Perturbative gravitational scattering and experimental mathematics

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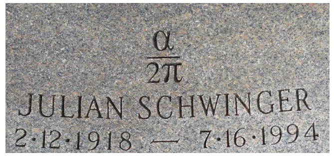
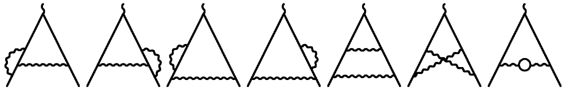
Calculation of multiloop Feynman integrals is surely a difficult problem.

A good milestone of the progressing/developing of methods and techniques is the calculation of the coefficients of the perturbative expansion of the electron anomalous magnetic moment (g-2) in powers of the fine structure constant α.

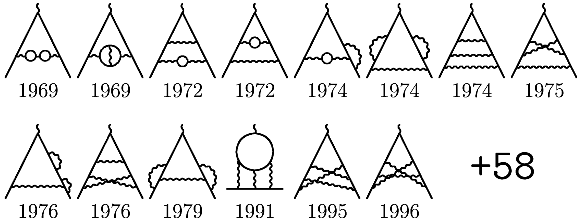
$$a_e^{QED} = C_1\left(\frac{\alpha}{\pi}\right) + C_2\left(\frac{\alpha}{\pi}\right)^2 + C_3\left(\frac{\alpha}{\pi}\right)^3 + C_4\left(\frac{\alpha}{\pi}\right)^4 + C_5\left(\frac{\alpha}{\pi}\right)^5 + \ldots$$

$$C_1 = \frac{1}{2}$$ (Julian Schwinger 1948)





$$C_2 = \frac{197}{144} + \frac{1}{12}\pi^2 - \frac{1}{2}\pi^2\ln 2 + \frac{3}{4}\zeta(3) = -0.328\,478\,965\ldots$$ (Petermann, Sommerfield 1957).



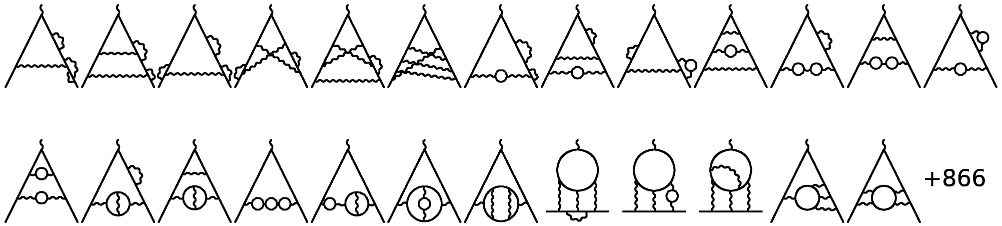
+58

$$C_3 = \frac{83}{72}\pi^2\zeta(3) - \frac{215}{24}\zeta(5) + \frac{100}{3}\left[\left(\mathrm{Li}_4\left(\frac{1}{2}\right) + \frac{\ln^4 2}{24}\right) - \frac{\pi^2\ln^2 2}{24}\right] - \frac{239}{2160}\pi^4$$
$$+ \frac{139}{18}\zeta(3) - \frac{298}{9}\pi^2\ln 2 + \frac{17101}{810}\pi^2 + \frac{28259}{5184} = 1.181\,241\,456\ldots$$

- The final analytical expression was obtained by S.L. and Ettore Remiddi in 1996. 
- Ettore Remiddi begun the analytical calculation of C_3 in 1969. I joined him and his group in Bologna in 1989 as Ph.D. student.

Three-loop needed computational help. . .


$$a_e^{QED} = C_1\left(\frac{\alpha}{\pi}\right) + C_2\left(\frac{\alpha}{\pi}\right)^2 + C_3\left(\frac{\alpha}{\pi}\right)^3 + C_4\left(\frac{\alpha}{\pi}\right)^4 + C_5\left(\frac{\alpha}{\pi}\right)^5 + \dots$$



891 diagrams 334 M.I.

$C_4 =$ -1.9122457649264455741526471674398300540608733906587253451713298480060384439806517061427608927000036315
83755841533147327005637851491285453919028043270502738223043455789570455627293099412966997602777822115 78
4720339064151908166527097970867438115012155147972274322164273431927975958607405005783738496070187432831
4024838025192249460742298558930463506140492252663431094424000235635688128062064549401322497759430042928
8836761748899236915180878086989705263578533753776964117024536196013497574494361268486175162606832387186
7473038315059627418780153055148794005369777983694642786843269184311758895811597435669504330483490736134
2658649953116387811743475385423488364085584441882237217456706871041823307430517443055739459611715508589
6114899526126606124699407311840392747234002346496953173548258481799822409737371077365740464513521123091
2425281111372153021544537210148111211598489708842232798797204842014451228284515165852365617865945926009
9173303172130286546721234534050034910470072892448720061604426132544906900043191519823004748818149431103
84953782994062967586787538524978194698979313216219797575067670114290489796208505... (S.L. (2017))

- This extremely high precision of the result was needed to fit analytically a (very complex) analytical ansatz to the numerical values by using the PSLQ algorithm.
- The successful fit is a strong reliability test of the result.
- 1100 digits is the final total precision; some intermediate fits needed up to 9600 digits of precision

