Intersection Numbers

from Electromagnetism to Quantum Field Theory (and Cosmology)

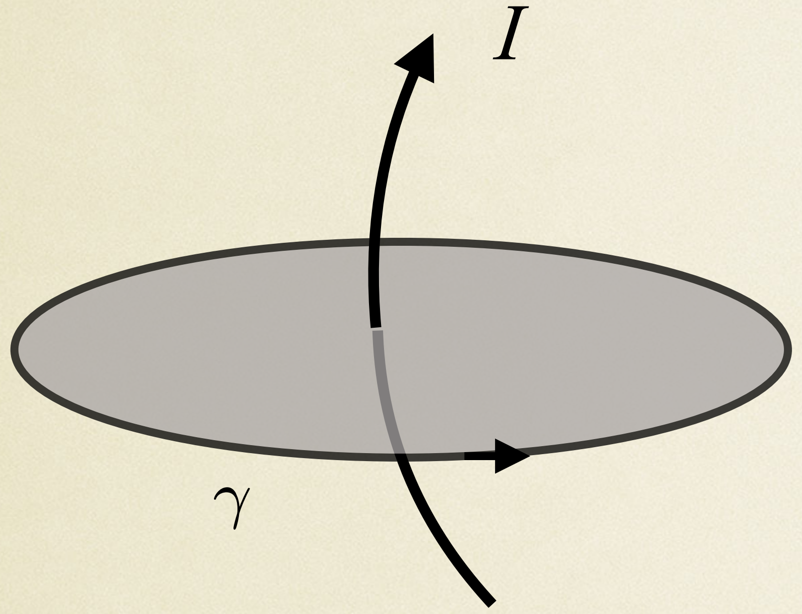
Pierpaolo Mastrolia

Workshop on EOB and Amplitudes in Gravity
Bologna, ***8-9 June 2023***

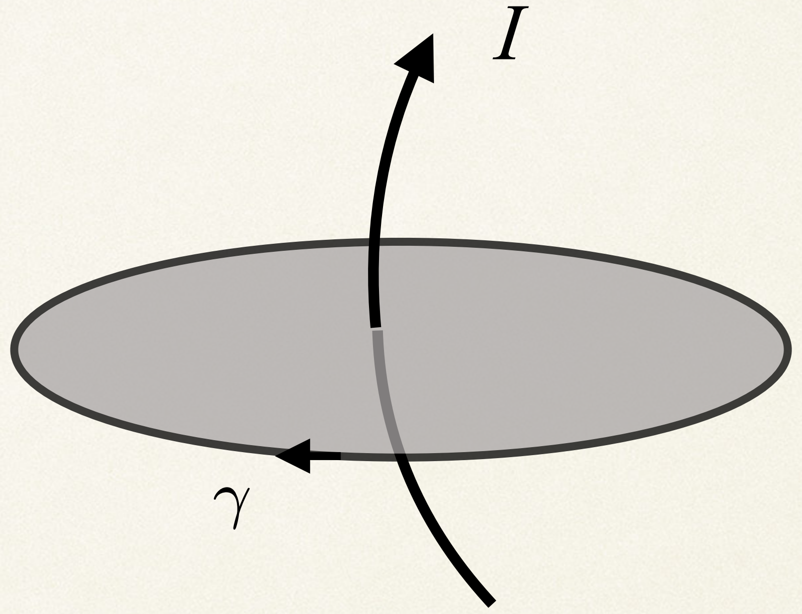
Physics and Astronomy Department
Galileo Galilei
University of Padova - Italy

Istituto Nazionale di Fisica Nucleare

Ampere's Law

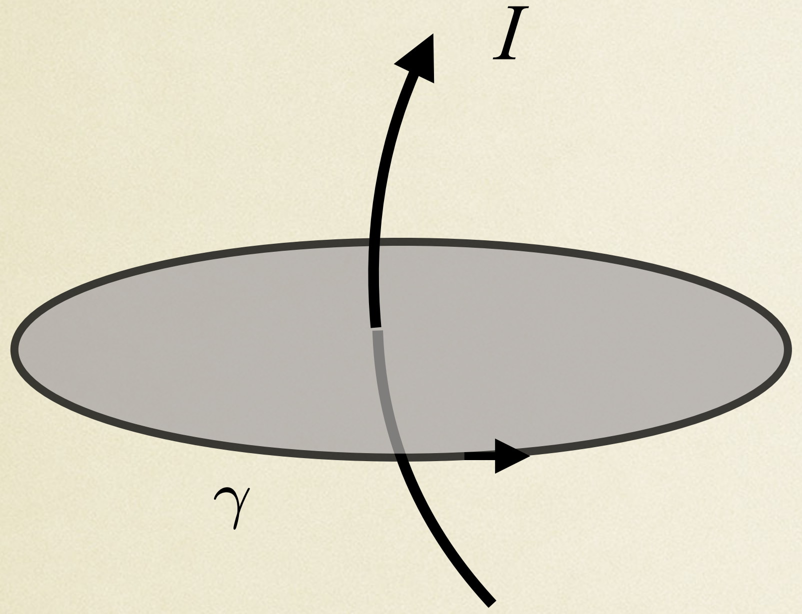


$$\oint_{\gamma} \mathbf{B} \cdot d\vec{\ell} = \mu_0 I.$$

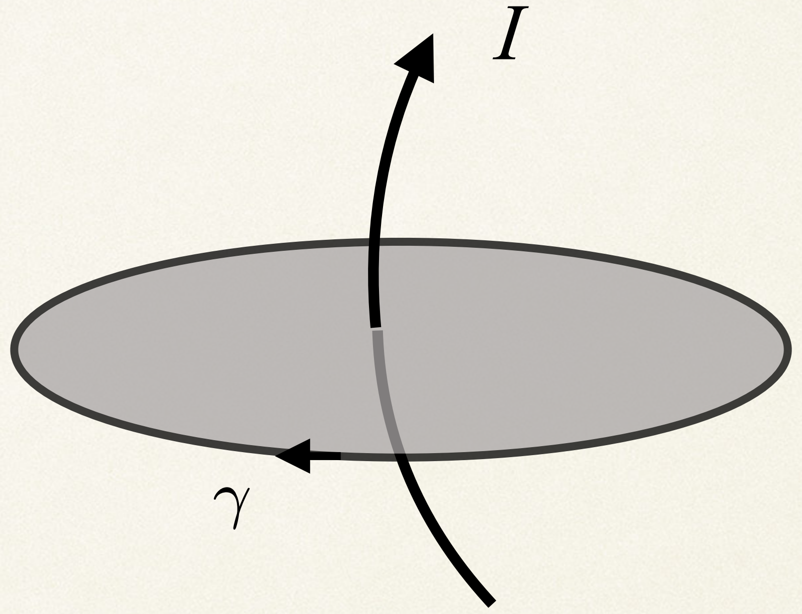


$$\oint_{\gamma} \mathbf{B} \cdot d\vec{\ell} = -\mu_0 I.$$

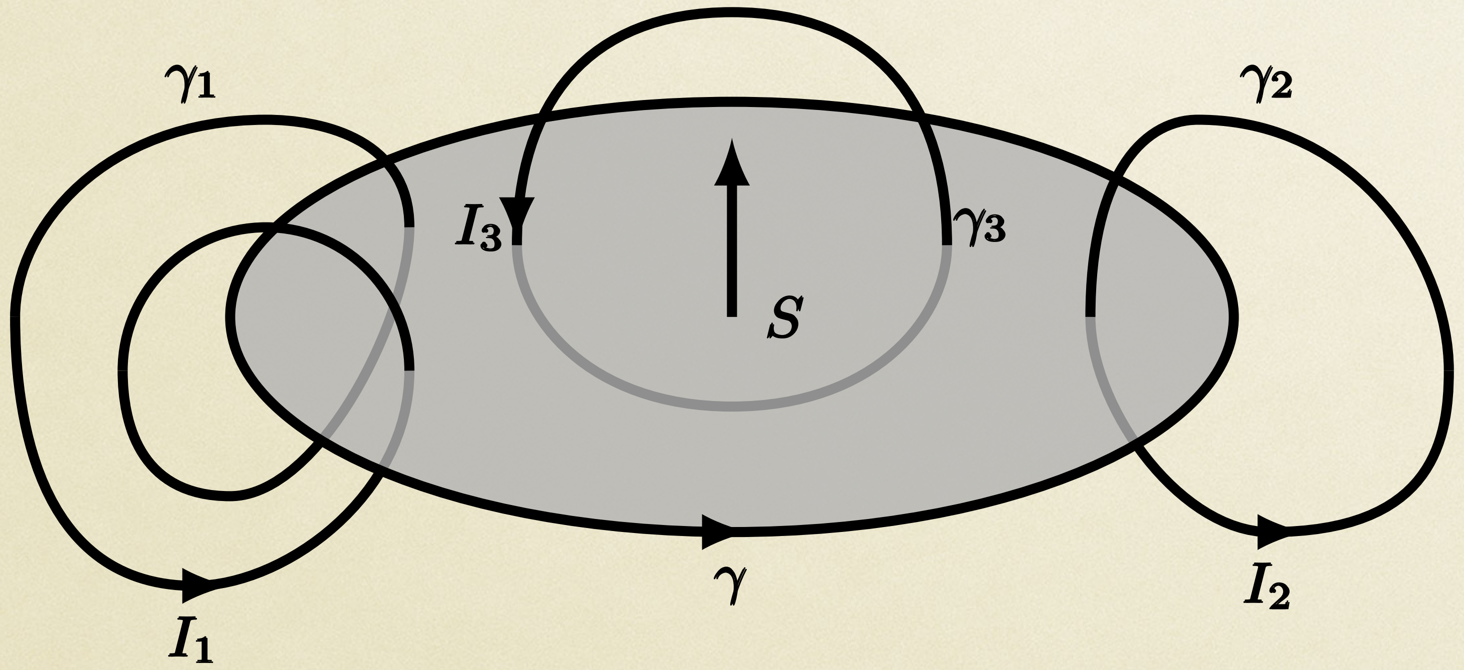
Ampere's Law



$$\oint_{\gamma} \mathbf{B} \cdot d\vec{\ell} \;=\; \mu_0 I.$$



$$\oint_{\gamma} \mathbf{B} \cdot d\vec{\ell} \;=\; -\,\mu_0 I.$$



- **Integral decomposition by geometry**

$$\oint_{\gamma} \mathbf{B} \cdot d\vec{\ell} \;=\; \mu_0 \sum_k (\pm n_k) I_k \qquad n_k \;=\; \mathrm{Link}(\gamma_k, \gamma)$$

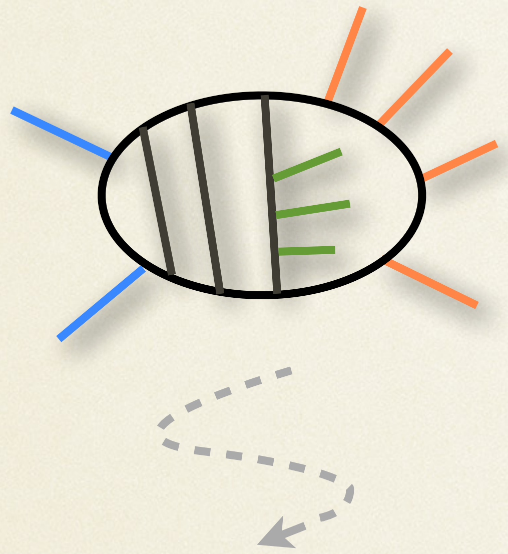
Master Contributions

Gauss' Linking Number

$\mathrm{Link}(\gamma_1, \gamma) = +2$, $\mathrm{Link}(\gamma_2, \gamma) = -1$, and $\mathrm{Link}(\gamma_3, \gamma) = 0$

Feynman Integrals

- **Momentum-space Representation**



=

$$I_{a_1,\dots,a_N} = \int \prod_{i=1}^{L} d^d k_i \left(\prod_{n=1}^{N} \frac{1}{D_n^{a_n}} \right)$$

L loops, $E{+}1$ external momenta,

$N = LE + \frac{1}{2}L(L{+}1)$ (generalised) denominators

total number of *reducible* and *irreducible* scalar products

't Hooft & Veltman

N-denominator
generic Integral

- **Integration-by-parts Identites** Tkachov; Chetyrkin & Tkachov

Laporta, Remiddi, Kuehn, Baikov, Smirnov, Melnikov, Gehrmann, Weinzierl, Anastasiou, Bonciani, &P.M. ...,

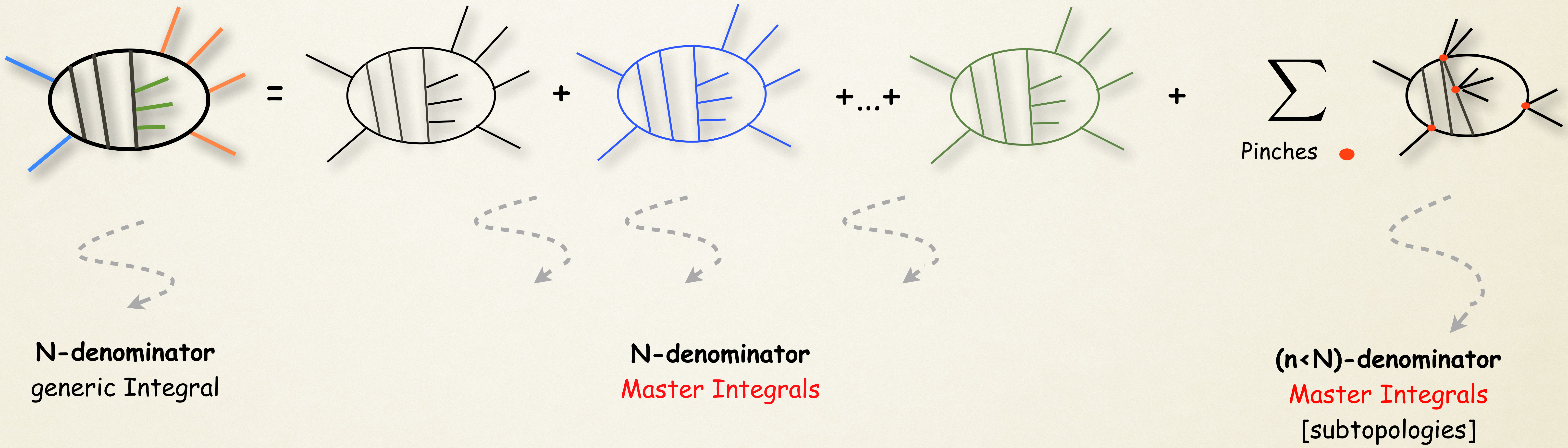
$$\int \prod_{i=1}^{L} d^d k_i \, \frac{\partial}{\partial k_j^\mu} \left(v_\mu \prod_{n=1}^{N} \frac{1}{D_n^{a_n}} \right) = 0$$

$v_\mu = v_\mu(p_i, k_j)$ arbitrary

- IBP identities $\sum_i b_i \, I_{a_1,\dots,a_i \pm 1,\dots,a_N} = 0$

Linear relations for Feynman Integrals identities

- **Relations among Integrals in dim. reg.**

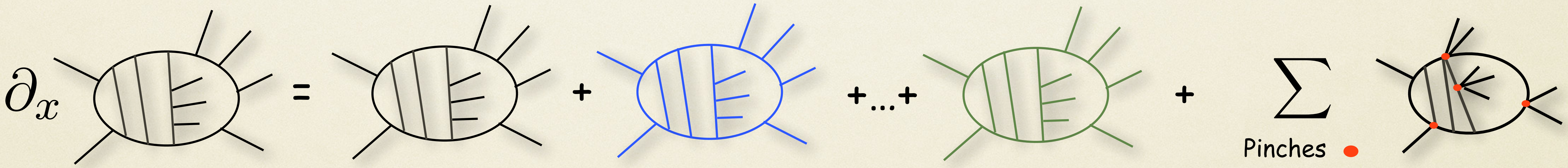


N-denominator
generic Integral

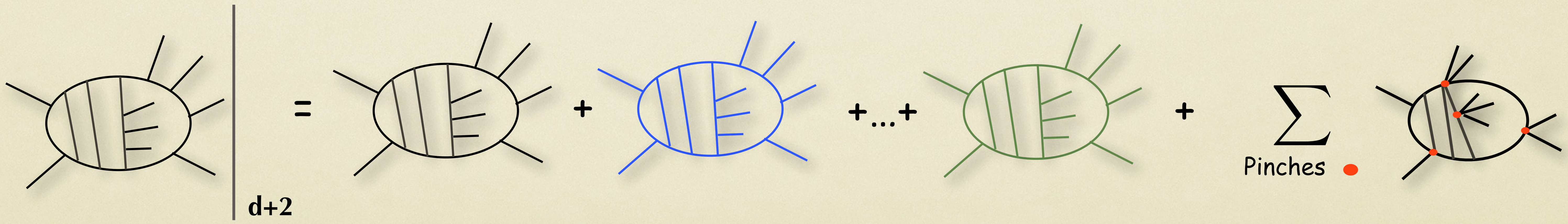
N-denominator
Master Integrals

(n<N)-denominator
Master Integrals
[subtopologies]

- **1st order Differential Equations for MIs**



- **Dimension-Shift relations and Gram determinant relations**



Outline

- **Vector Space Structure of (Feynman, GKZ, Euler-Mellin, …) twisted period Integrals**
  - Linear and Quadratic relations
- **Intersection Numbers**
  - 1-forms
  - n-forms (I): iterative method
  - n-forms (II): Partial Differential Equation
  - n-forms (III): Pfaffian Systems of D-modules
- **Applications**
  - Hypergeometric functions
  - Feynman Integrals
  - Matrix elements in Quantum Mechanics
  - Wick's theorem
  - Kontsevich-Witten tau-function
  - @ this workshop: AH-B-P integrals
- **Conclusions**

Based on:

- **PM**, Mizera
  Feynman Integral and Intersection Theory
  JHEP 1902 (2019) 139 [arXiv: 1810.03818]
- Frellesvig, Gasparotto, Laporta, Mandal, **PM**, Mattiazzi, Mizera
  Decomposition of Feynman Integrals in the Maximal Cut by Intersection Numbers
  JHEP 1095 (2019) 153 [arXiv: 1901.11510]
- Frellesvig, Gasparotto, Mandal, **PM**, Mattiazzi, Mizera
  Vector Space of Feynman Integrals and Multivariate Intersection Numbers
  Phys. Rev. Lett. 123 (2019) 20, 201602 [arXiv 1907.02000]
- Frellesvig, Gasparotto, Laporta, Mandal, **PM**, Mattiazzi, Mizera
  Decomposition of Feynman Integrals by Multivariate Intersection Numbers.
- *JHEP 03 (2021) 027* [arXiv 2008.04823]
- Chestnov, Gasparotto, Mandal, **PM**, Matsubara-Heo, Munch, Takayama
  Macaulay Matrix for Feynmna Integrals: linear relations and intersection numbers.
  JHEP09 (2022) 187 [arXiv: 2204.12983]
- Cacciatori & **PM,**
  Intersection Numbers in Quantum Mechanics and Field Theory.
  2211.03729 [heo-th].
- ***Brunello,*** Chestnov***, Crisanti***, Frellesvig, Gasparotto, Mandal & **PM**
  in progress

What we have found

Vector Space Structure (of Feynman Integrals and not only)

- **Vector decomposition** $$I = \sum_{i=1}^{\nu} c_i \, J_i$$ Master Integral = basis

$\nu =$ dimension of the vector space

- **Projections** $$c_i = I \cdot J_i \; , \qquad J_i \cdot J_j = \delta_{ij}$$

- **Completeness** $$\sum_i J_i \, J_i = \mathbb{I}_{\nu \times \nu}$$

Vector Space Structure (of Feynman Integrals and not only)

- **Vector decomposition**

$$I = \sum_{i=1}^{\nu} c_i \, J_i$$

Master Integral = basis

ν = dimension of the vector space

- **Projections**

$$c_i = I \cdot J_i \ , \qquad J_i \cdot J_j = \delta_{ij}$$

- **Completeness**

$$\sum_i J_i \, J_i = \mathbb{I}_{\nu \times \nu}$$

The two questions:
1) what is the vector space dimension ν ?
2) what is the *scalar product* "$\cdot$" between integrals ?

