EOB and Amplitudes in Gravity, *Bologna, June 8-9, 2023*

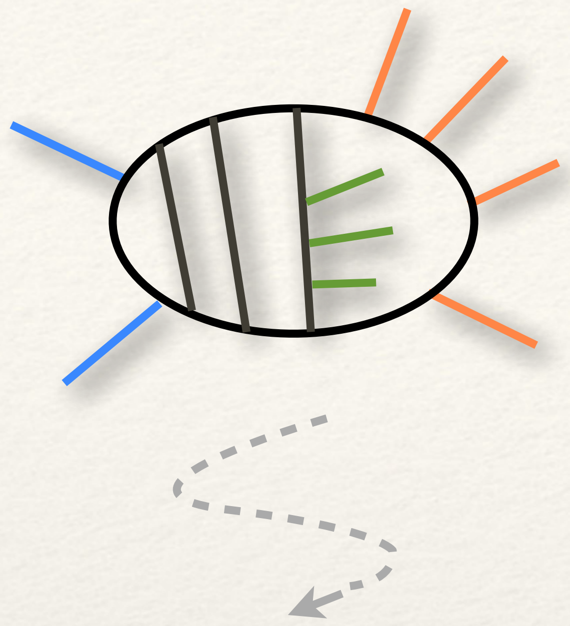
EFT-Diagrammatic Approach to Compact Binary Dynamics

Pierpaolo Mastrolia
University of Padova and INFN
June 8, 2023

From Elementary Particles & Feynman Integrals…

Feynman Integrals

- **Momentum-space Representation**



N-denominator
generic Integral

$$= I_{a_1,\ldots,a_N} = \int \prod_{i=1}^{L} d^d k_i \left(\prod_{n=1}^{N} \frac{1}{D_n^{a_n}} \right)$$

L loops, $E+1$ external momenta,

$N = LE + \frac{1}{2}L(L+1)$ (generalised) denominators

total number of *reducible* and *irreducible* scalar products

't Hooft & Veltman

- **Integration-by-parts Identites**

$$\int \prod_{i=1}^{L} d^d k_i \, \frac{\partial}{\partial k_j^\mu} \left(v_\mu \prod_{n=1}^{N} \frac{1}{D_n^{a_n}} \right) = 0$$

$v_\mu = v_\mu(p_i, k_j)$ arbitrary

Chetyrkin, Tkachov (1981)
Laporta, Remiddi (1996)
Remiddi (1996)
Caffo, Czyz, Laporta, Remiddi (1998)
Gehrmann, Remiddi (1999)
Laporta (2000)
Remiddi + Bonciani, Argeri & **P.M. …**
[Bologna Legacy]
… … … …

- IBP identities

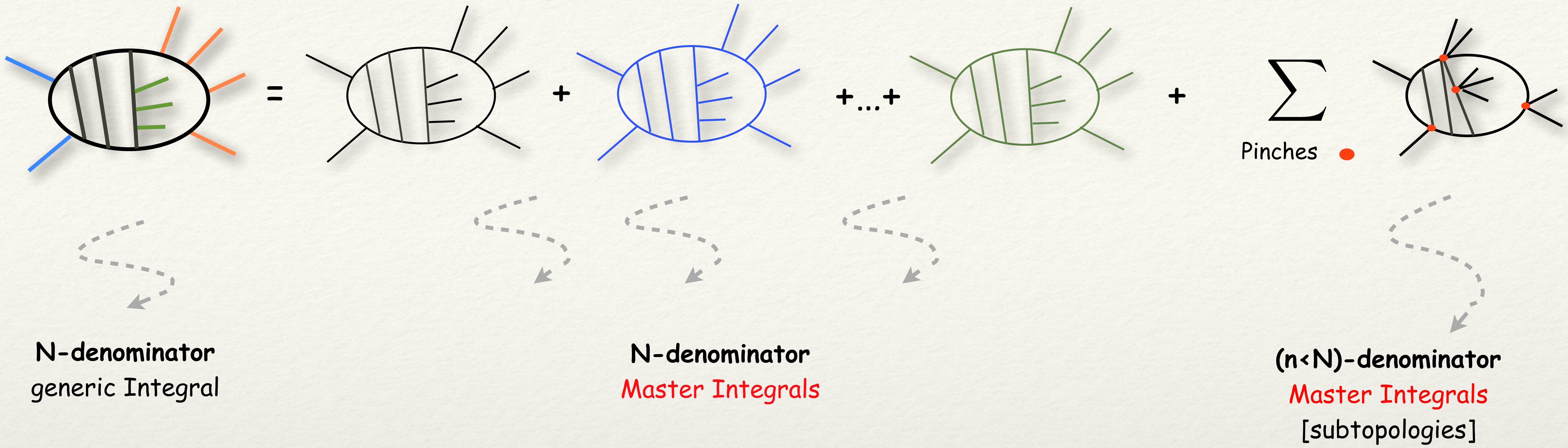
$$\sum_i b_i \, I_{a_1,\ldots,a_i \pm 1,\ldots,a_N} = 0$$

Linear relations for Feynman Integrals identities

- **Relations among Integrals in dim. reg.**

  Chetyrkin, Tkachov (1981)

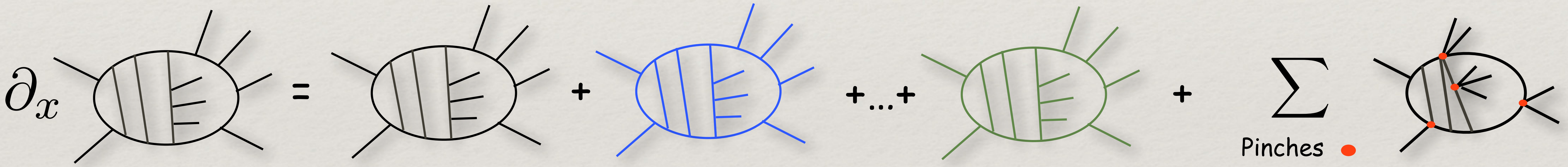
  Laporta (2000)



- **1st order Differential Equations for MIs**

  Kotikov (1991)

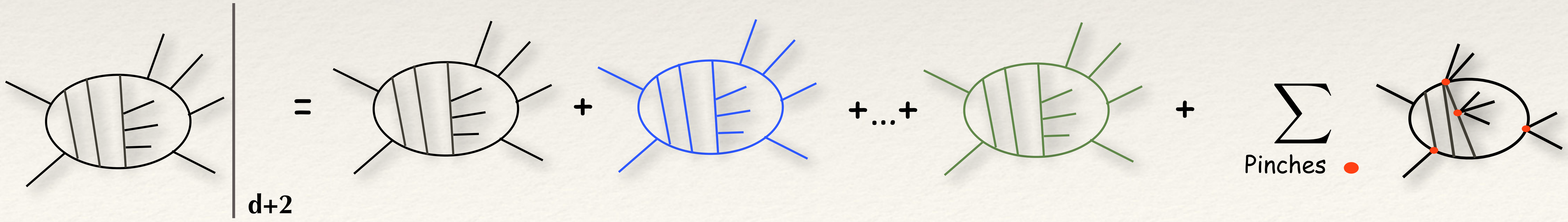
  Remiddi (1996)

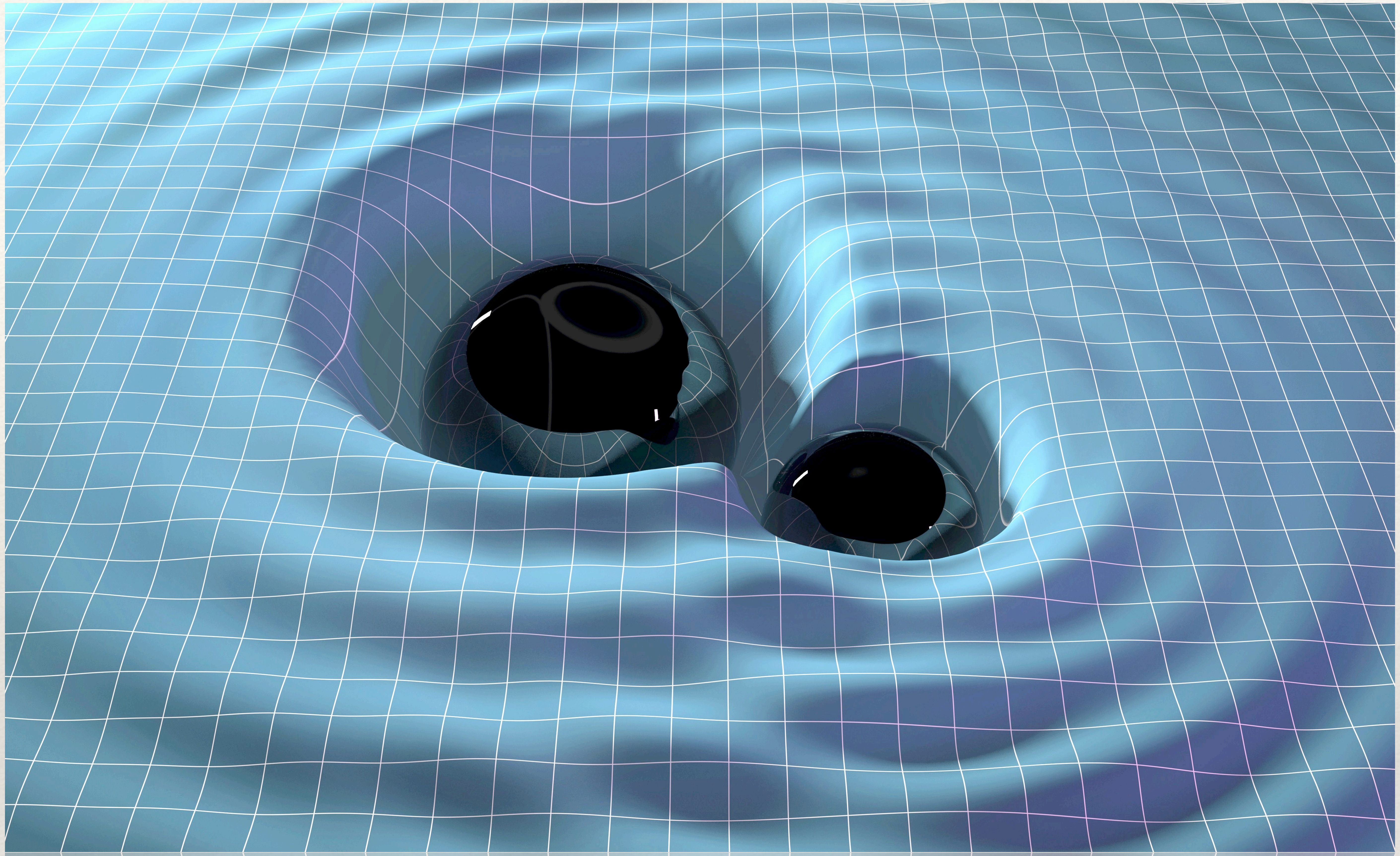


- **Dimension-Shift relations and Gram determinant relations**

  Tarasov (1998)

  Laporta (2000)

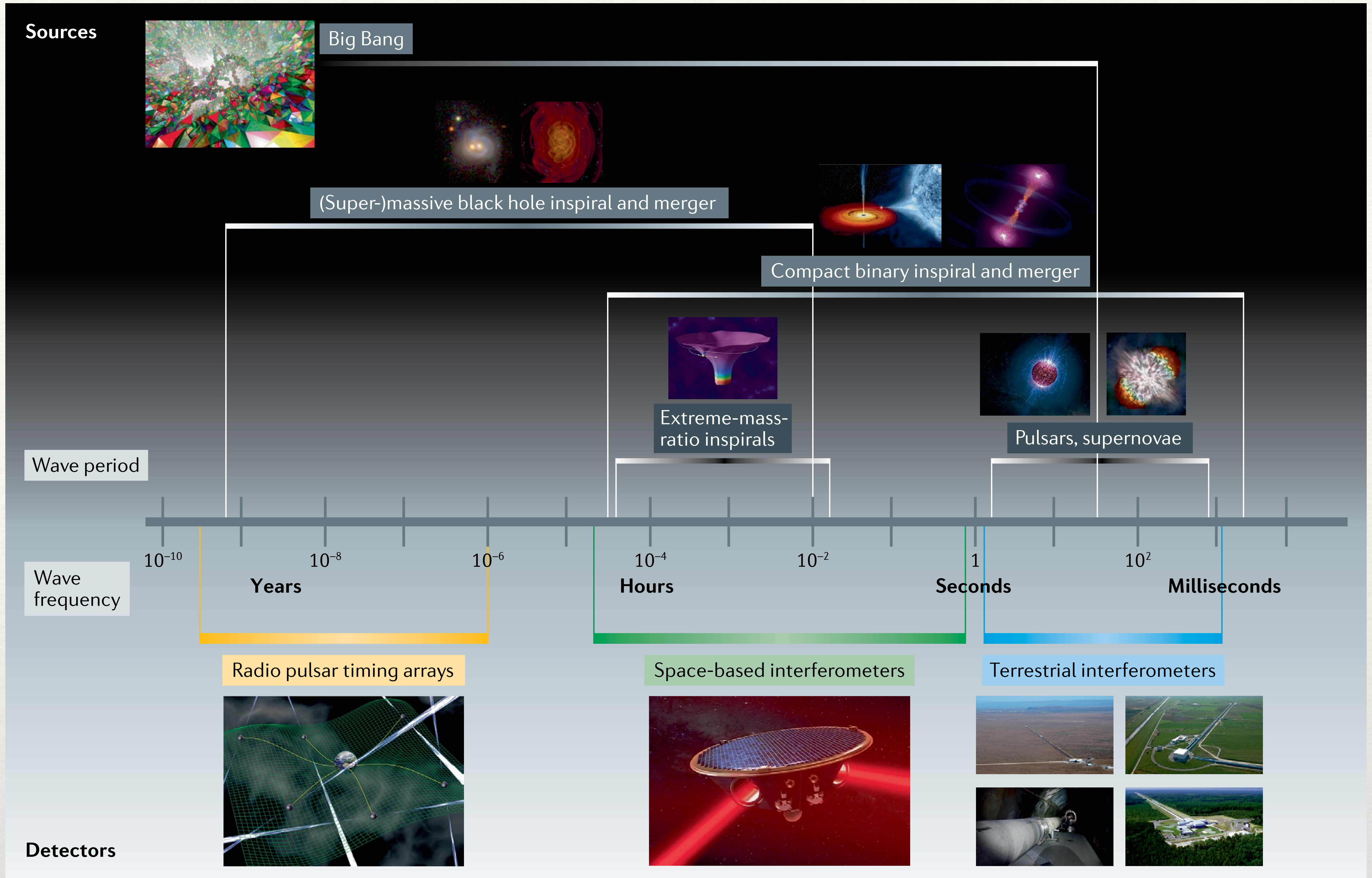




…to Astrophysical Systems and Gravitational Waves

Motivation

- Gravitational Waves a new window on the Universe
- Two-body dynamics and radiative effects to exploring the most extreme conditions of spacetime and matter
- Next generation detectors, ground-based and in space, need of accurate waveform templates
- Precision Physics vs Precision Calculations: Multi-Loop Calculus, Scattering Amplitudes and General Relativity



[Bailes et al. 2021]

Motivation

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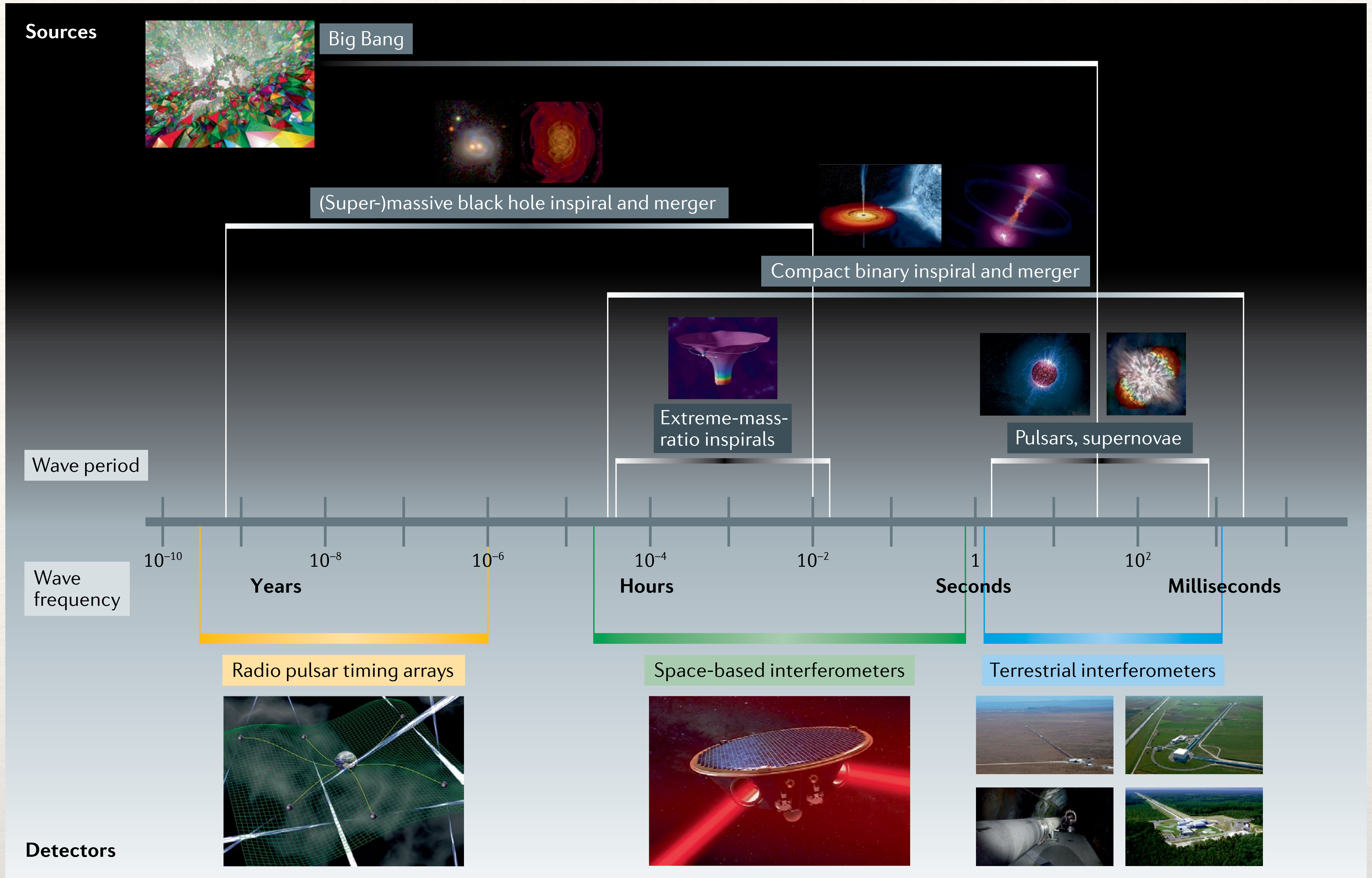


[Bailes et al. 2021]

Outline

1. Gravitational Waves Detection and Computational Techniques
2. Two-body problem in Classical GR and EFT Diagrammatic Approach
3. Conservative Effects from Near and Far Zone
4. Spin and Tidal Effects

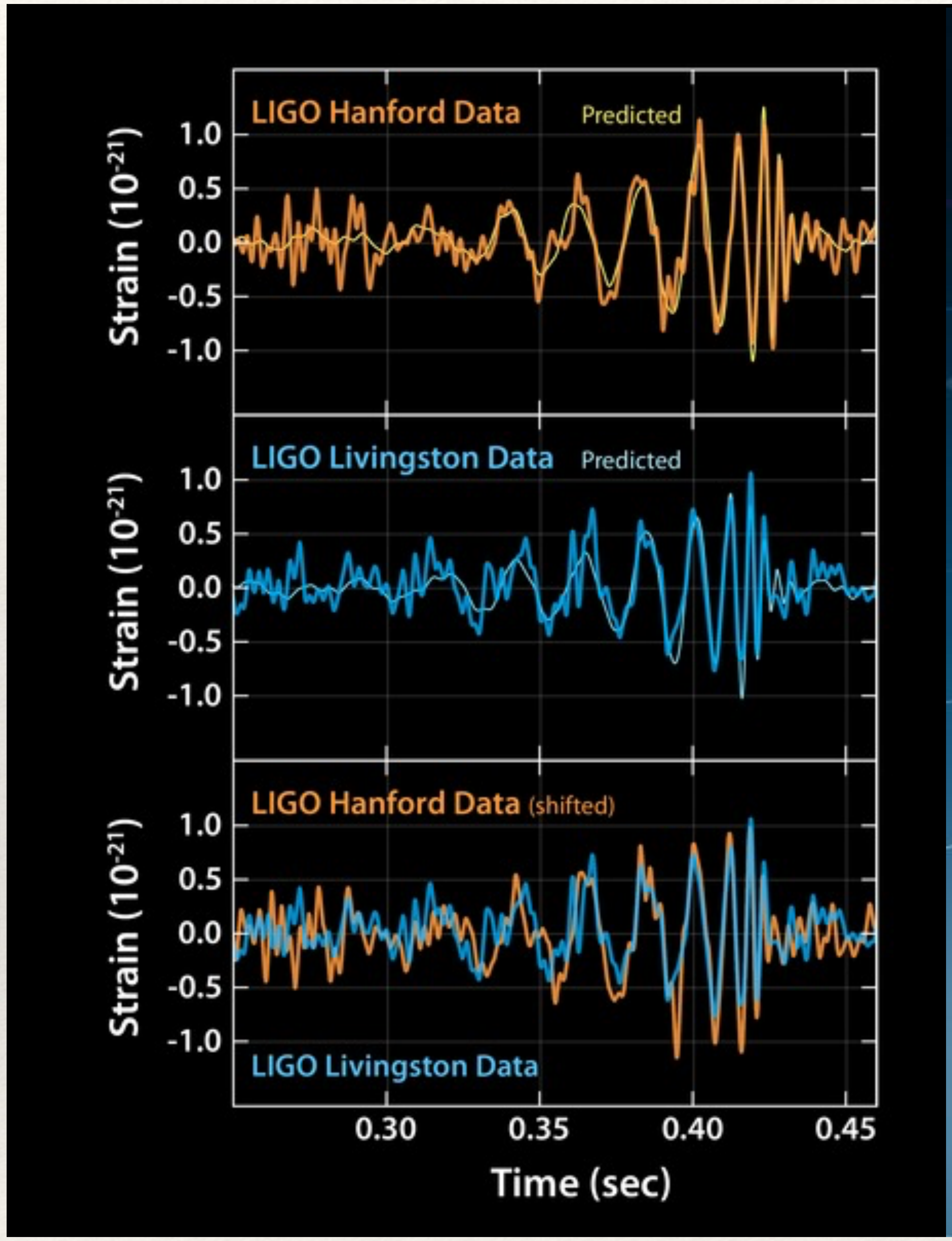
Based on collaborations with:

G. **Brunello**, J. Steinhoff, M.K. **Mandal**, R. **Patil,** H.O. **Silva**

D. Bini, T. Damour, A. **Geralico**, S. Laporta

S. Foffa, R. Sturani, C. Sturm, W.J. **Torres Bobadilla**

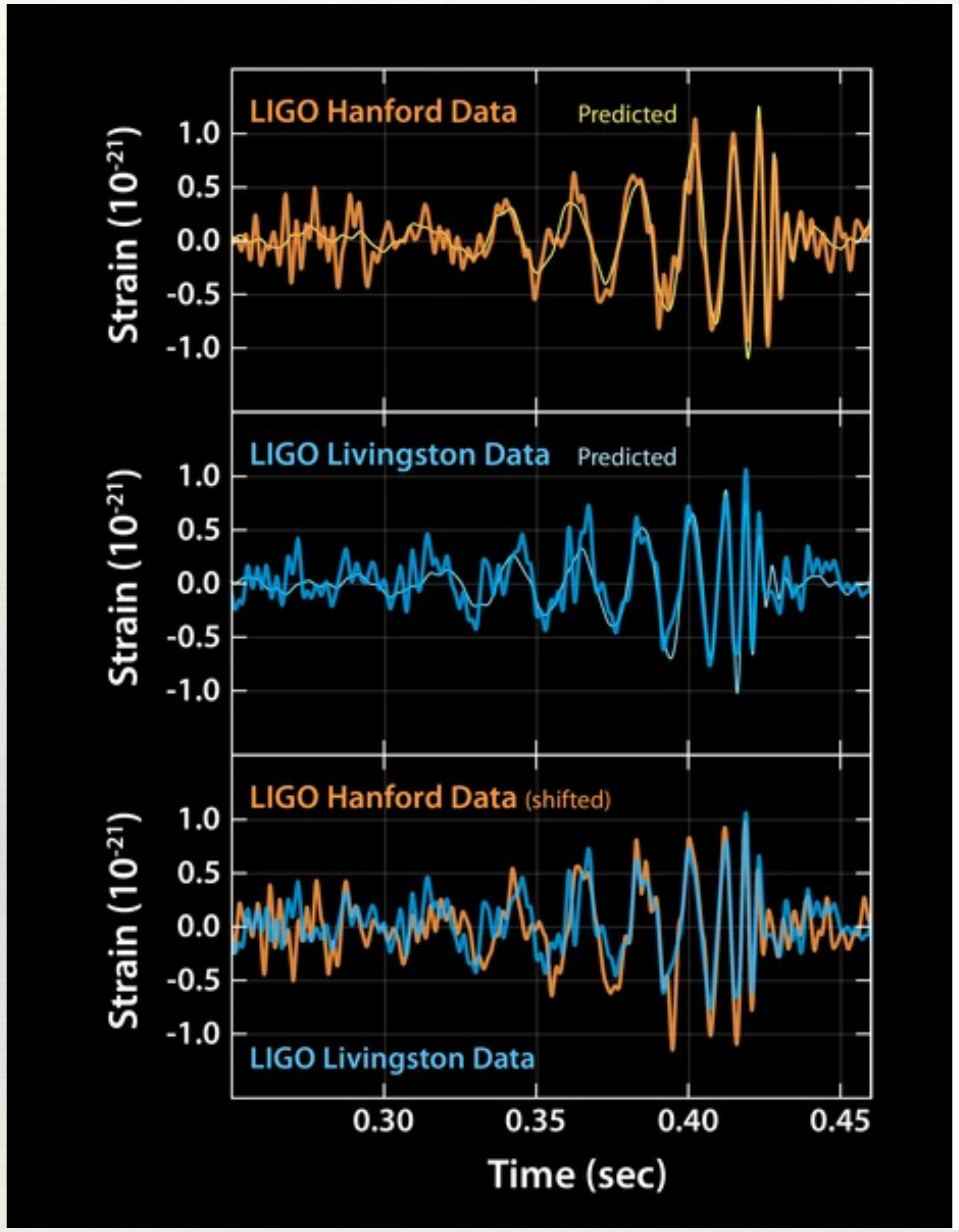
GW Detection

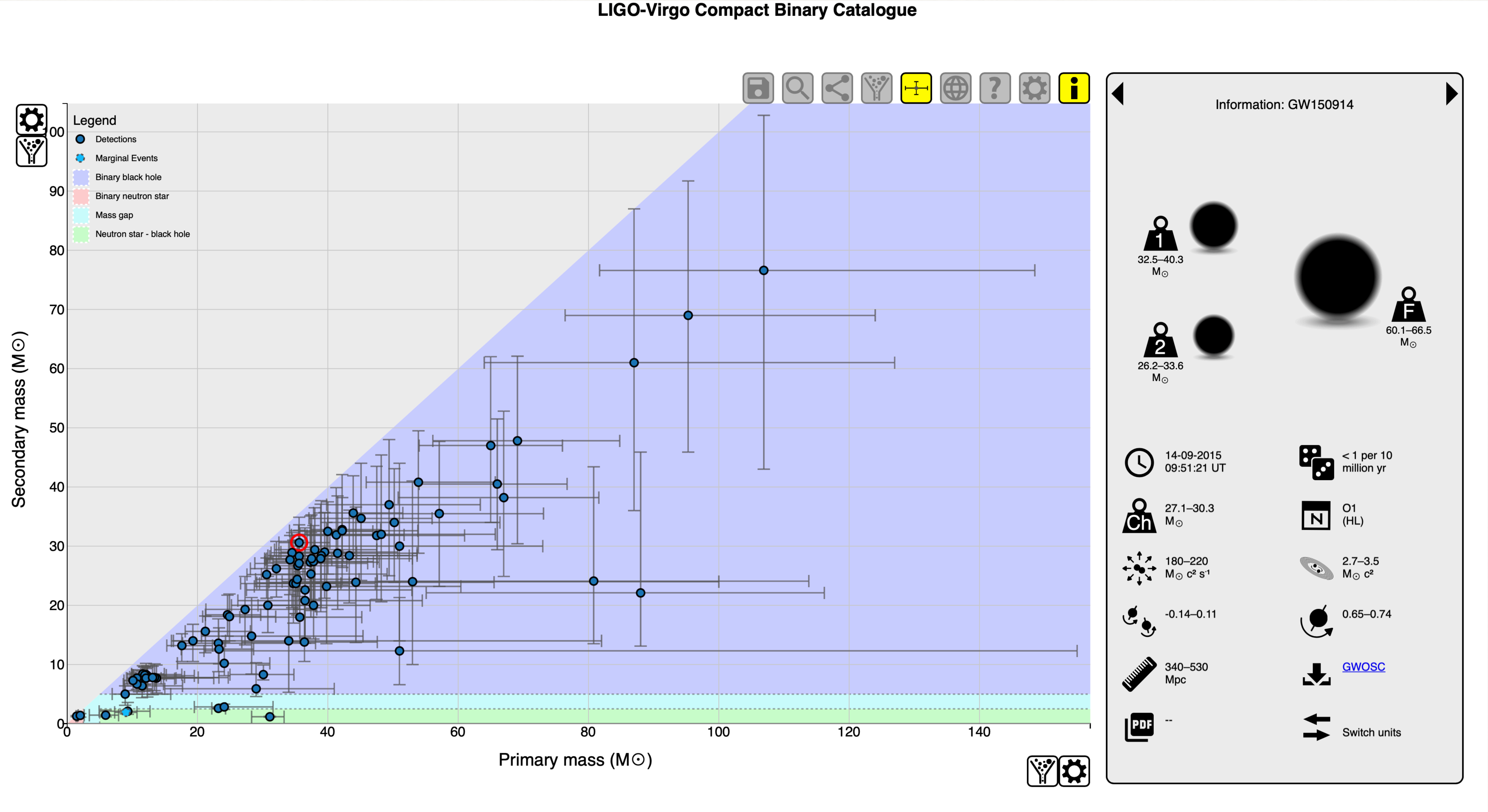


LIGO-Virgo Detection: **GW150914**

Bologna is here

GW Detection



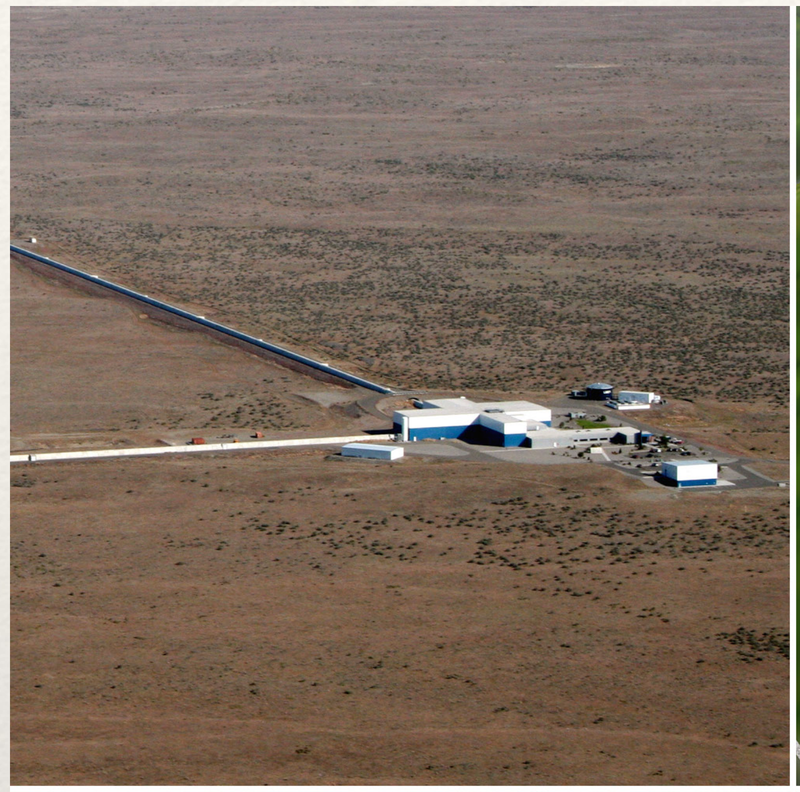


LIGO-Virgo Detection: **GW150914**

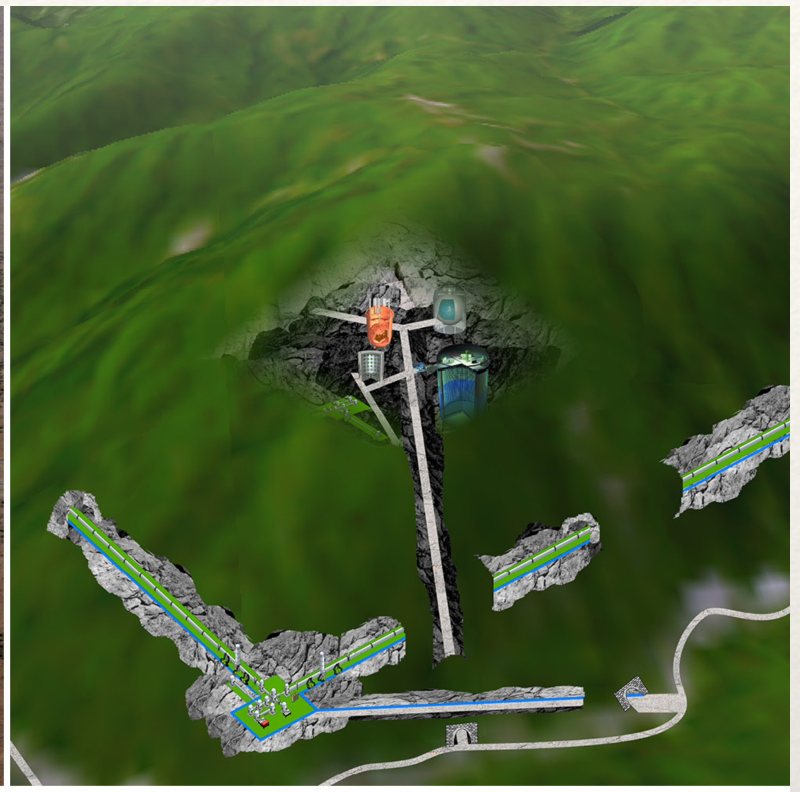
GW Detection

LIGO-Virgo-KAGRA Collaboration

Hanford



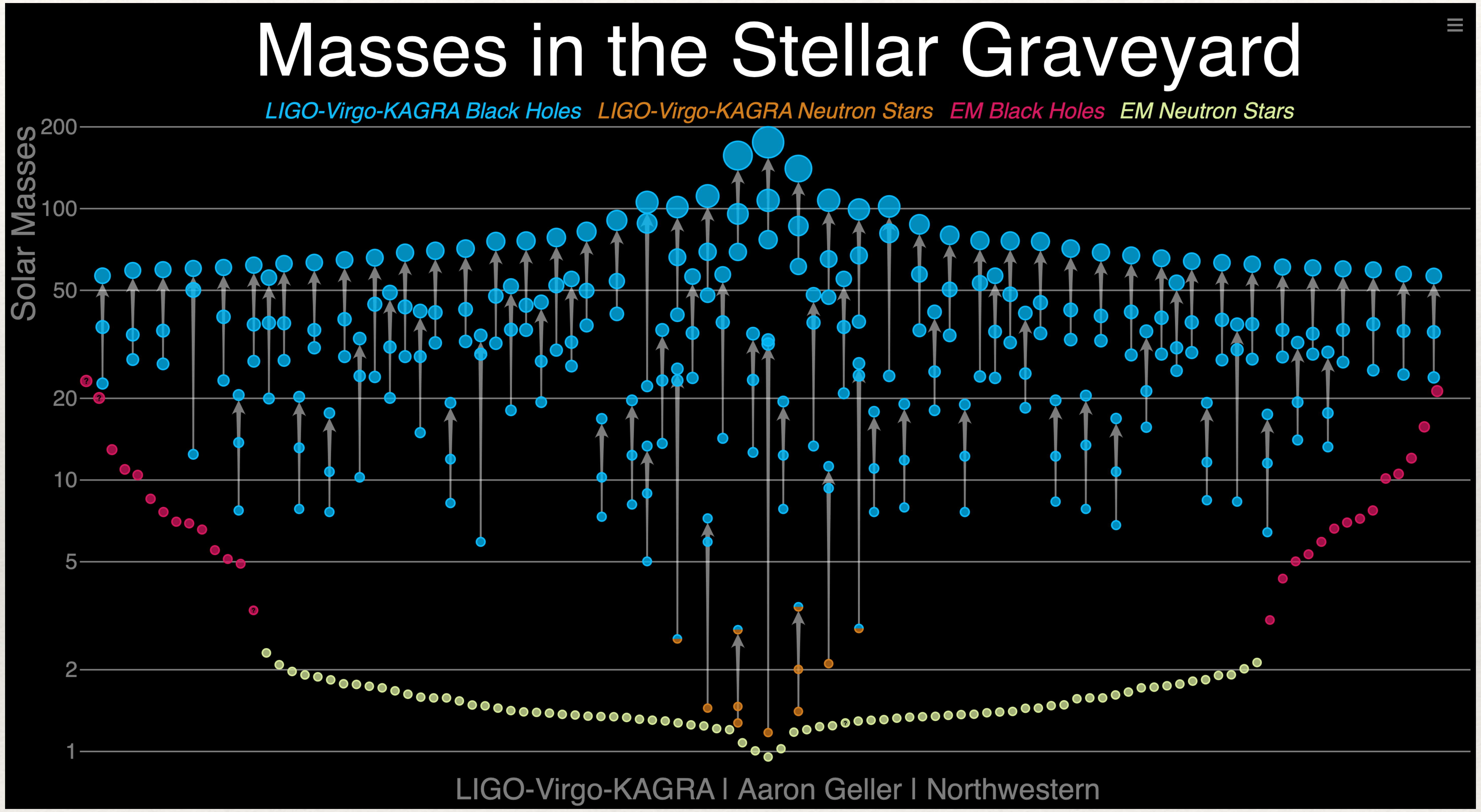
Kagra



Livingston

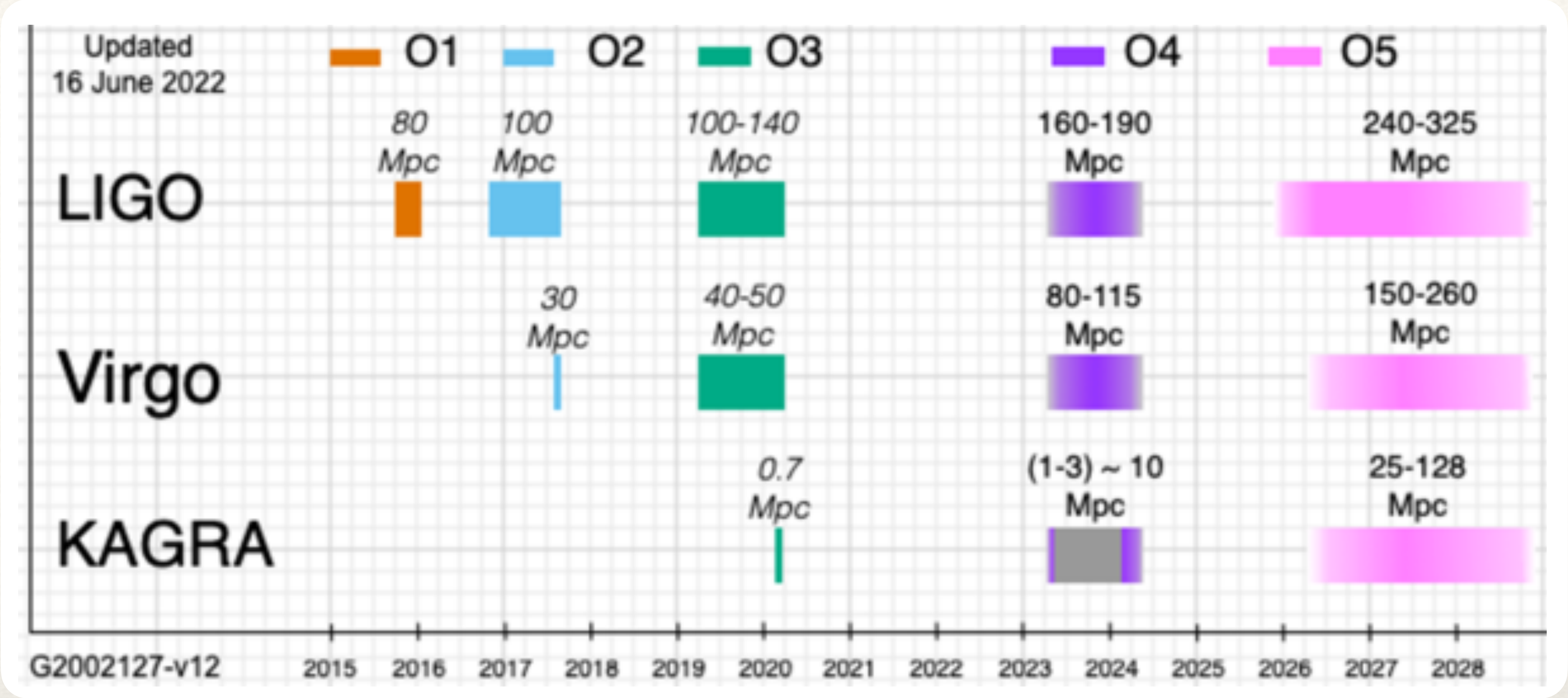
Virgo

O3b - Catalogue



GW Detection

:: Current GW Detectors: advanced programs



[credit: Ligo News]

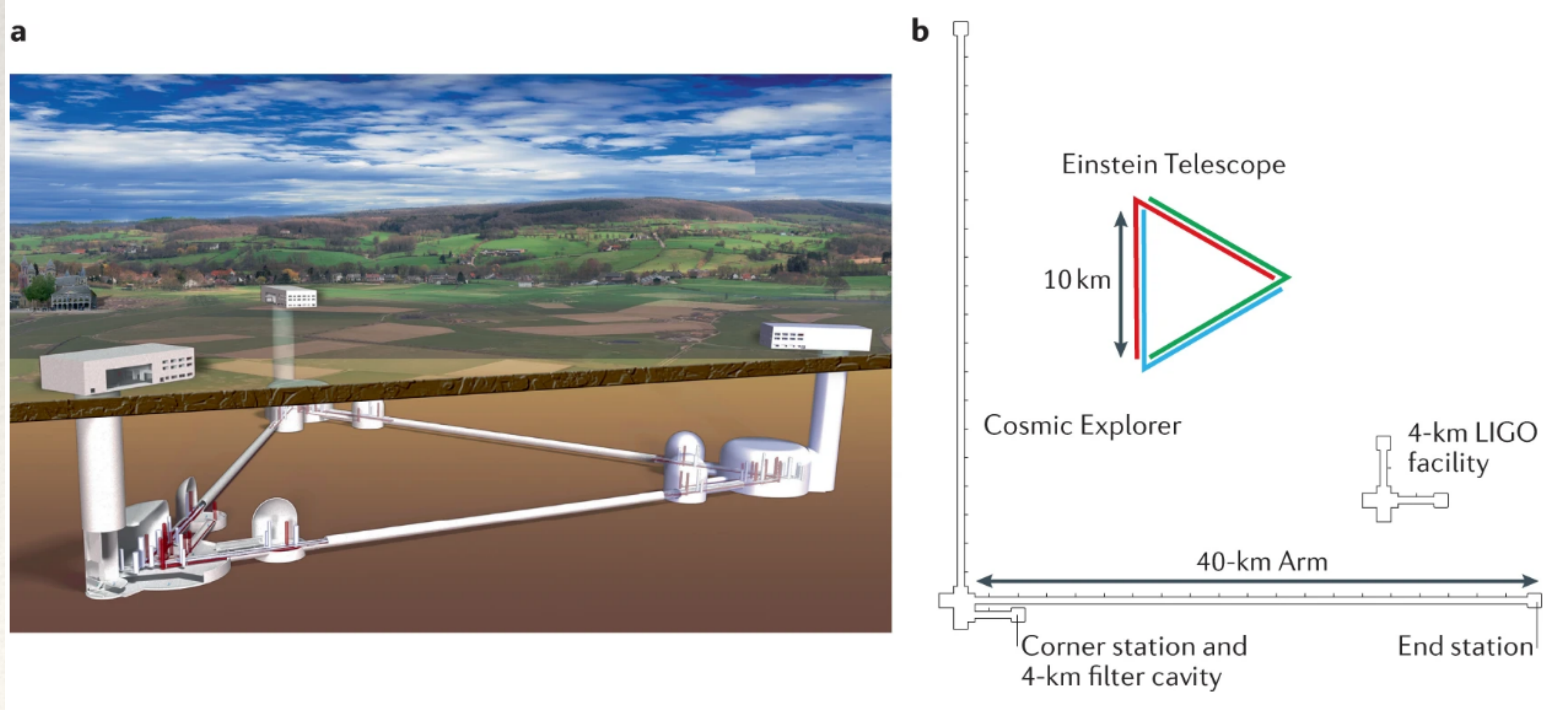
Prospects for observations within advanced Programs

updated [Abbot et al. 2020]