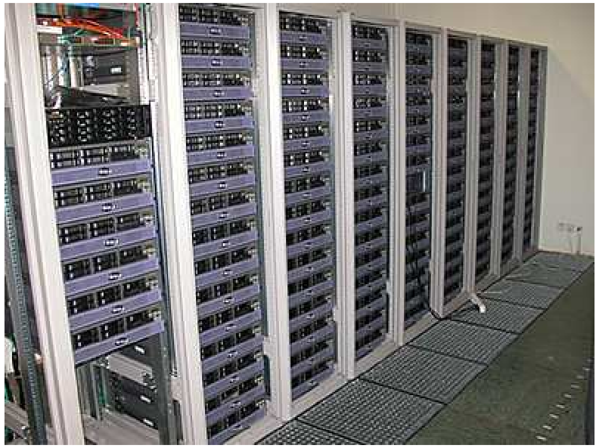
$$
\begin{aligned}
C_4 =& \frac{1243127611}{130636800} + \frac{30180451}{25920}\zeta(2) - \frac{255842141}{2721600}\zeta(3) - \frac{8873}{3}\zeta(2)\ln 2 + \frac{6768227}{2160}\zeta(4) + \frac{19063}{360}\zeta(2)\ln^2 2 + \frac{12097}{90}\left(a_4 + \frac{1}{24}\ln^4 2\right) - \frac{2862857}{6480}\zeta(5) - \frac{12720907}{64800}\zeta(3)\zeta(2) \\
& - \frac{221581}{2160}\zeta(4)\ln 2 + \frac{9656}{27}\left(a_5 + \frac{1}{12}\zeta(2)\ln^3 2 - \frac{1}{120}\ln^5 2\right) + \frac{191490607}{46656}\zeta(6) + \frac{10358551}{43200}\zeta^2(3) - \frac{40136}{27}a_6 + \frac{26404}{27}b_6 - \frac{700706}{675}a_4\zeta(2) - \frac{26404}{27}a_5\ln 2 \\
& + \frac{26404}{27}\zeta(5)\ln 2 - \frac{63749}{50}\zeta(3)\zeta(2)\ln 2 - \frac{40723}{135}\zeta(4)\ln^2 2 + \frac{13202}{81}\zeta(3)\ln^3 2 - \frac{253201}{2700}\zeta(2)\ln^4 2 + \frac{7657}{1620}\ln^6 2 + \frac{2895304273}{435456}\zeta(7) + \frac{670276309}{193536}\zeta(4)\zeta(3) + \frac{85933}{63}a_4\zeta(3) \\
& + \frac{7121162687}{967680}\zeta(5)\zeta(2) - \frac{142793}{18}a_5\zeta(2) - \frac{195848}{21}a_7 + \frac{195848}{63}b_7 - \frac{116506}{189}d_7 - \frac{4136495}{384}\zeta(6)\ln 2 - \frac{1053568}{189}a_6\ln 2 + \frac{233012}{189}b_6\ln 2 + \frac{407771}{432}\zeta^2(3)\ln 2 \\
& - \frac{8937}{2}a_4\zeta(2)\ln 2 + \frac{833683}{3024}\zeta(5)\ln^2 2 - \frac{3995099}{6048}\zeta(3)\zeta(2)\ln^2 2 - \frac{233012}{189}a_5\ln^2 2 + \frac{1705273}{1512}\zeta(4)\ln^3 2 + \frac{602303}{4536}\zeta(3)\ln^4 2 - \frac{1650461}{11340}\zeta(2)\ln^5 2 + \frac{52177}{15876}\ln^7 2 \\
& + \sqrt{3}\Big[- \frac{14101}{480}\mathrm{Cl}_4\left(\frac{\pi}{3}\right) - \frac{169703}{1440}\zeta(2)\mathrm{Cl}_2\left(\frac{\pi}{3}\right) + \frac{494}{27}\mathrm{Im}H_{0,0,0,1,-1,-1}\left(e^{i\frac{\pi}{3}}\right) + \frac{494}{27}\mathrm{Im}H_{0,0,0,1,-1,1}\left(e^{i\frac{2\pi}{3}}\right) + \frac{494}{27}\mathrm{Im}H_{0,0,0,1,1,-1}\left(e^{i\frac{2\pi}{3}}\right) \\
& + 19\,\mathrm{Im}H_{0,0,1,0,1,1}\left(e^{i\frac{2\pi}{3}}\right) + \frac{437}{12}\mathrm{Im}H_{0,0,0,1,1,1}\left(e^{i\frac{2\pi}{3}}\right) + \frac{29812}{297}\mathrm{Cl}_6\left(\frac{\pi}{3}\right) + \frac{4940}{81}a_4\mathrm{Cl}_2\left(\frac{\pi}{3}\right) - \frac{520847}{69984}\zeta(5)\pi - \frac{129251}{81}\zeta(4)\mathrm{Cl}_2\left(\frac{\pi}{3}\right) - \frac{892}{15}\mathrm{Im}H_{0,1,1,-1}\left(e^{i\frac{2\pi}{3}}\right)\zeta(2) \\
& - \frac{1784}{45}\mathrm{Im}H_{0,1,1,-1}\left(e^{i\frac{\pi}{3}}\right)\zeta(2) + \frac{1729}{54}\zeta(3)\mathrm{Im}H_{0,1,-1}\left(e^{i\frac{\pi}{3}}\right) + \frac{1729}{36}\zeta(3)\mathrm{Im}H_{0,1,1}\left(e^{i\frac{2\pi}{3}}\right) + \frac{837190}{729}\mathrm{Cl}_4\left(\frac{\pi}{3}\right)\zeta(2) + \frac{25937}{4860}\zeta(3)\zeta(2)\pi - \frac{223}{243}\zeta(4)\pi\ln 2 \\
& + \frac{892}{9}\mathrm{Im}H_{0,1,-1}\left(e^{i\frac{\pi}{3}}\right)\zeta(2)\ln 2 + \frac{446}{3}\mathrm{Im}H_{0,1,1}\left(e^{i\frac{2\pi}{3}}\right)\zeta(2)\ln 2 - \frac{7925}{81}\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\zeta(2)\ln^2 2 + \frac{1235}{486}\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\ln^4 2\Big] + \frac{13487}{60}\left(\mathrm{Re}H_{0,0,0,1,0,1}\left(e^{i\frac{\pi}{3}}\right) + \mathrm{Cl}_4\left(\frac{\pi}{3}\right)\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\right) \\
& + \frac{136781}{360}\mathrm{Cl}_2^2\left(\frac{\pi}{3}\right)\zeta(2) + \frac{651}{4}\mathrm{Re}H_{0,0,0,1,0,1,-1}\left(e^{i\frac{\pi}{3}}\right) + 651\,\mathrm{Re}H_{0,0,0,0,1,1,-1}\left(e^{i\frac{\pi}{3}}\right) - \frac{17577}{32}\mathrm{Re}H_{0,0,1,0,0,1,1}\left(e^{i\frac{2\pi}{3}}\right) - \frac{87885}{64}\mathrm{Re}H_{0,0,0,1,0,1,1}\left(e^{i\frac{2\pi}{3}}\right) \\
& - \frac{17577}{8}\mathrm{Re}H_{0,0,0,0,1,1,1}\left(e^{i\frac{2\pi}{3}}\right) + \frac{651}{4}\mathrm{Cl}_4\left(\frac{\pi}{3}\right)\mathrm{Im}H_{0,1,-1}\left(e^{i\frac{\pi}{3}}\right) + \frac{1953}{8}\mathrm{Cl}_4\left(\frac{\pi}{3}\right)\mathrm{Im}H_{0,1,1}\left(e^{i\frac{2\pi}{3}}\right) + \frac{31465}{176}\mathrm{Cl}_6\left(\frac{\pi}{3}\right)\pi + \frac{211}{4}\mathrm{Re}H_{0,1,0,1,-1}\left(e^{i\frac{\pi}{3}}\right)\zeta(2) \\
& + \frac{211}{2}\mathrm{Re}H_{0,0,1,1,-1}\left(e^{i\frac{\pi}{3}}\right)\zeta(2) + \frac{1899}{16}\mathrm{Re}H_{0,1,0,1,1}\left(e^{i\frac{2\pi}{3}}\right)\zeta(2) + \frac{1899}{8}\mathrm{Re}H_{0,0,1,1,1}\left(e^{i\frac{2\pi}{3}}\right)\zeta(2) + \frac{211}{4}\mathrm{Im}H_{0,1,-1}\left(e^{i\frac{\pi}{3}}\right)\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\zeta(2) \\
& + \frac{633}{8}\mathrm{Im}H_{0,1,1}\left(e^{i\frac{2\pi}{3}}\right)\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\zeta(2) - \frac{28276}{25}\zeta(2)\mathrm{Cl}_2\left(\frac{\pi}{2}\right)^2 + 104\left(4\mathrm{Re}H_{0,1,0,1,1}\left(e^{i\frac{\pi}{2}}\right)\zeta(2) + 4\mathrm{Im}H_{0,1,1}\left(e^{i\frac{\pi}{2}}\right)\mathrm{Cl}_2\left(\frac{\pi}{2}\right)\zeta(2) - 2\mathrm{Cl}_4\left(\frac{\pi}{2}\right)\zeta(2)\pi + \mathrm{Cl}_2^2\left(\frac{\pi}{2}\right)\zeta(2)\ln 2\right) \\
& + \sqrt{3}\Big[\pi\left(-\frac{28458503}{691200}B_3 + \frac{250077961}{18662400}C_3\right) + \frac{483913}{77760}\pi f_2(0,0,1) + \pi\left(\frac{4715}{1944}\ln 2\; f_2(0,0,1) + \frac{270433}{10935}f_2(0,2,0) - \frac{188147}{4860}f_2(0,1,1) + \frac{188147}{12960}f_2(0,0,2)\right) \\
& + \pi\left(\frac{826595}{248832}\zeta(2)f_2(0,0,1) - \frac{5525}{432}\ln 2\; f_2(0,0,2) + \frac{5525}{162}\ln 2\; f_2(0,1,1) - \frac{5525}{243}\ln 2\; f_2(0,2,0) + \frac{526015}{248832}f_2(0,0,3) - \frac{4675}{768}f_2(0,1,2) + \frac{1805965}{248832}f_2(0,2,1)\right. \\
& \left. - \frac{3710675}{1119744}f_2(0,3,0) - \frac{75145}{124416}f_2(1,0,2) - \frac{213635}{124416}f_2(1,1,1) + \frac{168455}{62208}f_2(1,2,0) + \frac{69245}{124416}f_2(2,1,0)\right)\Big] - \frac{4715}{1458}\zeta(2)f_1(0,0,1) + \zeta(2)\left(\frac{2541575}{82944}f_1(0,0,2)\right. \\
& \left. - \frac{556445}{6912}f_1(0,1,1) + \frac{54515}{972}f_1(0,2,0) - \frac{75145}{20736}f_1(1,0,1)\right) - \frac{541}{300}C_{81a} - \frac{629}{60}C_{81b} + \frac{49}{3}C_{81c} - \frac{327}{160}C_{83a} + \frac{49}{36}C_{83b} + \frac{37}{6}C_{83c}.
\end{aligned}
$$