Basics of Intersection Theory

Basics of Intersection Theory for deRham (co)-Homology

Aomoto, Brown, Cho, Goto, Kita, Matsubara-Heo, Mazumoto, Mimachi, Mizera, Ohara, Yoshida,…

Consider an integral I over the variables $\mathbf{z} = (z_1, z_2, \ldots, z_m)$

$$I = \underbrace{\int_{\mathcal{C}} u(\mathbf{z})}_{\text{twisted cycle}} \underbrace{\varphi_m(\mathbf{z})}_{\text{twisted cocycle}}$$

$\varphi_m(\mathbf{z})$ is a differential m-form

$u(\mathbf{z})$ is a multivalued function

$u(\partial\mathcal{C}) = 0$

Basics of Intersection Theory for deRham (co)-Homology

Aomoto, Brown, Cho, Goto, Kita, Matsubara-Heo, Mazumoto, Mimachi, Mizera, Ohara, Yoshida,…

Consider an integral I over the variables $\mathbf{z} = (z_1, z_2, \ldots, z_m)$

$$I = \underbrace{\int_{\mathcal{C}} u(\mathbf{z})}_{\text{twisted cycle}} \; \underbrace{\varphi_m(\mathbf{z})}_{\text{twisted cocycle}}$$

$\varphi_m(\mathbf{z})$ is a differential m-form

$u(\mathbf{z})$ is a multivalued function

$u(\partial\mathcal{C}) = 0$

- **The dawn of Integration by parts identities:**
  - **Equivalence Classes of DIFFERENTIAL FORMS**

    There could exist many forms φ_m that upon integration give the same result I
  - **Equivalence Classes of INTEGRATION CONTOURS**

    There could exist many contours $\mathcal{C}$ that do not alter the the result of I

Vector Space Structure of Twisted Period Integrals

Basics of Intersection Theory for deRham (co)-Homology

Consider the $(m-1)$-differential form φ_{m-1},

$$\boxed{0 =} \int_{\mathcal{C}} d\Big(u\ \varphi_{m-1}\Big) = \int_{\mathcal{C}} \Big(u\ d\varphi_{m-1} + du \wedge \varphi_{m-1}\Big) = \int_{\mathcal{C}} u\Big(d + \omega\ \wedge\Big)\varphi_{m-1} \boxed{= \int_{\mathcal{C}} u\ \nabla_\omega \varphi_{m-1}}$$

- **Covariant Derivative** $\omega \equiv d \log u \qquad \nabla_\omega \equiv d + \omega \wedge \ \equiv u^{-1} \cdot d \cdot u$
- **Integrals** $I \ = \int_{\mathcal{C}} u\ \varphi_m \ = \int_{\mathcal{C}} u\ \Big(\varphi_m + \nabla_\omega \varphi_{m-1}\Big) \qquad = \int_{\mathcal{C}+\partial\Gamma} u\,\varphi_m$

Basics of Intersection Theory for deRham (co)-Homology

Consider the $(m-1)$-differential form φ_{m-1},

$$\boxed{0 =} \int_{\mathcal{C}} d\Big(u\ \varphi_{m-1}\Big) = \int_{\mathcal{C}} \Big(u\ d\varphi_{m-1} + du \wedge \varphi_{m-1}\Big) = \int_{\mathcal{C}} u\Big(d + \omega\wedge\Big)\varphi_{m-1} \boxed{= \int_{\mathcal{C}} u\ \nabla_\omega \varphi_{m-1}}$$

- **Covariant Derivative** $\omega \equiv d\log u \qquad \nabla_\omega \equiv d + \omega\wedge \ \equiv u^{-1}\cdot d \cdot u$
- **Integrals** $I \ = \boxed{\int_{\mathcal{C}} u\ \varphi_m \ = \int_{\mathcal{C}} u\ \Big(\varphi_m + \nabla_\omega \varphi_{m-1}\Big)} \ = \int_{\mathcal{C}+\partial\Gamma} u\, \varphi_m$
- **Twisted Cohomology Group** $H^m_\omega(X) = \dfrac{\mathrm{Ker}(\nabla_\omega : \varphi_m \to \varphi_{m+1})}{\mathrm{Im}(\nabla_\omega : \varphi_{m-1} \to \varphi_m)}$

Basics of Intersection Theory for deRham (co)-Homology

Consider the $(m-1)$-differential form φ_{m-1},

$$\boxed{0 =} \int_{\mathcal{C}} d\Big(u\ \varphi_{m-1}\Big) = \int_{\mathcal{C}} \Big(u\ d\varphi_{m-1} + du \wedge \varphi_{m-1}\Big) = \int_{\mathcal{C}} u\Big(d + \omega\wedge\Big)\varphi_{m-1} \boxed{= \int_{\mathcal{C}} u\ \nabla_{\omega}\varphi_{m-1}}$$

- **Covariant Derivative** $\omega \equiv d \log u \qquad \nabla_{\omega} \equiv d + \omega\wedge \ \equiv u^{-1} \cdot d \cdot u$
- **Integrals** $I \ = \boxed{\int_{\mathcal{C}} u\ \varphi_m} = \int_{\mathcal{C}} u\ \Big(\varphi_m + \nabla_{\omega}\varphi_{m-1}\Big) \quad \boxed{= \int_{\mathcal{C}+\partial\Gamma} u\,\varphi_m}$
- **Twisted Homology Group** $H_p^{\omega}(X) = \dfrac{\mathrm{Ker}(\partial : \mathcal{C}_{p+1} \to \mathcal{C}_p)}{\mathrm{Im}(\partial : \mathcal{C}_p \to \mathcal{C}_{p-1})}$

Basics of Intersection Theory for deRham (co)-Homology

Consider the $(m-1)$-differential form φ_{m-1},

$$0 = \int_{\mathcal{C}} d\Big(u\ \varphi_{m-1}\Big) = \int_{\mathcal{C}} \Big(u\ d\varphi_{m-1} + du \wedge \varphi_{m-1}\Big) = \int_{\mathcal{C}} u\Big(d + \omega\wedge\Big)\varphi_{m-1} = \int_{\mathcal{C}} u\ \nabla_{\omega}\varphi_{m-1}$$

- **Covariant Derivative** $\omega \equiv d\log u \qquad \nabla_{\omega} \equiv d + \omega\wedge\ \equiv u^{-1}\cdot d \cdot u$
- **Integrals** $I\ = \int_{\mathcal{C}} u\ \varphi_m\ = \int_{\mathcal{C}} u\ \Big(\varphi_m + \nabla_{\omega}\varphi_{m-1}\Big)\ = \int_{\mathcal{C}+\partial\Gamma} u\,\varphi_m$

$$\boxed{0 =} \int_{\mathcal{C}} d\Big(u^{-1}\ \varphi_{m-1}\Big) = \int_{\mathcal{C}} \Big(u^{-1}\ d\varphi_{m-1} - u^{-2} du \wedge \varphi_{m-1}\Big) = \int_{\mathcal{C}} u^{-1}\Big(d - \omega\wedge\Big)\varphi_{m-1} \boxed{= \int_{\mathcal{C}} u^{-1}\ \nabla_{-\omega}\varphi_{m-1}}$$

- **Dual Covariant Derivative** $\nabla_{-\omega} \equiv d - \omega\wedge\ \equiv u\cdot d\cdot u^{-1}$
- **Dual Integrals** $\tilde{I}\ = \int_{\mathcal{C}} u^{-1}\ \phi_m\ = \int_{\mathcal{C}} u^{-1}\ \Big(\phi_m + \nabla_{-\omega}\phi_{m-1}\Big)\ = \int_{\mathcal{C}+\partial\Gamma} u^{-1}\ \phi_m$
- **Dual Twisted Co-Homology Groups** $H^m_{-\omega}(X) = \dfrac{\mathrm{Ker}(\nabla_{-\omega} : \varphi_m \to \varphi_{m+1})}{\mathrm{Im}(\nabla_{-\omega} : \varphi_{m-1} \to \varphi_m)} \qquad H_p^{-\omega}(X) = \dfrac{\mathrm{Ker}(\partial : \mathcal{C}_{p+1} \to \mathcal{C}_p)}{\mathrm{Im}(\partial : \mathcal{C}_p \to \mathcal{C}_{p-1})}$

Pairings of Cycles and Co-cycles

- **Basic building blocks**

$$\langle\varphi_L| \equiv \varphi_L(\mathbf{z}) \in H^m_\omega \qquad |\varphi_R\rangle \equiv \varphi_R(\mathbf{z}) \in H^m_{-\omega} \qquad |\mathcal{C}_R] \equiv \int_{\mathcal{C}_R} u(\mathbf{z}) \in H^\omega_m \qquad [\mathcal{C}_L| \equiv \int_{\mathcal{C}_L} u(\mathbf{z})^{-1} \in H^{-\omega}_m$$

- **Integrals ::** pairings of cycles and co-cycles

$$\langle\, \varphi_L \,|\, \mathcal{C}_R \,] \;\equiv\; \int_{\mathcal{C}_R} u(\mathbf{z})\; \varphi_L(\mathbf{z}) \;=\; I$$

- **Dual Integrals ::** pairings of cycles and co-cycles

$$[\, \mathcal{C}_L \,|\, \varphi_R \,\rangle \;\equiv\; \int_{\mathcal{C}_L} u(\mathbf{z})^{-1}\; \varphi_R(\mathbf{z}) \;=\; \tilde{I}$$

- **Intersection numbers for cycles ::** pairings of cycles

$$\left[\, \mathcal{C}_{\rm L} \,\middle|\, \mathcal{C}_{\rm R} \,\right] \equiv \text{intersection number}$$

- **Intersection numbers for co-cycles ::** pairings of co-cycles

$$\langle\, \varphi_{\rm L} \,|\, \varphi_{\rm R} \,\rangle \equiv \int_{\mathcal{C}} \iota(\varphi_{\rm L}) \wedge \varphi_{\rm R}$$

Identity Resolution

$$\dim H^{m}_{\pm\omega} = \dim H^{\pm\omega}_{m} \equiv \nu$$

- **Bases** $\{\langle e_i|\}_{i=1,\ldots,\nu} \in H^n_\omega$ and $\{|h_i\rangle\}_{i=1,\ldots,\nu} \in H^n_{-\omega}$

- **Forms** $$\mathbb{I}_c = \sum_{i,j=1}^{\nu} |h_i\rangle \left(\mathbf{C}^{-1}\right)_{ij} \langle e_j| \qquad \mathbf{C}_{ij} \equiv \langle e_i|h_j\rangle$$ Metric Matrix for Forms

- **Contours** $$\mathbb{I}_h = \sum_{i,j=1}^{\nu} |\gamma_i] \left(\mathbf{H}^{-1}\right)_{ij} [\eta_j| \qquad \mathbf{H}_{ij} \equiv [\eta_i|\gamma_j]$$ Metric Matrix for Contours

Identity Resolution

$$\mathrm{dim} H^{m}_{\pm\omega} = \mathrm{dim} H^{\pm\omega}_{m} \equiv \nu$$

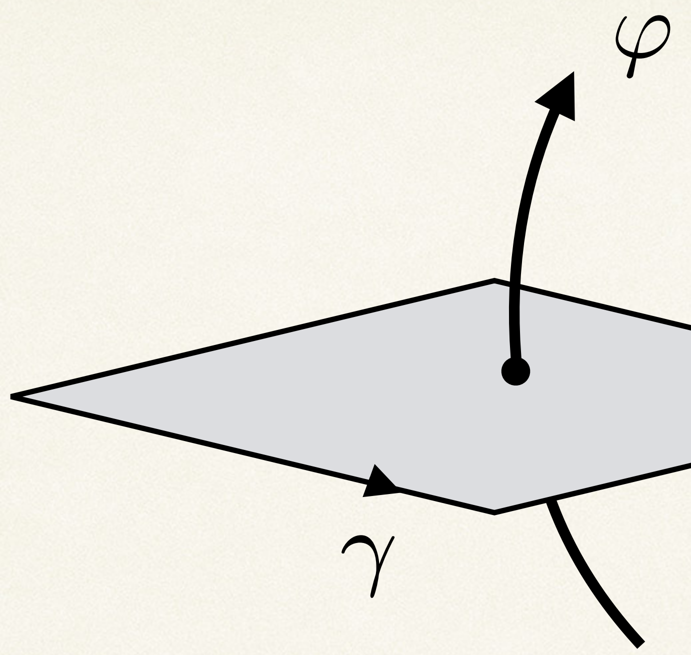
- **Bases** $\{\langle e_i|\}_{i=1,\ldots,\nu} \in H^n_\omega$ and $\{|h_i\rangle\}_{i=1,\ldots,\nu} \in H^n_{-\omega}$

- **Forms** $$\mathbb{I}_c = \sum_{i,j=1}^{\nu} |h_i\rangle \left(\mathbf{C}^{-1}\right)_{ij} \langle e_j|$$ $$\mathbf{C}_{ij} \equiv \langle e_i|h_j\rangle$$ Metric Matrix for Forms

- **Contours** $$\mathbb{I}_h = \sum_{i,j=1}^{\nu} |\gamma_i] \left(\mathbf{H}^{-1}\right)_{ij} [\eta_j|$$ $$\mathbf{H}_{ij} \equiv [\eta_i|\gamma_j]$$ Metric Matrix for Contours

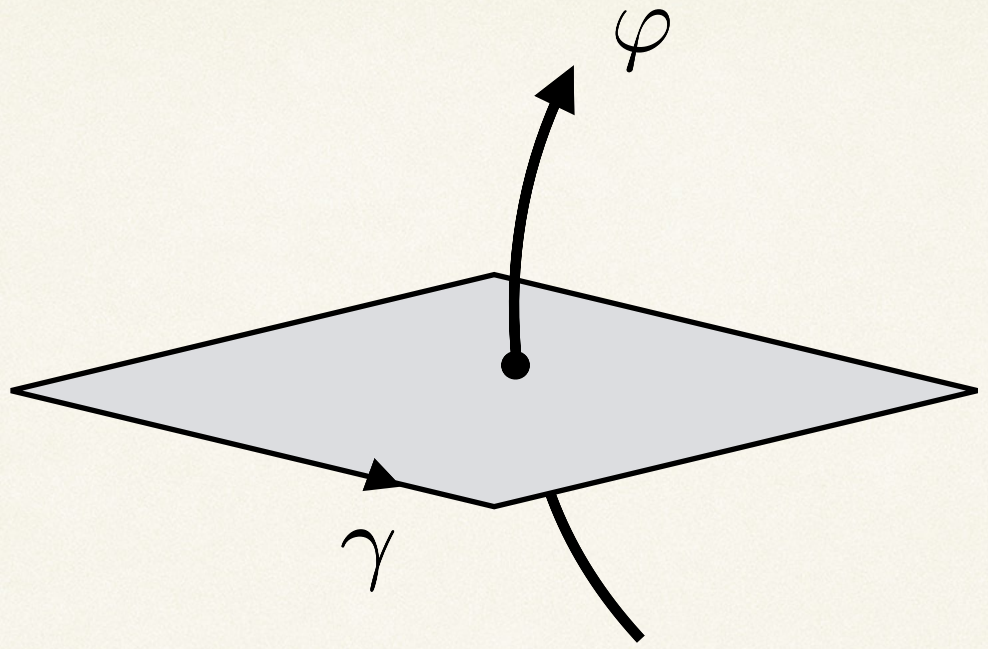
Linear Relations

Flux Decomposition

$\int_\gamma \varphi =$  $= \langle \varphi | \gamma]$

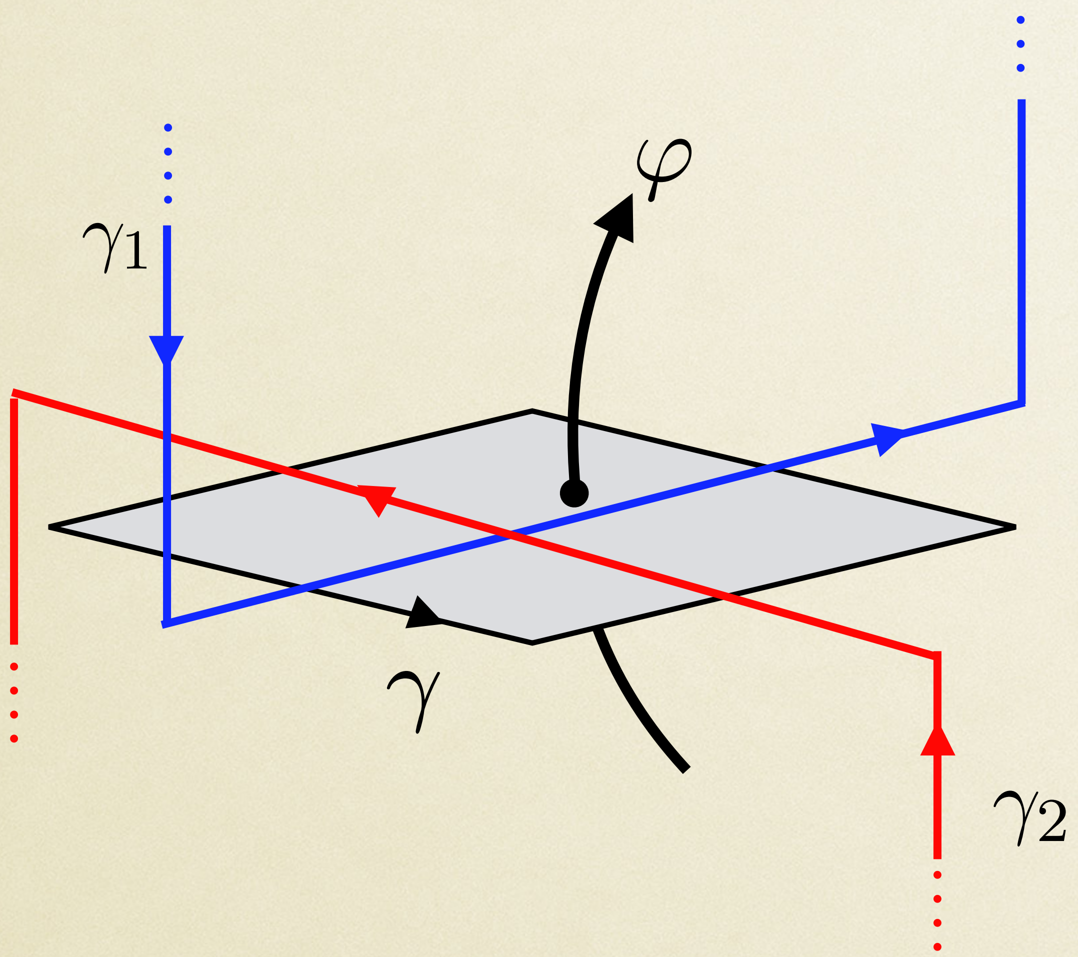
Flux Decomposition

$$\int_\gamma \varphi =$$



$$= \langle \varphi | \gamma]$$

- **Contour decomposition**



$$= \sum_{i=1} a_i \int_{\gamma_i} \varphi$$

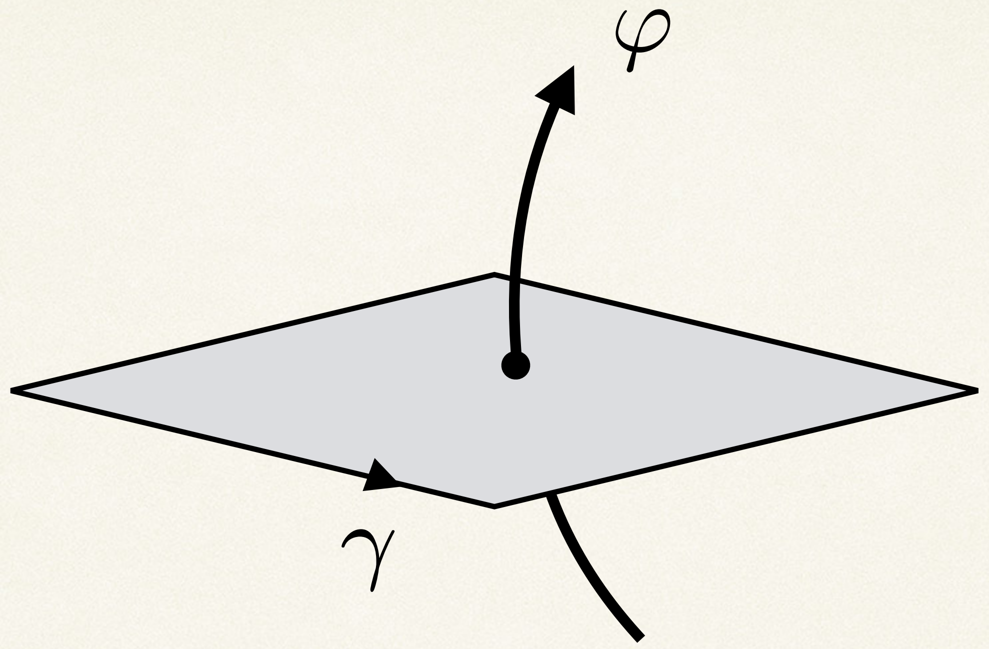
$$|\gamma] = \sum_i a_i \, |\gamma_i]$$

- Coefficients are Intersection Numbers (contours)