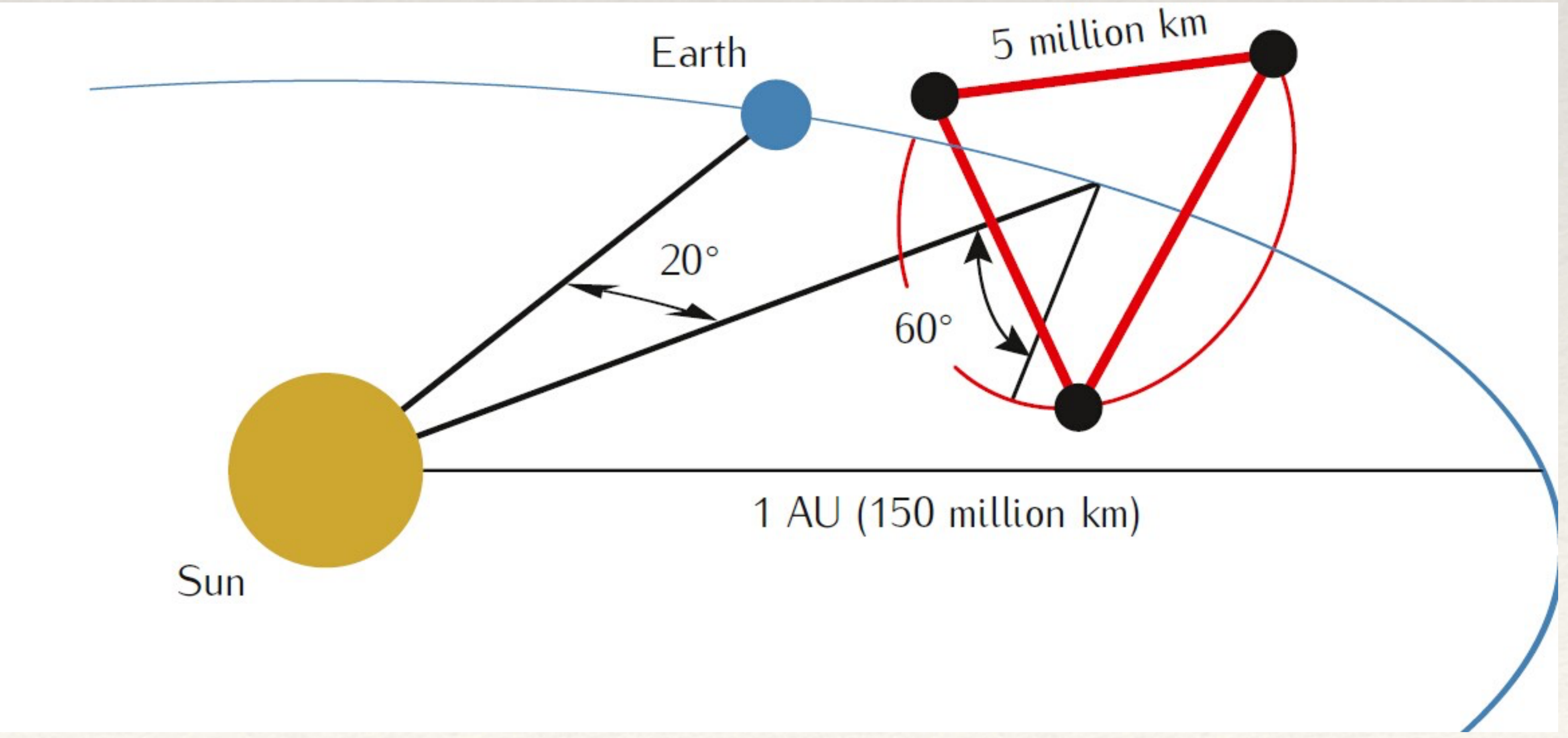
O3 --> O5 <==> *O*(10) --> *O*(100) GW detections/year

:: (some) Future GW Detectors [Bailes et al. 2021]

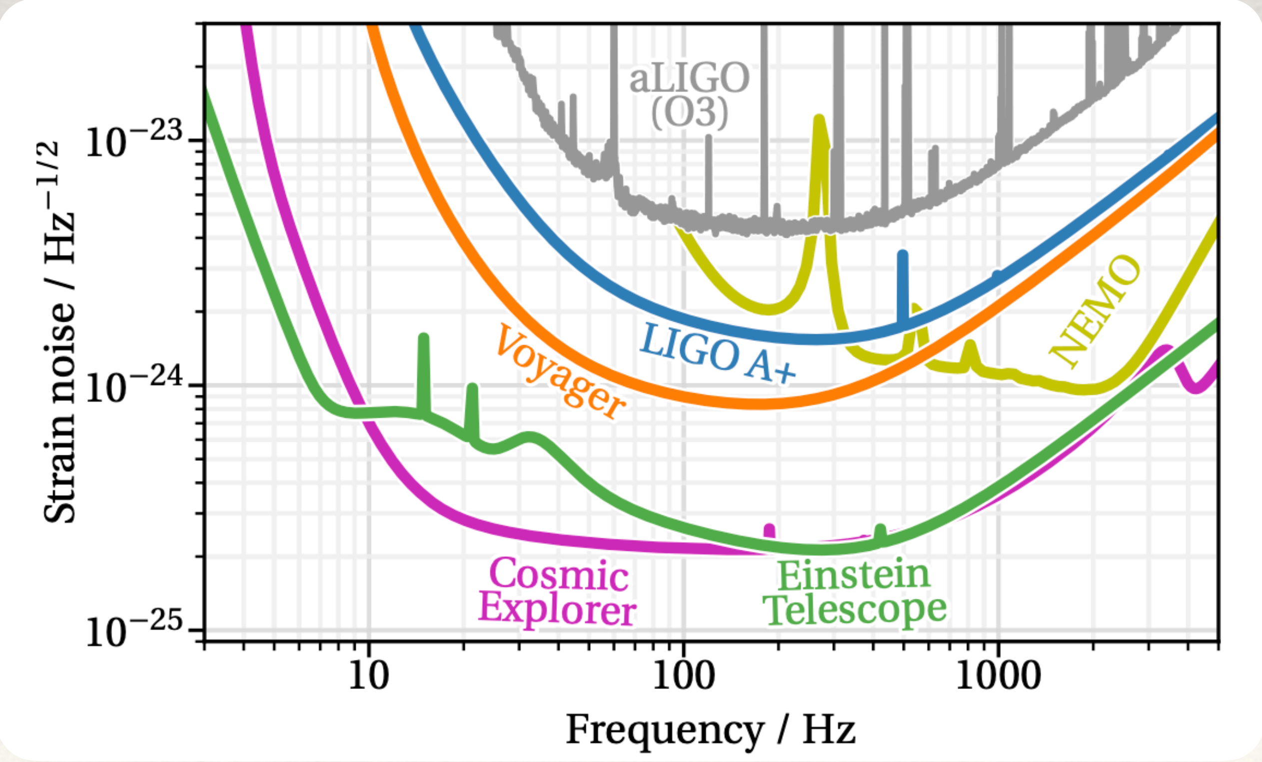
Einstein Telescope



Lisa Mission



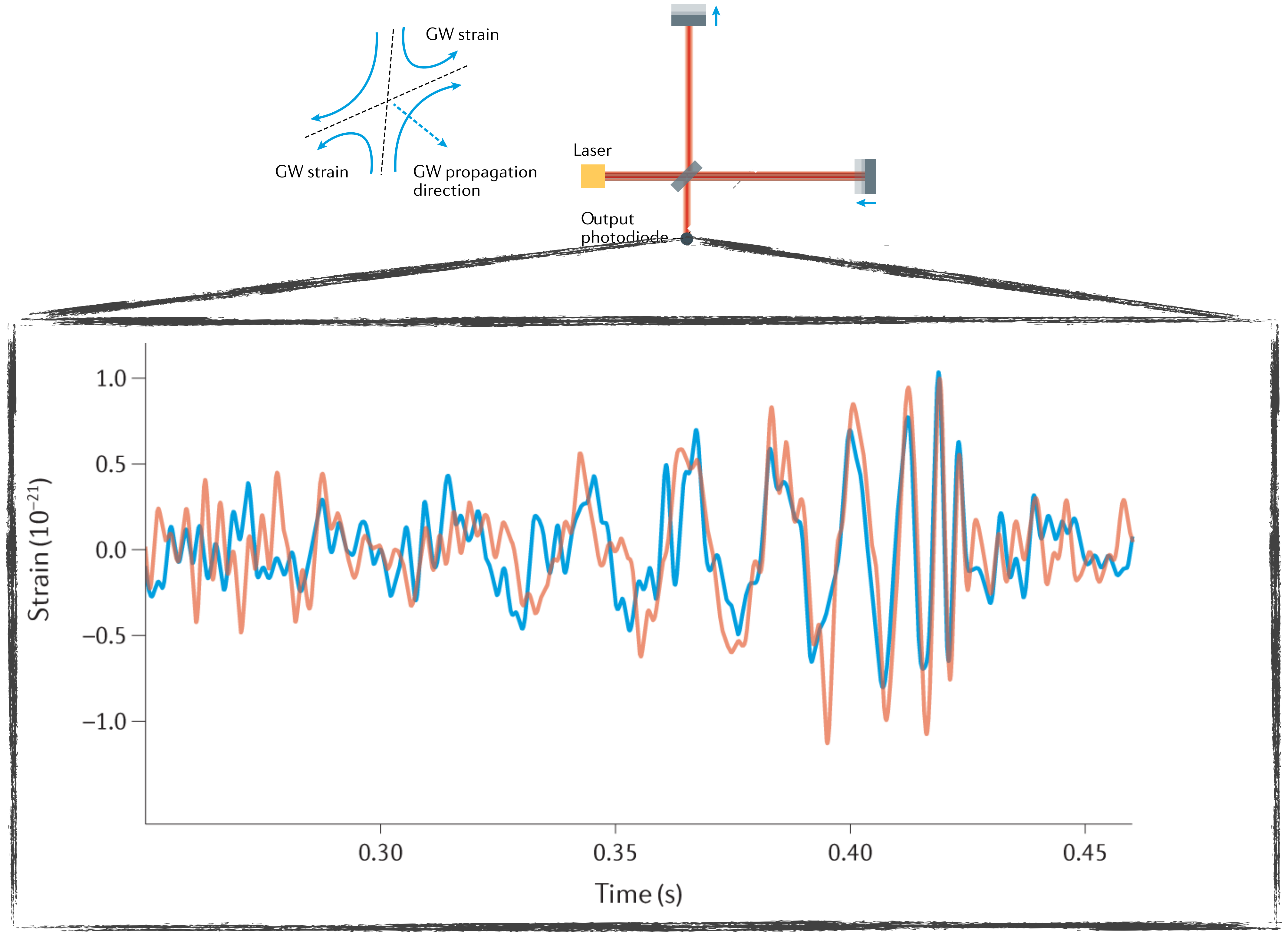
expected sensitivity



[credit: CE Consortium]

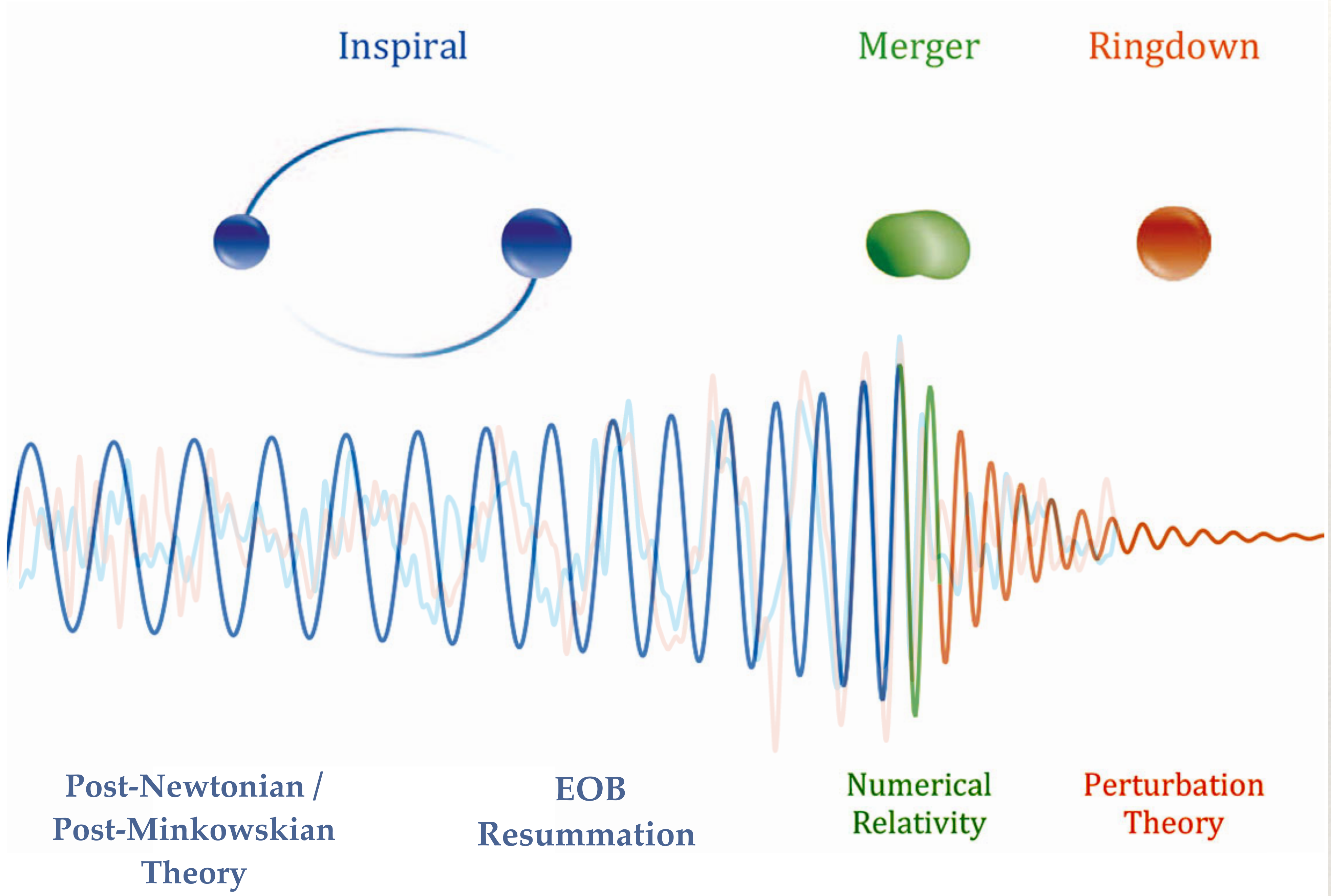
Two-Body Dynamics and GW Signal

Real Event



Two-Body Dynamics and GW Signal

▸ Waveform Model and Computing Techniques



Two-Body Dynamics and GW Signal

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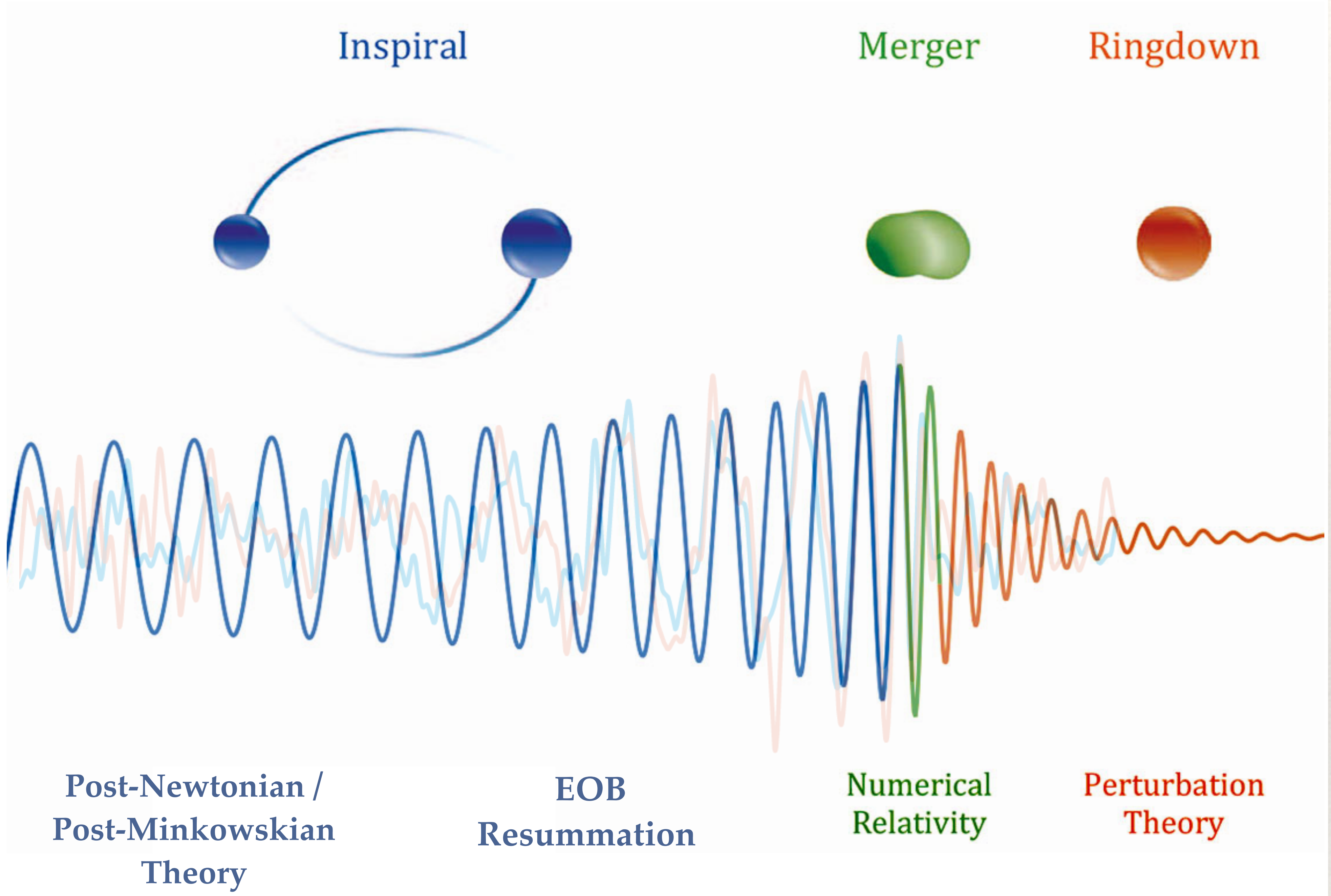
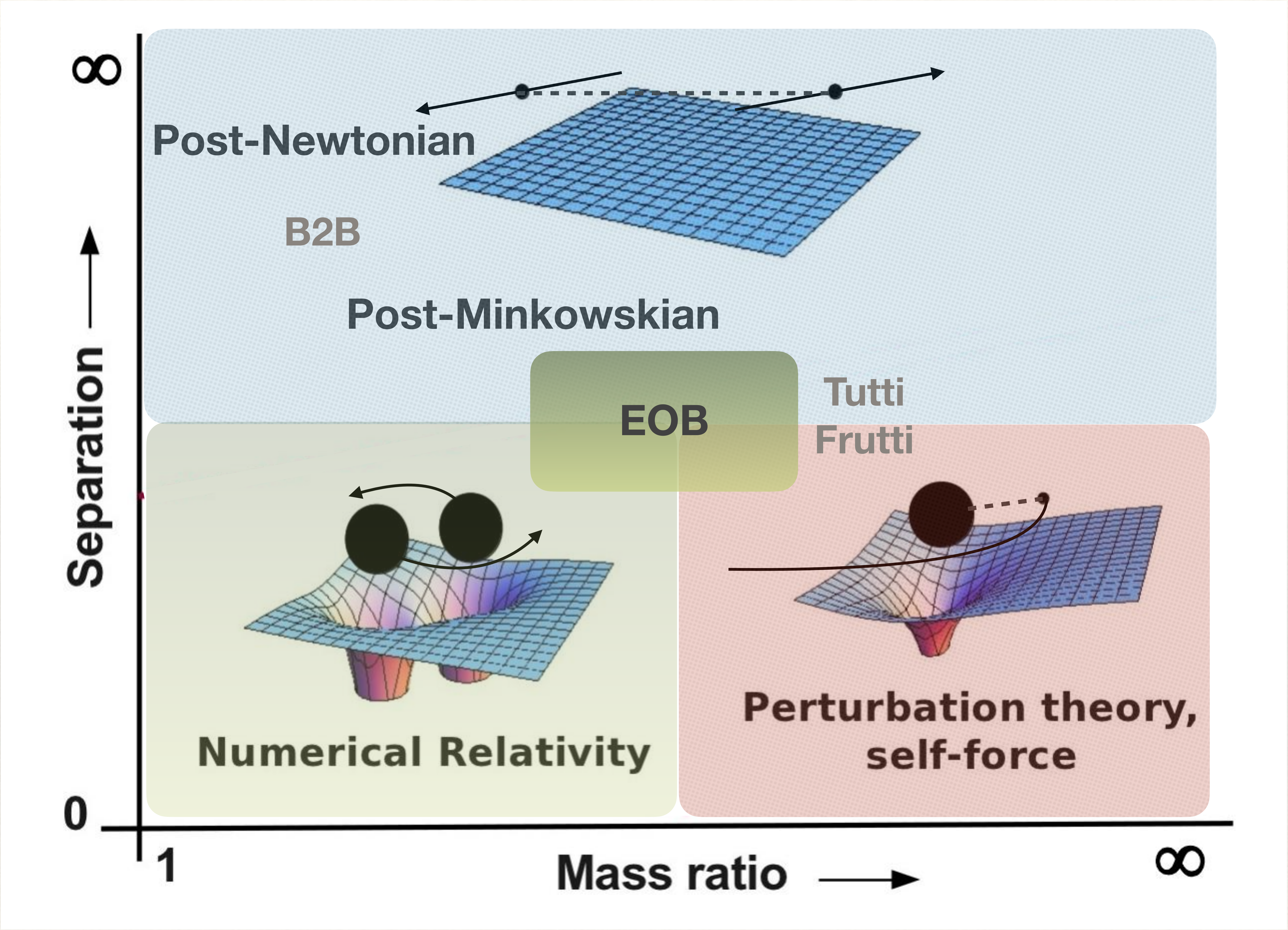


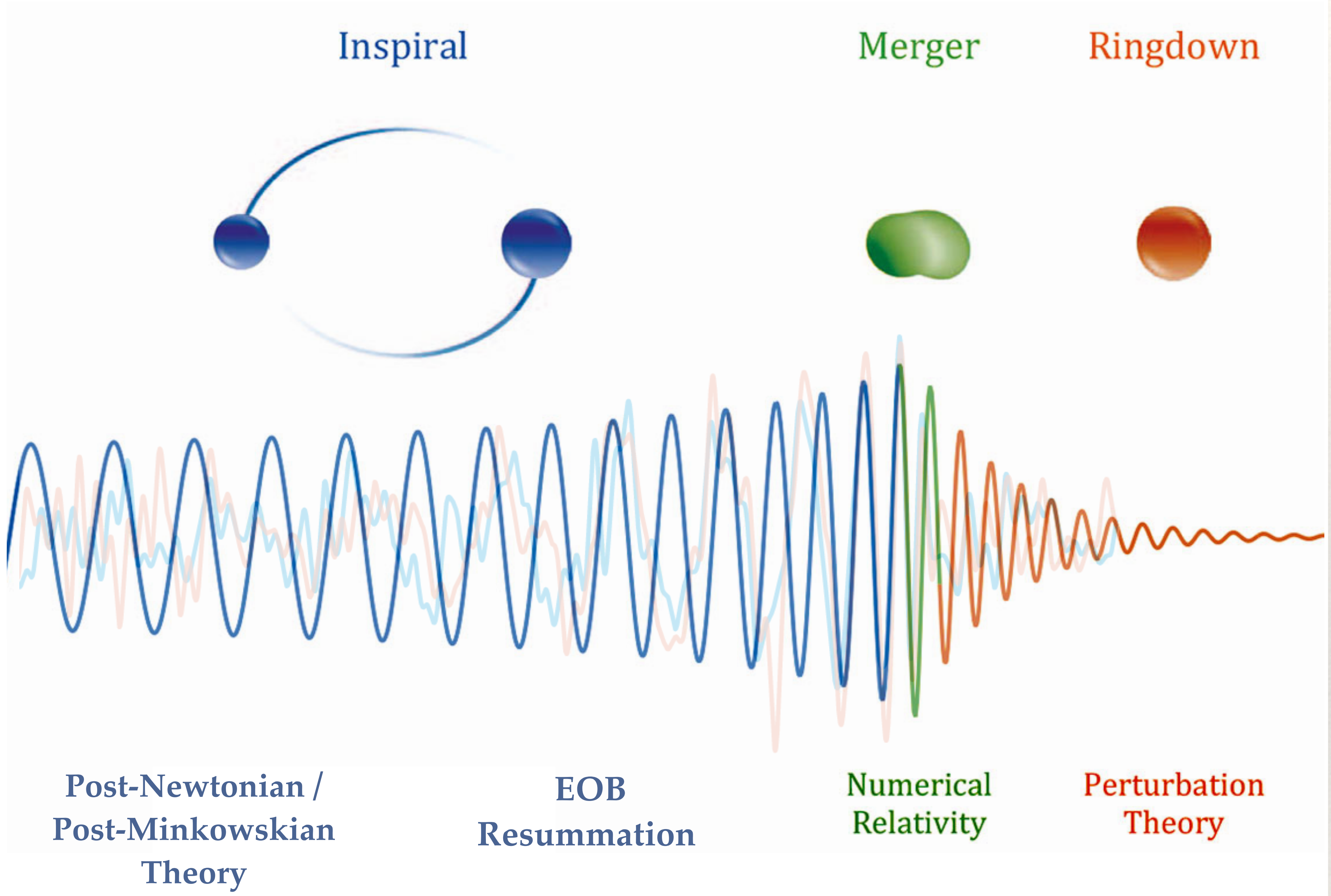
Standard Model of GW Physics



[adapted from: Barak]