(a) Home: Pentium I 32-bit 133MHz RAM 16MB HD 1.6GB, (1996)

(b) Home: Pentium III 32-bit 800MHz RAM 384MB HD 45GB, (2001)

(c) Home: Athlon64X2 64-bit 2.6GHz RAM 8GB HD 250+3×500 GB, (2007)

(d) DIFA BO PC: Intel I7-960 4cores 3.2GHz RAM 24GB (2010)



(e) The zBox2@UZH: 125 quad opteron 852, 500 cores, 580Gb ram, 65 Tb of disk (2006).

(f) The Schrödinger supercomputer at UZH, 4608 cores, 576 nodes 2× Xeon X5560 4C @ 2.8GHz RAM 24GB, 218KW, (2009)

Due incursioni in tre notti a Fisica Bottino magro ma molti danni

Uno dei ladri si è ferito con dei vetri rotti e ha lasciato tracce di sangue

Per due volte a pochi giorni di distanza dei ladri si sono intrufolati nelle aule della Facoltà di Fisica. Alcune decine di migliaia di euro il bottino, molti i danni.

It is the room with my machines (on the left). The thieves broke the glass, hurt themselves and made a mess. Luckily they were not interested in desktop computers running 4-loop g-2.

Feynman Integrals

L loop momenta k_j, N internal lines, N_p external momenta p_j

$$I(n_1, n_2, \ldots, n_N; \alpha_1, \ldots, \alpha_{N_{sp}}) = \int d^D k_1 \; d^D k_2 \ldots d^D k_L \frac{R_1^{\alpha_1} \ldots R_{N_{sp}}^{\alpha_{N_{sp}}}}{D_1^{n_1} D_2^{n_2} \ldots D_N^{n_N}}$$

$\{R_j\}$ irreducible scalar products ($p_i \cdot k_j$ or $k_i \cdot k_j$)

Integration by parts (IBP) identities

In D dimensions

$$\int d^D k_1 \ldots d^D k_L \frac{\partial}{\partial (k_j)_\mu} \left((p_l)_\mu \frac{R_1^{\alpha_1} \ldots R_{N_{sp}}^{\alpha_{N_{sp}}}}{D_1^{n_1} \ldots D_N^{n_N}} \right) = 0 \,, \quad j = 1, \ldots, L, \;\; l = 1, \ldots, N_p,$$

$$\int d^D k_1 \ldots d^D k_L \frac{\partial}{\partial (k_j)_\mu} \left((k_l)_\mu \frac{R_1^{\alpha_1} \ldots R_{N_{sp}}^{\alpha_{N_{sp}}}}{D_1^{n_1} \ldots D_N^{n_N}} \right) = 0 \,, \quad j, l = 1, \ldots, L \,,$$

(Chetyrkin Tkachov 1981)

Solution of the system of IBP identities (S.L. 2000)

- The algorithm of the solution is based on imposing a *lexicographic* ordering on the Feynman integrals in order to reduce "more difficult" integrals to "simple" integrals.
- The number of identities (size of the system) needed to reduce single integrals can be large. (it is possible to combine these identities by hand only in case of simple problems).
- There exist several private and public codes which perform this step.
- There are also other promising approach which aim to avoid the solutions of large systems (intersection theory, see Pierpaolo's talk).
- The integrals are reduced to a linear combinations of a *small* set of irreducible *master integrals*.
- The Master Integrals are to be evaluated numerically or analytically. Therefore, using IBP, one builds systems of difference or (canonical) differential equations satisfied by the Master Integrals.
- The systems of difference or differential equations are solved numerically or analytically, giving the value of the Master Integrals and therefore of the original quantities.

Example of large proble, the 4-loop g-2: 334 M.I., 30TB size of the system of IBP identities, 2GB of system of difference/differential equations.

The scattering integrals A_{mnk} (D.Bini, T Damour, A.Geralico, **S.L.**, P. Mastrolia, arXiv:2008.09389 [gr-qc]; arXiv:2012.12918 [gr-qc]; Phys. Rev. D **103** (2021) no.4, 044038)

[See Donato's talk]

$$A_{mnk} = \int_{-1}^{+1} dT \int_{-1}^{+1} dT' \frac{1}{|T-T'|} a_{mnk}(T,T')$$

$$a_{mnk}(T,T') = \sum_{p,q} R^{mnk}_{pq}(T,T') \left(\arctan(T) - \arctan(T')\right)^p \left(\mathrm{ArcTanh(T)} - \mathrm{ArcTanh(T')}\right)^q$$

$R^{mnk}_{pq}(T,T')$ rational functions of T, T'

We consider A_{242}

$$A_{242} = \tfrac{5}{2} c_{42} + \ldots \qquad c_{42} = \frac{1}{\pi}\left(\frac{1499}{20}K + \frac{59610947}{793800} + Q_{42}\right) \qquad K = \text{Catalan's constant} \qquad Q_{42} = \int_{-1}^{1} dT \int_{-1}^{1} dT' q_{42}[T,T']$$