$$a_i = [\gamma_i | \gamma] \; , \qquad [\gamma_i | \gamma_j] = \delta_{ij}$$

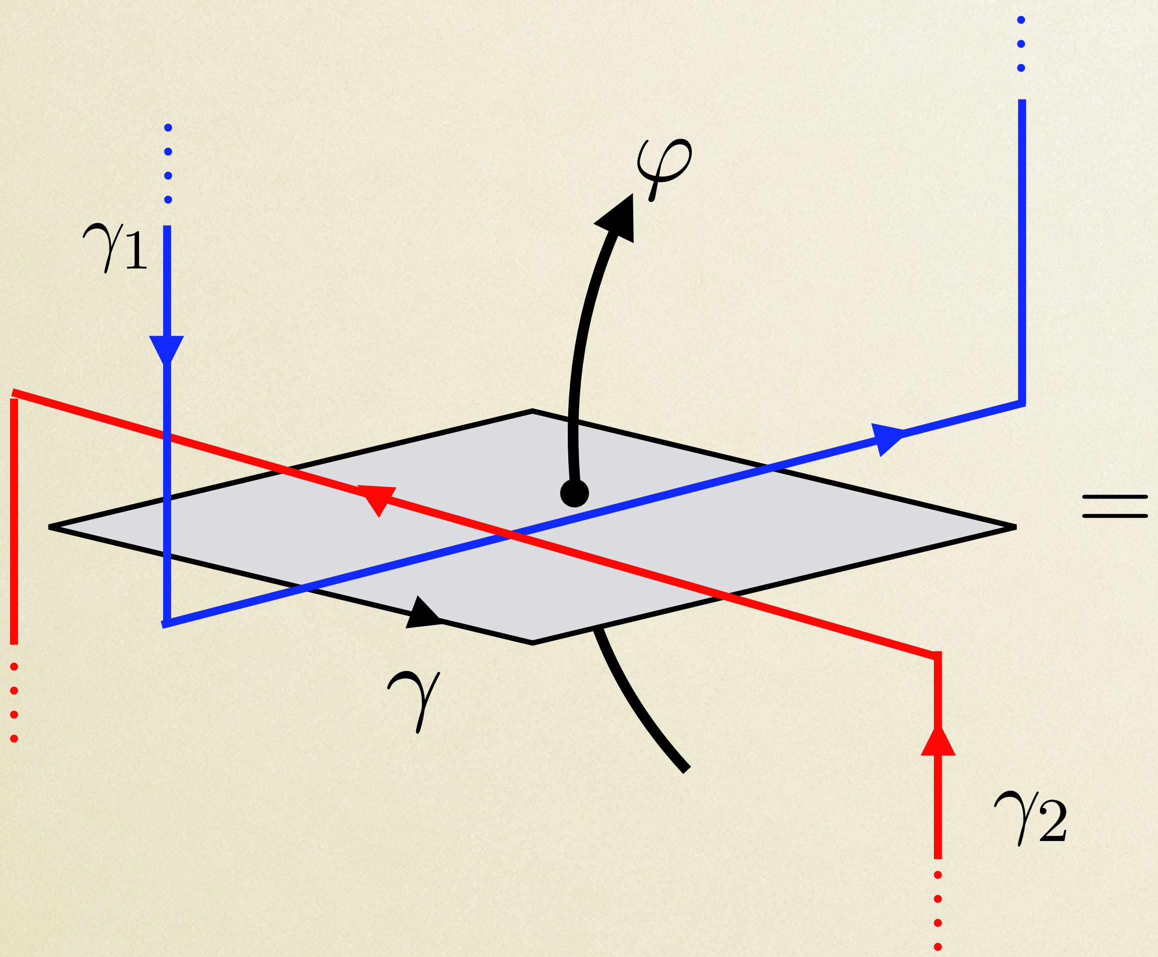
Flux Decomposition

$$\int_\gamma \varphi =$$



$$= \langle \varphi | \gamma]$$

- **Contour decomposition**



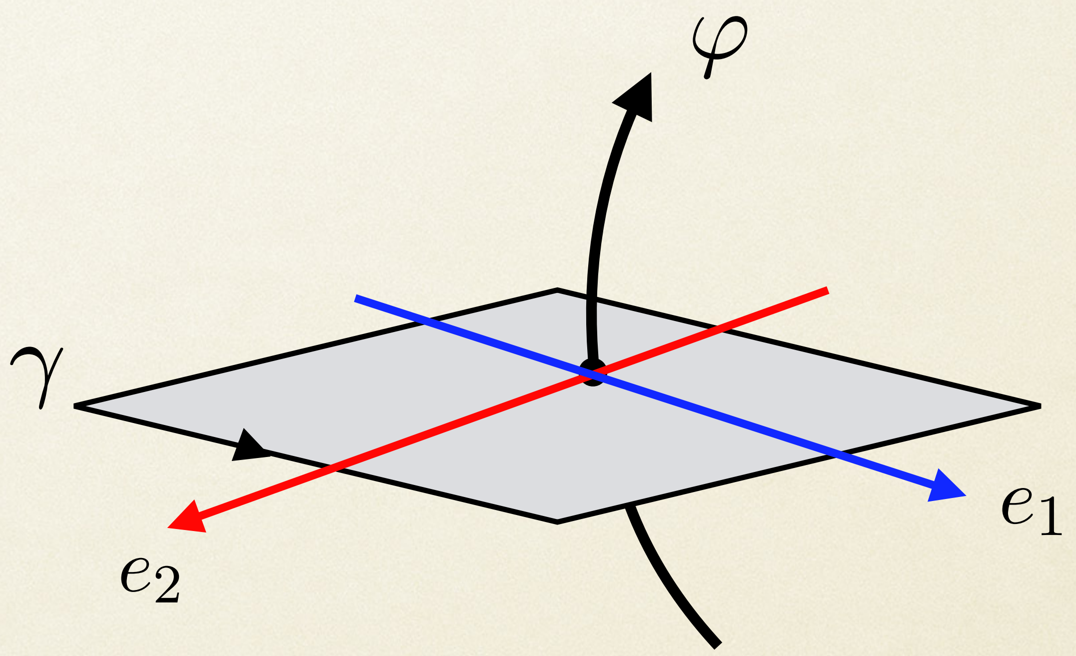
$$= \sum_{i=1} a_i \int_{\gamma_i} \varphi$$

$$|\gamma] = \sum_i a_i \, |\gamma_i]$$

- Coefficients are Intersection Numbers (contours)

$$a_i = [\gamma_i|\gamma] \ , \qquad [\gamma_i|\gamma_j] = \delta_{ij}$$

- **Form decomposition**



$$= \sum_i c_i \int_\gamma e_i$$

$$\langle \varphi | = \sum_i c_i \, \langle e_i |$$

- Coefficients are Intersection Numbers (forms)

$$c_i = \langle \varphi | e_i \rangle \ , \qquad \langle e_i | e_j \rangle = \delta_{ij}$$

Linear Relations / IBPs identity / Gauss contiguity relations

Mizera **& P.M.** (2018)

Frellesvig, Gasparotto, Laporta, Mandal, Mattiazzi, Mizera **& P.M.** (2019)

Frellesvig, Gasparotto, Mandal, Mattiazzi, Mizera **& P.M.** (2019)

Consider a set of ν MIs,

$$J_i = \int_{\mathcal{C}} u(\mathbf{z})\, e_i(\mathbf{z}) = \langle e_i|\mathcal{C}]\ , \qquad i = 1, \ldots, \nu\,,$$

- **Integral decomposition**

$$I = \langle \varphi_L | \mathcal{C}_R] = \sum_{i=1}^{\nu} c_i\, J_i\ .$$

Linear Relations / IBPs identity / Gauss contiguity relations

Mizera **& P.M.** (2018)

Frellesvig, Gasparotto, Laporta, Mandal, Mattiazzi, Mizera **& P.M.** (2019)

Frellesvig, Gasparotto, Mandal, Mattiazzi, Mizera **& P.M.** (2019)

Consider a set of ν MIs,

$$J_i = \int_{\mathcal{C}} u(\mathbf{z})\, e_i(\mathbf{z}) = \langle e_i | \mathcal{C}] \,, \qquad i = 1, \ldots, \nu \,,$$

- **Integral decomposition**

$$I = \langle \varphi_L | \mathcal{C}_R] = \sum_{i=1}^{\nu} c_i \, J_i$$

- **Decomposition of differential forms**.

● **Master Decomposition Formula**

$$\langle \varphi_L | \quad = \quad \langle \varphi_L | \mathbb{I}_c = \sum_{i=1}^{\nu} c_i \, \langle e_i | \,, \quad \text{with} \qquad c_i = \sum_{j=1}^{\nu} \langle \varphi_L | h_j \rangle \left(\mathbf{C}^{-1} \right)_{ji}$$

Quadratic Relations

Twisted Riemann Periods Relations (TRPR)

Cho, Matsumoto (1995)

$$\langle\varphi_L|\varphi_R\rangle \quad = \quad \langle\varphi_L|\mathbb{I}_h|\varphi_R\rangle = \sum_{i,j=1}^{\nu}\langle\varphi_L|\gamma_i]\left(\mathbf{H}^{-1}\right)_{ij}[\eta_j|\phi_R\rangle \;=\left(\mathbf{P}_{\omega}.\,\mathbf{H}^{-1}.\,\mathbf{P}_{-\omega}\right)_{LR}$$

$$[C_L|C_R] \quad = \quad [C_L|\mathbb{I}_c|C_R] = \sum_{i,j=1}^{\nu}[C_L|h_i\rangle\left(\mathbf{C}^{-1}\right)_{ij}\langle e_j|C_R] \;=\left(\mathbf{P}_{-\omega}.\,\mathbf{C}^{-1}.\,\mathbf{P}_{\omega}\right)_{LR}$$

Vector Space Structure of Feynman Integrals

Vector Space Dimensions

- **Space Dimensions = Number of Master Integrals**

$$\nu = \text{number of independent } \textit{master} \text{ integrals}$$

Chetyrkin, Tkachov (1981); Remiddi, Laporta (1996); Laporta (2000)

$$= \text{is finite}$$

Smirnov, Petuckhov (2010)

$$= \text{number of critical points of graph polynomials}$$

Lee, Pomeranski (2013)

$$= \text{is related to Euler characteritics } \chi_E$$

Aluffi, Marcolli (2008) Bitoun, Bogner, Klausen, Panzer (2018) Frellesvig, Gasparotto, Mandal, Mattiazzi, Mizera **& P.M.** (2019)

$$= \text{number of independent integration contours}$$

Lee, Pomeranski (2013) Bosma, Sogaard, Zhang (2017) Primo, Tancredi (2017)

$$= \text{number of independent forms}$$

Mizera **& P.M.** (2018)

Frellesvig, Gasparotto, Laporta, Mandal, Mattiazzi, Mizera **& P.M.** (2019)

$$= \text{dim} H^m_{\pm\omega}$$

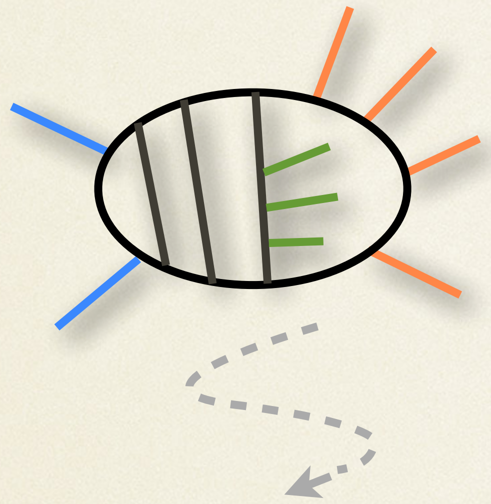
Frellesvig, Gasparotto, Mandal, Mattiazzi, Mizera **& P.M.** (2019)

$$= \text{dim}\Big(\mathbb{C}[\mathbf{z}]/ < \hat{\omega}_1, \ldots, \hat{\omega}_n > \Big) = \text{dim}\Big(\mathbb{C}[\mathbf{z}]/ < \mathcal{G} > \Big)$$

Frellesvig, Gasparotto, Laporta, Mandal, Mattiazzi, Mizera **& P.M.** (2020)

Parametric Representation(s)

- **Upon a change of integration variables**



N-denominator
generic Integral

$$= I_{a_1,\dots,a_N} = \int_{\mathcal{C}} u(\mathbf{z}) \;\; \varphi_N(\mathbf{z})$$

$$\varphi_N(\mathbf{z}) = \hat{\varphi}(\mathbf{z}) d^N\mathbf{z} \qquad \textbf{differential } N\textbf{-form}$$

$$d^N\mathbf{z} = dz_1 \wedge \ldots \wedge dz_N$$

$$\hat{\varphi}_N(\mathbf{z}) = f(\mathbf{z}) \prod_i z_i^{-a_i}$$

$$u(\mathbf{z}) = \mathcal{P}(\mathbf{z})^\gamma$$

$$\mathcal{P}(\mathbf{z}) = \textbf{graph-Polynomial}$$

$$\gamma(d) = \textbf{generic exponent}$$

- **Integration-by-parts:** two situations may occur

$$\int_{\mathcal{C}} d\Big(u(\mathbf{z}) \;\; \varphi_N(\mathbf{z})\Big) \quad \begin{cases} \neq 0 \,, \\ = 0 \,, \quad u(\partial\mathcal{C}) = 0. \end{cases}$$

- Schwinger representation, Lee-Pomeranski repr'n
- Baikov representation, or other repr'ns

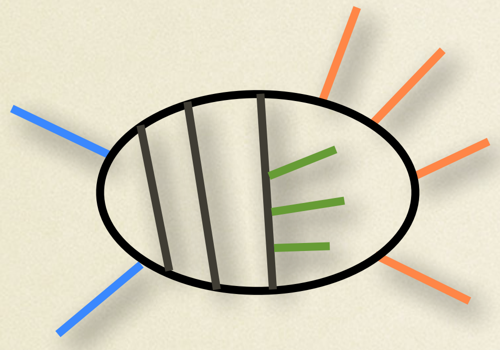
- IBP identities

$$\sum_i b_i \, I_{a_1,\dots,a_i \pm 1,\dots,a_N} = 0$$

Feynman Integrals :: *Baikov Representation*

- **Denominators as integration variables** Baikov (1996)

$\{D_1, \ldots, D_N\} \to \{z_1, \ldots, z_N\} \equiv \mathbf{z}$



$$= \quad I_{a_1,\ldots,a_N} \equiv \int_{\mathcal{C}} d\mathbf{z}\; B(\mathbf{z})^{\gamma} \prod_{i=1}^{N} \frac{1}{z_i^{a_n}}$$

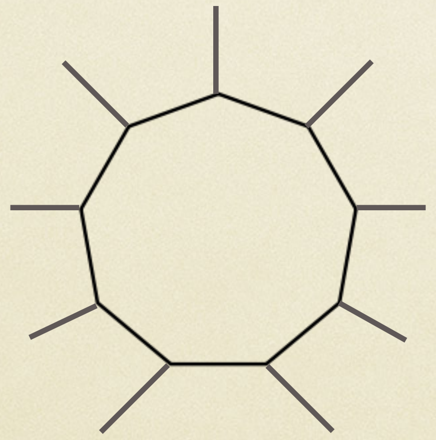
N-denominator
generic Integral

$$B(p_i, k_j) = \begin{vmatrix} k_1^2 & \ldots & (k_1 \cdot p_{E-1}) \\ \vdots & \ddots & \vdots \\ (p_{E-1} \cdot k_1) & \ldots & p_{E-1}^2 \end{vmatrix} = B(\mathbf{z})$$

$$\gamma \equiv (d - E - L - 1)/2$$

Gram determinant

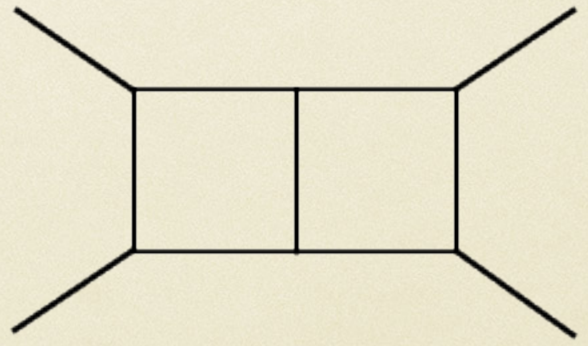
- **1-loop Nonagon**



$$N = LE + \frac{1}{2} L(L+1)$$

$$\int_{\mathcal{C}} dz_1 \wedge \cdots \wedge dz_9 \; \frac{B(\mathbf{z})^{\gamma}}{z_1^{n_1} \cdots z_9^{n_9}}$$

- **2-loop Box**

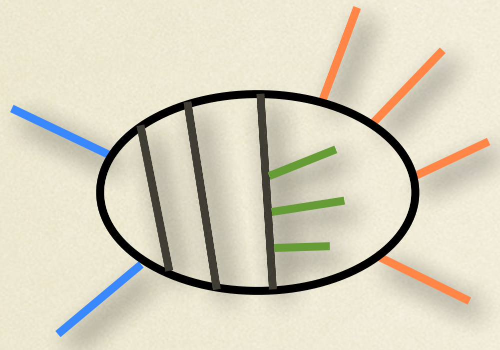


$B(\mathbf{z})$, $\mathcal{C}$, γ depend on the graph.