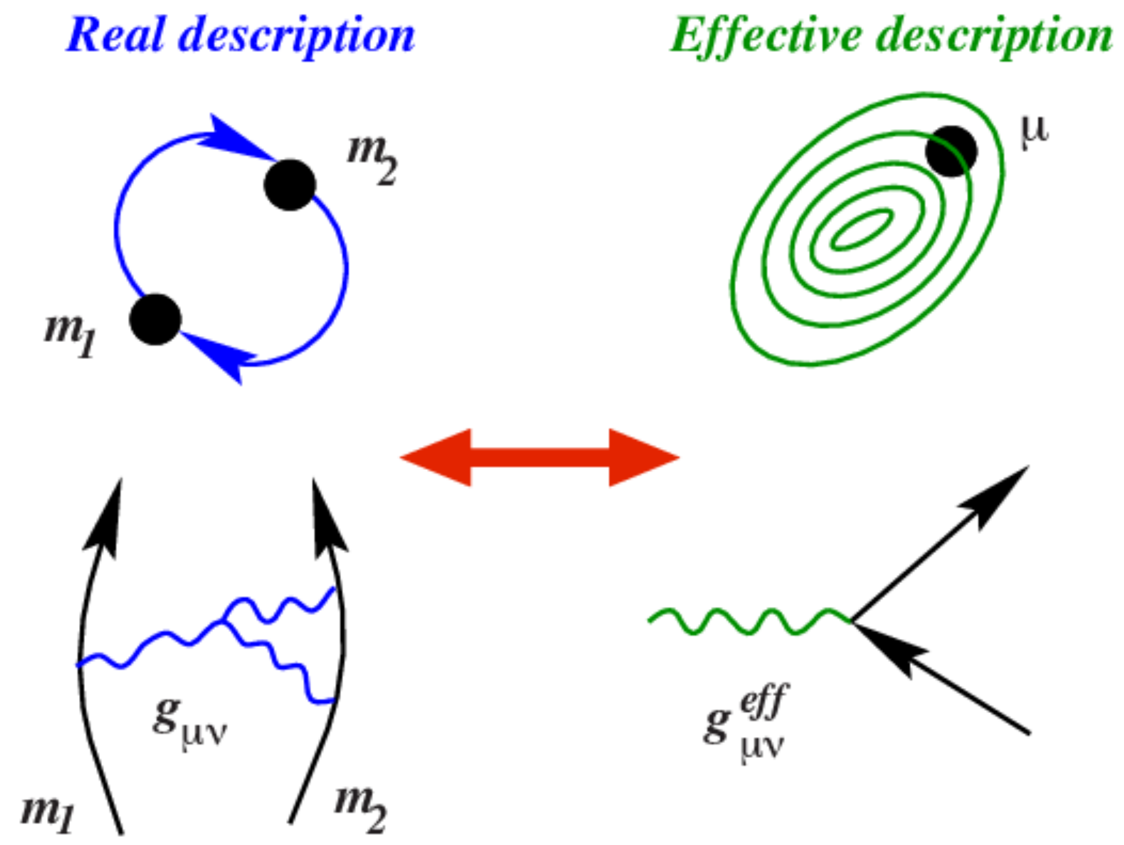
Two-Body Dynamics and GW Signal

Waveform Model and Computing Techniques



Effective One Body (EOB) Formalism [Buonanno Damour]

the contributions coming from different kinematic regions for combined and calibrated with Numerical Relativity



see BINI

- Post-Minkowskian Expansion [relativistic scattering]

$$G_N \frac{m}{r} << v^2 \sim 1$$

Expansion in powers of G_N

- Post-Newtonian Expansion [non relativistic system]

$$G_N \frac{m}{r} \sim v^2 << 1$$

Expansion in powers of v/c

- BH perturbation theory / self force

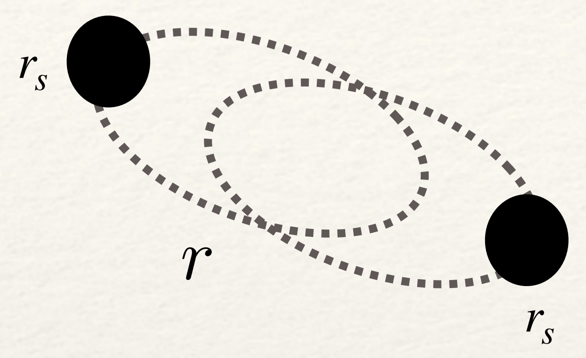
Expansion for small metric deformation

$$\delta g_{\mu\nu} \sim \epsilon = m_2/m_1 << 1$$

Effective Field Theory for General Relativity

Coalescing **B**inary **S**ystem

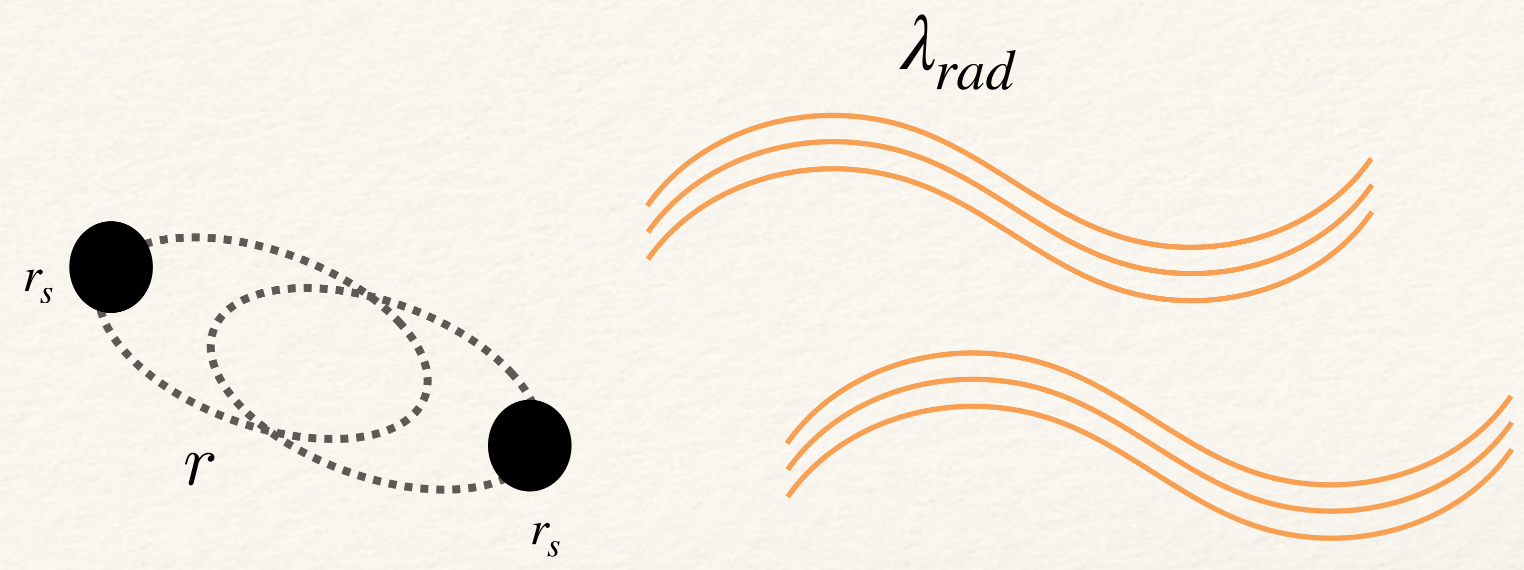
:: Double Hierarchy



Conservative system ::
~~GW emission~~

$$\boxed{r_s \ll r} \ll \lambda_{rad}$$

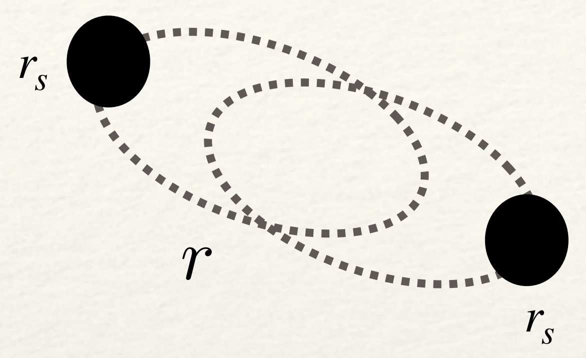
$$r_s \ll \boxed{r \ll \lambda_{rad}}$$



Dissipative system ::
GW emission

Coalescing Binary System

:: Double Hierarchy



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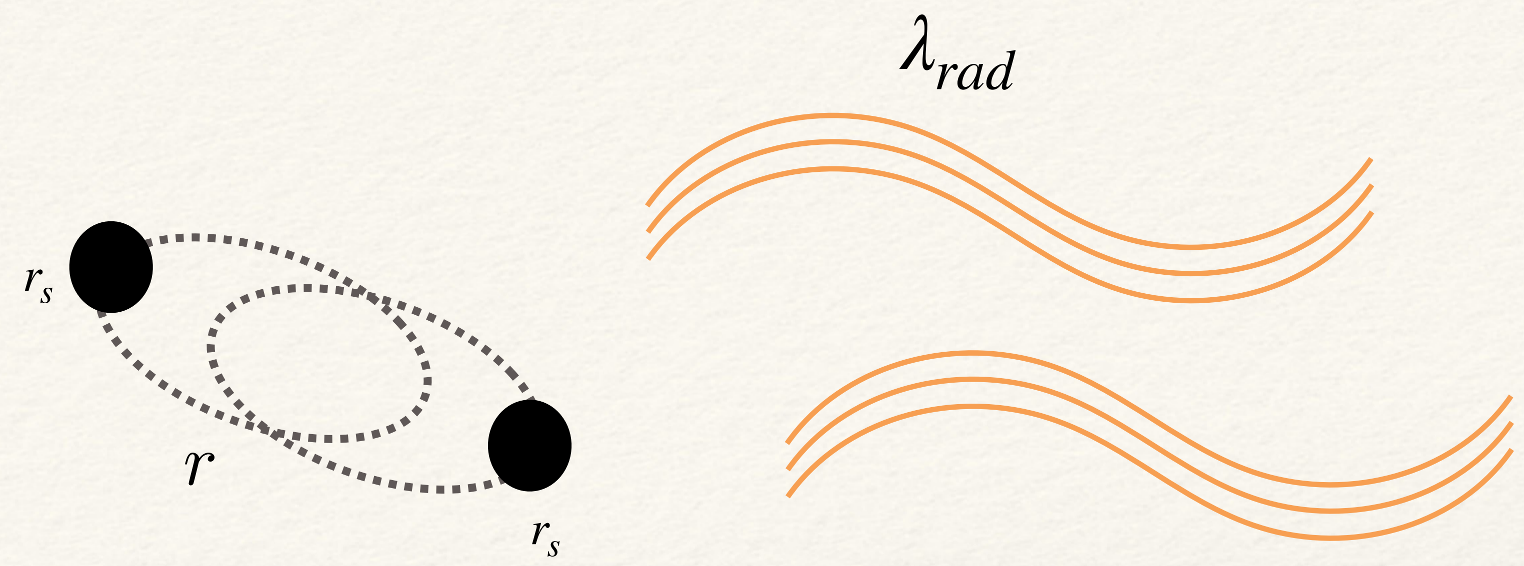
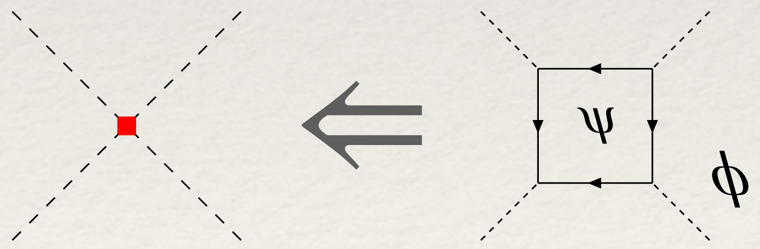


Dissipative system ::
GW emission

:: Effective Field Theory Approach

- Fundamental [*complete*] theory $S[\phi, \psi]$
  - Heavy fields ψ: Λ,
    short distance r_s
  - Light modes ϕ: $\omega \ll \Lambda$,
    large distance r

$$e^{\frac{iS_{eff}[\phi]}{\hbar}} = \int D\psi \; e^{\frac{iS[\phi,\psi]}{\hbar}}$$



- Effective [*incomplete*] theory $S_{eff}[\psi]$
  - Sensitive to the Lower-scale dynamics: $\omega \ll \Lambda$

GREFT / Action

$$S_{tot}[x_a, g] = S_{GR}[g] + S_{m_a}[x_a, g]$$

[Goldberger, Rothstein]

- **Einstein Hilbert + gauge fixing**

$$S_{GR}[g] = 2\Lambda^2 \int d^{d+1}x \sqrt{-g}\left(R - \frac{1}{2}\Gamma^{\mu}\Gamma_{\mu}\right)$$

$$\Lambda^{-1} = \sqrt{32\pi G_N}$$

- **Source/Worldline**

$$S_{m_a}[x_a, g] = S_{pp}[x_a, g] + \delta S_{m_a}[x_a, g]$$

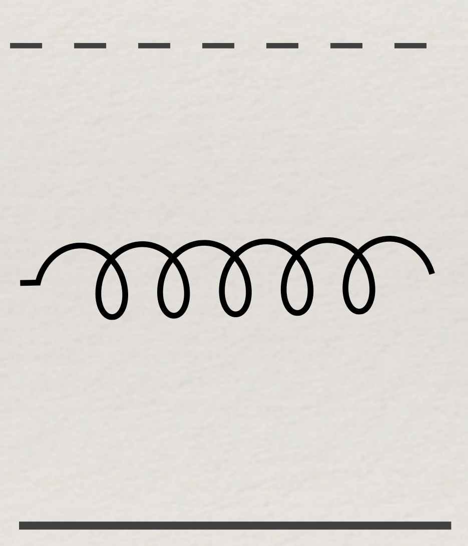
$$= -m_a \int d\tau_a = -m_a \int dt \sqrt{-g_{\mu\nu}(x_a)\dot{x}^{\mu}_a \dot{x}^{\nu}_a}$$

▸ Non-relativistic approximation [method of regions]: [Beneke Smirnov]

▸ Weak field expansion:

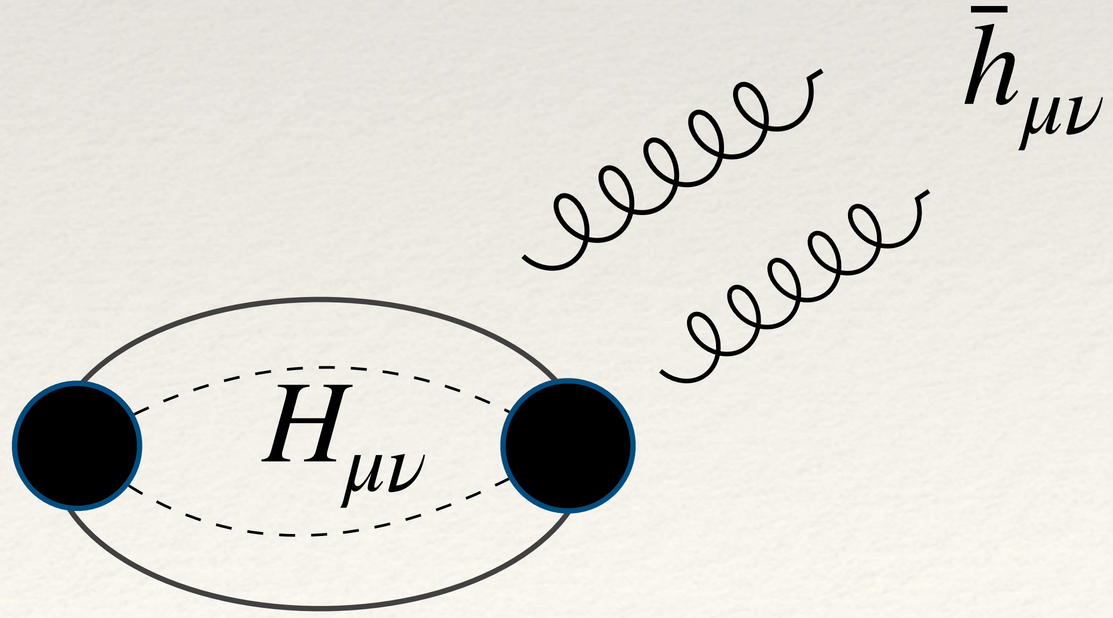
$$v \ll 1 \qquad g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \qquad h_{\mu\nu} = H_{\mu\nu} + \bar{h}_{\mu\nu}$$

- **Potential gravitons** $H_{\mu\nu}$: $(k_0, \mathbf{k}) \sim \left(\frac{v}{r}, \frac{1}{r}\right)$
- **Radiation gravitons** $\bar{h}_{\mu\nu}$: $(k_0, \mathbf{k}) \sim \left(\frac{v}{r}, \frac{v}{r}\right)$
- **Worldline/BH** x_a :



▸ Effective action by integrating out gravitons:

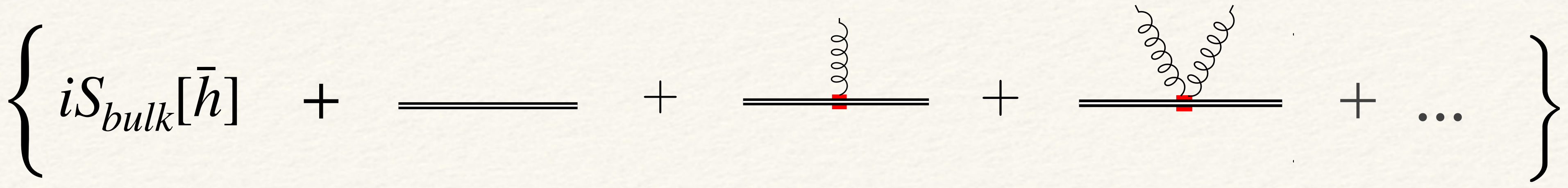
$$e^{iS_{eff}[x_a]} = \int D\bar{h} \int DH\, e^{iS_{tot}[x_a, H, \bar{h}]}$$

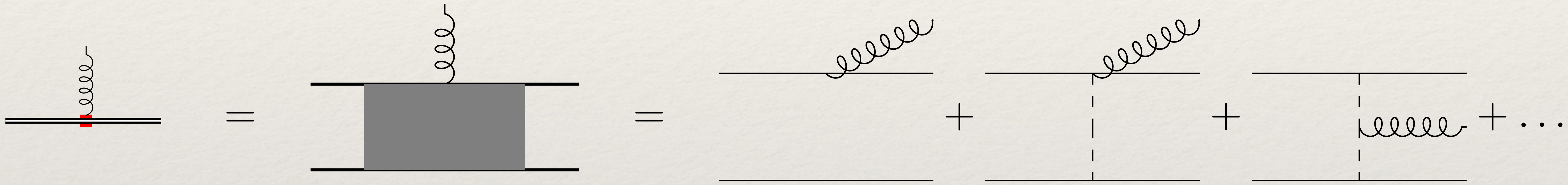


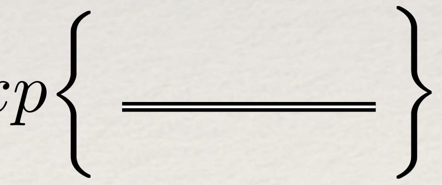
GREFT / Action / Near & Far Zone

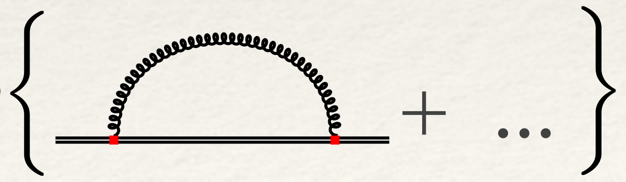
[Goldberger, Rothstein]

$$e^{iS_{eff}[x_a]} = \int D\bar{h} \int DH\ e^{iS_{tot}[x_a,H,\bar{h}]} = \int D\bar{h}\ e^{\left\{ iS_{bulk}[\bar{h}] + \right.}$$





▸**Near zone (r):** $S_{pot}[x_a, g] = S_{GR}[g] + S_{m_a}[x_a, g]$ $\quad g_{\mu\nu} = \eta_{\mu\nu} + H_{\mu\nu}$ $\quad \int DH\ e^{iS_{tot}[x_a,H,\bar{h}=0]} = exp\{$  $\}$

▸**Far zone (λ_{rad}):** $S_{rad}[g, \{Q_i\}] = S_{GR}[g] + S_{mult}[g, \{Q_i\}]$ $\quad g_{\mu\nu} = \eta_{\mu\nu} + \bar{h}_{\mu\nu}$ $\quad \int D\bar{h}\ e^{iS_{rad}[x_a,\bar{h}]} = exp\{$  $+ \ ... \}$

Conservative Dynamics :: **Near Zone** Spinless

Near Zone/EFT Diagrammatic Approach

[Goldberger, Rothstein]
[Gilmore, Ross]
[Foffa, Sturani]

$$S_{pot}[x_a, g] = S_{GR}[g] + S_{m_a}[x_a, g] \qquad S_{m_a}[x_a, g] = S_{pp}[x_a, g] + \delta S_{m_a}[x_a, g]$$

- **Kaluza-Klein parametrization:**

[Kol Smolkin] [Blanchet Damour]

$$g_{\mu\nu} = e^{2\phi/\Lambda} \begin{pmatrix} -1 & A_j/\Lambda \\ A_i/\Lambda & e^{-c_d \frac{\phi}{\Lambda}} \gamma_{ij} - A_i A_j/\Lambda^2 \end{pmatrix}$$

$$\gamma_{ij} = \delta_{ij} + \frac{\sigma_{ij}}{\Lambda} \qquad c_d = 2\frac{d-1}{d-2}$$