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q42[T_,Tp_]=(-1712/315*((1+T^2)^3*(1+Tp^2)^2*(1+Tp)^3*(-1+Tp)^3*((T^12+2424/107*T^10-196647/107*T^8+471752/107*T^6-196647/107*T^4+2424/107*T^2+1)*(-1+T)*(1+T)*Tp^14-9952/107*(T^12-69297/1244*T^10
+374541/1244*T^8-147514/311*T^6+374541/1244*T^4-69297/1244*T^2+1)*T*Tp^13+2317/107*(T^12-118308/331*T^10+4924023/2317*T^8-7588688/2317*T^6+4924023/2317*T^4-118308/331*T^2+1)*(-1+T)*(1+T)*Tp^12+55
4376/107*T*(T^12-311190/23099*T^10+822759/23099*T^8-1167144/23099*T^6+822759/23099*T^4-311190/23099*T^2+1)*Tp^11-199071/107*(-1+T)*(1+T)*(T^12-617012/22119*T^10+6136783/66357*T^8-922512/7373*T^6+
6136783/66357*T^4-617012/22119*T^2+1)*Tp^10-2996328/107*(T^12-822759/124847*T^10+2142126/124847*T^8-2782916/124847*T^6+2142126/124847*T^4-822759/124847*T^2+1)*T*Tp^9+668399/107*(T^12-11844312/668
399*T^10+31473861/668399*T^8-44618616/668399*T^6+31473861/668399*T^4-11844312/668399*T^2+1)*(-1+T)*(1+T)*Tp^8+4720448/107*T*(T^12+2087187/147514*T^4+1+2087187/147514*T^8-1299171/73757*T^6-437679/
73757*T^10-437679/73757*T^2)*Tp^7-668399/107*(T^12-11844312/668399*T^10+31473861/668399*T^8-44618616/668399*T^6+31473861/668399*T^4-11844312/668399*T^2+1)*(-1+T)*(1+T)*Tp^6-2996328/107*(T^12-8227
59/124847*T^10+2142126/124847*T^8-2782916/124847*T^6+2142126/124847*T^4-822759/124847*T^2+1)*T*Tp^5+199071/107*(-1+T)*(1+T)*(T^12-617012/22119*T^10+6136783/66357*T^8-922512/7373*T^6+6136783/66357
*T^4-617012/22119*T^2+1)*Tp^4+554376/107*T*(T^12-311190/23099*T^10+822759/23099*T^8-1167144/23099*T^6+822759/23099*T^4-311190/23099*T^2+1)*Tp^3-2317/107*(T^12-118308/331*T^10+4924023/2317*T^8-758
8688/2317*T^6+4924023/2317*T^4-118308/331*T^2+1)*(-1+T)*(1+T)*Tp^2-9952/107*(T^12-69297/1244*T^10+374541/1244*T^8-147514/311*T^6+374541/1244*T^4-69297/1244*T^2+1)*T*Tp+1+199071/107*T^10-2317/107*
T^12+2317/107*T^2-199071/107*T^4+668399/107*T^6-668399/107*T^8-T^14)*(-1+T)^2*(1+T)^2*Ap^2-2*(1+T^2)^2*(1+Tp^2)*(1+Tp)^2*((1+T^2)*(1+Tp^2)*(1+Tp)*(-1+Tp)*((T^12+2424/107*T^10-196647/107*T^8+47175
2/107*T^6-196647/107*T^4+2424/107*T^2+1)*(-1+T)*(1+T)*Tp^14-9952/107*(T^12-69297/1244*T^10+374541/1244*T^8-147514/311*T^6+374541/1244*T^4-69297/1244*T^2+1)*T*Tp^13+2317/107*(T^12-118308/331*T^10+
4924023/2317*T^8-7588688/2317*T^6+4924023/2317*T^4-118308/331*T^2+1)*(-1+T)*(1+T)*Tp^12+554376/107*T*(T^12-311190/23099*T^10+822759/23099*T^8-1167144/23099*T^6+822759/23099*T^4-311190/23099*T^2+1
)*Tp^11-199071/107*(-1+T)*(1+T)*(T^12-617012/22119*T^10+6136783/66357*T^8-922512/7373*T^6+6136783/66357*T^4-617012/22119*T^2+1)*Tp^10-2996328/107*(T^12-822759/124847*T^10+2142126/124847*T^8-27829
16/124847*T^6+2142126/124847*T^4-822759/124847*T^2+1)*T*Tp^9+668399/107*(T^12-11844312/668399*T^10+31473861/668399*T^8-44618616/668399*T^6+31473861/668399*T^4-11844312/668399*T^2+1)*(-1+T)*(1+T)*
Tp^8+4720448/107*T*(T^12+2087187/147514*T^4+1+2087187/147514*T^8-1299171/73757*T^6-437679/73757*T^10-437679/73757*T^2)*Tp^7-668399/107*(T^12-11844312/668399*T^10+31473861/668399*T^8-44618616/6683
99*T^6+31473861/668399*T^4-11844312/668399*T^2+1)*(-1+T)*(1+T)*Tp^6-2996328/107*(T^12-822759/124847*T^10+2142126/124847*T^8-2782916/124847*T^6+2142126/124847*T^4-822759/124847*T^2+1)*T*Tp^5+19907
1/107*(-1+T)*(1+T)*(T^12-617012/22119*T^10+6136783/66357*T^8-922512/7373*T^6+6136783/66357*T^4-617012/22119*T^2+1)*Tp^4+554376/107*T*(T^12-311190/23099*T^10+822759/23099*T^8-1167144/23099*T^6+822
759/23099*T^4-311190/23099*T^2+1)*Tp^3-2317/107*(T^12-118308/331*T^10+4924023/2317*T^8-7588688/2317*T^6+4924023/2317*T^4-118308/331*T^2+1)*(-1+T)*(1+T)*Tp^2-9952/107*(T^12-69297/1244*T^10+374541/
1244*T^8-147514/311*T^6+374541/1244*T^4-69297/1244*T^2+1)*T*Tp+1+199071/107*T^10-2317/107*T^12+2317/107*T^2-199071/107*T^4+668399/107*T^6-668399/107*T^8-T^14)*(-1+T)*(1+T)*A-745/107*(1-760971/149
0*T^12+177119/1490*T^2+8581593/1490*T^6-2650989/1490*T^4-10456923/1490*T^8+5209761/1490*T^10-6169/1490*T^14-63/1490*T^16+(74439419/1490*T^3+2407290749/4470*T^7-388694303/4470*T^13-233420617/894*T
^5+26841191/4470*T^15-2580200959/4470*T^9-1519825/894*T+1444499851/4470*T^11)*Tp^13+(8232051727/4470*T^7-45964133/1490*T-972635147/894*T^5+1444499851/4470*T^3+4404368009/4470*T^11-233420617/894*T
^13+26321663/1490*T^15-1613541481/894*T^9)*Tp^5+(-233420617/894*T^11+74439419/1490*T^13+26841191/4470*T-1519825/894*T^15-388694303/4470*T^3+1444499851/4470*T^5+2407290749/4470*T^9-2580200959/4470
*T^7)*Tp^3+(8232051727/4470*T^9+4404368009/4470*T^5-1613541481/894*T^7-45964133/1490*T^15-972635147/894*T^11-233420617/894*T^3+1444499851/4470*T^13+26321663/1490*T)*Tp^11+(1896578458/2235*T^10+29
165202/745*T^14+2064512186/2235*T^6-760971/1490*T^16-700792039/2235*T^12-2661519844/2235*T^8-875415041/2235*T^4+150824246/2235*T^2-2650989/1490)*Tp^4+(5209761/1490-4636041523/2235*T^6-4767868877/
2235*T^10-136459827/745*T^14+2064512186/2235*T^12+8581593/1490*T^16+1896578458/2235*T^4+6081157589/2235*T^8-65603867/447*T^2)*Tp^10+(6081157589/2235*T^8+5209761/1490*T^16+8581593/1490+2064512186/
2235*T^4-4767868877/2235*T^6-136459827/745*T^2-4636041523/2235*T^10+1896578458/2235*T^12-65603867/447*T^14)*Tp^6+(177119/1490*T^14-6169/1490*T^2-760971/1490*T^4-63/1490-10456923/1490*T^8+8581593/