Feynman Integrals :: *Baikov Representation*

- **Denominators as integration variables** Baikov (1996)

$\{D_1, \ldots, D_N\} \to \{z_1, \ldots, z_N\} \equiv \mathbf{z}$



$$= \quad I_{a_1,\ldots,a_N} \equiv \int_{\mathcal{C}} d\mathbf{z}\; B(\mathbf{z})^{\gamma} \prod_{i=1}^{N} \frac{1}{z_i^{a_n}}$$

N-denominator
generic Integral

$$B(p_i, k_j) = \begin{vmatrix} k_1^2 & \ldots & (k_1 \cdot p_{E-1}) \\ \vdots & \ddots & \vdots \\ (p_{E-1} \cdot k_1) & \ldots & p_{E-1}^2 \end{vmatrix} = B(\mathbf{z})$$

$$\gamma \equiv (d - E - L - 1)/2$$

Gram determinant

- **Integration-by-parts Identities** Zhang, Larsen; Lee; Frellesvig, Papadopoulos

$$B(\partial\mathcal{C}) = 0$$

Fundamental property

$$\int_{\mathcal{C}} d\left(B(\mathbf{z})^{\gamma} \prod_{i=1}^{N} \frac{1}{z_i^{a_n}} \right) = 0$$

Three special applications:

i) Dimensional Recurrence Relation

Frellesvig, Gasparotto, Laporta, Mandal, Mattiazzi, Mizera **& P.M.** (2019)

- **MIs in (d+2n) dimensions**

$$J_i^{(d+2n)} \equiv K(d+2n)\, E_i^{(d+2n)} \qquad E_i^{(d+2n)} \equiv \langle B^n e_i | \mathcal{C}] = \int_{\mathcal{C}} u\,(B^n\, e_i)\ , \qquad u = B^{\gamma}\,, \qquad \gamma \equiv (d{-}E{-}L{-}1)/2$$

- **Master Decomposition Formula @ special basis choice**

$$\langle B^{\nu} e_i | = \sum_{n=0}^{\nu-1} c_n\, \langle B^n e_i | \qquad n = 0, 1, \ldots, \nu-1$$

- **Recurrence Relations for Master Forms**

$$\sum_{n=0}^{\nu} c_n\, \langle B^n e_i | = 0\ , \qquad c_{\nu} \equiv -1$$

- **Recurrence Relations for Master Integrals**

$$\sum_{n=0}^{\nu} \alpha_n\, J_i^{(d+2n)} = 0 \qquad \alpha_n \equiv c_n / K(d+2n)$$

ii) Differential Equations

Mizera **& P.M.** (2018)
Frellesvig, Gasparotto, Laporta, Mandal, Mattiazzi, Mizera **& P.M.** (2019)

- **External Derivative**

$$\partial_x I = \partial_x \langle \varphi | \mathcal{C}] = \partial_x \int_{\mathcal{C}} u \varphi = \int_{\mathcal{C}} u \left(\frac{\partial_x u}{u} \wedge + \partial_x \right) \varphi = \langle (\partial_x + \sigma) \varphi | \mathcal{C}]$$

- **External (connection) dLog-form**

$$\nabla_{x,\sigma} \equiv \partial_x + \sigma \qquad \sigma = \partial_x \log u$$

- **Derivative of Master Forms**

$$\partial_x \langle e_i | = \langle (\partial_x + \sigma \wedge) e_i | = \langle (\partial_x + \sigma \wedge) e_i \underbrace{| h_k \rangle \, \left(\mathbf{C}^{-1} \right)_{kj} \, \langle e_j |}_{=1} = \boldsymbol{\Omega}_{ij} \langle e_j |$$

- **System of DEQ for Master Forms**

$$\partial_x \langle e_i | = \boldsymbol{\Omega}_{ij} \, \langle e_j | \ , \qquad \boldsymbol{\Omega} = \boldsymbol{\Omega}(d, x)$$

An analogous System of DEQ can be derived for dual forms: $u \to u^{-1} \quad \Longrightarrow \quad \nabla_{x,\sigma} \to \nabla_{x,-\sigma}$

iii) Secondary Equation

Matsubara-Heo, Takayama (2019)

Weinzierl (2020)

Chestnov, Gasparotto, Munch Matsubara-Heo, Takayama **& P.M.** (2022)

- **DEQ for forms**

$$\partial_x \langle e_i| = \Omega_{ij} \, \langle e_j| \qquad \Omega_{ij} = \langle (\partial_x + \sigma_x) e_i | h_k \rangle \, (\mathbf{C}^{-1})_{kj}$$

- **DEQ dual-forms**

$$\partial_x |h_i\rangle = \tilde{\Omega}_{ji} \, |h_j\rangle \qquad \tilde{\Omega}_{ji} = (\mathbf{C}^{-1})_{jk} \, \langle e_k | (\partial_x - \sigma_x) h_i \rangle$$

- **Secondary Equation for the Intersection Matrix**

$$\mathbf{C}_{ij} \equiv \langle e_i | h_j \rangle$$

$$\partial_x \mathbf{C} = \mathbf{\Omega}.\mathbf{C} + \mathbf{C}.\tilde{\mathbf{\Omega}} \, , \qquad \partial_x \mathbf{C}^{-1} = \tilde{\mathbf{\Omega}}.\mathbf{C}^{-1} - \mathbf{C}^{-1}.\mathbf{\Omega}$$

Intersection Numbers for 1-forms

Intersection Numbers :: *1-forms*

Cho and Matsumoto (1998)

- **1-form** $\langle\varphi| \equiv \hat{\varphi}(z)\ dz$ $\hat{\varphi}(z)$ rational function

- **Zeroes and Poles of** ω

$$\omega \equiv d\log u$$

$$\nu = \{\text{the number of solutions of } \omega = 0\}$$

$$\mathcal{P} \equiv \{\, z \mid z \text{ is a pole of } \omega \,\}$$

$\mathcal{P}$ can also include the pole at infinity if $\mathrm{Res}_{z=\infty}(\omega) \neq 0$.

- **Intersection Numbers**

1-forms φ_L and φ_R

$$\langle\varphi_L|\varphi_R\rangle = \sum_{p\in\mathcal{P}} \mathrm{Res}_{z=p}\Big(\psi_p\, \varphi_R\Big)$$

ψ_p is a function (0-form), solution to the differential equation $\nabla_\omega\psi = \varphi_L$, around p

Intersection Numbers for n-forms :: Iterative Method

Intersection Numbers for *Logarithmic* n-Forms

Matsumoto (1998), Mizera (2017)

If $\langle\varphi_L|$ and $\langle\varphi_R|$ are dLog n-forms (hence contain only simple poles)

$$\langle\varphi_L|\varphi_R\rangle = \int dz_1 \cdots dz_n \, \delta(\omega_1) \cdots \delta(\omega_n) \; \hat{\varphi}_L \, \hat{\varphi}_R =$$

$$= \sum_{(z_1^*,\ldots,z_n^*)} \det{}^{-1} \begin{bmatrix} \frac{\partial\omega_1}{\partial z_1} & \cdots & \frac{\partial\omega_1}{\partial z_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial\omega_n}{\partial z_1} & \cdots & \frac{\partial\omega_n}{\partial z_n} \end{bmatrix} \widehat{\varphi}_L \, \widehat{\varphi}_R \Bigg|_{(z_1,\ldots,z_n)=(z_1^*,\ldots z_n^*)}$$

[Global Residue Theorem]

$(z_1^*,\ldots z_n^*)$ *critical points*, namely the solutions of the system $\omega_i = 0, \quad i = 1, \ldots n.$

In the 1-variate case: $\langle\varphi_L|\varphi_R\rangle = \mathrm{Res}_{z\in\mathcal{P}_{\omega_1}}\left(\frac{\hat{\varphi}_L\,\hat{\varphi}_R}{\omega}\right) = \int dz_1\,\delta(\omega_1)\,\hat{\varphi}_L\,\hat{\varphi}_R = \sum_{(z_1^*)} \frac{\hat{\varphi}_L\,\hat{\varphi}_R}{\partial\omega_1/\partial z_1}$

[Residue Theorem]

- **Efficiently implemented also *via* Companion Matrix**

credit **Salvatori**

Nested Integrations

- **Multivariate integral decomposition**

$$I = \int dz_n \ldots \int dz_3 \int dz_2 \int dz_1 \, f(z_n, \ldots, z_3, z_2, z_1)$$

$$I = \sum_{i=1}^{\nu} c_i \, J_i$$

- **Independent (Master) Integrals**

$$J_i \equiv \int dz_n \ldots \int dz_3 \int dz_2 \int dz_1 \, f_i(z_n, \ldots, z_1)$$

Cascade of Master Integrals

$$I = \int dz_n \ldots \int dz_3 \int dz_2 \underbrace{\int dz_1 \, f(z_n, \ldots, z_3, z_2, z_1)}_{\exists \; \nu^{(1)} \; \text{master integrals in } \mathrm{z}_1}$$

$$I = \int dz_n \ldots \int dz_3 \int dz_2 \sum_{i_1=1}^{\nu^{(1)}} c_{i_1}(z_n, \ldots, z_3, z_2) \, J_{i_1}(z_n, \ldots, z_3, z_2)$$

Cascade of Master Integrals

$$I = \int dz_n \dots \int dz_3 \int dz_2 \underbrace{\int dz_1 \, f(z_n, \dots, z_3, z_2, z_1)}_{\exists \ \nu^{(1)} \ \text{master integrals in } \mathrm{z}_1}$$

$$I = \int dz_n \dots \int dz_3 \underbrace{\int dz_2 \sum_{i_1=1}^{\nu^{(1)}} c_{i_1}(z_n, \dots, z_3, z_2) \, J_{i_1}(z_n, \dots, z_3, z_2)}_{\exists \ \nu^{(2)} \ \text{master integrals in } \mathrm{z}_2}$$

$$I = \int dz_n \dots \int dz_3 \sum_{i_2=1}^{\nu^{(2)}} c_{i_2}(z_n, \dots, z_3) \, J_{i_2}(z_n, \dots, z_3)$$

Cascade of Master Integrals

$$I = \int dz_n \ldots \int dz_3 \int dz_2 \underbrace{\int dz_1\, f(z_n, \ldots, z_3, z_2, z_1)}_{\exists\ \nu^{(1)}\ \text{master integrals in } \mathrm{z}_1}$$

$$I = \int dz_n \ldots \int dz_3 \underbrace{\int dz_2 \sum_{i_1=1}^{\nu^{(1)}} c_{i_1}(z_n, \ldots, z_3, z_2)\, J_{i_1}(z_n, \ldots, z_3, z_2)}_{\exists\ \nu^{(2)}\ \text{master integrals in } \mathrm{z}_2}$$

$$I = \int dz_n \ldots \underbrace{\int dz_3 \sum_{i_2=1}^{\nu^{(2)}} c_{i_2}(z_n, \ldots, z_3)\, J_{i_2}(z_n, \ldots, z_3)}_{\exists\ \nu^{(3)}\ \text{master integrals in } \mathrm{z}_3}$$

$$\vdots$$

$$I = \underbrace{\int dz_n \sum_{i_n=1}^{\nu^{(n-1)}} c_{i_n}(z_n)\, J_{i_n}(z_n)}_{\exists\ \nu\ \text{master integrals in } \mathrm{z}_\mathrm{n}}$$

$$I = \sum_{i=1}^{\nu} c_i\, J_i$$

Multivariate Intersection Numbers (I)

Ohara (1998)   Mizera (2019)

Frellesvig, Gasparotto, Mandal, Mattiazzi, Mizera **& P.M.** (2019)

- by ***Induction***:
- **(n-1)-form Vector Space: known!**

$$\nu_{\mathbf{n-1}} \qquad \langle e_i^{(\mathbf{n-1})}| \qquad |h_i^{(\mathbf{n-1})}\rangle \qquad \left(\mathbf{C}_{(\mathbf{n-1})}\right)_{ij} \equiv {}_{\mathbf{n-1}}\langle e_i^{(\mathbf{n-1})}|h_j^{(\mathbf{n-1})}\rangle$$

- **n-form decomposition: n = (n-1) +** (n)

$$\langle\varphi_L^{(\mathbf{n})}| = \sum_{i=1}^{\nu_{\mathbf{n-1}}} \langle e_i^{(\mathbf{n-1})}| \wedge \langle\varphi_{L,i}^{(n)}|\,, \qquad \langle\varphi_{L,i}^{(n)}| = \langle\varphi_L^{(\mathbf{n})}|h_j^{(\mathbf{n-1})}\rangle\,\left(\mathbf{C}^{-1}_{(\mathbf{n-1})}\right)_{ji}\,, \qquad \langle\varphi_{L,i}^{(n)}|\,(C_{(\mathbf{n-1})})_{ij} = \langle\varphi_L^{(\mathbf{n})}|h_j^{(\mathbf{n-1})}\rangle$$

$$|\varphi_R^{(\mathbf{n})}\rangle = \sum_{i=1}^{\nu_{\mathbf{n-1}}} |h_i^{(\mathbf{n-1})}\rangle \wedge |\varphi_{R,i}^{(n)}\rangle\,, \qquad |\varphi_{R,i}^{(n)}\rangle = \left(\mathbf{C}^{-1}_{(\mathbf{n-1})}\right)_{ij}\,\langle e_j^{(\mathbf{n-1})}|\varphi_R^{(\mathbf{n})}\rangle\,, \qquad (C_{(\mathbf{n-1})})_{ij}\,|\varphi_{R,j}^{(n)}\rangle = \langle e_i^{(\mathbf{n-1})}|\varphi_R^{(\mathbf{n})}\rangle$$

Multivariate Intersection Numbers (I)

Ohara (1998) Mizera (2019)

Frellesvig, Gasparotto, Mandal, Mattiazzi, Mizera **& P.M.** (2019)

- by *Induction*:
- **(n-1)-form Vector Space: known!**

$$\nu_{\mathbf{n-1}} \qquad \langle e_i^{(\mathbf{n-1})}| \qquad |h_i^{(\mathbf{n-1})}\rangle \qquad \left(\mathbf{C}_{(\mathbf{n-1})}\right)_{ij} \equiv {}_{\mathbf{n-1}}\langle e_i^{(\mathbf{n-1})}|h_j^{(\mathbf{n-1})}\rangle$$

- **n-form decomposition: n = (n-1) +** (n)

$$\langle\varphi_L^{(\mathbf{n})}| = \sum_{i=1}^{\nu_{\mathbf{n-1}}} \langle e_i^{(\mathbf{n-1})}| \wedge \langle\varphi_{L,i}^{(n)}| \,, \qquad \langle\varphi_{L,i}^{(n)}| = \langle\varphi_L^{(\mathbf{n})}|h_j^{(\mathbf{n-1})}\rangle \left(\mathbf{C}^{-1}_{(\mathbf{n-1})}\right)_{ji} \,, \qquad \langle\varphi_{L,i}^{(n)}|\,(C_{(\mathbf{n-1})})_{ij} = \langle\varphi_L^{(\mathbf{n})}|h_j^{(\mathbf{n-1})}\rangle$$

$$|\varphi_R^{(\mathbf{n})}\rangle = \sum_{i=1}^{\nu_{\mathbf{n-1}}} |h_i^{(\mathbf{n-1})}\rangle \wedge |\varphi_{R,i}^{(n)}\rangle \,, \qquad |\varphi_{R,i}^{(n)}\rangle = \left(\mathbf{C}^{-1}_{(\mathbf{n-1})}\right)_{ij} \langle e_j^{(\mathbf{n-1})}|\varphi_R^{(\mathbf{n})}\rangle \,, \qquad (C_{(\mathbf{n-1})})_{ij}\,|\varphi_{R,j}^{(n)}\rangle = \langle e_i^{(\mathbf{n-1})}|\varphi_R^{(\mathbf{n})}\rangle$$

Intersection Numbers for n-forms :: Recursive Formula

$$\begin{aligned} \langle\varphi_L^{(\mathbf{n})}|\varphi_R^{(\mathbf{n})}\rangle &= \sum_{i,j} \langle\varphi_L^{(\mathbf{n})}\overbrace{|h_j^{(\mathbf{n-1})}\rangle (C_{(\mathbf{n-1})})^{-1}_{ji} \langle e_i^{(\mathbf{n-1})}|}^{=1}\varphi_R^{(\mathbf{n})}\rangle \\ &= \sum_{i,j} \langle\varphi_{L,i}^{(n)}|(C_{(\mathbf{n-1})})_{ij}\,\varphi_{R,j}^{(n)}\rangle \end{aligned}$$

$$\partial_{z_n}\psi_i^{(n)} + \psi_j^{(n)}\,\hat{\mathbf{\Omega}}_{ji}^{(n)} = \hat{\varphi}_{L,i}^{(n)} \,,$$

$\hat{\mathbf{\Omega}}^{(n)}$ is a $\nu_{\mathbf{n-1}} \times \nu_{\mathbf{n-1}}$ matrix, whose entries are given by

$$\hat{\mathbf{\Omega}}_{ji}^{(n)} = \langle(\partial_{z_n} + \hat{\omega}_n)e_j^{(\mathbf{n-1})}|h_k^{(\mathbf{n-1})}\rangle \left(\mathbf{C}^{-1}_{(\mathbf{n-1})}\right)_{ki}$$

Multivariate Intersection Numbers (I)

Ohara (1998) Mizera (2019)
Frellesvig, Gasparotto, Mandal, Mattiazzi, Mizera **& P.M.** (2019)

- **Property of Intersection Number**

invariance under differential forms redefinition within the same equivalence classes,

$$\langle \varphi_L | \varphi_R \rangle = \langle \varphi'_L | \varphi'_R \rangle \ , \qquad \varphi'_L = \varphi_L + \nabla_\omega \xi_L \ , \qquad \varphi'_R = \varphi_R + \nabla_{-\omega} \xi_R$$

- **Global Residue Thm** Weinzierl (2020)

choose ξ_L and ξ_R, to build φ'_L and φ'_R that contain only simple poles. and if $\hat{\mathbf{\Omega}}^{(n)}$ is reduced to Fuchsian form

▽

the computation of multivariate intesection number can benefit of the evaluation of intersection numbers for dlog forms at each step of the iteration.