Graviton = Scalar + Vector + Sym. Tensor
$10 \qquad 1 + 3 + 6$

$$g_{\mu\nu} \Rightarrow \phi \quad A^i \quad \sigma^{ij}$$

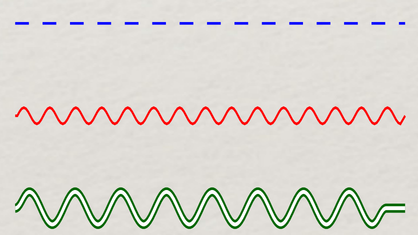
- **Feynman rules for:** $\phi \quad A^i \quad \sigma^{ij} \quad x_a$

see PEGORIN

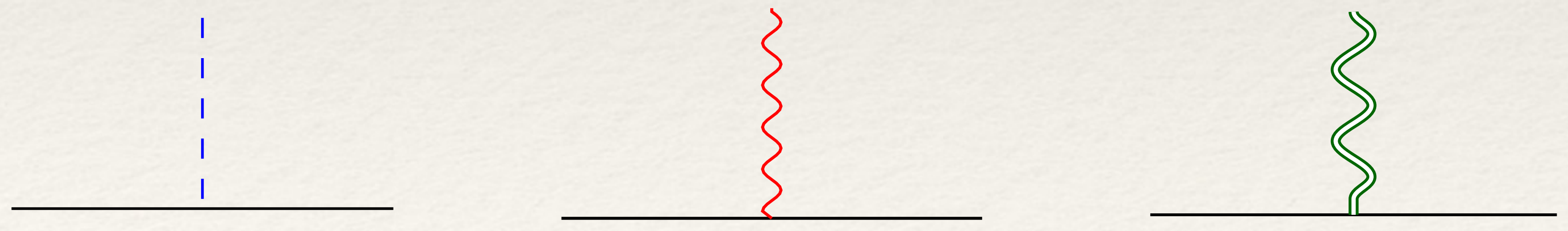
Static / non-propagating source: x_a

Propagators:

ϕ
A^i
σ^{ij}

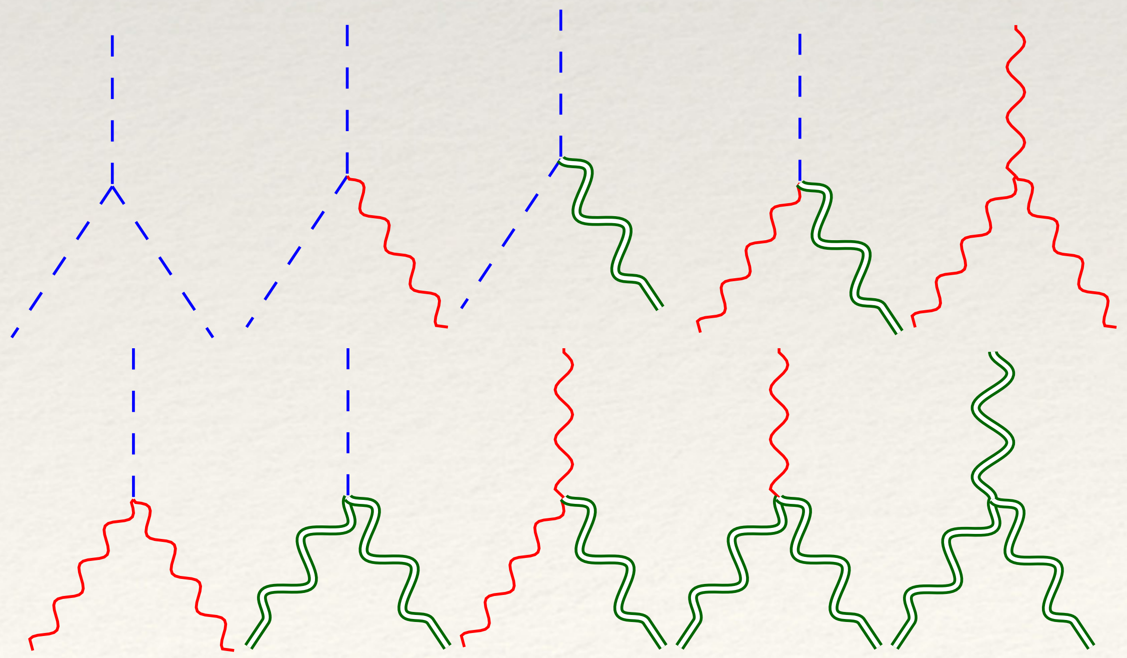


Source couplings:



$+ \ldots$

Self-interactions:

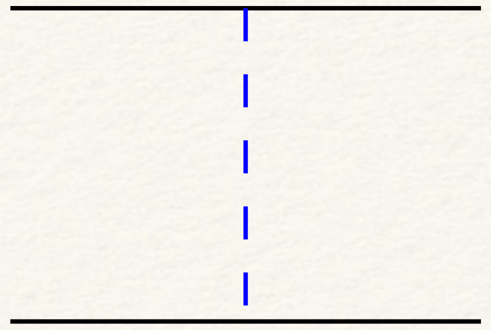


$+ \ldots$

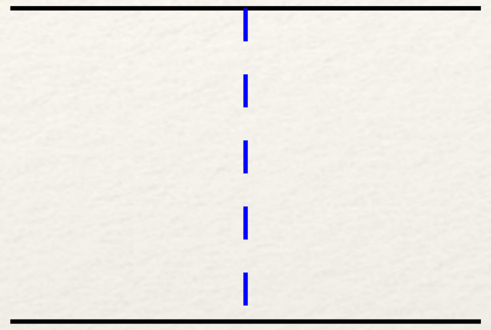
Newton Potential

Diagrammatic approach

- Just 1 diagram:

$$\mathcal{M}_{0PN} =$$  $$= \frac{i m_1 m_2}{2 c_d \Lambda^2} \frac{1}{\mathbf{p}^2}$$

- Fourier transform: from amplitude to the effective action:

$$\mathcal{L}_{0PN} = -\, i \lim_{d \to 3} \int \frac{d^d \mathbf{p}}{(2\pi)^d} e^{i\mathbf{p}(x_1 - x_2)} \Bigg($$  $$\Bigg) = \frac{G_N m_1 m_2}{r}$$

Newton Potential

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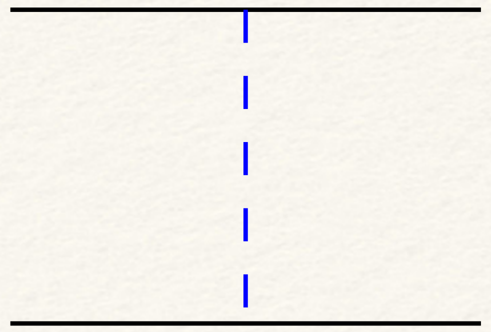
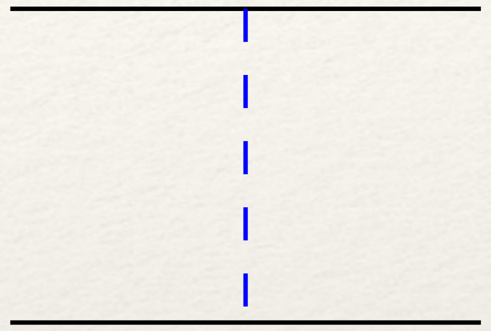
- Just 1 diagram: $\mathscr{M}_{0PN} =$  $= \dfrac{i m_1 m_2}{2 c_d \Lambda^2} \dfrac{1}{\mathbf{p}^2}$

- Fourier transform: from amplitude to the effective action: $\mathscr{L}_{0PN} = -\, i \lim_{d \to 3} \int \dfrac{d^d \mathbf{p}}{(2\pi)^d} e^{i\mathbf{p}(x_1 - x_2)} \Bigg($  $\Bigg) = \dfrac{G_N m_1 m_2}{r}$

Corrections to the Newtonian potential:

- Non-relativistic velocities: $v^2 \ll 1$

- Dynamics in Post-Minkowskian perturbative scheme

- At nPM order: G_N^n

Astrophysicists/Cosmologists’ whishlist

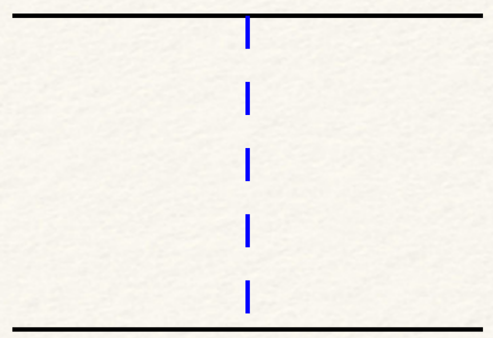
		0PN	1PN	2PN	3PN	4PN	5PN	6PN		
	G	(1	+ v^2	+ v^4	+ v^6	+ v^8	+ v^{10}	+ v^{12}	+ …)	1PM
1979-81 →	G^2	(1	+ v^2	+ v^4	+ v^6	+ v^8	+ v^{10}	+ v^{12}	+ …)	2PM
2019 →	G^3	(1	+ v^2	+ v^4	+ v^6	+ v^8	+ v^{10}	+ v^{12}	+ …)	3PM
2021 →	G^4	(1	+ v^2	+ v^4	+ v^6	+ v^8	+ v^{10}	+ v^{12}	+ …)	4PM
	G^5	(1	+ v^2	+ v^4	+ v^6	+ v^8	+ v^{10}	+ v^{12}	+ …)	5PM
	G^6	(1	+ v^2	+ v^4	+ v^6	+ v^8	+ v^{10}	+ v^{12}	+ …)	6PM
	G^7	(1	+ v^2	+ v^4	+ v^6	+ v^8	+ v^{10}	+ v^{12}	+ …)	7PM

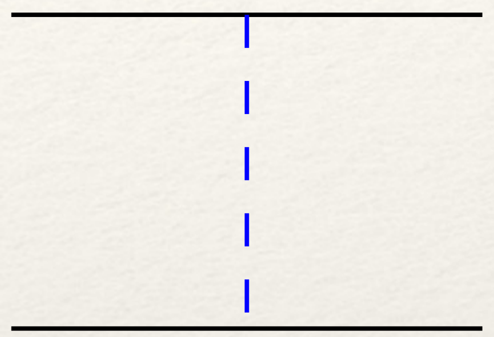
[credit: Bern et al.]

…Westphal, Damour, Cheung, Rothstein, Solon, Bern, Roiban, Shen, Zeng, Parra-Martinez, Ruf, Hermann, Buonanno, Porto, Dlapa, Kalin, Liu, Neef, Bjerrum-Bohr, Vanhove, Plante, Cristofoli, Damgaard, Guevara, Ochirov, Vines, Di Vecchia, Veneziano, Heisenberg, Russo, Plefka, Jakobsen, Mogull, Brandhuber, Travaglini, De Angelis, Accetulli-Huber, Luna, Kosmopoulos, and collaborators…

Newton Potential

Diagrammatic approach

- Just 1 diagram: $\mathcal{M}_{0PN} =$  $= \dfrac{i m_1 m_2}{2 c_d \Lambda^2} \dfrac{1}{\mathbf{p}^2}$

- Fourier transform: from amplitude to the effective action: $\mathcal{L}_{0PN} = -i \lim_{d \to 3} \int \dfrac{d^d \mathbf{p}}{(2\pi)^d} e^{i\mathbf{p}(x_1 - x_2)} \Big($  $\Big) = \dfrac{G_N m_1 m_2}{r}$

Corrections to the Newtonian potential:

- Non-relativistic velocities: $v^2 \ll 1$

- Virial theorem: $\dfrac{G_N m}{r} \approx v^2$

- Dynamics in Post-Newtonian perturbative scheme

- At nPN order: $G_N^{n-\ell} v^{2\ell}$

Astrophysicists/Cosmologists' whishlist

	0PN		1PN		2PN		3PN		4PN		5PN		6PN		7PN			
G	$(1$	$+$	v^2	$+$	v^4	$+$	v^6	$+$	v^8	$+$	v^{10}	$+$	v^{12}	$+$	v^{14}	$+$	$\cdots)$	1PM
G^2			$(1$	$+$	v^2	$+$	v^4	$+$	v^6	$+$	v^8	$+$	v^{10}	$+$	v^{12}	$+$	$\cdots)$	2PM
G^3					$(1$	$+$	v^2	$+$	v^4	$+$	v^6	$+$	v^8	$+$	v^{10}	$+$	$\cdots)$	3PM
G^4							$(1$	$+$	v^2	$+$	v^4	$+$	v^6	$+$	v^8	$+$	$\cdots)$	4PM
G^5									$(1$	$+$	v^2	$+$	v^4	$+$	v^6	$+$	$\cdots)$	5PM
G^6											$(1$	$+$	v^2	$+$	v^4	$+$	$\cdots)$	6PM
G^7													$(1$	$+$	v^2	$+$	$\cdots)$	7PM

1687, 1938, 1980, 2000, 2014, 2021

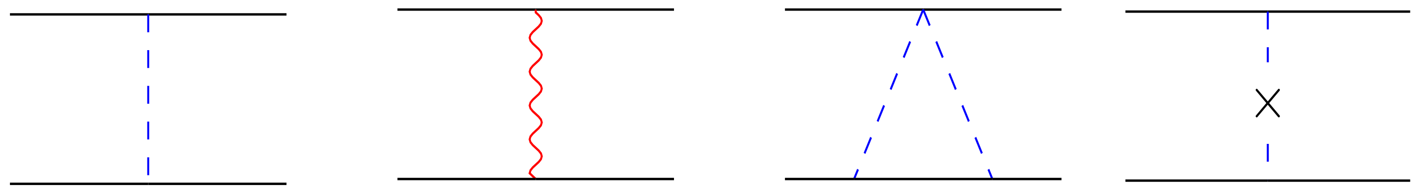
[credit: Bern et al.]

...Jaranowski, Schaefer, Damour, Blanchet, Faye, Porto, Rothstein, Goldberger, Foffa, Sturani, Bini, Buonanno, Geralico, Sturm, Torres Bobadilla, Bluemlein, Maier, Marquard, Levi, Steinhoff, Vines, Antonelli, Kavanagh, Khalil, Galley, von Hippel, McLeod, Edison, Kim, Morales, Yin, Mandal, Patil, Teng, P.M. ...and collaborators ....

Post-Newtonian Corrections/EFT Potential

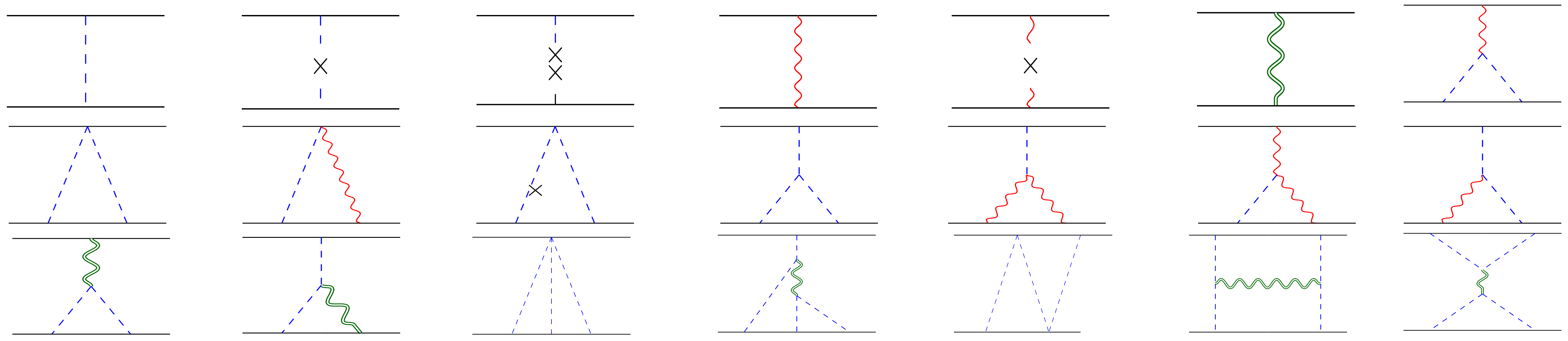
1PN corrections:

Einstein, Infeld, Hoffman (1938)



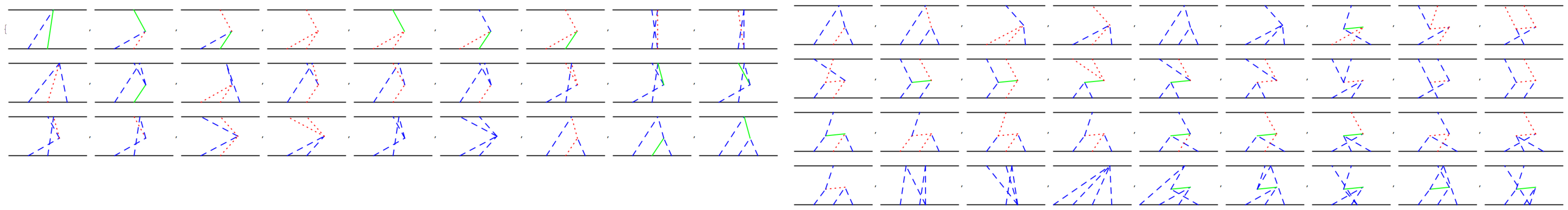
2PN corrections:

Ohta-Okamura-Kimura-Hiida (1974)
Gilmore, Ross (2008)



3PN corrections:

Jaranowski,Schaefer (1997); Damour, Jaranowski, Schaefer (1997); Blachę, Faye (2000); Damour, Jaranowski Schaefer (2001); Foffa Sturani (2011)



4PN: corrections:

Damour, Jaranowski, Schaefer (2014);
Bernard, Blanchet, Bohe, Faye, Marsa (2016);
Foffa, Sturani, Sturm & P.M. (2016);
Foffa, Porto, Rothstein, Sturani (2019)
Blumlein, Maier, Marquard,Schaefer (2020)

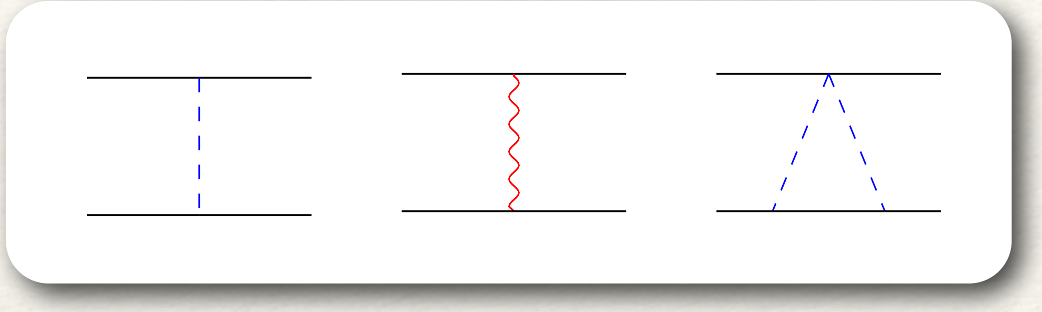
5PN: corrections:

Bini, Damour, Geralico (2019);
Foffa, Sturani, Sturm, Torres Bobadilla & P.M. (2019);
Blumlein, Maier, Marquard, Schaefer (2020,2021)