1490*T^10-2650989/1490*T^12+5209761/1490*T^6+T^16)*Tp^16+(-136459827/745*T^6+105267023/447*T^8-6169/1490*T^16-1063759/447*T^14-65603867/447*T^10+177119/1490-6402919/745*T^2+29165202/745*T^12+1508
24246/2235*T^4)*Tp^2+(8232051727/4470*T^5-2580200959/4470*T^3+2407290749/4470*T^13+13780259497/4470*T^9-1613541481/894*T^11+262089643/4470*T-918449041/298*T^7-44219729/894*T^15)*Tp^7+(-44219729/8
94*T-2580200959/4470*T^13+13780259497/4470*T^7-918449041/298*T^9-1613541481/894*T^5+8232051727/4470*T^11+262089643/4470*T^15+2407290749/4470*T^3)*Tp^9+(-6402919/745*T^14-6169/1490+105267023/447*T
^8-1063759/447*T^2+29165202/745*T^4-65603867/447*T^6-136459827/745*T^10+150824246/2235*T^12+177119/1490*T^16)*Tp^14+(262089643/4470*T^9+26841191/4470*T^13-11813/1490*T-44219729/894*T^7-45964133/1
490*T^11-1519825/894*T^3+26321663/1490*T^5-310829/1490*T^15)*Tp^15+(-700792039/2235*T^4-2661519844/2235*T^8+150824246/2235*T^14-2650989/1490*T^16+2064512186/2235*T^10-875415041/2235*T^12+29165202
/745*T^2-760971/1490+1896578458/2235*T^6)*Tp^12+(-11813/1490*T^15-1519825/894*T^13+26321663/1490*T^11-44219729/894*T^9+26841191/4470*T^3-45964133/1490*T^5-310829/1490*T+262089643/4470*T^7)*Tp+(10
5267023/447*T^2+6081157589/2235*T^6-2661519844/2235*T^4-10456923/1490*T^16+105267023/447*T^14-1540913531/447*T^8-10456923/1490+6081157589/2235*T^10-2661519844/2235*T^12)*Tp^8)*(T-Tp)*(T*Tp-1))*(-
1+Tp)^2*(-1+T)*(1+T)*Ap+(1+T^2)^3*(1+Tp^2)^2*(1+Tp)^3*(-1+Tp)^3*((T^12+2424/107*T^10-196647/107*T^8+471752/107*T^6-196647/107*T^4+2424/107*T^2+1)*(-1+T)*(1+T)*Tp^14-9952/107*(T^12-69297/1244*T^10
+374541/1244*T^8-147514/311*T^6+374541/1244*T^4-69297/1244*T^2+1)*T*Tp^13+2317/107*(T^12-118308/331*T^10+4924023/2317*T^8-7588688/2317*T^6+4924023/2317*T^4-118308/331*T^2+1)*(-1+T)*(1+T)*Tp^12+55
4376/107*T*(T^12-311190/23099*T^10+822759/23099*T^8-1167144/23099*T^6+822759/23099*T^4-311190/23099*T^2+1)*Tp^11-199071/107*(-1+T)*(1+T)*(T^12-617012/22119*T^10+6136783/66357*T^8-922512/7373*T^6+
6136783/66357*T^4-617012/22119*T^2+1)*Tp^10-2996328/107*(T^12-822759/124847*T^10+2142126/124847*T^8-2782916/124847*T^6+2142126/124847*T^4-822759/124847*T^2+1)*T*Tp^9+668399/107*(T^12-11844312/668
399*T^10+31473861/668399*T^8-44618616/668399*T^6+31473861/668399*T^4-11844312/668399*T^2+1)*(-1+T)*(1+T)*Tp^8+4720448/107*T*(T^12+2087187/147514*T^4+1+2087187/147514*T^8-1299171/73757*T^6-437679/
73757*T^10-437679/73757*T^2)*Tp^7-668399/107*(T^12-11844312/668399*T^10+31473861/668399*T^8-44618616/668399*T^6+31473861/668399*T^4-11844312/668399*T^2+1)*(-1+T)*(1+T)*Tp^6-2996328/107*(T^12-8227
59/124847*T^10+2142126/124847*T^8-2782916/124847*T^6+2142126/124847*T^4-822759/124847*T^2+1)*T*Tp^5+199071/107*(-1+T)*(1+T)*(T^12-617012/22119*T^10+6136783/66357*T^8-922512/7373*T^6+6136783/66357
(...omitted...)
/.{A->ArcTanh[T],Ap->ArcTanh[Tp],B->ArcTan[T],Bp->ArcTan[Tp]};
```

example: $\dfrac{(1-T^2)^3(1-T'^2)^3(T^6T'^8+T^8T'^6)}{(1+T^2)^8(1+T'^2)^8(1+TT')^3(T-T')^3} \times [\mathrm{ArcTan(T)} - \mathrm{ArcTanh(T')}]^2$

A_{mnk} are defined as 2-variable integrals. The dimensionality of the integrals is sufficiently small so we can avoid the use of IBP. We will

1. Compute a very high precision value with direct numerical integration, and use PSLQ to fit the analytical expression
2. Compute analytically the integral reducing it to the well-known family of Harmonic Polylogarithms, applying some tricks if necessary.

1. compute an extremely high-precision value of the given integral (see the *double exponential* method of numerical integration).
2. guess the right analytical ansatz for the basis
3. fit an analytical expression by using the "PSLQ algorithm"

The PSLQ algorithm

- Obtained by Ferguson and Bailey in 1992
- It is a multi-integers extension of the well-known elementary Euclid's algorithm for the calculation of the GCD of two integers
- PSLQ finds an integer relation between floating point numbers or,alternatively, bounds on size of coefficients.
- The first application was testing if numbers were algebraic.
- PSLQ requires high precision; at least (number of digits of coefficients) * (number of elements of the basis of real numbers)
- Several implementations of PSLQ are publicy available, e.g. the C code from Paul Zimmermann https://members.loria.fr/PZimmermann/software/pslq-1.0.c
- PSLQ is implemented in Mathematica and Maple.
- The fastest implementation of PSLQ is a *parallel* modification of the algorithm by Bailey and Broadhurst 1999 which was absolutely essential for the fit of the 4-loop g-2 numerical value (basis of 300- 400 terms).

Integrating analytically in T', and integrating numerically in T using the *double exponential method* one gets an high-precision numerical value

$$
\begin{aligned}
Q_{42} = &-802.88505705078664275588629506903445997073686505843065496417889590242\\
&6423211047940727300850918742678716230784351105139654447667985252511824\\
&6835094053176716319764506087580278153759319186028784338146646397532417\\
&4112925070424154423398282949175201983750364845886298740596803561516671\\
&0264902370076396362812881360509480526317135104676476166581563009251879\\
&76335624637819549332955452819667306495545 37\ldots\\
= &-\frac{59610947}{793800}1-\frac{402163}{2520}\pi+\frac{4497}{80}\pi\ln 2-\frac{1499}{20}K-\frac{11871}{160}\pi\zeta(3)
\end{aligned}
$$

black: ansatz brown: PSLQ algorithm

Mathematica code