- **Special dual basis choice** CaronHuot Pokraka (2019-2021)

Relative Dirac-delta basis elements trivialise the evaluation of the intersection numbers

- **Multi-pole ansatz** Fontana Peraro (2022)

Solving $\nabla_\omega \psi = \varphi_L$, bypassing the pole factorisation, and using FF reconstruction methods.
(avoiding irrational functions which would disappear in the intersection numbers)

Contiguity relations for Special Functions

Hypergeometric ${}_3F_2$

$$u(\mathbf{z}) = ((1-z_1)z_1(1-z_2)z_2(1-xz_1z_2))^{\gamma}\,; \qquad \omega \equiv d\log u(\mathbf{z}) = \sum_{i=1}^{2} \hat{\omega}_i\, dz_i;$$

a. *Number of MIs ::* I choose the ordering as $\{z_1, z_2\}$.

$$\nu_{12} = 3, \quad \{\omega_1 = 0, \omega_2 = 0\} \qquad e^{(12)} = \{\frac{1}{z_2(z_1-\frac{1}{x})}, \frac{1}{(z_1-1)\ (z_2-1)}, \frac{1}{z_1(z_2-xz_1)}\}$$

b. *Choice of bases ::*

$$\nu_2 = 2, \quad \{\omega_2 = 0\} \qquad e^{(2)} = \{\frac{1}{z_2}, \frac{1}{z_2-1}\}$$

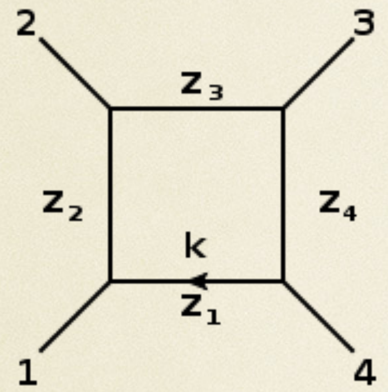
$$\partial_x \langle \hat{e}_i^{(12)}| \;=\; \langle \partial_x \hat{e}_i^{(12)} + \sigma \hat{e}_i^{(12)}| = \Omega_{ij} \langle \hat{e}_j^{(12)}| \qquad \sigma = \frac{d\,\mathrm{Log} u}{dx} = \frac{\gamma z_1 z_2}{x z_1 z_2 - 1}$$

$$\mathbf{\Omega} = \gamma \begin{pmatrix} -\frac{x-2}{4(x-1)x} & \frac{3x+10}{20(x-1)x} & \frac{13\ x-10}{20(x-1)x} \\ \frac{3}{4(x-1)x} & \frac{20x+19}{20(x-1)x} & \frac{9}{20(x-1)\ x} \\ 0 & 0 & \frac{1}{x} \end{pmatrix}$$

- Canonical

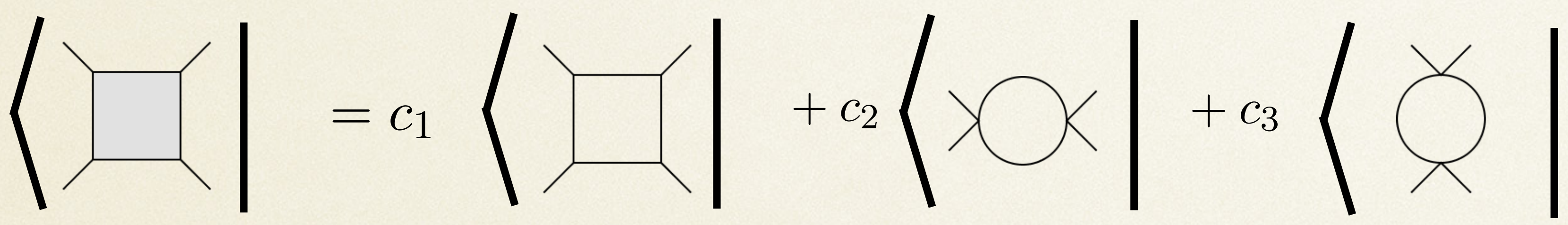
Feynman Integrals Decomposition

Example: 1-Loop Box Integrals

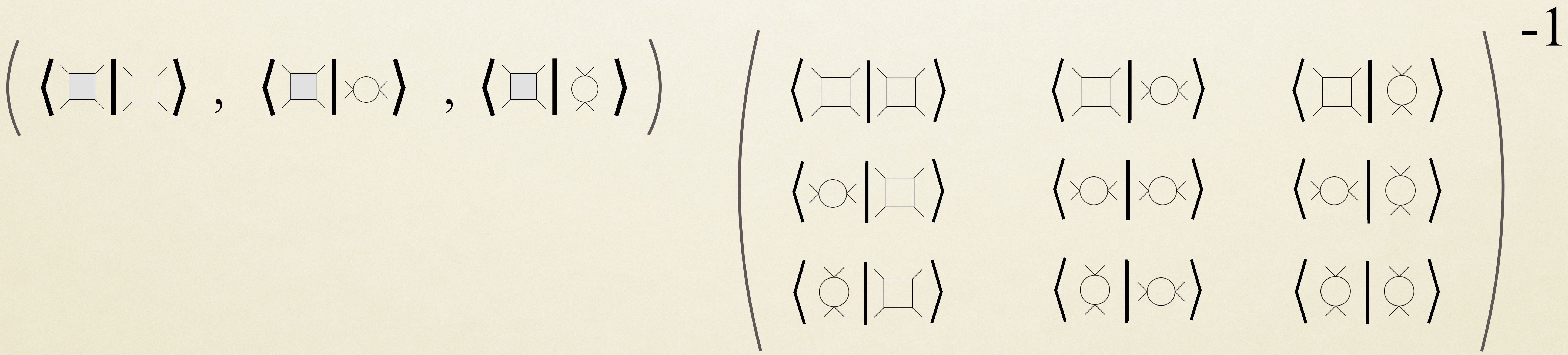


$$u(\mathbf{z}) = ((st - sz_4 - tz_3)^2 - 2tz_1(s(t+2z_3-z_2-z_4)+tz_3) + s^2 z_2^2 + t^2 z_1^2 - 2sz_2(t(s-z_3)+z_4(s+2t)))^{\frac{d-5}{2}}$$

- **Integral Decomposition**



$(c_1, c_2, c_3) =$

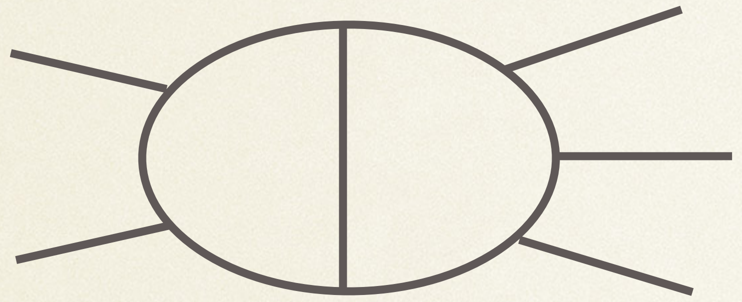


- see Extra slides for details

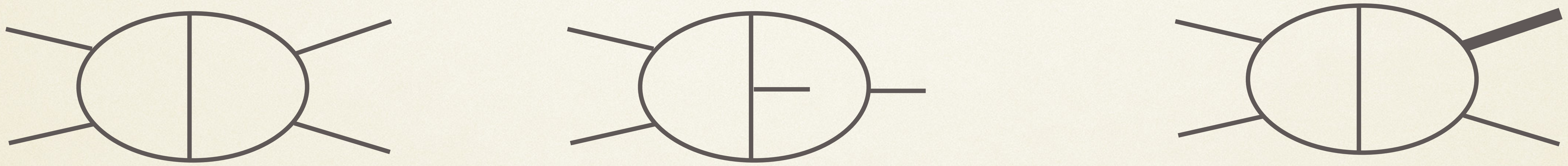
Recent Applications

Brunello, Chestnov, **Crisanti**, Frellesvig, Gasparotto, Mandal & P.M. (in progress)

- **2-loop 5-point integrals**



- **2-loop 4-point integrals**

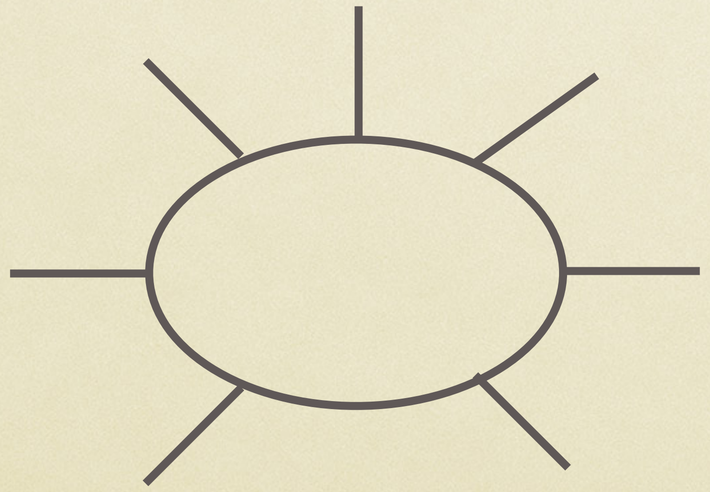


credit **Brunello**

- **1-Loop 6-point**

credit **Brunello**

- **1-Loop 7-point**



credit **Brunello**

Intersection Numbers for n-forms :: nPDE

Chestnov, Frellesvig, Gasparotto, Mandal **& P.M.** (2022)

Multivariate Intersection Numbers (II)

Matsumoto (1998)

Chestnov, Frellesvig, Gasparotto, Mandal **& P.M.** (2022)

$$\langle \varphi_L^{(\mathbf{n})} \,|\, \varphi_R^{(\mathbf{n})} \rangle = (2\pi \mathrm{i})^{-n} \int_X (u\, \varphi_{L,c}^{(\mathbf{n})}) \wedge (u^{-1} \varphi_R^{(\mathbf{n})}) = \sum_{p \in \mathbb{P}_\omega} \mathrm{Res}_{z=p}(\psi\, \varphi_R^{(\mathbf{n})})$$

- **nPDE**

$$\boxed{\nabla_{\omega_1} \nabla_{\omega_2} \ldots \nabla_{\omega_n} \psi = \varphi_L^{(\mathbf{n})}}$$

Intersection Numbers for n-forms: Pfaffian systems

Chestnov, Gasparotto, Mandal, Munch, Matsubara-Heo, Takayama **& P.M.** (2022)

Multivariate Intersection Numbers (III) from Pfaffian D-module systems

Chestnov, Gasparotto, Mandal, Munch, Matsubara-Heo, Takayama **& P.M.** (2022)

Let $\{e_i\}_{i=1}^r$ be a basis for $\mathbb{H}^n$ and $\{h_i\}_{i=1}^r$ a basis for $\mathbb{H}^{n\vee}$

$\varphi \in \mathbb{H}^n$ in terms of $\{e_i\}_{i=1}^r$

- **Secondary Equations**

$$\partial_x \mathbf{C} = \mathbf{\Omega}.\mathbf{C} + \mathbf{C}.\tilde{\mathbf{\Omega}} \,, \qquad \partial_x \mathbf{C}^{-1} = \tilde{\mathbf{\Omega}}.\mathbf{C}^{-1} - \mathbf{C}^{-1}.\mathbf{\Omega}$$

1) Build them from Macaulay Matrix for D-module

Multivariate Intersection Numbers (III) from Pfaffian D-module systems

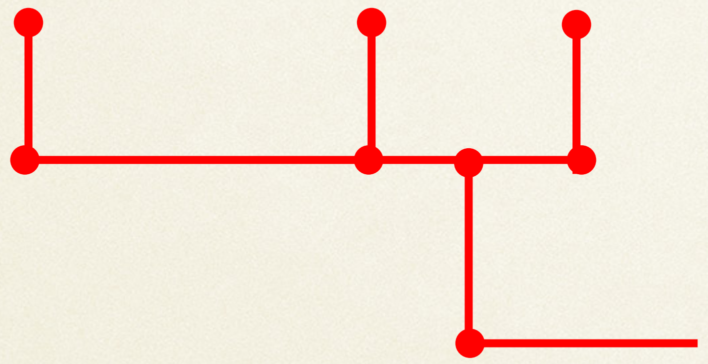
Chestnov, Gasparotto, Mandal, Munch, Matsubara-Heo, Takayama **& P.M.** (2022)

Let $\{e_i\}_{i=1}^r$ be a basis for $\mathbb{H}^n$ and $\{h_i\}_{i=1}^r$ a basis for $\mathbb{H}^{n\vee}$

$\varphi \in \mathbb{H}^n$ in terms of $\{e_i\}_{i=1}^r$

- **Secondary Equations**

$$\partial_x \mathbf{C} = \mathbf{\Omega}.\mathbf{C} + \mathbf{C}.\tilde{\mathbf{\Omega}} \,, \qquad \partial_x \mathbf{C}^{-1} = \tilde{\mathbf{\Omega}}.\mathbf{C}^{-1} - \mathbf{C}^{-1}.\mathbf{\Omega}$$



2) Rational Solutions of Secondary Equations [integrable connections]

Barkatou et al. @ MAPLE

Direct determination of Intersection Matrices

Multivariate Intersection Numbers (III) from Pfaffian D-module systems

Chestnov, Gasparotto, Mandal, Munch, Matsubara-Heo, Takayama **& P.M.** (2022)

Let $\{e_i\}_{i=1}^{r}$ be a basis for $\mathbb{H}^n$ and $\{h_i\}_{i=1}^{r}$ a basis for $\mathbb{H}^{n\vee}$

$\varphi \in \mathbb{H}^n$ in terms of $\{e_i\}_{i=1}^{r}$

- **Secondary Equations**

$$\partial_x \mathbf{C} = \mathbf{\Omega}.\mathbf{C} + \mathbf{C}.\tilde{\mathbf{\Omega}}\,, \qquad \partial_x \mathbf{C}^{-1} = \tilde{\mathbf{\Omega}}.\mathbf{C}^{-1} - \mathbf{C}^{-1}.\mathbf{\Omega}$$

- **Master Decomposition**

$$\langle \varphi | = \sum_{\lambda=1}^{r} c_\lambda \langle e_\lambda | \,,$$

$$\begin{bmatrix} e_1 \\ \vdots \\ e_{r-1} \\ \varphi \end{bmatrix} = C^{\mathrm{aux}} \cdot C^{-1} \begin{bmatrix} e_1 \\ \vdots \\ e_{r-1} \\ e_r \end{bmatrix} \quad \Longrightarrow \quad C^{\mathrm{aux}} \cdot C^{-1} = \left[\begin{array}{ccc|c} & & & 0 \\ & \mathrm{id}_{r-1} & & \vdots \\ & & & 0 \\ \hline c_1 & \cdots & c_{r-1} & c_r \end{array}\right]$$

Coefficients from matrix multiplication

Intersections Numbers @ QM and QFT

Cacciatori **& P.M.** (2022)

(Special) Applications of Intersection Numbers for 1-forms

- **Looking at a known landscapes with new eyes**

1. Identify a univariate twisted period integral $\int_\Gamma \mu\,\varphi$

   If μ is not multivalued, replace it with the regulated twist $u = u(\rho)$ by introducing a regulator ρ, so that, for a suitable value ρ_0, $u(\rho_0) = \mu$.

2. After choosing the bases of forms $e_i \equiv \hat{e}_i\,dz$ and dual forms $h_i \equiv \hat{h}_i\,dz$, with $\hat{h}_i = \hat{e}_i$, such that $\hat{e}_1 = \hat{h}_1 = 1$, decompose φ

   - **Master Decomposition formula** $\varphi = c_1\,e_1 + c_2\,e_2 + \ldots + c_\nu\,e_\nu$

3. Translate the decomposition of φ to the one of the corresponding integral, (eventually, taking the $\rho \to \rho_0$ limit)

$$\int_\Gamma \mu\,\varphi = c_1\,E_1 + c_2\,E_2 + \ldots + c_\nu\,E_\nu\,, \quad \text{with} \quad E_1 \equiv \int_\Gamma \mu\,dz\,, \quad \text{and} \quad E_j = \int_\Gamma \mu\,e_j\,, \quad (j \neq 1)\,,$$

   and compare the result with the literature.

Orthogonal Polynomials and Matrix Elements in QM

Case i) $$I_{nm} \equiv \int_\Gamma P_n(z)\, P_m(z)\, f(z)\, dz\,,$$

Case ii) $$I_{nm} \equiv \langle n|\mathscr{O}|m\rangle = \int_\Gamma \psi_n^*(z)\, \mathscr{O}(z)\, \psi_m(z)\, f(z)\, dz$$

- **Master Decomposition formula**

For the considered cases, we obtain: $\varphi = c_1\, e_1\,,$ in terms of just one basic form, $e_1 = dz$

corresponding to: $I_{nm} = c_1\, E_1$ (one master integral)

i) Orthogonal Polynomials

Laguerre $L_n^{(\rho)}$, Legendre P_n, Tchebyshev T_n, Gegenbauer $C_n^{(\rho)}$, and Hermite H_n polynomials:

$$I_{nm} \equiv \int_\Gamma \mu\, P_n P_m\, dz = f_n\, \delta_{nm} \; = \int_\Gamma \mu\, \varphi = c_1\, E_1 \qquad\qquad \varphi \equiv P_n\, P_m\, dz$$

Type	u	ν	$\hat{e}_i$	**C**-matrix	ρ_0	E_1	c_1
$L_n^{(\rho)}$	$z^\rho \exp(-z)$	1	1	ρ	–	$\Gamma(1+\rho)$	$(\rho+1)(\rho+2)\cdots(\rho+n)/n!$
P_n	$(z^2-1)^\rho$	1	1	$2\rho/(4\rho^2-1)$	0	2	$1/(2n+1)$
T_n	$(1-z^2)^\rho$	1	1	$2\rho/(4\rho^2-1)$	$-1/2$	π	$1/2$
$C_n^{(\rho)}$	$(1-z^2)^{\rho-1/2}$	1	1	$(1-2\rho)/(4\rho(\rho-1))$	–	$\sqrt{\pi}\Gamma(1/2+\rho)/\Gamma(1+\rho)$	$\rho(2\rho(2\rho+1)\cdots(2\rho+n-1))/((n+\rho)n!)$
H_n	$z^\rho \exp(-z^2)$	2	$1, 1/z$	$\mathrm{diagonal}(1/2, 1/\rho)$	0	$\sqrt{\pi}$	$2^n n!$

Let us observe that, in the case of Hermite polynomials, $\nu = 2$, yielding $\varphi = c_1\, e_1 + c_2\, e_2$, but $c_2 = 0$, due to the adopted basis

ii) Matrix Elements in QM

Harmonic Oscillator. (for unitary mass and pulsation, $m = 1 = \omega$)

$$\langle z|n\rangle = \psi_n(z) = e^{-\frac{z^2}{2}}\, W_n(z)\ , \quad \text{with} \qquad W_n(z) \equiv N_n H_n(z)\ , \qquad\qquad N_n \equiv 1/\sqrt{(2^n n!\sqrt{\pi})}$$

- **Position operator**

$$\langle m|z^k|n\rangle = \int_{-\infty}^{\infty} dz\, \psi_m(z)\, z^k\, \psi_n(z) = \int_\Gamma \mu\, \varphi = c_1 E_1\ , \quad \text{with} \qquad \mu \equiv e^{-z^2}\ , \quad \text{and} \quad \varphi \equiv W_m(z)\, z^k\, W_n(z)\, dz.$$