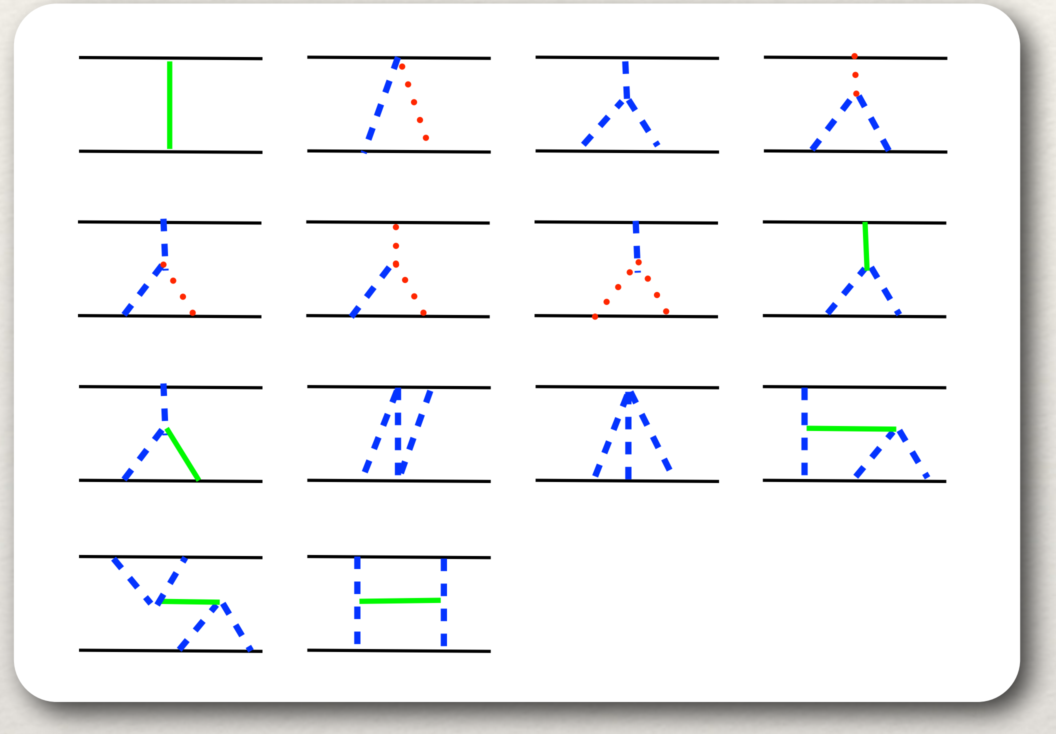
A closer look to 4PN anatomy

- Loop nr. $0 \leq \ell \leq n-1$

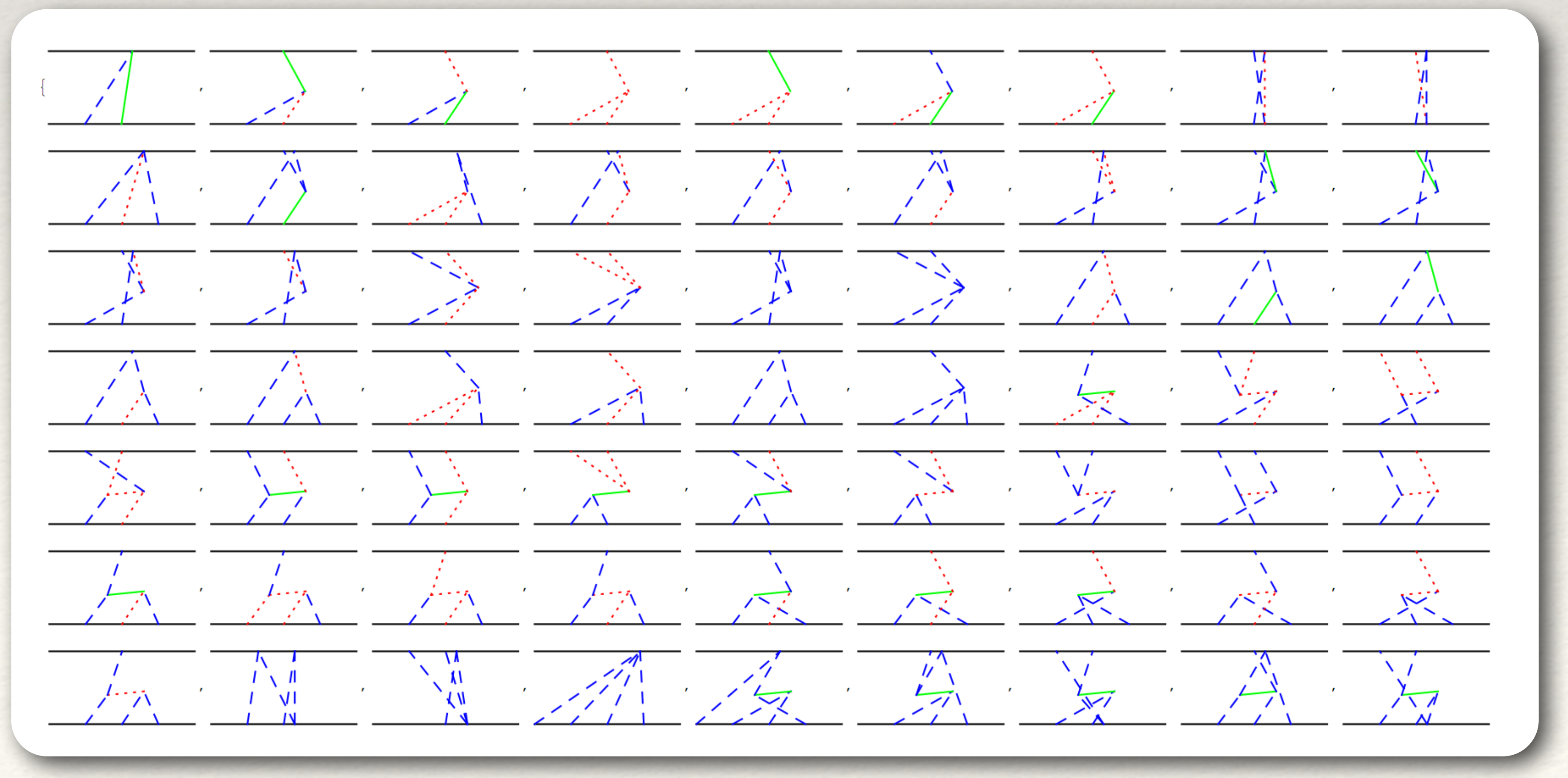
1PN



2PN

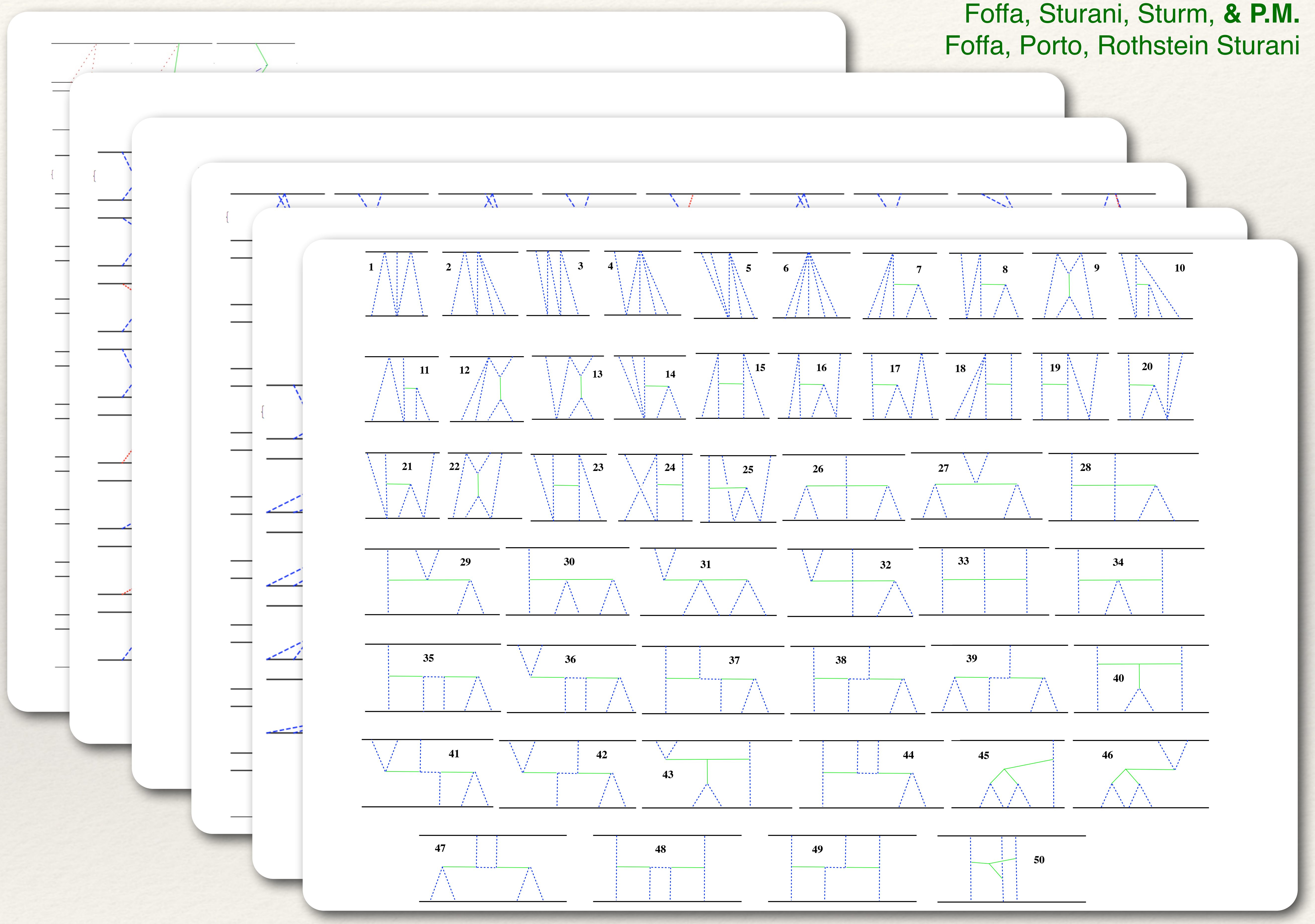


3PN



4PN : 605 GREFT diagrams (up-to 4-loops)

Foffa & Sturani
Foffa, Sturani, Sturm, **& P.M.**
Foffa, Porto, Rothstein Sturani

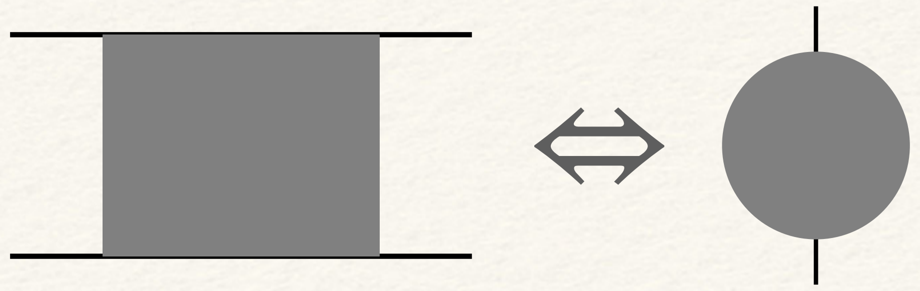


GREFT Diagrams & 2pt-QFT Integrals / a key observation

Foffa, Sturani, Sturm, & **P.M.** (2016)

Computational techniques:

- From Effective diagrams to QFT Amplitudes:



- World-lines are not propagating
- EFTGravity Amplitudes of order G_N^{ℓ} mapped into $(\ell - 1)$—loop 2-point functions with massless internal lines:

$$\mathscr{M} = \sum_i c_i \, I_i^{MI}$$

- Dimensional Regularization $d = 3 + \epsilon$
- Integration-by-parts (IBP) decomposition
- Master Integrals and Differential Equations

Chetyrkin, Tkachov (1981)
Laporta, Remiddi (1996)
Remiddi (1996)
Caffo, Czyz, Laporta, Remiddi (1998)
Gehrmann, Remiddi (1999)
Laporta (2000)
Remiddi + Bonciani, Argeri & **P.M. …**

[Bologna Legacy]

… … … …

- Amplitudes evaluation with QFT multi-loop techniques

- From QFT Amplitudes to Effective Lagrangians:

$$\mathscr{L}_{eff}[x_a] = -\, i \lim_{d\to 3} \int \frac{d^d\mathbf{p}}{(2\pi)^d} e^{i\mathbf{p}\cdot\mathbf{r}} \Big(\quad \Big)$$

GREFT Diagrams & 2pt-QFT Integrals / a key observation

Foffa, Sturani, Sturm, & **P.M.** (2016)

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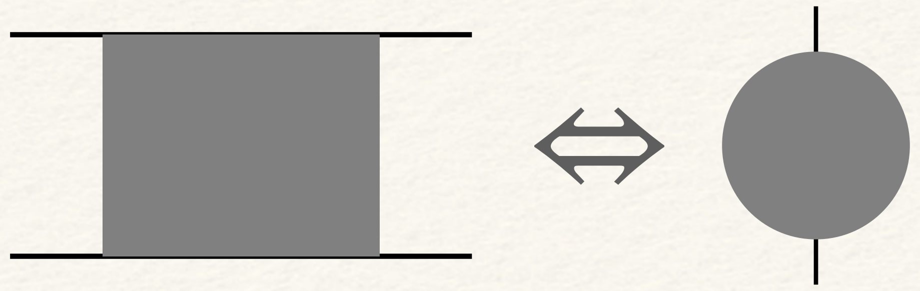

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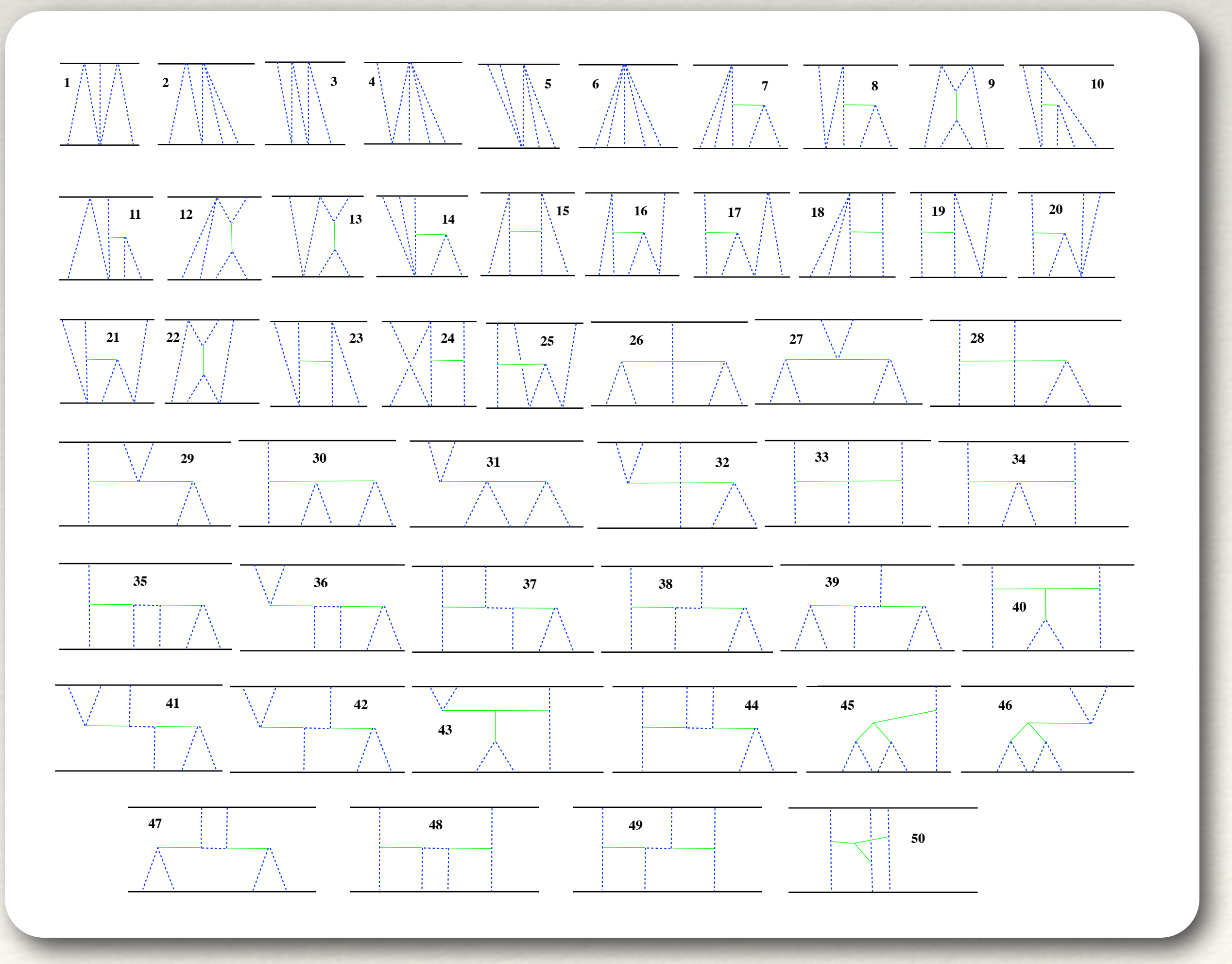
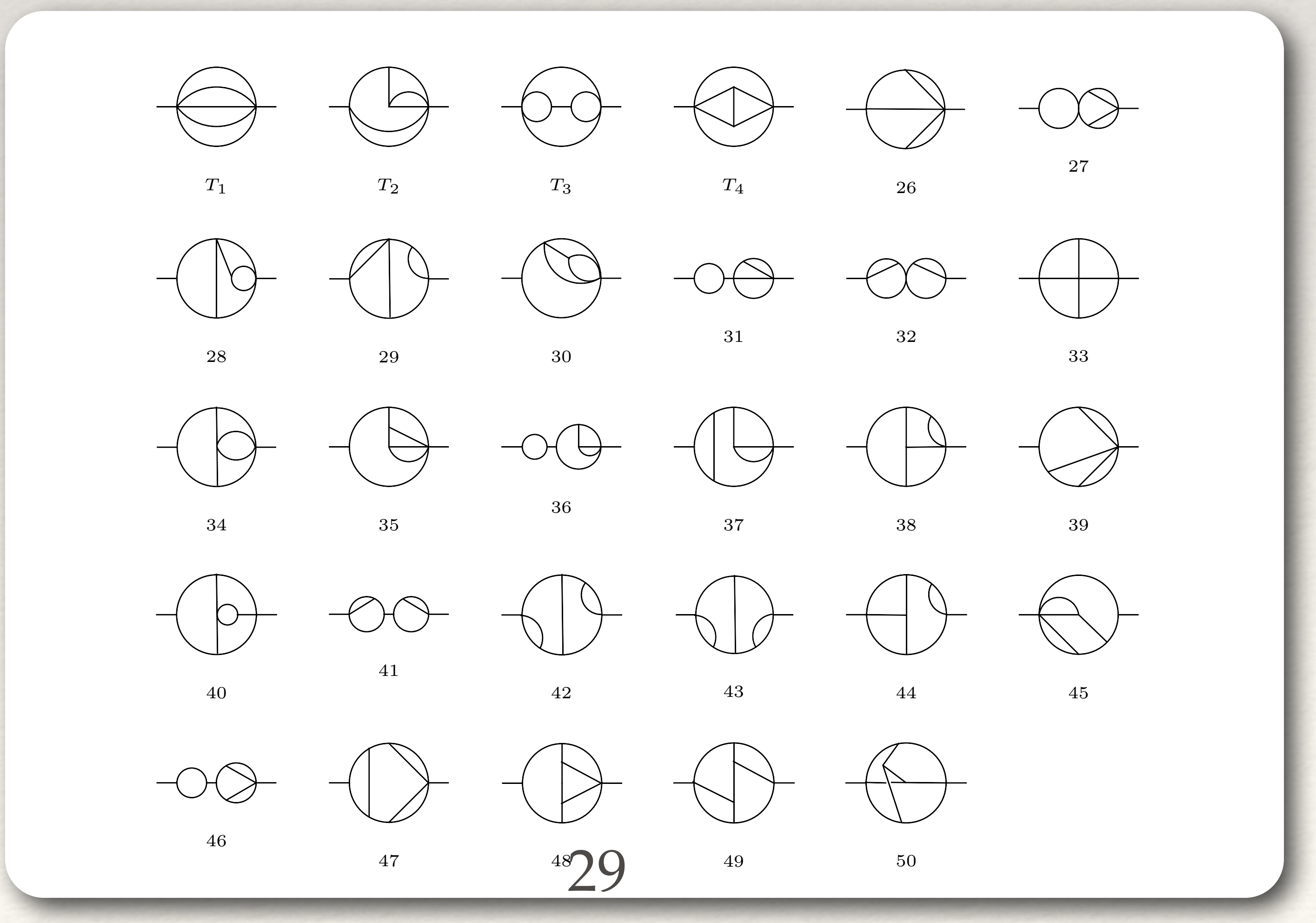
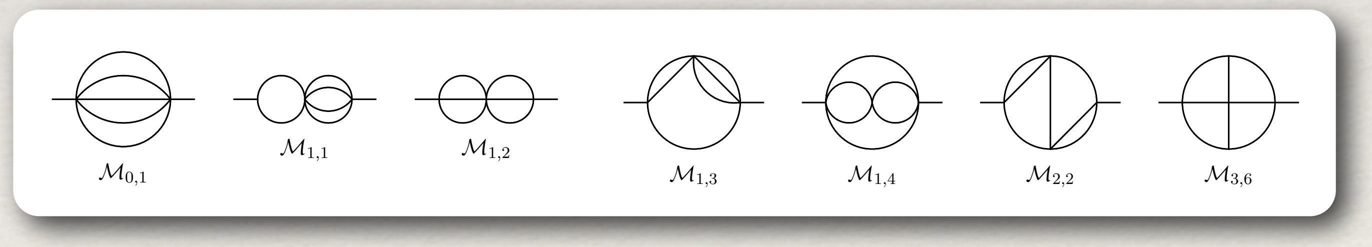
4PN static O(G^5): 50 4-loop GREFT diagrams



29 4-loop QFT diagrams

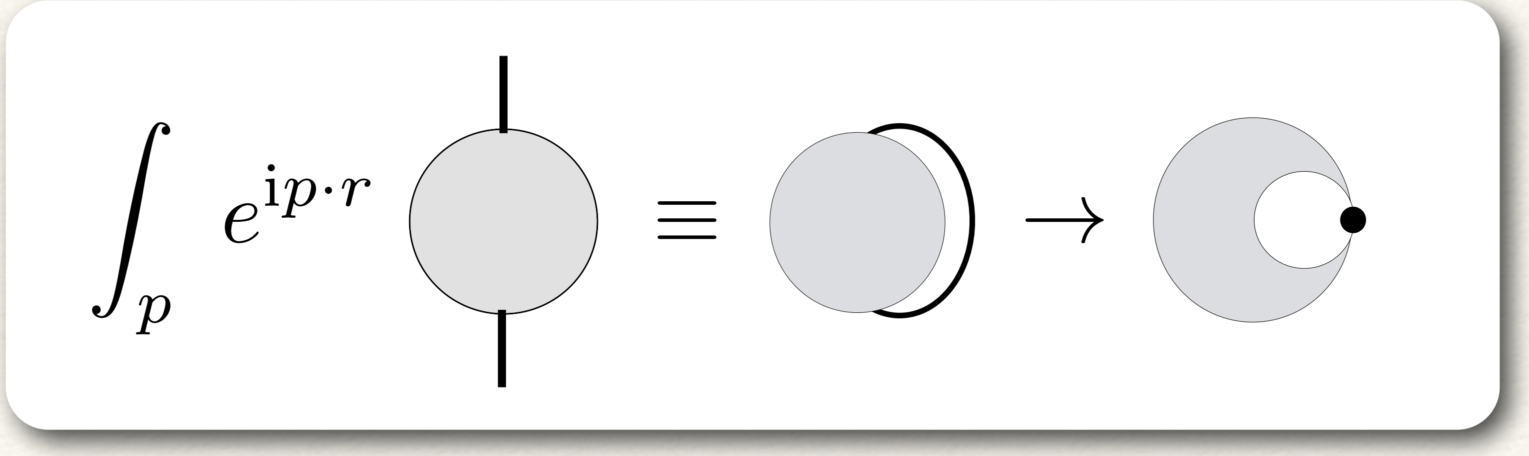


7 4-loop Master Integrals

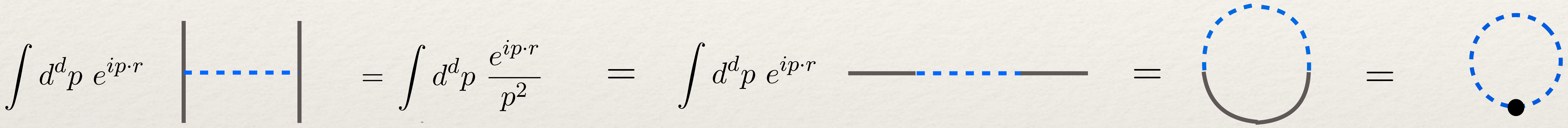


GREFT Diagrams & 2pt-QFT Integrals / Factorization Th'm

Foffa, Sturani, Sturm, Torres-Bobadilla & **P.M.** (2019)

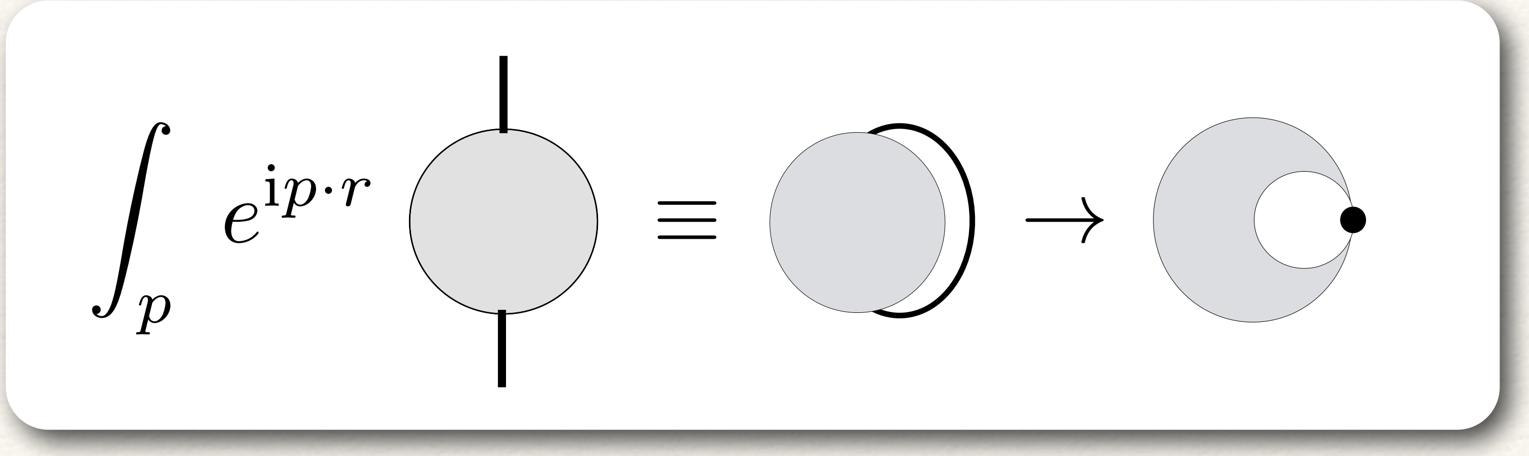


Newton Potential (reloaded):