```
vector= FindIntegerNullVector[ N[{Q42, 1, Pi, Log[2]*Pi, Catalan, Zeta[3]*Pi}, 200]]
Out[-]= {3175200, 238443788, 506725380, -178485930, 237981240, 235579995}
vector.{Q42, 1, Pi, Log[2]*Pi, Catalan, Zeta[3]*Pi}
Out[-]= 2.11107 × 10^-385
```

$$c_{42} = \frac{1}{\pi}\left(\cancel{\frac{1499}{20}K}+\cancel{\frac{59610947}{793800}}+\left[-\cancel{\frac{59610947}{793800}}1-\frac{402163}{2520}\pi+\frac{4497}{80}\pi\ln 2-\cancel{\frac{1499}{20}K}-\frac{11871}{160}\pi\zeta(3)\right]\right) = -\frac{402163}{2520}+\frac{4497}{80}\ln 2-\frac{11871}{160}\zeta(3)$$

$$d_{42} = \frac{1105777}{6048}-\frac{4497}{32}\ln 2+\frac{5}{2}c_{42} = \frac{1105777}{6048}-\cancel{\frac{4497}{32}\ln 2}+\frac{5}{2}\left(-\frac{402163}{2520}+\cancel{\frac{4497}{80}\ln 2}-\frac{11871}{160}\zeta(3)\right) = -\frac{186743}{864}-\frac{11871}{64}\zeta(3)$$

- Integrand of (T, T'): integration over T' possible analytically, through partial fractioning of denominators. Integral of the type $dT' \frac{\mathrm{ArcTanh(T')}}{(T'-a)^q}$. Result of integration contains at most $\mathrm{Li_3(z)}$ with $z = -\frac{1+T}{1-T}$ of $z = -\left(\frac{1+T}{1-T}\right)$
- integral in t contain polynomial denominators with powers of T, $T \pm 1$, $T^2 + 1$
- where needed terms with powers in the denominator are integrated by parts
- terms with T and $T \pm 1$ only can be cast in the form of Harmonic Polylogarithms

(Vermaseren Remiddi 1999)
HArmonic Polylogarithms are defined as iterated integrals

$$H_{i_1,i_2,i_3,\ldots,i_n}(x) = \int_0^x dt_1 f_{i_1}(t) H_{i_2,\ldots,i_n}(t_1)$$

$f_0 = \frac{1}{x}$
$f_{\pm 1}(x) = \frac{1}{1 \mp x}$
Regularization at $x = 0$ so that $H_{0,0,\ldots,0}(x) \equiv \ln^n(x)/n!$

The family contains the *Nielsen* polylogarithms $\mathrm{S}_{n,p}(x)$ (which contain the usual polylogarithms $\mathrm{Li_n(x)}$)

$$\hat{J} = \int_0^1 dT \frac{-2\,\ln^3\left(\frac{1-T}{1+T}\right) - 3\,\mathrm{Li}_3\left[-\left(\frac{1-T}{1+T}\right)^2\right]}{1+T^2} \;\rightarrow\; J(x) \equiv i\int_0^1 dT\,(1-x^2)\frac{-2\,\ln^3\left(\frac{1-T}{1+T}\right) - 3\,\mathrm{Li}_3\left[\left(\frac{(1-T)(1-x)}{(1+T)(1+x)}\right)^2\right]}{2x(T+x)(T+1/x)}$$

$\hat{J} = J(i)$ and that $J(1) = 0$. By differentiating and reintegrating three times over x, $J(x)$ can be expressed in terms of HPLs at weight $w \leq 4$, namely:

$$\begin{aligned} i\,J(x) = {} & \frac{23}{240}\pi^4 - 21\ln(2)\,\zeta(3) + \pi^2\ln^2(2) - \ln^4(2) - 24a_4 + \frac{21}{2}H_{-1}(x)\zeta(3) - \frac{3}{2}H_0(x)\zeta(3) + \frac{21}{2}H_1(x)\zeta(3) + \frac{1}{2}\pi^2 H_{0,-1}(x) \\ & + \frac{1}{2}\pi^2 H_{0,1}(x) - \frac{3}{2}\pi^2 H_{-1,-1}(x) - \frac{3}{2}\pi^2 H_{-1,1}(x) - \frac{3}{2}\pi^2 H_{1,-1}(x) - \frac{3}{2}\pi^2 H_{1,1}(x) + 12H_{0,1,-1}(x)\ln(2) \\ & + 12H_{0,1,1}(x)\ln(2) - 12H_{0,-1,-1,-1}(x) + 6H_{0,-1,-1,0}(x) - 12H_{0,-1,1,-1}(x) + 6H_{0,-1,1,0}(x) - 12H_{0,1,-1,-1}(x) \\ & + 6H_{0,1,-1,0}(x) - 12H_{0,1,1,-1}(x) + 6H_{0,1,1,0}(x) - 6H_{-1,-1,-1,0}(x) - 6H_{-1,-1,1,0}(x) - 6H_{-1,1,-1,0}(x) \\ & - 6H_{-1,1,1,0}(x) - 6H_{1,-1,-1,0}(x) - 6H_{1,-1,1,0}(x) - 6H_{1,1,-1,0}(x) - 6H_{1,1,1,0}(x) + 12H_{0,-1,-1}(x)\ln(2) + 12H_{0,-1,1}(x)\ln(2) \end{aligned}$$

$\hat{J} = J(i)$ in terms of the values at the fourth root of unity, i, of HPLs of weight $w \leq 4$.

$$\hat{J} = J(i) = -\frac{1}{2}\pi^2\,\mathrm{K} + \frac{9}{2}\pi\zeta(3)\,.$$

Independent sets of HPLs, at the point $x = i$, up to weight four

$H_{-1}(i)$	$\frac{\ln 2}{2} + i\frac{\pi}{4}$
$H_{0}(i)$	$i\frac{\pi}{2}$
$H_{1}(i)$	$-\frac{\ln 2}{2} + i\frac{\pi}{4}$
$H_{0,-1}(i)$	$\frac{\pi^2}{48} + i\mathrm{K}$
$H_{0,1}(i)$	$-\frac{\pi^2}{48} + i\mathrm{K}$
$H_{-1,-1}(i)$	$-\frac{\pi^2}{32} + \frac{\ln^2 2}{8} + \frac{1}{8}i\pi\ln 2$
$H_{-1,1}(i)$	$-\frac{\pi^2}{32} - \frac{\ln^2 2}{8} - \frac{3}{8}i\pi\ln 2 + i\mathrm{K}$