Type	u	ν	$\hat{e}_i$	**C**-matrix	ρ_0	E_1
W_n	$z^\rho \exp(-z^2)$	2	$1, 1/z$	$\text{diagonal}(1/2, 1/\rho)$	0	$\sqrt{\pi}$

$$\begin{aligned}
\langle n|m\rangle &= \delta_{nm}\ ,\\
\langle n|z^{2k+1}|n\rangle &= 0\ ,\\
\langle n|z^4|n\rangle &= \frac{3}{4}(2n^2 + 2n + 1)\ ,\\
\langle n|z^3|n-3\rangle &= \sqrt{n(n-1)(n-2)/8}\ ,\\
\langle n|z^3|n-1\rangle &= \sqrt{9n^3/8}\ .
\end{aligned}$$

- **Hamiltonian operator**

$$\langle n|H|n\rangle = (n + 1/2) \qquad\qquad H \equiv (1/2)(-\nabla^2 + z^2) \qquad\qquad \varphi = \textstyle\sum_{k=0}^{n} b_k\, z^{2k}$$

ii) Matrix Elements in QM

Hydrogen Atom. (for unitary Bohr radius $a_0 = 1$)

$$\langle z|n,\ell\rangle = R_{n,\ell}(z) = e^{-\frac{z}{n}}\, W_{n,\ell}(z)\;, \quad \text{with} \qquad W_{n,\ell}(z) \equiv N_{n\ell}\left(\frac{2z}{n}\right)^{\ell} L^{2\ell+1}_{(n-\ell-1)}\left(\frac{2z}{n}\right) \qquad N_{n\ell} = (2/n)^{3/2}\sqrt{(n-\ell-1)!/(2n\,(n+\ell)!)}$$

- **Position operator**

$$\langle n_1,\ell|z^k|n_2,\ell\rangle = \int_0^\infty dz\, z^2\, R_{n_1,\ell}(z)\, z^k\, R_{n_2,\ell}(z) \;= \int_\Gamma \mu\,\varphi = c_1\, E_1\;, \quad \text{with} \quad \mu \equiv z^2\, e^{-z\left(\frac{1}{n_1}+\frac{1}{n_2}\right)}, \text{ and } \varphi \equiv W_{n_1,\ell}(z)\, z^k\, W_{n_2,\ell}(z)$$

Type	u	ν	$\hat{e}_i$	C-matrix	ρ_0	E_1
$W_{n,\ell}$	$z^{\rho+2}\exp(-z(n_1+n_2)/(n_1 n_2))$	1	1	$(n_1 n_2/(n_1+n_2))^2(2+\rho)$	0	$2(n_1 n_2/(n_1+n_2))^3$

$$\langle n_1,\ell|n_2,\ell\rangle = \delta_{n_1 n_2}\;,$$
$$\langle n,\ell|z|n,\ell\rangle = \frac{1}{2}[3n^2 - \ell(\ell+1)]\;,$$
$$\langle n,\ell|z^{-1}|n,\ell\rangle = \frac{1}{n^2}\;,$$

$$\langle n,\ell|z^{-2}|n,\ell\rangle = \frac{2}{n^3(2\ell+1)}\;,$$
$$\langle n,\ell|z^{-3}|n,\ell\rangle = \frac{2}{n^3\ell(\ell+1)(2\ell+1)}$$

Green's Function and Kontsevich-Witten tau-function

Case i) $$G_n \equiv \frac{\int \mathscr{D}\phi\, \phi(x_1)\cdots\phi(x_n)\, \exp[-S_E]}{\int \mathscr{D}\phi\ \exp[-S_E]}$$

Weinzierl (2020)

Case ii) $$Z_{KW} \equiv \frac{\int d\Phi \exp\left[-\mathrm{Tr}\left(-\frac{i}{3!}\Phi^3 + \frac{\Lambda}{2}\Phi^2\right)\right]}{\int d\Phi \exp\left[-\mathrm{Tr}\left(\frac{\Lambda}{2}\Phi^2\right)\right]}$$

$$c_1 = \frac{\int_\Gamma \mu\, \varphi}{\int_\Gamma \mu\, e_1}\,, \quad \text{equivalently rewritten as} \quad \int_\Gamma \mu\, \varphi = c_1\, E_1$$

- **Master Decomposition formula**

- **Toy models univariate integrals**

i) Green's Function

Single field, ϕ^4-theory

real scalar field $\phi(x)$ $\qquad$ $S_E \equiv S_0 + \varepsilon S_1$, with $S_0 = (\gamma/2)\,\phi^2(x)$, and $S_1 = \phi^4(x)$

$$\int \mathscr{D}\phi\, \phi(x_1)\cdots\phi(x_n)\, e^{-S_E} = G_n \int \mathscr{D}\phi\; e^{-S_E}$$

$$\int_\Gamma \mu\,\varphi = G_n E_1\,, \quad \text{with} \quad \mu \equiv e^{-S_E}\,, \quad \varphi \equiv \phi(x_1)\cdots\phi(x_n)\,\mathscr{D}\phi\,, \quad E_1 \equiv \int_\Gamma \mu\, e_1\,, \quad \text{and} \quad e_1 \equiv \mathscr{D}\phi$$

Free theory. The n-point Green's function $G_n^{(0)}$ $\qquad$ $\phi(x) \equiv z$ $\qquad$ $\mu \equiv e^{-S_0}$ $\qquad$ $\varphi = z^n\, dz$

Type	u	ν	$\hat{e}_i$	**C**-matrix	ρ_0	E_1	c_1
$G_n^{(0)}$	$z^\rho \exp(-\gamma z^2/2)$	2	$1, 1/z$	diagonal$(1/\gamma, 1/\rho)$	0	not needed	$(n-1)!!/\gamma^{n/2}$

for even n

i) Green's Function

Single field, ϕ^4-theory

real scalar field $\phi(x)$ $\quad S_E \equiv S_0 + \varepsilon S_1\,,$ with $S_0 = (\gamma/2)\,\phi^2(x)\,,$ and $\;S_1 = \phi^4(x)$

$$\int \mathscr{D}\phi\,\phi(x_1)\cdots\phi(x_n)\,e^{-S_E} = G_n \int \mathscr{D}\phi\;e^{-S_E}$$

$$\int_\Gamma \mu\,\varphi = G_n\,E_1\,, \quad \text{with} \quad \mu \equiv e^{-S_E}\,, \quad \varphi \equiv \phi(x_1)\cdots\phi(x_n)\,\mathscr{D}\phi\,, \quad E_1 \equiv \int_\Gamma \mu\,e_1\,, \quad \text{and} \quad e_1 \equiv \mathscr{D}\phi$$

Free theory. The n-point Green's function $G_n^{(0)}$ $\qquad \phi(x) \equiv z \qquad \mu \equiv e^{-S_0} \qquad \varphi = z^n\,dz$

Type	u	ν	$\hat{e}_i$	**C**-matrix	ρ_0	E_1	c_1
$G_n^{(0)}$	$z^\rho \exp(-\gamma\,z^2/2)$	2	$1, 1/z$	diagonal$(1/\gamma, 1/\rho)$	0	not needed	$(n-1)!!/\gamma^{n/2}$

for even n

- **2-point function: the propagator** $\quad G_2^{(0)} = 1/\gamma$

Perturbation Theory. The n-point correlation function G_n in the full theory can be computed perturbatively, in the small coupling limit, $\varepsilon \to 0$, and expressed in terms of $G_n^{(0)}$. For example, the determination of the next-to-leading order (NLO) corrections to the 2-point function, proceeds as follows,

$$G_2 = \frac{\int dz\;z^2\;e^{-S_0-\epsilon S_1}}{\int dz\;e^{-S_0-\epsilon S_1}} = \frac{\int dz\;z^2\;e^{-S_0}(1-\epsilon S_1+\ldots)}{\int dz\;e^{-S_0}(1-\epsilon S_1+\ldots)} = \left(G_2^{(0)} - \epsilon\,G_6^{(0)} + \ldots\right)\left(1 + \epsilon\,G_4^{(0)} + \ldots\right) = G_2^{(0)} + \epsilon\left(G_2^{(0)}G_4^{(0)} - G_6^{(0)}\right) + \mathcal{O}(\epsilon^2)$$

$$= \frac{1}{\gamma}\left(1 - 12\epsilon\frac{1}{\gamma^2}\right) + \mathcal{O}(\epsilon^2)$$

i) Green's Function

Single field, ϕ^4-theory

real scalar field $\phi(x)$ $\qquad S_E \equiv S_0 + \varepsilon S_1$, with $S_0 = (\gamma/2)\,\phi^2(x)$, and $S_1 = \phi^4(x)$

Exact theory. $\qquad \phi(x) \equiv z \qquad \mu \equiv e^{-S_E} \qquad \varphi = z^n\, dz$

$$u \equiv z^\rho\, \mu \qquad \nu = 4,$$

$$\{\hat{e}_1, \hat{e}_2, \hat{e}_3, \hat{e}_4\} = \{1, 1/z, z, z^2\},$$

$$\{\hat{h}_i\}_{i=1}^4 = \{\hat{e}_i\}_{i=1}^4,$$

$$\mathbf{C} = \begin{pmatrix} 0 & 0 & 0 & \frac{1}{4\gamma} \\ 0 & \frac{1}{\rho} & 0 & 0 \\ 0 & 0 & \frac{1}{4\gamma} & 0 \\ \frac{1}{4\gamma} & 0 & 0 & -\frac{\gamma}{16\epsilon^2} \end{pmatrix}$$

For instance, let us consider the decomposition:

$$\varphi = z^4\, dz = c_1\, e_1 + c_2\, e_2 + c_3\, e_3 + c_4\, e_4 \qquad c_1 = \frac{1}{4\epsilon}\,, \quad c_2 = 0\,, \quad c_3 = 0\,, \qquad c_4 = -\frac{\gamma}{4\epsilon}$$

$$\int_\Gamma dz\, z^4\, e^{-S_E} = c_1 \int_\Gamma dz\, e^{-S_E} + c_4 \int_\Gamma dz\, z^2\, e^{-S_E} \qquad G_4 = c_1 + c_4 G_2 \qquad G_2 = \frac{1}{\gamma}\Big(1 - 4\epsilon G_4\Big)$$

ii) Kontsevich-Witten tau-function

$$Z_{KW} \equiv \frac{\int d\Phi \, \exp\left[-\operatorname{Tr}\left(-\frac{i}{3!}\Phi^3 + \frac{\Lambda}{2}\Phi^2 \right)\right]}{\int d\Phi \, \exp\left[-\operatorname{Tr}\left(\frac{\Lambda}{2}\Phi^2 \right)\right]}$$

- **Univariate Model** Itzykson-Zuber (1992)

$Z_{KW} = \sum_{n=0}^{\infty} Z_{KW}^{(n)}$.

$$\int_\Gamma \mu \, \varphi = c_1 \, E_1$$

$c_1 = Z_{KW}^{(n)}$

$\varphi \equiv N_n \, z^{6n}$, $N_n \equiv \varepsilon^{2n}$ $\varepsilon \equiv i/(3!)(\Lambda/2)^{-3/2}$

Type	u	ν	$\hat{e}_i$	**C**-matrix	ρ_0	E_1	c_1
$Z_{KW}^{(n)}$	$z^\rho \exp(-z^2)$	2	$1, 1/z$	diagonal$(1/2, 1/\rho)$	0	not needed	$(-2/9)^n \, (\Lambda^{-3n}/(2n)!) \prod_{j=0}^{3n-1} (j+1/2)$

Intersections Numbers @ this workshop

[Piementel] **+ Brunello** & P.M.

AH-B-P like integral

$$I = \int dz_1 \wedge dz_2 \frac{(z_1 z_2)^\epsilon}{(z_1+y_1+1)(z_2+y_2+1)(z_1+z_2+y_1+y_2)}$$

curtesy **Piementel**

Brunello & P.M.

- **Twisted Period Integrals**

$$I = \int_{\mathcal{C}} u(z_1,z_2)\varphi(z_1,z_2) \qquad u = (z_1 z_2)^\epsilon (D_1 D_2 D_3)^\gamma \qquad D_1 = (z_1+y_1+1)\,, \quad D_2 = (z_2+y_2+1)\,, \quad D_3 = (z_1+z_2+y_1+y_2)$$

γ is a regulator

$$\omega = d\log(u) = \omega_1 dz_1 + \omega_2 dz_2 \qquad \omega_1 = \frac{\gamma(2y_1+y_2+2z_1+z_2+1)}{(y_1+z_1+1)(y_1+y_2+z_1+z_2)} + \frac{\epsilon}{z_1} \qquad \omega_2 = \frac{\gamma(y_1+2y_2+z_1+2z_2+1)}{(y_2+z_2+1)(y_1+y_2+z_1+z_2)} + \frac{\epsilon}{z_2}$$

AH-B-P like integral

$$I = \int dz_1 \wedge dz_2 \frac{(z_1 z_2)^{\epsilon}}{(z_1+y_1+1)(z_2+y_2+1)(z_1+z_2+y_1+y_2)}$$

curtesy **Piementel**

Brunello & P.M.

- **Twisted Period Integrals**

$$I = \int_{\mathcal{C}} u(z_1,z_2)\varphi(z_1,z_2) \qquad u = (z_1 z_2)^{\epsilon}(D_1 D_2 D_3)^{\gamma} \qquad D_1 = (z_1+y_1+1)\,, \quad D_2 = (z_2+y_2+1)\,, \quad D_3 = (z_1+z_2+y_1+y_2)$$

γ is a regulator

$$\omega = d\log(u) = \omega_1 dz_1 + \omega_2 dz_2 \qquad \omega_1 = \frac{\gamma(2y_1+y_2+2z_1+z_2+1)}{(y_1+z_1+1)(y_1+y_2+z_1+z_2)} + \frac{\epsilon}{z_1} \qquad \omega_2 = \frac{\gamma(y_1+2y_2+z_1+2z_2+1)}{(y_2+z_2+1)(y_1+y_2+z_1+z_2)} + \frac{\epsilon}{z_2}$$

- **Number of MIs = dimH and bases choice**

$$\omega_2 = 0 \qquad \nu_2 = 2 \qquad e^{(2)} = h^{(2)} = \left\{\frac{1}{D_1}, \frac{1}{D_2}\right\}$$

- **2 MIs in the internal layer**

$$\begin{cases}\omega_1 = 0\\ \omega_2 = 0\end{cases} \qquad \nu = 3 \qquad e^{(21)} = h^{(21)} = \left\{\frac{1}{D_1 D_3}, \frac{1}{D_2 D_3}, \frac{1}{D_1 D_2 D_3}\right\}$$

- **3 MIs in the external layer**

AH-B-P like integral

Brunello & P.M.

$$I = \int dz_1 \wedge dz_2 \frac{(z_1 z_2)^\epsilon}{(z_1+y_1+1)(z_2+y_2+1)(z_1+z_2+y_1+y_2)}$$

curtesy **Piementel**

- **Twisted Period Integrals**

$$I = \int_{\mathcal{C}} u(z_1,z_2)\varphi(z_1,z_2) \qquad u = (z_1 z_2)^\epsilon (D_1 D_2 D_3)^\gamma \qquad D_1 = (z_1+y_1+1)\,, \quad D_2 = (z_2+y_2+1)\,, \quad D_3 = (z_1+z_2+y_1+y_2)$$

γ is a regulator

$$\omega = d\log(u) = \omega_1 dz_1 + \omega_2 dz_2 \qquad \omega_1 = \frac{\gamma(2y_1+y_2+2z_1+z_2+1)}{(y_1+z_1+1)(y_1+y_2+z_1+z_2)} + \frac{\epsilon}{z_1} \qquad \omega_2 = \frac{\gamma(y_1+2y_2+z_1+2z_2+1)}{(y_2+z_2+1)(y_1+y_2+z_1+z_2)} + \frac{\epsilon}{z_2}$$

- **Number of MIs = dimH and bases choice**

$$\omega_2 = 0 \qquad \nu_2 = 2 \qquad e^{(2)} = h^{(2)} = \left\{\frac{1}{D_1}, \frac{1}{D_2}\right\}$$

- **2 MIs in the internal layer**

$$\begin{cases} \omega_1 = 0 \\ \omega_2 = 0 \end{cases} \qquad \nu = 3 \qquad e^{(21)} = h^{(21)} = \left\{\frac{1}{D_1 D_3}, \frac{1}{D_2 D_3}, \frac{1}{D_1 D_2 D_3}\right\}$$

- **3 MIs in the external layer**

- **Intersection Matrix**

$$C = \begin{pmatrix} \frac{2(\gamma+\epsilon)^2}{\gamma^2(2\gamma+\epsilon)(3\gamma+2\epsilon)} & \frac{1}{\gamma(3\gamma+2\epsilon)} & \frac{1}{\gamma^2} \\ \frac{1}{\gamma(3\gamma+2\epsilon)} & \frac{2(\gamma+\epsilon)^2}{\gamma^2(2\gamma+\epsilon)(3\gamma+2\epsilon)} & \frac{1}{\gamma^2} \\ \frac{1}{\gamma^2} & \frac{1}{\gamma^2} & \frac{3}{\gamma^2} \end{pmatrix}$$

AH-B-P like integral

Brunello & P.M.

$$I = \int dz_1 \wedge dz_2 \frac{(z_1 z_2)^\epsilon}{(z_1+y_1+1)(z_2+y_2+1)(z_1+z_2+y_1+y_2)}$$

curtesy **Piementel**

- **3 MIs**

$$e^{(21)} = \left\{ \frac{1}{D_1 D_3}, \frac{1}{D_2 D_3}, \frac{1}{D_1 D_2 D_3} \right\}$$

- **System of Differential Equations**

$$\partial_x \langle e_i | = \Omega_{ij} \langle e_j |$$

- **Master Decomposition Formula**

$$\Omega_{ij} = \langle (\partial_x + \sigma_x) e_i | h_k \rangle \, (\mathbf{C}^{-1})_{kj}$$