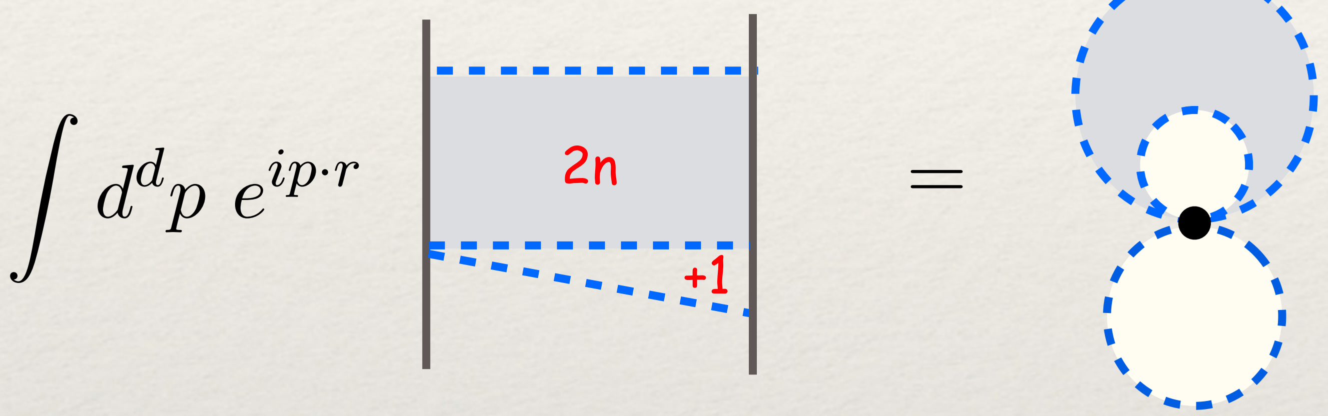


GREFT Diagrams & 2pt-QFT Integrals / Factorization Th'm

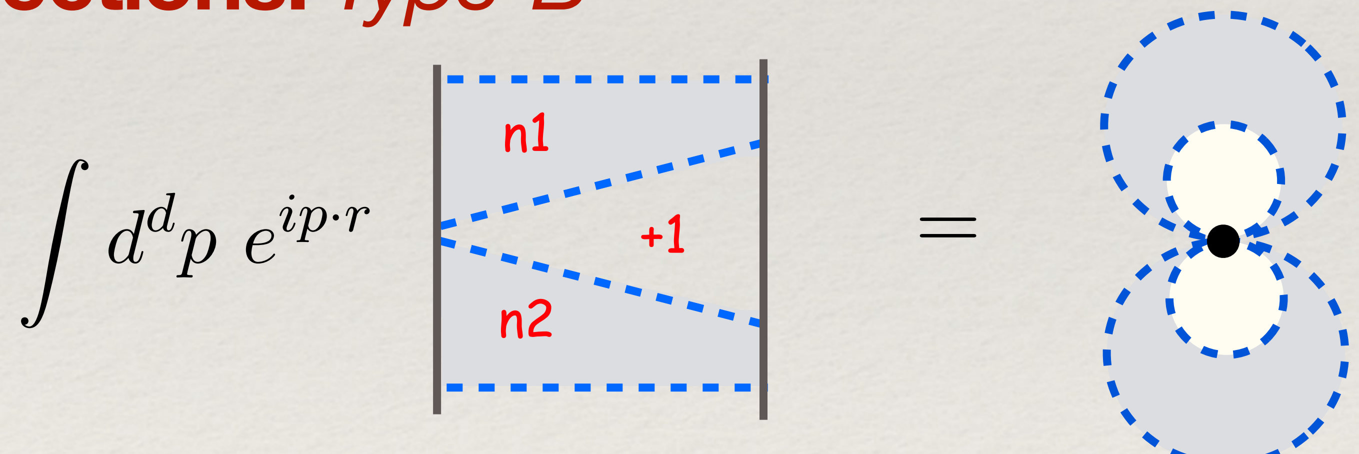
Foffa, Sturani, Sturm, Torres-Bobadilla & **P.M.** (2019)



(2n+1)-PN corrections: *Type-A*



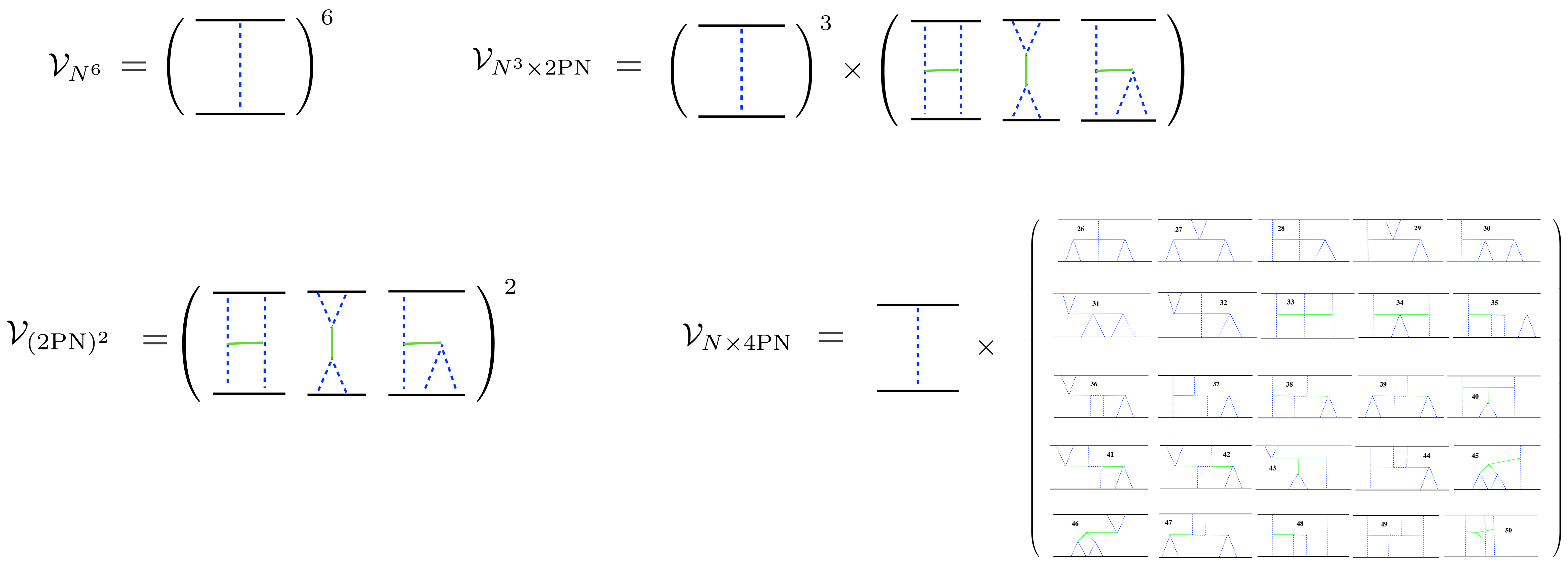
(2n+1)-PN corrections: *Type-B*



- static (2n+1)-PN Potential as product of lower-PN Potential terms

5PN static O(G^6): 154 5-loop GREFT diagrams

Foffa, Sturani, Sturm, Torres-Bobadilla & **P.M.**



$$\mathcal{V}^{(5\mathrm{PN})}_{\mathrm{static}} = \mathcal{V}_{N^6} + \mathcal{V}_{N^3\times 2\mathrm{PN}} + \mathcal{V}_{N\times 4\mathrm{PN}} + \mathcal{V}_{(2\mathrm{PN})^2}$$

5PN O(G^5 v^2): 1220 4-loop GREFT diagrams

Foffa, Sturani,, Torres-Bobadilla (2020)

- **Factorization Th'm: NO 5-loop diagram explicitly computed**

- Results confirmed and completed by explicit evaluation of 2pt-QFT 5-loop Integrals

Blümelein, Maier, Marquard, Schäfer (2019-21)

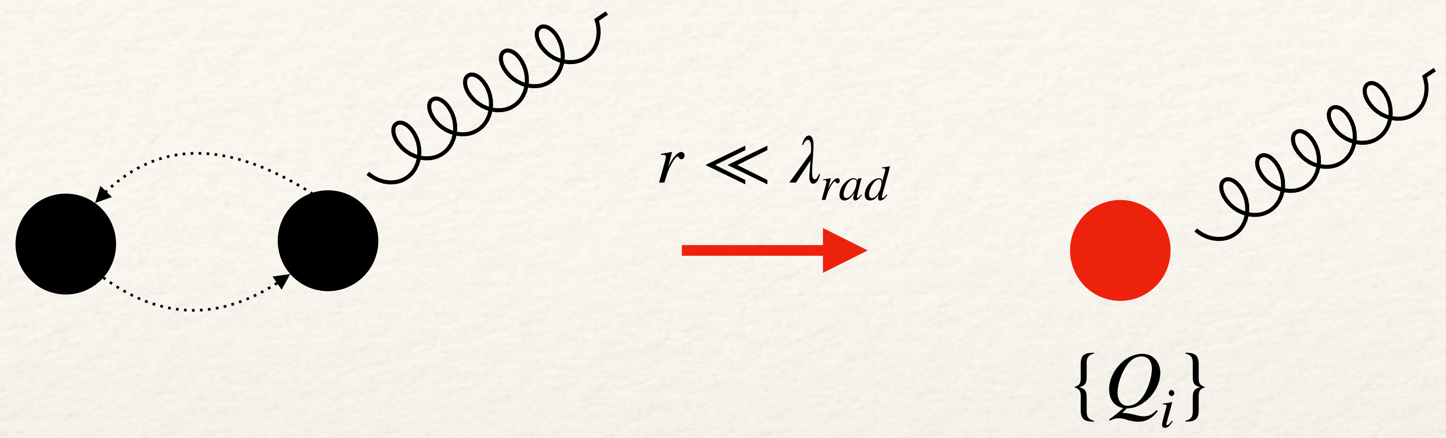
Conservative Dynamics :: **Far Zone** Spinless

Far Zone/EFT Diagrammatic Approach

Thorne (1980)
Goldberger, Rothstein (2005)
Goldberger, Ross (2009)
Galley, Tiglio (2009,2012)
Foffa, Sturani (2012); Ross (2012)
Galley, Leibovich, Porto, Ross (2015)
Leibovich, Maia, Rothstein, Yang (2019)
Blanchet et al.(2021)
....

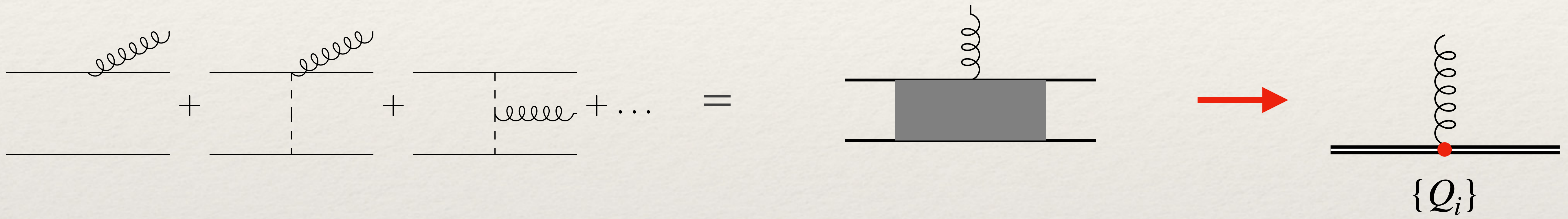
$$S_{rad}[g, \{Q_i\}] = S_{GR}[g] + S_{mult}[g, \{Q_i\}]$$

- **Far zone contributions** to the conservative dynamics are needed, starting at $4PN$ order



Multipole source emitting gravitons

- **Long-wavelength EFT:**

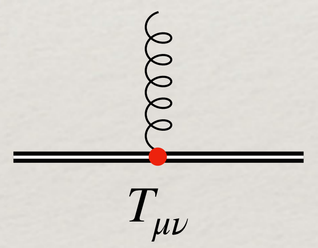


EFT matching

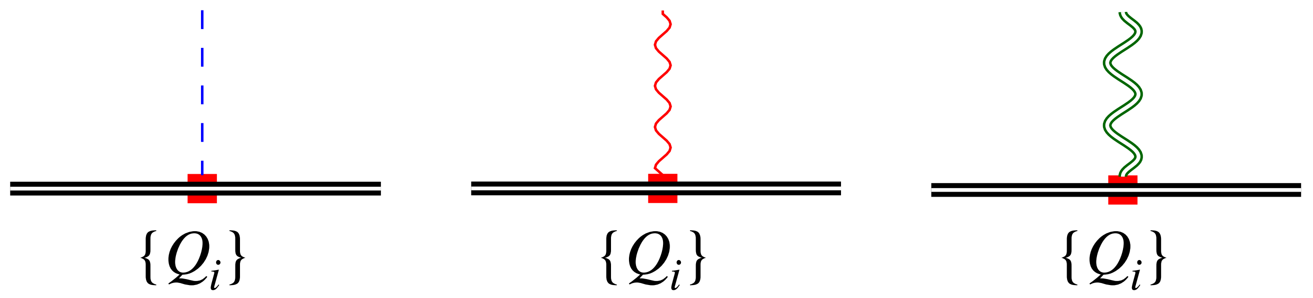
- **Multipole Action:**

Binary system as a linear source $T_{\mu\nu}$ of size r emitting $\bar{h}_{\mu\nu}$:

$$S_{mult} = -\frac{1}{2}\int d^4x T^{\mu\nu}\bar{h}_{\mu\nu}$$



$$S_{mult}[\bar{h}, \{Q_i\}] = \int dt \left[\frac{1}{2}E\bar{h}_{00} - \frac{1}{2}\epsilon_{ijk}L^i\bar{h}_{0j,k} - \frac{1}{2}Q^{ij}\mathscr{E}_{ij} - \frac{1}{6}O^{ijk}\mathscr{E}_{ij,k} - \frac{2}{3}J^{ij}B_{ij} + \dots\right]$$



- $\mathscr{E}_{ij}$, B_{ij} are the electric and magnetic components of the Riemann tensor

$$\mathscr{E}_{ij} = R_{0i0j} \approx -\frac{1}{2}\left(\bar{h}_{00,ij} + \ddot{\bar{h}}_{ij} - \dot{\bar{h}}_{0i,j} - \dot{\bar{h}}_{0j,i} + \mathcal{O}(\bar{h}^2)\right)$$

$$B_{ij} = \frac{1}{2}\epsilon_{ikl}R_{0jkl} \approx \frac{1}{4}\epsilon_{ikl}\left(\dot{\bar{h}}_{jk,l} - \dot{\bar{h}}_{jl,k} + \bar{h}_{0l,jk} - \bar{h}_{0k,jl} + \mathcal{O}(\bar{h}^2)\right)$$

- $\{Q_i\}$: multipole moments $E, L^i, Q^{ij}, O^{ijk}, J^{ij}$

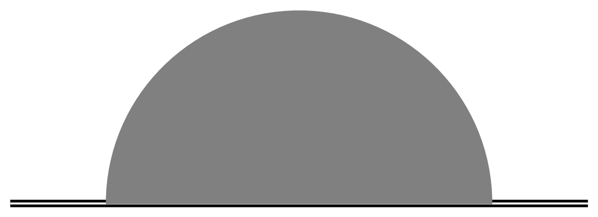
Far Zone/EFT Diagrammatic Approach

Thorne (1980)
Goldberger, Rothstein (2005)
Goldberger, Ross (2009)
Galley, Tiglio (2009,2012)
Foffa, Sturani (2012); Ross (2012)
Galley, Leibovich, Porto, Ross (2015)
Leibovich, Maia, Rothstein, Yang (2019)
Blanchet et al.(2021)
Almeida, Foffa, Sturani (2021,2022)
Blumlein, Maier, Marquard, Schaefer (2021)
Edison, Levi (2022)
Brunello, Mandal, Patil & P.M. in progress

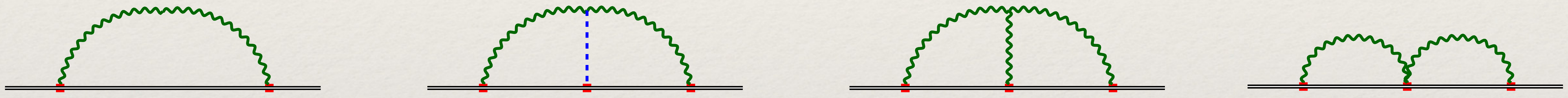
Hereditary Effects

- Contributions to the conservative dynamics by integrating out radiation gravitons:

$$S_{eff}[\{Q_i\}] = -i \lim_{d\to 3}$$



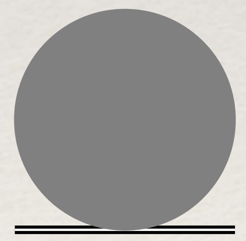
- Hereditary Effects: GWs emitted by the source and then back-scattered into the system:



Back-scattering — Tail-Effects — Memory effects — Double emission

- EFTGravity Amplitude mapped into multi-loop 1-point functions with massive internal lines:

$$= \int \prod_{i=1}^{n} \left[\frac{dk_0^i}{2\pi}\right] \Bigg(\quad \Bigg)$$



Non propagating sources

Radiation gravitons propagator:

$$\frac{1}{\mathbf{k}^2 - k_0^2}$$

k_0 mass

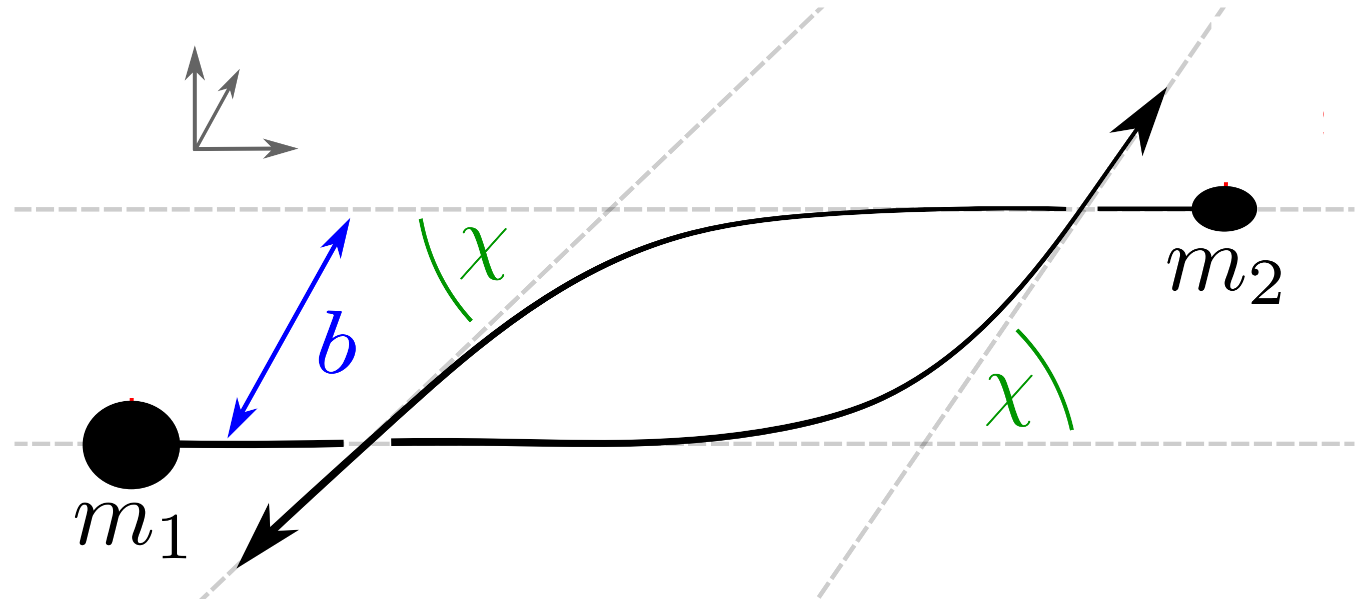
$$\mathcal{M} = \sum_i c_i \, I_i^{MI}$$

- Dimensional Regularization $d = 3 + \epsilon$
- Integration-by-parts (IBP) decomposition
- Master Integrals evaluation

see BRUNELLO

Scattering Angle

$$\chi = -2\int_{r_{\min}}^{\infty} dr \frac{\partial p_r}{\partial L} - \pi$$



[credit: Antornelli et al.]

$$M = m_1 + m_2 \qquad \mu = \frac{m_1 m_2}{m_1 + m_2} \qquad \nu = \frac{\mu}{M}$$

$$p_r = p_r(r, E, L, S_{(a)}) = p_r(r, \nu, b, S_{(a)})$$

$$\mathbf{p}^2 = p_r^2 + \frac{L^2}{r^2} = p_\infty^2 - V_{eff}\,, \qquad V_{eff}(r) = -\sum_{n\geq 1} f_n(E)\left(\frac{G_N}{r}\right)^n, \qquad p_r = \sqrt{p_\infty^2 - \frac{L^2}{r} - V_{eff}(r)}\,, \qquad V_{eff}(r\to\infty)\to 0\,.$$

$$H^{cons.} = H^{loc} + H^{nonloc.,cons.} \quad \Rightarrow \quad \chi = \chi^{loc} + \chi^{nonloc.}$$

▸**PM-expansion:**

$$\frac{1}{2}\chi(b, E) = \sum_n \chi_b^{(n)}(E)\left(\frac{GM}{b}\right)^n = \sum_n \chi_j^{(n)}(E)\frac{1}{j^n}\,,$$

▸**PN-expansion:**

$$\chi_b^{(n)} = \sum_{k\geq 0}\chi_b^{(n,k)}\left(\frac{\mathrm{v}^2}{c^2}\right)^k$$

$$\chi_j^{(n)} = \hat{p}_\infty^n \chi_b^{(n)}\,, \qquad \hat{p}_\infty = p_\infty/\mu. \qquad j = \frac{L}{G_N M \mu}$$

$$E = M\,\Gamma \qquad \Gamma = \sqrt{1 + 2\nu(\gamma - 1)} \qquad \gamma = \frac{1}{\sqrt{1 - v_\infty^2}} \qquad p_\infty = \frac{m_1 m_2}{E}\sqrt{\gamma^2 - 1} = \mu^2\frac{\gamma^2 - 1}{\Gamma^2}$$

Scattering Angle / far zone (no spin) : 6PN & 7PN

Bini Damour Geralico (2020)

Bini Damour Geralico Laporta & P.M.(2020)

χ^{loc} analytically known

$$\chi^{nonloc.} = \frac{\partial}{\partial L}\int_{-\infty}^{\infty} dt\; H^{nonloc.}(t)$$

▸**PM-expansion:**

$$\frac{1}{2}\chi^{nonloc.} = \nu p_\infty^4\left(\frac{A_0}{j^4} + \frac{A_1}{p_\infty j^5} + \frac{A_2}{p_\infty^2 j^6} + \dots\right) \qquad p_\infty \equiv \sqrt{\gamma^2 - 1}\,, \quad \text{and} \quad j = \frac{L}{G_N M \mu}$$

▸**PN-expansion:**

$$A_m = \sum_{n\geq 0}\left(A_{mn} + A_{mn}^{\ln}\log(p_\infty/2)\right)p_\infty^n\,, \qquad A_{mn} = \sum_{k\geq 0} A_{mnk}\,\nu^k$$

$$A_{mnk} = \int_{-1}^{+1}\int_{-1}^{+1}\frac{dT\,dT'}{|T - T'|}a_{mnk}(T, T')$$

O(G^6) Coefficients

O(200) coefficients:
4 of them coefficients only numerically [Bini Damour Geralico]

▸**Analytic evaluation**

$A_{220},\ A_{240},\ A_{241},\ A_{242}$ [Bini Damour Geralico Laporta & P.M.]