$H_{1,-1}(i)$	$-\frac{\pi^2}{32} - \frac{\ln^2 2}{8} + \frac{3}{8}i\pi\ln 2 - i\mathrm{K}$
$H_{1,1}(i)$	$-\frac{\pi^2}{32} + \frac{\ln^2 2}{8} - \frac{1}{8}i\pi\ln 2$
$H_{0,-1,-1}(i)$	$\frac{29}{64}\zeta(3) - \frac{1}{4}\mathrm{K}\pi - i\mathrm{Q}_3$
$H_{0,-1,1}(i)$	$\frac{27}{64}\zeta(3) - \frac{1}{4}\mathrm{K}\pi + i\frac{\pi^3}{32} - 3i\mathrm{Q}_3 - 2i\mathrm{K}\ln 2$
$H_{0,1,-1}(i)$	$\frac{27}{64}\zeta(3) - \frac{1}{4}\mathrm{K}\pi - i\frac{\pi^3}{32} + 3i\mathrm{Q}_3 + 2i\mathrm{K}\ln 2$
$H_{0,1,1}(i)$	$\frac{29}{64}\zeta(3) - \frac{1}{4}\mathrm{K}\pi + i\mathrm{Q}_3$
$H_{0,-1,-1,-1}(i)$	$\frac{61}{15360}\pi^4 - \frac{35}{128}\zeta(3)\ln 2 + \frac{5}{384}\pi^2\ln^2 2 - \frac{5}{384}\ln^4 2 - \frac{5}{16}a_4 + \frac{\pi\mathrm{Q}_3}{4} + i\mathrm{Q}_4$
$H_{0,-1,-1,0}(i)$	$-\frac{\pi^4}{4608} + \frac{\mathrm{Cl}_2^2\left(\frac{\pi}{2}\right)}{2} + \frac{\pi\mathrm{Q}_3}{2} + \frac{29}{128}i\pi\zeta(3) - \frac{7}{48}i\mathrm{K}\pi^2$
$H_{0,-1,1,-1}(i)$	$-\frac{97}{9216}\pi^4 + \frac{91}{128}\zeta(3)\ln 2 - \frac{13}{384}\pi^2\ln^2 2 + \frac{13}{16}a_4 - \frac{13}{384}\ln^4 2 + \frac{3}{4}\pi\mathrm{Q}_3 + \frac{\mathrm{K}^2}{2} - \frac{7}{8}i\pi\zeta(3) + \frac{1}{16}i\pi^3\ln 2 - 2i\mathrm{K}\ln^2 2$ $-\frac{5}{16}i\mathrm{K}\pi^2 + 5i\mathrm{Cl}_4\left(\frac{\pi}{2}\right) - 6i\mathrm{Q}_3\ln 2 - 9i\mathrm{Q}_4$
$H_{0,-1,1,0}(i)$	$-\frac{71}{4608}\pi^4 + \frac{3}{2}\pi\mathrm{Q}_3 + \mathrm{K}\pi\ln 2 + \frac{\mathrm{Cl}_2^2\left(\frac{\pi}{2}\right)}{2} + \frac{27}{128}i\pi\zeta(3) + \frac{1}{8}i\pi^3\ln 2 - \frac{5}{48}i\mathrm{K}\pi^2 - 3i\mathrm{Cl}_4\left(\frac{\pi}{2}\right)$
$H_{0,1,-1,-1}(i)$	$\frac{169}{9216}\pi^4 - \frac{77}{128}\zeta(3)\ln 2 + \frac{9}{128}\pi^2\ln^2 2 - \frac{27}{16}a_4 - \frac{9}{128}\ln^4 2 - \frac{3}{4}\pi\mathrm{Q}_3 - \frac{1}{2}\mathrm{K}\pi\ln 2 - \frac{21}{128}i\pi\zeta(3) - \frac{1}{32}i\pi^3\ln 2$ $+i\mathrm{K}\ln^2 2 + i\mathrm{Cl}_4\left(\frac{\pi}{2}\right) + 2i\mathrm{Q}_3\ln 2 + i\mathrm{Q}_4$
$H_{0,1,-1,0}(i)$	$\frac{73}{4608}\pi^4 + \frac{\mathrm{Cl}_2^2\left(\frac{\pi}{2}\right)}{2} - \mathrm{K}\pi\ln 2 - \frac{3}{2}\pi\mathrm{Q}_3 + \frac{27}{128}i\pi\zeta(3) - \frac{1}{8}i\pi^3\ln 2 - \frac{7}{48}i\mathrm{K}\pi^2 + 3i\mathrm{Cl}_4\left(\frac{\pi}{2}\right)$
$H_{0,1,1,-1}(i)$	$\frac{61}{9216}\pi^4 + \frac{21}{128}\zeta(3)\ln 2 + \frac{13}{384}\pi^2\ln^2 2 - \frac{13}{384}\ln^4 2 - \frac{13}{16}a_4 - \frac{1}{2}\mathrm{K}\pi\ln 2 + \frac{\mathrm{Cl}_2^2\left(\frac{\pi}{2}\right)}{2} - \frac{1}{4}\pi\mathrm{Q}_3 + \frac{133}{128}i\pi\zeta(3) - \frac{1}{32}i\pi^3\ln 2$ $+i\mathrm{K}\ln^2 2 + \frac{3}{16}i\mathrm{K}\pi^2 - 5i\mathrm{Cl}_4\left(\frac{\pi}{2}\right) + 4i\mathrm{Q}_3\ln 2 + 7i\mathrm{Q}_4$

$H_{0,1,1,0}(i)$	$-\frac{\pi^4}{4608}+\frac{\mathrm{Cl}_2^2\left(\frac{\pi}{2}\right)}{2}-\frac{1}{2}\pi\mathrm{Q}_3-\frac{5}{48}i\mathrm{K}\pi^2+\frac{29}{128}i\pi\zeta(3)$
$H_{-1,-1,-1,0}(i)$	$-\frac{31}{15360}\pi^4+\frac{1}{2}\zeta(3)\ln 2-\frac{1}{32}\pi^2\ln^2 2+\frac{5}{384}\ln^4 2+\frac{5}{16}a_4+\frac{29}{256}i\pi\zeta(3)+\frac{1}{96}i\pi\ln^3 2-\frac{1}{96}i\pi^3\ln 2-\frac{1}{32}i\mathrm{K}\pi^2$ $-\frac{1}{8}i\mathrm{K}\ln^2 2-\frac{1}{2}i\mathrm{Q}_3\ln 2-i\mathrm{Q}_4$
$H_{-1,-1,1,0}(i)$	$-\frac{115}{9216}\pi^4+\frac{13}{16}\zeta(3)\ln 2-\frac{1}{12}\pi^2\ln^2 2+\frac{9}{128}\ln^4 2+\frac{27}{16}a_4+\frac{1}{4}\mathrm{K}\pi\ln 2+\frac{1}{2}\pi\mathrm{Q}_3+\frac{27}{256}i\pi\zeta(3)-\frac{1}{96}i\pi\ln^3 2$ $+\frac{1}{24}i\pi^3\ln 2-\frac{1}{32}i\mathrm{K}\pi^2-\frac{1}{8}i\mathrm{K}\ln^2 2-i\mathrm{Cl}_4\left(\frac{\pi}{2}\right)-\frac{1}{2}i\mathrm{Q}_3\ln 2-i\mathrm{Q}_4$