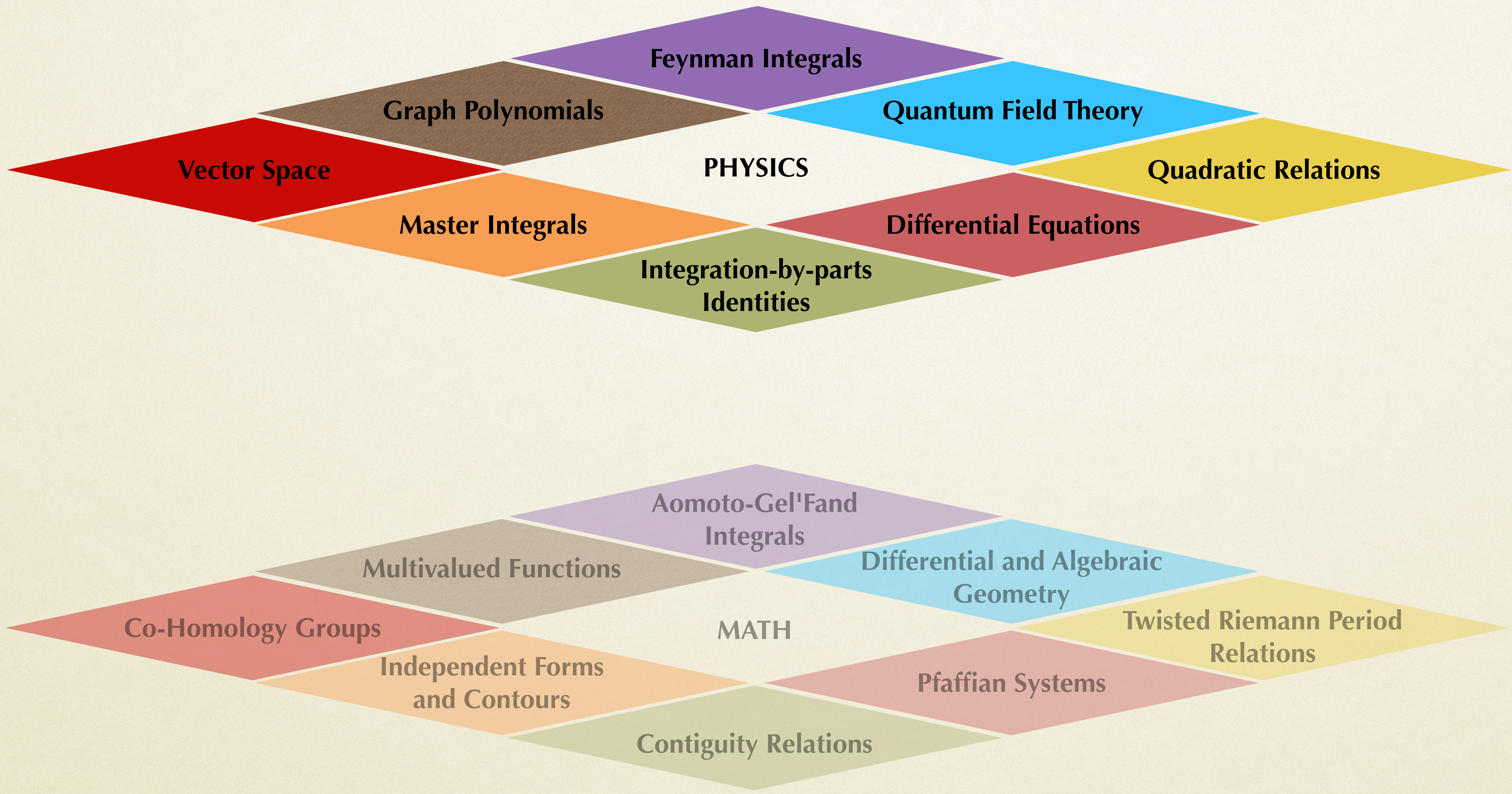
after taking the limit $\gamma \to 0$:

$$\Omega_{y_1} = \begin{pmatrix} \frac{\epsilon}{y_1+1} & 0 & 0 \\ 0 & \frac{\epsilon}{y_1} & 0 \\ 0 & \frac{\epsilon}{y_1(y_1+1)} & \frac{\epsilon}{y_1+1} \end{pmatrix}$$

$$\Omega_{y_2} = \begin{pmatrix} \frac{\epsilon}{y_2} & 0 & 0 \\ 0 & \frac{\epsilon}{y_2+1} & 0 \\ \frac{\epsilon}{y_2(y_2+1)} & 0 & \frac{\epsilon}{y_2+1} \end{pmatrix}$$

- **Canonical system**

To Conclude:



Feynman Integrals

Graph Polynomials

Quantum Field Theory

Vector Space

PHYSICS

Quadratic Relations

Master Integrals

Differential Equations

Integration-by-parts Identities

Quantum Field Theory

Aomoto-Gel'Fand Integrals

Multivalued Functions

Differential and Algebraic Geometry

Co-Homology Groups

MATH

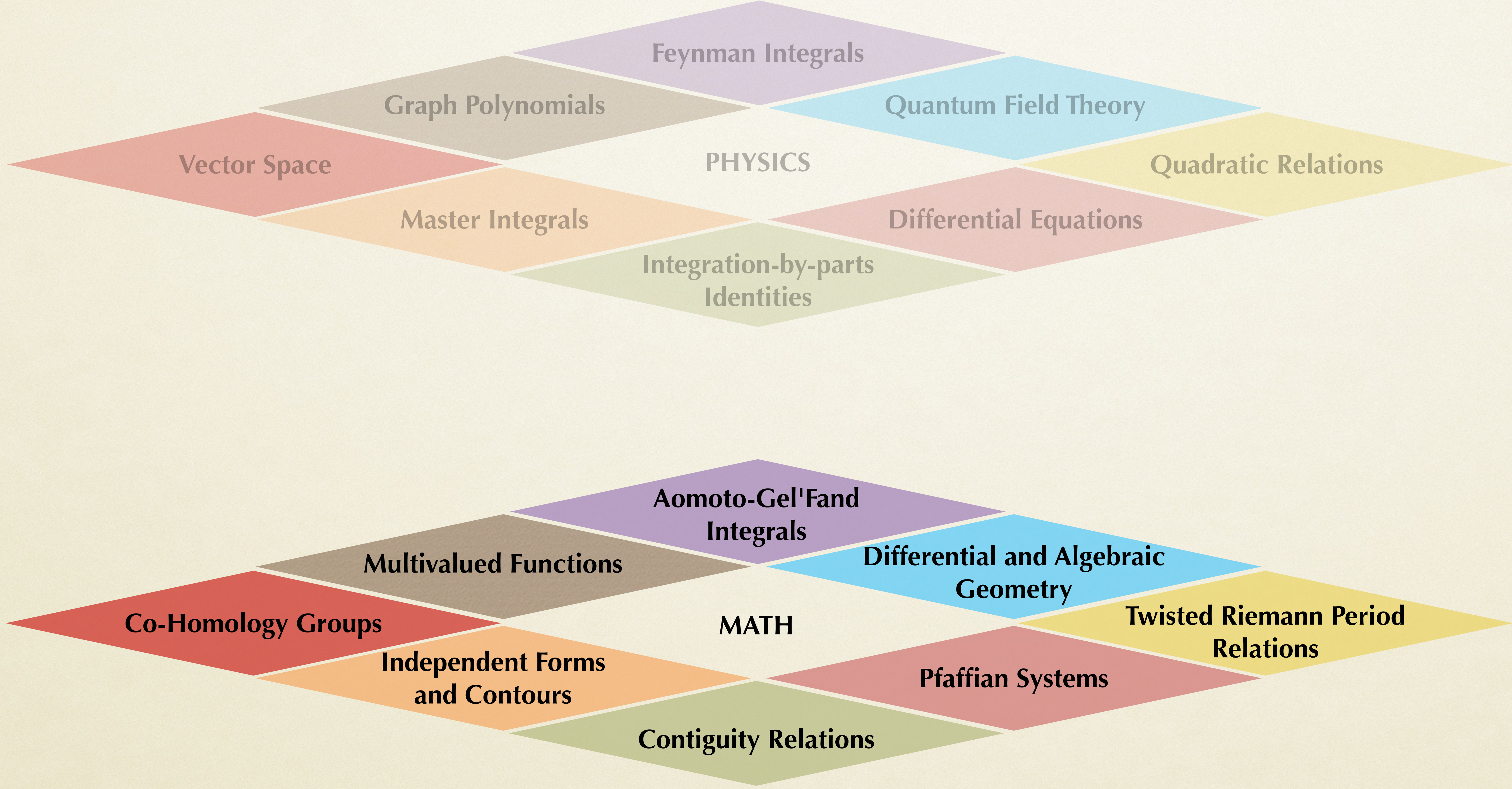
Twisted Riemann Period Relations

Independent Forms and Contours

Pfaffian Systems

Contiguity Relations

Twisted de Rham Theory



Feynman Integrals
Graph Polynomials
Quantum Field Theory
Vector Space
PHYSICS
Quadratic Relations
Master Integrals
Differential Equations
Integration-by-parts Identities

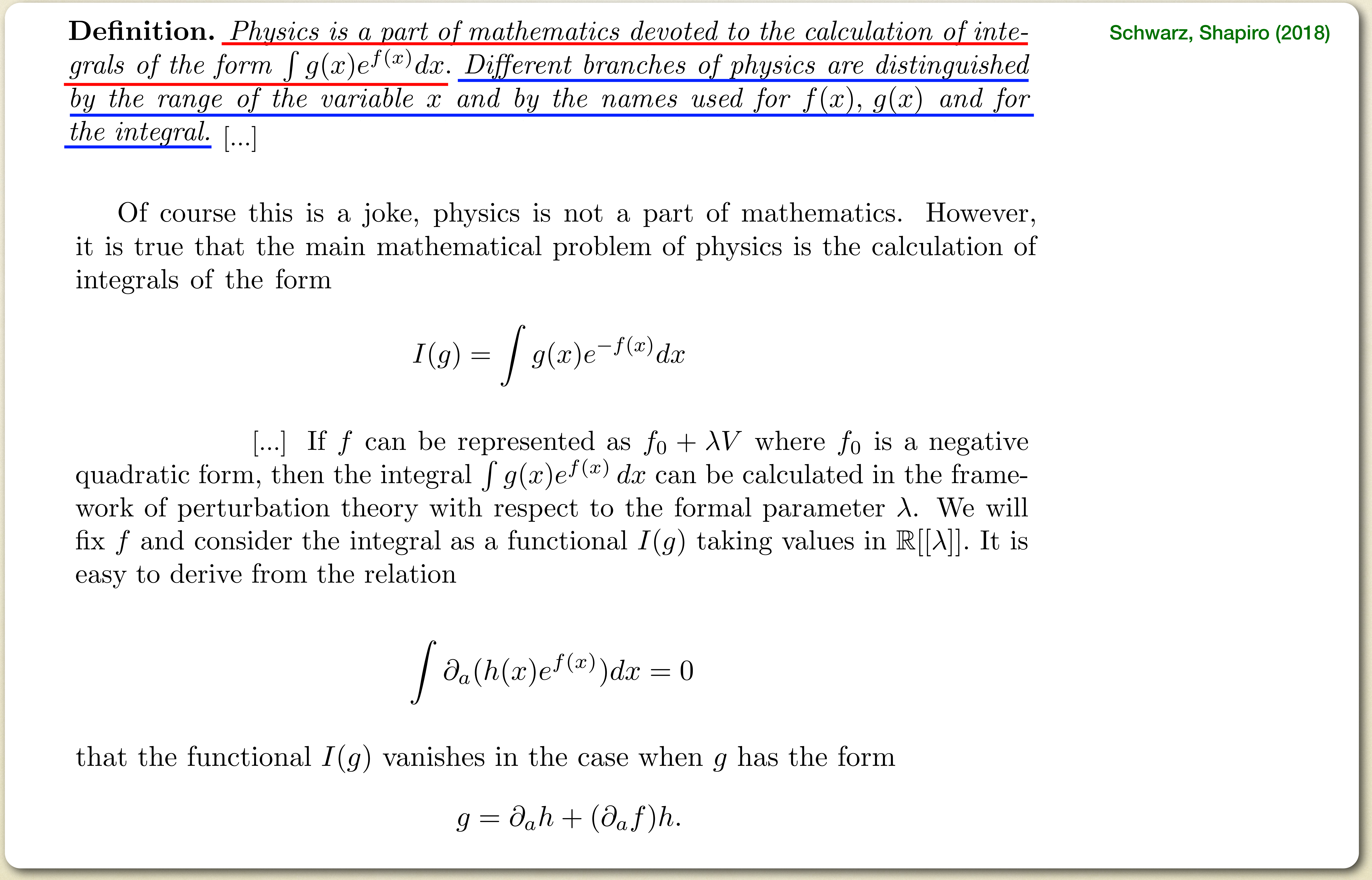
Quantum Field Theory

Aomoto-Gel'Fand Integrals
Multivalued Functions
Differential and Algebraic Geometry
Co-Homology Groups
MATH
Twisted Riemann Period Relations
Independent Forms and Contours
Pfaffian Systems
Contiguity Relations

Twisted de Rham Theory

Summary

- **Novel Mathematical Sctructure for Quantum Field Theory Integrals came into view**
    - Intersection Theory for Twisted de Rham co-homology
    - Rich theory :: Differential and Algebraic Geometry, Topology, Number Theory, Combinatorics
- **Novel Concepts: Vector Space Structures**
    - Space dimensions = Dimension of co-homology group = number of independent Integrals
    - Intersection Numbers ~ **Scalar Product** for Feynman Integrals
- **New Methods for Multivariate Intersection number**
    - Iterative method
    - Higher-Order PDE method
    - Secondary equation (Pfaffians via Macaulay)
- **General algorithm for Physics and Math applications**
    - key: Co-Homolgy Group Isomorphisms
    - Feynman Integrals, Euler-Mellin Integrals, D-Module and GKZ hypergeometric theory, Orthogonal Polynomials, QM matrix elements, Correlator functions in QFT.
- **Modern Multi-Loop diagrammatic techniques and Amplitudes calculus useful beyond Particle Physics**
    - Triggering interdisciplinarity
- **Emerging Picture**
    - Interwintwinement between Fundamental Physics, Geometry and Statistics: fluxes ~ period integrals ~ statistical moments
    - Interesting implications in QM, QFT (and Cosmology): invariance and independent moments of distributions, perturbation vs non-perturbative approaches



Schwarz, Shapiro (2018)

Definition. *Physics is a part of mathematics devoted to the calculation of integrals of the form* $\int g(x)e^{f(x)}dx$*. Different branches of physics are distinguished by the range of the variable* x *and by the names used for* $f(x)$*,* $g(x)$ *and for the integral.* [...]

Of course this is a joke, physics is not a part of mathematics. However, it is true that the main mathematical problem of physics is the calculation of integrals of the form

$$I(g) = \int g(x)e^{-f(x)}dx$$

[...] If f can be represented as $f_0 + \lambda V$ where f_0 is a negative quadratic form, then the integral $\int g(x)e^{f(x)}\,dx$ can be calculated in the framework of perturbation theory with respect to the formal parameter λ. We will fix f and consider the integral as a functional $I(g)$ taking values in $\mathbb{R}[[\lambda]]$. It is easy to derive from the relation

$$\int \partial_a(h(x)e^{f(x)})dx = 0$$

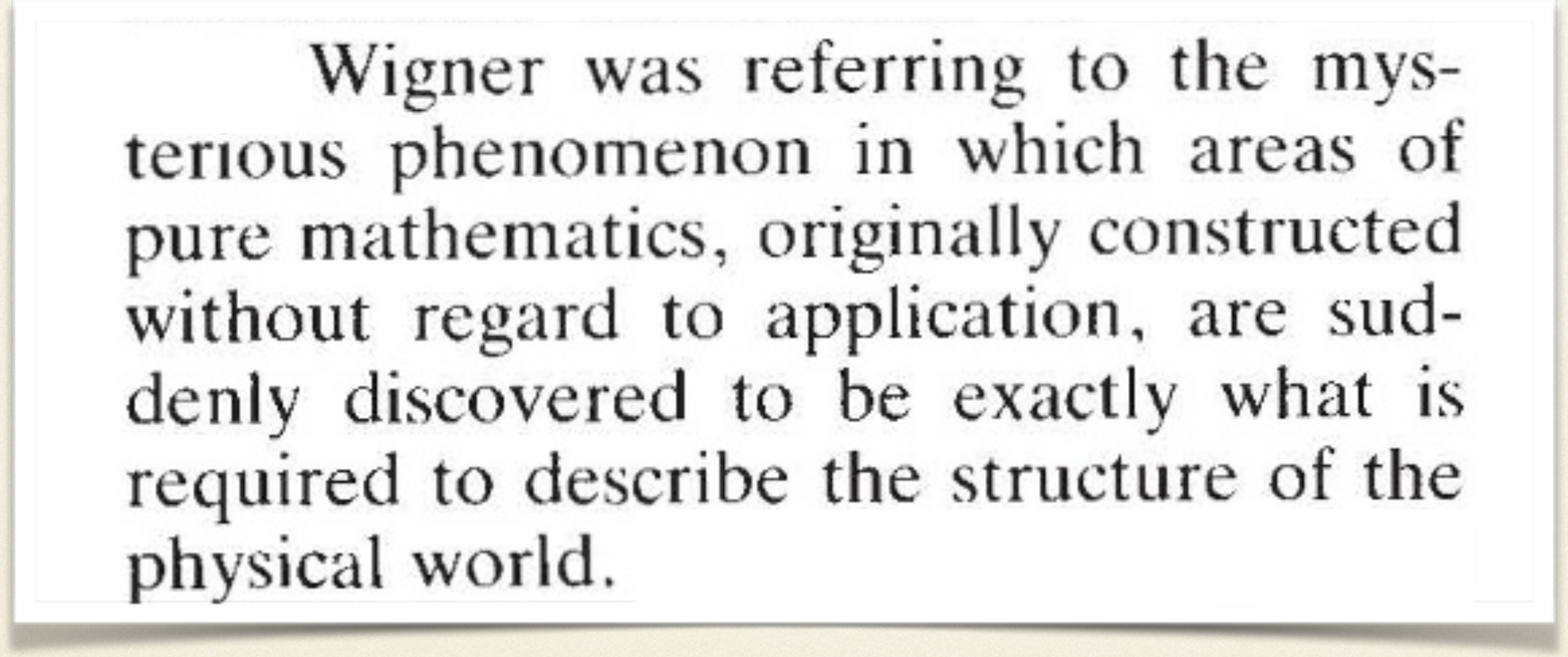
that the functional $I(g)$ vanishes in the case when g has the form

$$g = \partial_a h + (\partial_a f)h.$$

Addressing a common math problem might be useful to make progress in different disciplines

The unreasonable effectiveness of mathematics

E. Wigner



Wigner was referring to the mysterious phenomenon in which areas of pure mathematics, originally constructed without regard to application, are suddenly discovered to be exactly what is required to describe the structure of the physical world.

