}^a)(e^{-i\beta\boldsymbol{\pi}} - s_a \boldsymbol{\gamma}^a \boldsymbol{\pi})$

Thank you :)

Additional Slides

Curvatures are defined so that $[\boldsymbol{\nabla}_\mu, \boldsymbol{\nabla}_\nu]\psi = \frac{1}{2}R_{ij\mu\nu}\boldsymbol{\sigma}^{ij}\psi + iqF_{\mu\nu}\psi$

It is straightforward to demonstrate that $qF_{\mu\nu} = -(\nabla_\mu P_\nu - \nabla_\nu P_\mu)$

$$R^i{}_{j\mu\nu} = -(\nabla_\mu R^i{}_{j\nu} - \nabla_\nu R^i{}_{j\mu} + R^i{}_{k\mu} R^k{}_{j\nu} - R^i{}_{k\nu} R^k{}_{j\mu})$$

Non-zero solutions of zero-Riemann equation describe inertial accelerations as they are not gravitational acceleration: COVARIANT INERTIA

An example of non-zero curvatureless tensorial connection

$$R_{r\varphi\varphi} = -r|\sin\theta|^2$$
$$R_{\theta\varphi\varphi} = -r^2\cos\theta\sin\theta$$

$$R_{t\varphi\theta} = r\sin\theta\partial_\theta\alpha$$
$$R_{t\varphi r} = r\sin\theta\partial_r\alpha$$
$$R_{r\theta\theta} = -r(1+\partial_\theta\gamma)$$
$$R_{\theta rr} = r\partial_r\gamma$$

A physical example for Hydrogen Atom (orbital 1S½)

$$\psi = \frac{1}{\sqrt{1+\Gamma}} r^{\Gamma-1} e^{-\alpha m r} e^{-im\Gamma t} \begin{pmatrix} 1+\Gamma \\ 0 \\ i\alpha\cos\theta \\ i\alpha\sin\theta e^{i\varphi} \end{pmatrix}$$

with $\Gamma = \sqrt{1-\alpha^2}$ they give

$$\phi = r^{\Gamma-1} e^{-\alpha m r} (1-\alpha^2|\sin\theta|^2)^{\frac{1}{4}}$$

$$\beta = -\arctan\left(\frac{\alpha}{\Gamma}\cos\theta\right)$$

and defining $\Delta(\theta) = 1/\sqrt{1-\alpha^2|\sin\theta|^2}$

$$R_{t\varphi\theta} = -\alpha r \sin\theta\cos\theta|\Delta(\theta)|^2$$

$$R_{r\theta\theta} = -r(1-\Gamma|\Delta(\theta)|^2)$$

$$R_{r\varphi\varphi} = -r|\sin\theta|^2$$

$$R_{\theta\varphi\varphi} = -r^2\sin\theta\cos\theta$$

Singular spinors: $\overline{\psi}\psi = 0$

$i\overline{\psi}\boldsymbol{\pi}\psi = 0$

the spinor is $\psi = \frac{1}{\sqrt{2}}(\mathbb{I}\cos\frac{\alpha}{2} - \boldsymbol{\pi}\sin\frac{\alpha}{2})\boldsymbol{L}^{-1}\begin{pmatrix}1\\0\\0\\1\end{pmatrix}$

Singular spinors: dipole (Weyl) has field equations

$$R_\mu U^\mu = 0$$

$$(-B_\mu + 2XW_\mu \pm 2P_\mu)U^\mu = 0$$

$$[(-B_\mu + 2XW_\mu \pm 2P_\mu)\varepsilon^{\mu\rho\alpha\nu} + g^{\rho[\alpha}R^{\nu]}]U_\rho = 0$$

Singular spinors: flagpole (Majorana) has field equations

$$(g_{\sigma[\pi}B_{\kappa]}-R^{\mu}\varepsilon_{\mu\sigma\pi\kappa})M^{\pi\kappa}=0$$

$$\frac{1}{2}(B_{\mu}\varepsilon^{\mu\sigma\pi\kappa}+g^{\sigma[\pi}R^{\kappa]})M_{\pi\kappa}-2mU^{\sigma}=0$$

or equivalently

$$R_{\mu}U^{\mu}=0$$

$$B_{\mu}U^{\mu}=0$$

$$(-B_{\mu}\varepsilon^{\mu\rho\alpha\nu}+g^{\rho[\alpha}R^{\nu]})U_{\rho}+2mM^{\alpha\nu}=0$$

Adv. Appl. Clifford Algebras 32, 3 (2022)