$H_{-1,1,-1,0}(i)$	$\frac{91}{9216}\pi^4-\frac{1}{2}\zeta(3)\ln 2+\frac{17}{96}\pi^2\ln^2 2-\frac{13}{384}\ln^4 2-\frac{13}{16}a_4-\pi\mathrm{Q}_3+\frac{\mathrm{Cl}_2^2\left(\frac{\pi}{2}\right)}{2}-\frac{3}{4}\mathrm{K}\pi\ln 2+\frac{335}{256}i\pi\zeta(3)-\frac{1}{96}i\pi\ln^3 2$ $-\frac{3}{64}i\pi^3\ln 2+\frac{13}{96}i\mathrm{K}\pi^2+\frac{9}{8}i\mathrm{K}\ln^2 2-5i\mathrm{Cl}_4\left(\frac{\pi}{2}\right)+\frac{9}{2}i\mathrm{Q}_3\ln 2+9i\mathrm{Q}_4$
$H_{-1,1,1,0}(i)$	$-\frac{79}{9216}\pi^4+\frac{1}{16}\zeta(3)\ln 2-\frac{7}{48}\pi^2\ln^2 2+\frac{13}{384}\ln^4 2+\frac{13}{16}a_4+\frac{\mathrm{Cl}_2^2\left(\frac{\pi}{2}\right)}{2}+\frac{1}{2}\pi\mathrm{Q}_3+\frac{1}{2}\mathrm{K}\pi\ln 2-\frac{195}{256}i\pi\zeta(3)+\frac{1}{96}i\pi\ln^3 2$ $+\frac{3}{64}i\pi^3\ln 2-\frac{31}{96}i\mathrm{K}\pi^2-\frac{7}{8}i\mathrm{K}\ln^2 2+5i\mathrm{Cl}_4\left(\frac{\pi}{2}\right)-\frac{7}{2}i\mathrm{Q}_3\ln 2-7i\mathrm{Q}_4$
$H_{1,-1,-1,0}(i)$	$\frac{55}{9216}\pi^4-\frac{7}{96}\pi^2\ln^2 2-\frac{1}{16}\zeta(3)\ln 2-\frac{13}{384}\ln^4 2-\frac{13}{16}a_4+\frac{1}{2}\mathrm{K}\pi\ln 2+\frac{1}{2}\pi\mathrm{Q}_3-\frac{\mathrm{Cl}_2^2\left(\frac{\pi}{2}\right)}{2}-\frac{279}{256}i\pi\zeta(3)-\frac{1}{96}i\pi\ln^3 2$ $-\frac{7}{96}i\mathrm{K}\pi^2-\frac{7}{8}i\mathrm{K}\ln^2 2+5i\mathrm{Cl}_4\left(\frac{\pi}{2}\right)-\frac{7}{2}i\mathrm{Q}_3\ln 2-7i\mathrm{Q}_4$
$H_{1,-1,1,0}(i)$	$\frac{29}{9216}\pi^4+\frac{1}{2}\zeta(3)\ln 2+\frac{5}{48}\pi^2\ln^2 2+\frac{13}{384}\ln^4 2+\frac{13}{16}a_4-\frac{3}{4}\mathrm{K}\pi\ln 2-\frac{\mathrm{Cl}_2^2\left(\frac{\pi}{2}\right)}{2}-\pi\mathrm{Q}_3+\frac{167}{256}i\pi\zeta(3)+\frac{1}{96}i\pi\ln^3 2$ $-\frac{1}{32}i\pi^3\ln 2+\frac{37}{96}i\mathrm{K}\pi^2+\frac{9}{8}i\mathrm{K}\ln^2 2-5i\mathrm{Cl}_4\left(\frac{\pi}{2}\right)+\frac{9}{2}i\mathrm{Q}_3\ln 2+9i\mathrm{Q}_4$
$H_{1,1,-1,0}(i)$	$\frac{91}{9216}\pi^4-\frac{13}{16}\zeta(3)\ln 2+\frac{5}{96}\pi^2\ln^2 2-\frac{9}{128}\ln^4 2-\frac{27}{16}a_4+\frac{1}{4}\mathrm{K}\pi\ln 2+\frac{1}{2}\pi\mathrm{Q}_3+\frac{111}{256}i\pi\zeta(3)-\frac{1}{192}i\pi^3\ln 2$ $+\frac{1}{96}i\pi\ln^3 2-\frac{1}{8}i\mathrm{K}\ln^2 2-\frac{1}{32}i\mathrm{K}\pi^2-i\mathrm{Cl}_4\left(\frac{\pi}{2}\right)-\frac{1}{2}i\mathrm{Q}_3\ln 2-i\mathrm{Q}_4$
$H_{1,1,1,0}(i)$	$\frac{71}{15360}\pi^4-\frac{1}{2}\zeta(3)\ln 2-\frac{5}{384}\ln^4 2-\frac{5}{16}a_4+\frac{29}{256}i\pi\zeta(3)+\frac{1}{192}i\pi^3\ln 2-\frac{1}{32}i\mathrm{K}\pi^2-\frac{1}{8}i\mathrm{K}\ln^2 2-\frac{1}{96}i\pi\ln^3 2$ $-\frac{1}{2}i\mathrm{Q}_3\ln 2-i\mathrm{Q}_4$
$\mathrm{Li}_4(1/2)$	a_4
$\mathrm{Im}\mathrm{Li}_2(i)$	K
$\mathrm{Im}\mathrm{Li}_4(i)$	$\mathrm{Cl}_4\left(\frac{\pi}{2}\right)$
$\mathrm{Im}H_{0,1,1}(i)$	Q_3
$\mathrm{Im}H_{0,1,1,1}(i)$	Q_4

Example: Q_{42}

- We expect it to containg object up to weight 4

- The size of the complete basis for value at $x = i$ of HPLs is $1 + 2 + 4 + 8 + 16 = 31$
- Blinding fitting the complete basis, the minimum number of digits for a succesful fit of Q_{42} turns out to be 295.
- Selecting a smaller basis allows to fit with less digits.

$$v_0 = \left\{1\right\}$$

$$v_1 = \left\{\pi, \ln 2\right\}$$

$$v_2 = \left\{\pi^2, \pi \ln 2, \ln^2 2, K\right\}$$

$$v_3 = \left\{\zeta(3), \pi^3, \pi^2 \ln 2, \pi \ln^2 2, \ln^3 2, \pi K, \ln 2K, \mathrm{ImS}_{1,2}(i)\right\}$$

$$v_4 = \left\{\pi^4, \zeta(3) \ln 2, \pi^2 \ln^2 2, \ln^4(2), \mathrm{Li}_4\left(\frac{1}{2}\right), \pi\zeta(3), \pi^3 \ln 2, \mathrm{K}^2, \pi^2\mathrm{K}, \pi\mathrm{K} \ln 2, \mathrm{K} \ln^2 2, \pi \ln^3 2, \right.$$
$$\left. \mathrm{ImLi}_4(\mathrm{i}), \ln 2\mathrm{ImS}_{1,2}(\mathrm{i}), \pi\mathrm{ImS}_{1,2}(\mathrm{i}), \pi\mathrm{ImS}_{1,3}(\mathrm{i})\right\}$$

- There are several techniques and methods from multiloop Fynman diagrams evaluation, useful for numerical calculations, analytical fits and analytical calculations,
- The use of these tools will help to push further the limit of the calculations in perturbative gravity.

Thank You!