M. Berry

Extra Slides

Intersection Numbers for n-forms :: nPDE

Chestnov, Frellesvig, Gasparotto, Mandal **& P.M.** (2022)

Multivariate Intersection Numbers (II)

Matsumoto (1998)

Chestnov, Frellesvig, Gasparotto, Mandal **& P.M.** (2022)

$$\langle \varphi_L^{(\mathbf{n})} \mid \varphi_R^{(\mathbf{n})} \rangle = (2\pi \mathrm{i})^{-n} \int_X (u\, \varphi_{L,c}^{(\mathbf{n})}) \wedge (u^{-1} \varphi_R^{(\mathbf{n})}) = \sum_{p \in \mathbb{P}_\omega} \mathrm{Res}_{z=p}(\psi\, \varphi_R^{(\mathbf{n})})$$

- **nPDE**

$$\boxed{\nabla_{\omega_1} \nabla_{\omega_2} \ldots \nabla_{\omega_n} \psi = \varphi_L^{(\mathbf{n})}}$$

Multivariate Intersection Numbers (II)

Matsumoto (1998)

Chestnov, Frellesvig, Gasparotto, Mandal **& P.M.** (2022)

$$\langle \varphi_L^{(\mathbf{n})} \,|\, \varphi_R^{(\mathbf{n})} \rangle = (2\pi \mathrm{i})^{-n} \int_X (u\, \varphi_{L,c}^{(\mathbf{n})}) \wedge (u^{-1} \varphi_R^{(\mathbf{n})}) = \sum_{p \in \mathbb{P}_\omega} \mathrm{Res}_{z=p}(\psi\, \varphi_R^{(\mathbf{n})})$$

- **nPDE**

$$\boxed{\nabla_{\omega_1} \nabla_{\omega_2} \ldots \nabla_{\omega_n} \psi = \varphi_L^{(\mathbf{n})}}$$

Proof.

$$\eta := \bar{h}_1 \ldots \bar{h}_n \, (u\, \psi) \, (u^{-1} \varphi_R^{(\mathbf{n})}) \qquad \mathrm{d}_{z_1} \ldots \mathrm{d}_{z_n} \eta = (u\, \varphi_{L,c}) \wedge (u^{-1}\, \varphi_R) \ ,$$

$$\bar{h}_i := 1 - h_i$$

$$h_i \equiv h(z_i) := \begin{cases} 1 & \text{for } |z_i| < \epsilon \ , \\ 0 & \text{otherwise,} \end{cases}$$

$$\varphi_{L,c} := \bar{h}_1 \ldots \bar{h}_n \, \varphi_L + \ldots + (-1)^n \, \psi \, \mathrm{d}h_1 \wedge \ldots \wedge \mathrm{d}h_n \equiv \nabla_{\omega_1} \ldots \nabla_{\omega_n} (\bar{h}_1 \ldots \bar{h}_n \psi)$$

Multivariate Intersection Numbers (II)

Matsumoto (1998)

Chestnov, Frellesvig, Gasparotto, Mandal **& P.M.** (2022)

$$\langle \varphi_L^{(\mathbf{n})} \,|\, \varphi_R^{(\mathbf{n})} \rangle = (2\pi \mathrm{i})^{-n} \int_X (u\,\varphi_{L,c}^{(\mathbf{n})}) \wedge (u^{-1} \varphi_R^{(\mathbf{n})}) = \sum_{p \in \mathbb{P}_\omega} \mathrm{Res}_{z=p}(\psi\, \varphi_R^{(\mathbf{n})})$$

- **nPDE**

$$\boxed{\nabla_{\omega_1} \nabla_{\omega_2} \ldots \nabla_{\omega_n} \psi = \varphi_L^{(\mathbf{n})}}$$

Proof.

$$\eta := \bar{h}_1 \ldots \bar{h}_n \,(u\,\psi)\, (u^{-1} \varphi_R^{(\mathbf{n})}) \qquad \mathrm{d}_{z_1} \ldots \mathrm{d}_{z_n} \eta = (u\,\varphi_{L,c}) \wedge (u^{-1}\, \varphi_R) \;,$$

$$\bar{h}_i := 1 - h_i$$

$$h_i \equiv h(z_i) := \begin{cases} 1 & \text{for } |z_i| < \epsilon \;, \\ 0 & \text{otherwise,} \end{cases}$$

$$\varphi_{L,c} := \bar{h}_1 \ldots \bar{h}_n \, \varphi_L + \ldots + (-1)^n \, \psi \, \mathrm{d}h_1 \wedge \ldots \wedge \mathrm{d}h_n \equiv \nabla_{\omega_1} \ldots \nabla_{\omega_n} (\bar{h}_1 \ldots \bar{h}_n \psi)$$

$$\int_X (u\,\varphi_{L,c}^{(\mathbf{n})}) \wedge (u^{-1} \varphi_R^{(\mathbf{n})}) = \sum_{p \in \mathbb{P}_\omega} \int_{D_p} \mathrm{d}_{z_1} \ldots \mathrm{d}_{z_n} \eta \quad = (-1)^n \sum_{p \in \mathbb{P}_\omega} \int_{D_p} (u\,\psi)\, \mathrm{d}h_1 \wedge \ldots \wedge \mathrm{d}h_n \wedge (u^{-1} \varphi_R^{(\mathbf{n})})$$

$$= \sum_{p \in \mathbb{P}_\omega} \int_{\circlearrowleft_1 \wedge \ldots \wedge \circlearrowleft_n} \psi\, \varphi_R^{(\mathbf{n})} \quad = (2\pi \mathrm{i})^n \sum_{p \in \mathbb{P}_\omega} \mathrm{Res}_{z=p}(\psi\, \varphi_R^{(\mathbf{n})})$$

Intersection Numbers and Pfaffian systems

Chestnov, Gasparotto, Mandal, Munch, Matsubara-Heo, Takayama **& P.M.** (2022)

GKZ Hypergeometric Systems

Bernstein, Saito, Sturmfels, Takayama, Matsubara-Heo,
Agostini, Fevola, Sattelberger, Tellen,
De La Crux,...

- **Euler-Mellin Integral / A-Hypergeometric function**

$$f_\Gamma(z) = \int_\Gamma g(z;x)^{\beta_0}\, x_1^{-\beta_1} \cdots x_n^{-\beta_n}\, \frac{\mathrm{d}x}{x} \quad , \quad \frac{\mathrm{d}x}{x} := \frac{\mathrm{d}x_1}{x_1} \wedge \cdots \wedge \frac{\mathrm{d}x_n}{x_n}$$

$$g(z;x) = \sum_{i=1}^{N} z_i\, x^{\alpha_i} \qquad x^{\alpha_i} := x_1^{\alpha_{i,1}} \cdots x_n^{\alpha_{i,n}} \qquad A = \begin{pmatrix} a_1 & \dots & a_N \end{pmatrix} \quad (n+1) \times N \text{ matrix} \qquad a_i := (1, \alpha_i)$$

$$\mathrm{Ker}(A) = \{u = (u_1, \dots, u_N) \in \mathbb{Z}^N \,|\, \sum_{j=1}^{N} u_j\, a_j = \mathbf{0}\}$$

- **GKZ system of PDEs**

$$E_j\, f_\Gamma(z) = 0 \ ,$$
$$\Box_u\, f_\Gamma(z) = 0 \ ,$$

$$E_j = \sum_{i=1}^{N} a_{j,i}\, z_i\, \frac{\partial}{\partial z_i} - \beta_j \,, \qquad j = 1, \dots, n+1$$

$$\Box_u = \prod_{u_i>0} \left(\frac{\partial}{\partial z_i}\right)^{u_i} - \prod_{u_i<0} \left(\frac{\partial}{\partial z_i}\right)^{-u_i} \quad , \quad \forall u \in \mathrm{Ker}(A) \,.$$

GKZ D-Module and De Rham Cohomolgy group

E_j $\square_u$ can be regarded as elements of a Weyl algebra

$$\mathcal{D}_N = \mathbb{C}[z_1, \ldots, z_N]\langle \partial_1, \ldots, \partial_N \rangle \quad , \quad [\partial_i, \partial_j] = 0 \quad , \quad [\partial_i, z_j] = \delta_{ij}$$

GKZ system as the left $\mathcal{D}_N$-module $\mathcal{D}_N / H_A(\beta)$

$$H_A(\beta) = \sum_{j=1}^{n+1} \mathcal{D}_N \cdot E_j + \sum_{u \in \mathrm{Ker}(A)} \mathcal{D}_N \cdot \square_u$$

- **Standard Monomials** $\mathrm{Std} := \{\partial^k\}$ found by Groebner basis Hibi, Nishiyama, Takayama (2017)

The holonomic rank equals the number of independent solutions to the system of PDEs

$$r = n! \cdot \mathrm{vol}(\Delta_A)$$

- **Isomorphism**

$$\mathcal{D}_N / H_A(\beta) \simeq \mathbb{H}^n$$

GKZ D-module

nth-Cohomology group

Generalised Feynman Integrals

$$I(d_0,\nu;z) \;:=\; c(d_0,\nu) f_\Gamma(\beta)$$

$$\beta = (\epsilon, -\epsilon\delta, \ldots, -\epsilon\delta) - (d_0/2, \nu_1, \ldots, \nu_n)$$

Let $0 < \epsilon, \delta \ll 1$, $d_0 \in 2\cdot\mathbb{N}$, $L \in \mathbb{N}$ and $\nu := (\nu_1, \ldots, \nu_n) \in \mathbb{Z}^n$

$$f_\Gamma(\beta) \;:=\; \int_\Gamma \mathcal{G}(z;x)^{\epsilon - d_0/2}\, x_1^{\nu_1+\epsilon\delta} \,\cdots\, x_n^{\nu_n+\epsilon\delta}\, \frac{\mathrm{d}x}{x} \;,$$

$$c(d_0,\nu) \;:=\; \frac{\Gamma(d_0/2-\epsilon)}{\Gamma\big((L+1)(d_0/2-\epsilon) - |\nu| - n\epsilon\delta\big)\prod_{i=1}^n \Gamma(\nu_i+\epsilon\delta)} \quad , \quad |\nu| := \nu_1 + \ldots + \nu_n$$

Generalised Feynman Integrals

$$I(d_0,\nu;z) \;:=\; c(d_0,\nu)f_\Gamma(\beta)$$

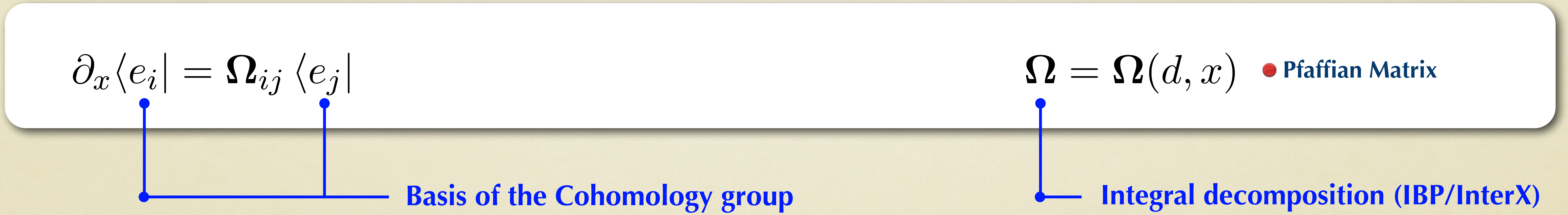
$$\beta = (\epsilon, -\epsilon\delta, \dots, -\epsilon\delta) - (d_0/2, \nu_1, \dots, \nu_n)$$

Let $0 < \epsilon, \delta \ll 1$, $d_0 \in 2\cdot\mathbb{N}$, $L \in \mathbb{N}$ and $\nu := (\nu_1, \dots, \nu_n) \in \mathbb{Z}^n$

$$f_\Gamma(\beta) \;:=\; \int_\Gamma \mathcal{G}(z;x)^{\epsilon-d_0/2}\, x_1^{\nu_1+\epsilon\delta} \,\cdots\, x_n^{\nu_n+\epsilon\delta}\, \frac{\mathrm{d}x}{x} \;,$$

$$c(d_0,\nu) \;:=\; \frac{\Gamma(d_0/2-\epsilon)}{\Gamma\big((L+1)(d_0/2-\epsilon)-|\nu|-n\epsilon\delta\big)\prod_{i=1}^{n}\Gamma(\nu_i+\epsilon\delta)} \quad , \quad |\nu| := \nu_1+\ldots+\nu_n$$

Pfaffian Systems: for **Master Integrals** (alias **Master forms**)



$$\partial_x \langle e_i| = \mathbf{\Omega}_{ij}\,\langle e_j|$$

$$\mathbf{\Omega} = \mathbf{\Omega}(d,x)$$

● **Pfaffian Matrix**

Basis of the Cohomology group

Integral decomposition (IBP/InterX)

Generalised Feynman Integrals

$$I(d_0, \nu; z) \ := \ c(d_0, \nu) f_\Gamma(\beta)$$

$$\beta = (\epsilon, -\epsilon\delta, \ldots, -\epsilon\delta) - (d_0/2, \nu_1, \ldots, \nu_n)$$

Let $0 < \epsilon, \delta \ll 1$, $d_0 \in 2 \cdot \mathbb{N}$, $L \in \mathbb{N}$ and $\nu := (\nu_1, \ldots, \nu_n) \in \mathbb{Z}^n$

$$f_\Gamma(\beta) \ := \ \int_\Gamma \mathcal{G}(z; x)^{\epsilon - d_0/2} \, x_1^{\nu_1 + \epsilon\delta} \ \cdots \ x_n^{\nu_n + \epsilon\delta} \, \frac{\mathrm{d}x}{x} \ ,$$

$$c(d_0, \nu) \ := \ \frac{\Gamma(d_0/2 - \epsilon)}{\Gamma\big((L+1)(d_0/2 - \epsilon) - |\nu| - n\epsilon\delta\big) \prod_{i=1}^{n} \Gamma(\nu_i + \epsilon\delta)} \quad , \quad |\nu| := \nu_1 + \ldots + \nu_n$$

Pfaffian Systems: for Master Integrals (alias Master forms) & for D-operators (alias Std mon's)

$$\partial_x \langle e_i| = \mathbf{\Omega}_{ij} \, \langle e_j|$$

$$\mathbf{\Omega} = \mathbf{\Omega}(d, x)$$

● **Pfaffian Matrix**

Basis of the D-Operators

Macaulay Matrix method

Chestnov, Gasparotto, Mandal, Munch, Matsubara-Heo, Takayama **& P.M.** (2022)

Master Decomposition Formula & Pfaffian

Let $\{e_i\}_{i=1}^r$ be a basis for $\mathbb{H}^n$ and $\{h_i\}_{i=1}^r$ a basis for $\mathbb{H}^{n\vee}$

$\varphi \in \mathbb{H}^n$ in terms of $\{e_i\}_{i=1}^r$

$$\langle\varphi| = \sum_{\lambda=1}^{r} c_\lambda \langle e_\lambda| \,, \quad c_\lambda = \sum_{\mu=1}^{r} \langle\varphi|h_\mu\rangle \left(C^{-1}\right)_{\mu\lambda} \qquad C_{\lambda\mu} := \langle e_\lambda|h_\mu\rangle$$

$$\begin{cases} \partial_{z_i} \langle e_\lambda| = \left(P_i\right)_{\lambda\nu} \langle e_\nu| \\ \partial_{z_i} |h_\mu\rangle = |h_\xi\rangle \left(P_i^\vee\right)_{\xi\mu} \end{cases} \Longrightarrow \partial_{z_i} C = P_i \cdot C + C \cdot \left(P_i^\vee\right)^{\mathrm{T}}$$

● **Secondary Equation 1**

Master Decomposition Formula & Pfaffian

Let $\{e_i\}_{i=1}^r$ be a basis for $\mathbb{H}^n$ and $\{h_i\}_{i=1}^r$ a basis for $\mathbb{H}^{n\vee}$

$\varphi \in \mathbb{H}^n$ in terms of $\{e_i\}_{i=1}^r$

$$\langle\varphi| = \sum_{\lambda=1}^{r} c_\lambda \langle e_\lambda|\,, \quad c_\lambda = \sum_{\mu=1}^{r} \langle\varphi|h_\mu\rangle (C^{-1})_{\mu\lambda} \qquad C_{\lambda\mu} := \langle e_\lambda|h_\mu\rangle$$

$$\begin{cases} \partial_{z_i} \langle e_\lambda| = (P_i)_{\lambda\nu} \langle e_\nu| \\ \partial_{z_i} |h_\mu\rangle = |h_\xi\rangle \left(P_i^\vee\right)_{\xi\mu} \end{cases} \Longrightarrow \partial_{z_i} C = P_i \cdot C + C \cdot \left(P_i^\vee\right)^{\mathrm{T}}$$

- **Secondary Equation 1**

auxiliary basis $e^{\text{aux}} := \{\boxed{e_1, \ldots, e_{r-1}}, \boxed{\varphi}\}$

$$\begin{cases} \partial_{z_i} \langle \overset{\text{aux}}{e_\lambda}| = (\overset{\text{aux}}{P_i})_{\lambda\nu} \langle \overset{\text{aux}}{e_\nu}| \\ \partial_{z_i} |h_\mu\rangle = |h_\xi\rangle \left(P_i^\vee\right)_{\xi\mu} \end{cases} \Longrightarrow \partial_{z_i} \overset{\text{aux}}{C} = \overset{\text{aux}}{P_i} \cdot \overset{\text{aux}}{C} + \overset{\text{aux}}{C} \cdot \left(P_i^\vee\right)^{\mathrm{T}} \qquad \overset{\text{aux}}{C_{\lambda\mu}} := \langle \overset{\text{aux}}{e_\lambda}|h_\mu\rangle$$

- **Secondary Equation 2**

Master Decomposition Formula & Pfaffian

Let $\{e_i\}_{i=1}^{r}$ be a basis for $\mathbb{H}^n$ and $\{h_i\}_{i=1}^{r}$ a basis for $\mathbb{H}^{n\vee}$

$\varphi \in \mathbb{H}^n$ in terms of $\{e_i\}_{i=1}^{r}$

$$\langle\varphi| = \sum_{\lambda=1}^{r} c_\lambda \langle e_\lambda| \,, \quad c_\lambda = \sum_{\mu=1}^{r} \langle\varphi|h_\mu\rangle (C^{-1})_{\mu\lambda} \qquad\qquad C_{\lambda\mu} := \langle e_\lambda|h_\mu\rangle$$

$$\begin{cases} \partial_{z_i} \langle e_\lambda| = (P_i)_{\lambda\nu} \langle e_\nu| \\ \partial_{z_i} |h_\mu\rangle = |h_\xi\rangle \left(P_i^\vee\right)_{\xi\mu} \end{cases} \implies \partial_{z_i} C = P_i \cdot C + C \cdot \left(P_i^\vee\right)^{\mathrm{T}}$$

- **Secondary Equation 1**

auxiliary basis $e^{\text{aux}} := \{e_1, \ldots, e_{r-1}, \varphi\}$

$$\begin{cases} \partial_{z_i} \langle \overset{\text{aux}}{e_\lambda}| = (\overset{\text{aux}}{P_i})_{\lambda\nu} \langle \overset{\text{aux}}{e_\nu}| \\ \partial_{z_i} |h_\mu\rangle = |h_\xi\rangle \left(P_i^\vee\right)_{\xi\mu} \end{cases} \implies \partial_{z_i} \overset{\text{aux}}{C} = \overset{\text{aux}}{P_i} \cdot \overset{\text{aux}}{C} + \overset{\text{aux}}{C} \cdot \left(P_i^\vee\right)^{\mathrm{T}} \qquad\qquad \overset{\text{aux}}{C}_{\lambda\mu} := \langle \overset{\text{aux}}{e_\lambda}|h_\mu\rangle$$

- **Secondary Equation 2**

Rational Solutions of PDE
[integrable connections]

Barkatou et al. @ MAPLE

Direct determination of Intersection Matrices

Multivariate Intersection Numbers (III) from Pfaffians

Chestnov, Gasparotto, Mandal, Munch, Matsubara-Heo, Takayama **& P.M.** (2022)

Let $\{e_i\}_{i=1}^r$ be a basis for $\mathbb{H}^n$ and $\{h_i\}_{i=1}^r$ a basis for $\mathbb{H}^{n\vee}$

$\varphi \in \mathbb{H}^n$ in terms of $\{e_i\}_{i=1}^r$

$$\langle\varphi| = \sum_{\lambda=1}^{r} c_\lambda \langle e_\lambda| \,,$$

$$\begin{bmatrix} e_1 \\ \vdots \\ e_{r-1} \\ \varphi \end{bmatrix} = C^{\text{aux}} \cdot C^{-1} \begin{bmatrix} e_1 \\ \vdots \\ e_{r-1} \\ e_r \end{bmatrix} \quad \Longrightarrow \quad C^{\text{aux}} \cdot C^{-1} = \left[\begin{array}{ccc|c} & & & 0 \\ & \mathrm{id}_{r-1} & & \vdots \\ & & & 0 \\ \hline c_1 & \cdots & c_{r-1} & c_r \end{array}\right]$$

Coefficients from matrix multiplication