Extended to O(G^7) Coefficients

Multipole Radiation Formula

$$H^{nonloc.}(t) \propto \dddot{Q}_{ij}(t)\ \mathrm{PF}_T\int_{-\infty}^{\infty}\frac{d\tau}{|\tau|}\dddot{Q}_{ij}(t+\tau) \quad \text{+ higher-multipole terms} \qquad \text{time scale } T \equiv 2r_{12}/c$$

Quadrupole moment

$$Q_{ij} \equiv \sum_a m_a\left(x_a^i x_a^j - \frac{1}{3}\delta^{ij}\mathbf{x}_a^2\right) \text{ + PN corrections}$$

Hadamard Partie Finie

$$\mathrm{Pf}_T\int_0^{+\infty}\frac{\mathrm{d}v}{v}g(v) \equiv \int_0^T\frac{\mathrm{d}v}{v}[g(v) - g(0)] + \int_T^{+\infty}\frac{\mathrm{d}v}{v}g(v)$$

(... similar to the plus-distribution formula)

see BINI

see LAPORTA

TABLE I: Numerical values of the Q_{nk} integrals with 200-digit accuracy.

Q_{20}	524.7672921802125843427359557031017584761419995573690119377287112384988398300977120939070371581960608317062389952056770520679467837449664751347301110104558831841701708293472120711241061131658613485679
Q_{40}	544.4939915701706772258458158548215701355843583332648304959367083415682948158610574285653029862870842521159234233393649815247226338079050337694321196917178747431442826770414846949399926914472804761699
Q_{41}	−1029.52887537403849684626420906288951311349891044967686745420133893415513339408657109916000809600270006833111792420815144840143345012667124339258875382660056039521310075062073051406462130065024513617
Q_{42}	−802.885057050786642755886295069034459970736865058430654964178895902426423211047940727300850918742678716230784351105139654447667985252511824683509405317671631976450608758027815375931918602878433814664

TABLE II: PSLQ reconstruction of the various integrals.

Q_{20}	$\frac{25883}{1800} + \frac{22333}{140}\mathrm{K} - \frac{625463}{3360}\pi - \frac{361911}{560}\pi\ln 2 + \frac{99837}{160}\pi\zeta(3)$
d_{20}	$-\frac{32981}{112} - \frac{9216}{7}\ln 2 + \frac{99837}{64}\zeta(3)$
Q_{40}	$\frac{750674317}{1905120} + \frac{442237}{5040}\mathrm{K} - \frac{571787}{103680}\pi\frac{7207043}{6720}\pi\ln 2 - \frac{190489}{320}\pi\zeta(3)$
d_{40}	$\frac{725051}{1296} + \frac{19920}{7}\ln 2 - \frac{190489}{128}\zeta(3)$
Q_{41}	$-\frac{703435949}{1587600} - \frac{5747}{24}\mathrm{K} + \frac{1154149}{17280}\pi + \frac{1897771}{3360}\pi\ln 2 - \frac{306219}{640}\pi\zeta(3)$
d_{41}	$\frac{607867}{8064} + \frac{20224}{21}\ln 2 - \frac{306219}{256}\zeta(3)$
Q_{42}	$-\frac{59610947}{793800} - \frac{1499}{20}\mathrm{K} - \frac{402163}{2520}\pi + \frac{4497}{80}\pi\ln 2 - \frac{11871}{160}\pi\zeta(3)$
d_{42}	$-\frac{186743}{864} - \frac{11871}{64}\zeta(3)$

1. Numerical reconstruction w/200 digits

TABLE III: Independent sets of HPLs, at the point $x = i$, up to weight four.

$H_{-1}(i)$	$\frac{\ln 2}{2} + i\frac{\pi}{4}$
$H_0(i)$	$i\frac{\pi}{2}$
$H_1(i)$	$-\frac{\ln 2}{2} + i\frac{\pi}{4}$
$H_{0,-1}(i)$	$\frac{\pi^2}{48} + i\mathrm{K}$
$H_{0,1}(i)$	$-\frac{\pi^2}{48} + i\mathrm{K}$
$H_{-1,-1}(i)$	$-\frac{\pi^2}{32} + \frac{\ln^2 2}{8} + \frac{1}{8}i\pi\ln 2$
$H_{-1,1}(i)$	$-\frac{\pi^2}{32} - \frac{\ln^2 2}{8} - \frac{3}{8}i\pi\ln 2 + i\mathrm{K}$
$H_{1,-1}(i)$	$-\frac{\pi^2}{32} - \frac{\ln^2 2}{8} + \frac{3}{8}i\pi\ln 2 - i\mathrm{K}$
$H_{1,1}(i)$	$-\frac{\pi^2}{32} + \frac{\ln^2 2}{8} - \frac{1}{8}i\pi\ln 2$
$H_{0,-1,-1}(i)$	$\frac{29}{64}\zeta(3) - \frac{1}{4}\mathrm{K}\pi - i\mathrm{Q}_3$
$H_{0,-1,1}(i)$	$\frac{27}{64}\zeta(3) - \frac{1}{4}\mathrm{K}\pi + i\frac{\pi^3}{32} - 3i\mathrm{Q}_3 - 2i\mathrm{K}\ln 2$
$H_{0,1,-1}(i)$	$\frac{27}{64}\zeta(3) - \frac{1}{4}\mathrm{K}\pi - i\frac{\pi^3}{32} + 3i\mathrm{Q}_3 + 2i\mathrm{K}\ln 2$
$H_{0,1,1}(i)$	$\frac{29}{64}\zeta(3) - \frac{1}{4}\mathrm{K}\pi + i\mathrm{Q}_3$
$H_{0,-1,-1,-1}(i)$	$\frac{61}{15360}\pi^4 - \frac{35}{128}\zeta(3)\ln 2 + \frac{5}{384}\pi^2\ln^2 2 - \frac{5}{384}\ln^4 2 - \frac{5}{16}a_4 + \frac{\pi\mathrm{Q}_3}{4} + i\mathrm{Q}_4$
$H_{0,-1,-1,0}(i)$	$-\frac{\pi^4}{4608} + \frac{\mathrm{K}^2}{2} + \frac{\pi\mathrm{Q}_3}{2} + \frac{29}{128}i\pi\zeta(3) - \frac{7}{48}i\mathrm{K}\pi^2$
$H_{0,-1,1,-1}(i)$	$-\frac{97}{9216}\pi^4 + \frac{91}{128}\zeta(3)\ln 2 - \frac{13}{384}\pi^2\ln^2 2 + \frac{13}{16}a_4 - \frac{13}{384}\ln^4 2 + \frac{3}{4}\pi\mathrm{Q}_3 + \frac{\mathrm{K}^2}{2} - \frac{7}{8}i\pi\zeta(3) + \frac{1}{16}i\pi^3\ln 2 - 2i\mathrm{K}\ln^2 2$ $-\frac{5}{16}i\mathrm{K}\pi^2 + 5i\beta(4) - 6i\mathrm{Q}_3\ln 2 - 9i\mathrm{Q}_4$
$H_{0,-1,1,0}(i)$	$-\frac{71}{4608}\pi^4 + \frac{3}{2}\pi\mathrm{Q}_3 + \mathrm{K}\pi\ln 2 + \frac{\mathrm{K}^2}{2} + \frac{27}{128}i\pi\zeta(3) + \frac{1}{8}i\pi^3\ln 2 - \frac{5}{48}i\mathrm{K}\pi^2 - 3i\beta(4)$
$H_{0,1,-1,-1}(i)$	$\frac{169}{9216}\pi^4 - \frac{77}{128}\zeta(3)\ln 2 + \frac{9}{128}\pi^2\ln^2 2 - \frac{27}{16}a_4 - \frac{9}{128}\ln^4 2 - \frac{3}{4}\pi\mathrm{Q}_3 - \frac{1}{2}\mathrm{K}\pi\ln 2 - \frac{21}{128}i\pi\zeta(3) - \frac{1}{32}i\pi^3\ln 2$ $+i\mathrm{K}\ln^2 2 + i\beta(4) + 2i\mathrm{Q}_3\ln 2 + i\mathrm{Q}_4$
$H_{0,1,-1,0}(i)$	$\frac{73}{4608}\pi^4 + \frac{\mathrm{K}^2}{2} - \mathrm{K}\pi\ln 2 - \frac{3}{2}\pi\mathrm{Q}_3 + \frac{27}{128}i\pi\zeta(3) - \frac{1}{8}i\pi^3\ln 2 - \frac{7}{48}i\mathrm{K}\pi^2 + 3i\beta(4)$
$H_{0,1,1,-1}(i)$	$\frac{61}{9216}\pi^4 + \frac{21}{128}\zeta(3)\ln 2 + \frac{13}{384}\pi^2\ln^2 2 - \frac{13}{384}\ln^4 2 - \frac{13}{16}a_4 - \frac{1}{2}\mathrm{K}\pi\ln 2 + \frac{\mathrm{K}^2}{2} - \frac{1}{4}\pi\mathrm{Q}_3 + \frac{133}{128}i\pi\zeta(3) - \frac{1}{32}i\pi^3\ln 2$ $+i\mathrm{K}\ln^2 2 + \frac{3}{16}i\mathrm{K}\pi^2 - 5i\beta(4) + 4i\mathrm{Q}_3\ln 2 + 7i\mathrm{Q}_4$
$H_{0,1,1,0}(i)$	$-\frac{\pi^4}{4608} + \frac{\mathrm{K}^2}{2} - \frac{1}{2}\pi\mathrm{Q}_3 - \frac{5}{48}i\mathrm{K}\pi^2 + \frac{29}{128}i\pi\zeta(3)$
$H_{-1,-1,-1,0}(i)$	$-\frac{31}{15360}\pi^4 + \frac{1}{2}\zeta(3)\ln 2 - \frac{1}{32}\pi^2\ln^2 2 + \frac{5}{384}\ln^4 2 + \frac{5}{16}a_4 + \frac{29}{256}i\pi\zeta(3) + \frac{1}{96}i\pi\ln^3 2 - \frac{1}{96}i\pi^3\ln 2 - \frac{1}{32}i\mathrm{K}\pi^2$ $-\frac{1}{8}i\mathrm{K}\ln^2 2 - \frac{1}{2}i\mathrm{Q}_3\ln 2 - i\mathrm{Q}_4$
$H_{-1,-1,1,0}(i)$	$-\frac{115}{9216}\pi^4 + \frac{13}{16}\zeta(3)\ln 2 - \frac{1}{12}\pi^2\ln^2 2 + \frac{9}{128}\ln^4 2 + \frac{27}{16}a_4 + \frac{1}{4}\mathrm{K}\pi\ln 2 + \frac{1}{2}\pi\mathrm{Q}_3 + \frac{27}{256}i\pi\zeta(3) - \frac{1}{96}i\pi\ln^3 2$ $+\frac{1}{24}i\pi^3\ln 2 - \frac{1}{32}i\mathrm{K}\pi^2 - \frac{1}{8}i\mathrm{K}\ln^2 2 - i\beta(4) - \frac{1}{2}i\mathrm{Q}_3\ln 2 - i\mathrm{Q}_4$
$H_{-1,1,-1,0}(i)$	$\frac{91}{9216}\pi^4 - \frac{1}{2}\zeta(3)\ln 2 + \frac{17}{96}\pi^2\ln^2 2 - \frac{13}{384}\ln^4 2 - \frac{13}{16}a_4 - \pi\mathrm{Q}_3 + \frac{\mathrm{K}^2}{2} - \frac{3}{4}\mathrm{K}\pi\ln 2 + \frac{335}{256}i\pi\zeta(3) - \frac{1}{96}i\pi\ln^3 2$ $-\frac{3}{64}i\pi^3\ln 2 + \frac{13}{96}i\mathrm{K}\pi^2 + \frac{9}{8}i\mathrm{K}\ln^2 2 - 5i\beta(4) + \frac{9}{2}i\mathrm{Q}_3\ln 2 + 9i\mathrm{Q}_4$
$H_{-1,1,1,0}(i)$	$-\frac{79}{9216}\pi^4 + \frac{1}{16}\zeta(3)\ln 2 - \frac{7}{48}\pi^2\ln^2 2 + \frac{13}{384}\ln^4 2 + \frac{13}{16}a_4 + \frac{\mathrm{K}^2}{2} + \frac{1}{2}\pi\mathrm{Q}_3 + \frac{1}{2}\mathrm{K}\pi\ln 2 - \frac{195}{256}i\pi\zeta(3) + \frac{1}{96}i\pi\ln^3 2$ $+\frac{3}{64}i\pi^3\ln 2 - \frac{31}{96}i\mathrm{K}\pi^2 - \frac{7}{8}i\mathrm{K}\ln^2 2 + 5i\beta(4) - \frac{7}{2}i\mathrm{Q}_3\ln 2 - 7i\mathrm{Q}_4$
$H_{1,-1,-1,0}(i)$	$\frac{55}{9216}\pi^4 - \frac{7}{96}\pi^2\ln^2 2 - \frac{1}{16}\zeta(3)\ln 2 - \frac{13}{384}\ln^4 2 - \frac{13}{16}a_4 + \frac{1}{2}\mathrm{K}\pi\ln 2 + \frac{1}{2}\pi\mathrm{Q}_3 - \frac{\mathrm{K}^2}{2} - \frac{279}{256}i\pi\zeta(3) - \frac{1}{96}i\pi\ln^3 2$ $-\frac{7}{96}i\mathrm{K}\pi^2 - \frac{7}{8}i\mathrm{K}\ln^2 2 + 5i\beta(4) - \frac{7}{2}i\mathrm{Q}_3\ln 2 - 7i\mathrm{Q}_4$
$H_{1,-1,1,0}(i)$	$\frac{29}{9216}\pi^4 + \frac{1}{2}\zeta(3)\ln 2 + \frac{5}{48}\pi^2\ln^2 2 + \frac{13}{384}\ln^4 2 + \frac{13}{16}a_4 - \frac{3}{4}\mathrm{K}\pi\ln 2 - \frac{\mathrm{K}^2}{2} - \pi\mathrm{Q}_3 + \frac{167}{256}i\pi\zeta(3) + \frac{1}{96}i\pi\ln^3 2$ $-\frac{1}{32}i\pi^3\ln 2 + \frac{37}{96}i\mathrm{K}\pi^2 + \frac{9}{8}i\mathrm{K}\ln^2 2 - 5i\beta(4) + \frac{9}{2}i\mathrm{Q}_3\ln 2 + 9i\mathrm{Q}_4$
$H_{1,1,-1,0}(i)$	$\frac{91}{9216}\pi^4 - \frac{13}{16}\zeta(3)\ln 2 + \frac{5}{96}\pi^2\ln^2 2 - \frac{9}{128}\ln^4 2 - \frac{27}{16}a_4 + \frac{1}{4}\mathrm{K}\pi\ln 2 + \frac{1}{2}\pi\mathrm{Q}_3 + \frac{111}{256}i\pi\zeta(3) - \frac{1}{192}i\pi^3\ln 2$ $+\frac{1}{96}i\pi\ln^3 2 - \frac{1}{8}i\mathrm{K}\ln^2 2 - \frac{1}{32}i\mathrm{K}\pi^2 - i\beta(4) - \frac{1}{2}i\mathrm{Q}_3\ln 2 - i\mathrm{Q}_4$
$H_{1,1,1,0}(i)$	$\frac{71}{15360}\pi^4 - \frac{1}{2}\zeta(3)\ln 2 - \frac{5}{384}\ln^4 2 - \frac{5}{16}a_4 + \frac{29}{256}i\pi\zeta(3) + \frac{1}{192}i\pi^3\ln 2 - \frac{1}{32}i\mathrm{K}\pi^2 - \frac{1}{8}i\mathrm{K}\ln^2 2 - \frac{1}{96}i\pi\ln^3 2$ $-\frac{1}{2}i\mathrm{Q}_3\ln 2 - i\mathrm{Q}_4$
$\mathrm{Li}_4(1/2)$	a_4
$\mathrm{ImLi}_2(i)$	K
$\mathrm{ImLi}_4(i)$	$\beta(4)$
$\mathrm{Im}H_{0,1,1}(i)$	Q_3
$\mathrm{Im}H_{0,1,1,1}(i)$	Q_4

2. Analytic integration w/ **HPL's**

Far-Zone GR**EFT** / validation

- Mass polynomiality of the scattering angle: $\chi_4^{cons,tot} = \chi_4^{Schw} + \nu\chi_4^{\nu}$ $\qquad \nu = \frac{\mu}{M}$

- Compatible with "Tutti Frutti" method and PM-Amplitudes-based calculations
  [Damour] [Bern et al.]
  [Damour, Bini, Geralico]

- GREFT calculations point at possible quadratic behaviour: $\chi_4^{cons,tot} = \chi_4^{Schw} + \nu\chi_4^{\nu} + \nu^2\chi_4^{\nu^2}\,, \qquad \chi_4^{\nu^2} \neq 0$
  [Bluemlein et al.]
  [Almeida et al.] [Porto et al.]
  [Brunello et al.]

- **known unknown: FarZone-GREFT is an challenging theoretical puzzle:**
  - Which effects do the GREFT diagrams contain?
  - Interplay between conservative and dissipative effects?
  - Double counting or missing contribution?
  - FarZone/Radiation and proper choice of Green-Functions

Conservative Dynamics :: **Near Zone** with *Spin* and *Tidal Effects*

Near Zone with **Spin**/PN Corrections

Porto (2013)
Levi, Steinhoff (2015)
……
Kim, Levi, Yin (2022)
Mandal, Patil, Steinhoff & P.M. (2022)
Levi, Morales, Yin (2022)
Levi, Yin (2022)

EFT Action for Spinning compact object

$$S_{pot}[x_a, g] = S_{GR}[g] + S_{m_a}[x_a, g] \qquad S_{m_a}[x_a, g] = S_{pp}[x_a, g] + \delta S_{m_a}[x_a, g]$$

$$S_{m_a}[x_a, g] = \sum_{a=1,2} \int d\tau \left(-m_{(a)} c \sqrt{u_{(a)}^2} - \frac{1}{2} S_{(a)\mu\nu} \Omega_{(a)}^{\mu\nu} - \frac{S_{(a)\mu\nu} u_{(a)}^{\nu}}{u_{(a)}^2} \frac{du_{(a)}^{\mu}}{d\tau} + \mathcal{L}_{(a)}^{(R)} + \mathcal{L}_{(a)}^{(R^2)} + \dots \right)$$

$$u_{(a)}^{\mu} \equiv \dot{x}_a^{\mu}$$

Wilson coefficients that describe the internal structure

$$\mathcal{L}_{(a)}^{(R)} = -\frac{1}{2m_{(a)}c} \left(C_{\mathrm{ES}^2}^{(0)} \right)_{(a)} \frac{E_{\mu\nu}}{u_{(a)}} \left[S_{(a)}^{\mu} S_{(a)}^{\nu} \right]_{\mathrm{STF}} + \dots$$

$$\mathcal{L}_{(a)}^{(R^2, S^0)} = \frac{1}{2} \left(C_{\mathrm{E}^2}^{(2)} \right)_{(a)} \frac{G_N^2 m_{(a)}}{c^5} \frac{E_{\mu\nu} E^{\mu\nu}}{u_{(a)}^3} S_{(a)}^2 + \dots$$

$$\mathcal{L}_{(a)}^{(R^2, S^2)} = \frac{1}{2} \left(C_{\mathrm{E}^2\mathrm{S}^2}^{(0)} \right)_{(a)} \frac{G_N^2 m_{(a)}}{c^5} \frac{E_{\mu\alpha} E_{\nu}{}^{\alpha}}{u_{(a)}^3} \left[S_{(a)}^{\mu} S_{(a)}^{\nu} \right]_{\mathrm{STF}} + \dots$$

Electric and Magnetic components of Riemann tensor

$$E_{\mu\nu} \equiv R_{\mu\alpha\nu\beta} u^{\alpha} u^{\beta}$$

$$B_{\mu\nu} \equiv \frac{1}{2} \epsilon_{\alpha\beta\gamma\mu} R^{\alpha\beta}{}_{\delta\nu} u^{\gamma} u^{\delta}$$

STF = Symmetrized Trace-Free

PN order	0	1	1.5	2	2.5	3	3.5	4	4.5	5	5.5	6	6.5
S^0	0PN	1PN		2PN		3PN		4PN		5PN		6PN	
S^1			LO		NLO		N2LO		N3LO		N4LO		
S^2				LO		NLO		N2LO		N3LO			
S^3							LO		NLO				
S^4								LO		NLO			
S^5											LO		NLO
S^6												LO	

(L+1)PM/loop order
- tree
- 1-loop
- 2-loop
- 3-loop
- 4-loop
- 5-loop
- 6-loop
- 7-loop

[credit: Vines/Roiban]

Near Zone with Spin/EFT Diagrammatic Approach

Kim, Levi, Yin (2022)
Mandal, Patil, Steinhoff & P.M. (2022)

$$S_{pot}[x_a, g] = S_{GR}[g] + S_{m_a}[x_a, g] \qquad S_{m_a}[x_a, g] = S_{pp}[x_a, g] + \boxed{\delta S_{m_a}[x_a, g]}$$

Spin dependence

- **Kaluza-Klein parametrization:** [Kol Smolkin]

$$g_{\mu\nu} = e^{2\phi/\Lambda} \begin{pmatrix} -1 & A_j/\Lambda \\ A_i/\Lambda & e^{-c_d \frac{\phi}{\Lambda}} \gamma_{ij} - A_i A_j/\Lambda^2 \end{pmatrix}$$

Graviton = Scalar + Vector + Sym. Tensor
10 = 1 + 3 + 6

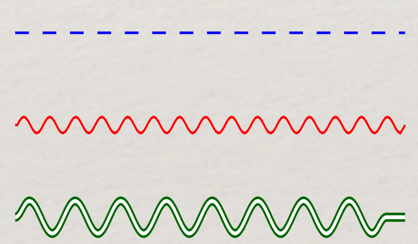
$$g_{\mu\nu} \Rightarrow \quad \phi \quad A^i \quad \sigma^{ij}$$

$$\gamma_{ij} = \delta_{ij} + \frac{\sigma_{ij}}{\Lambda} \qquad c_d = 2\frac{d-1}{d-2}$$

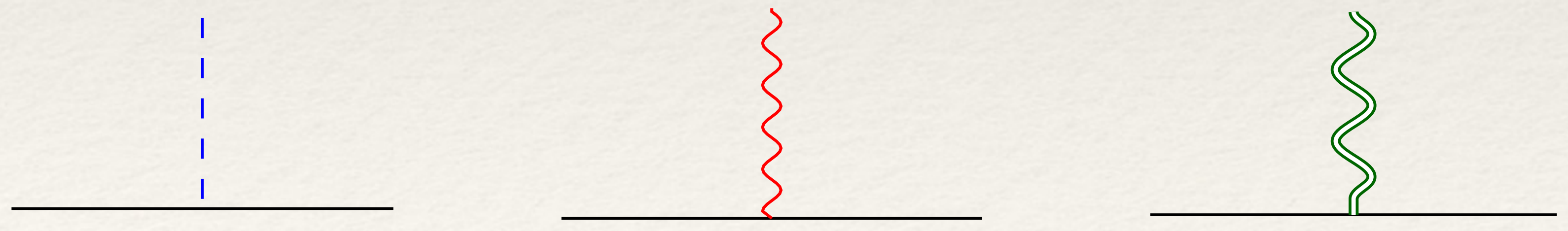
- **Feynman rules for:** $\phi \quad A^i \quad \sigma^{ij} \quad x_a$

Static / non-propagating source: x_a

Propagators: ϕ, A^i, σ^{ij}



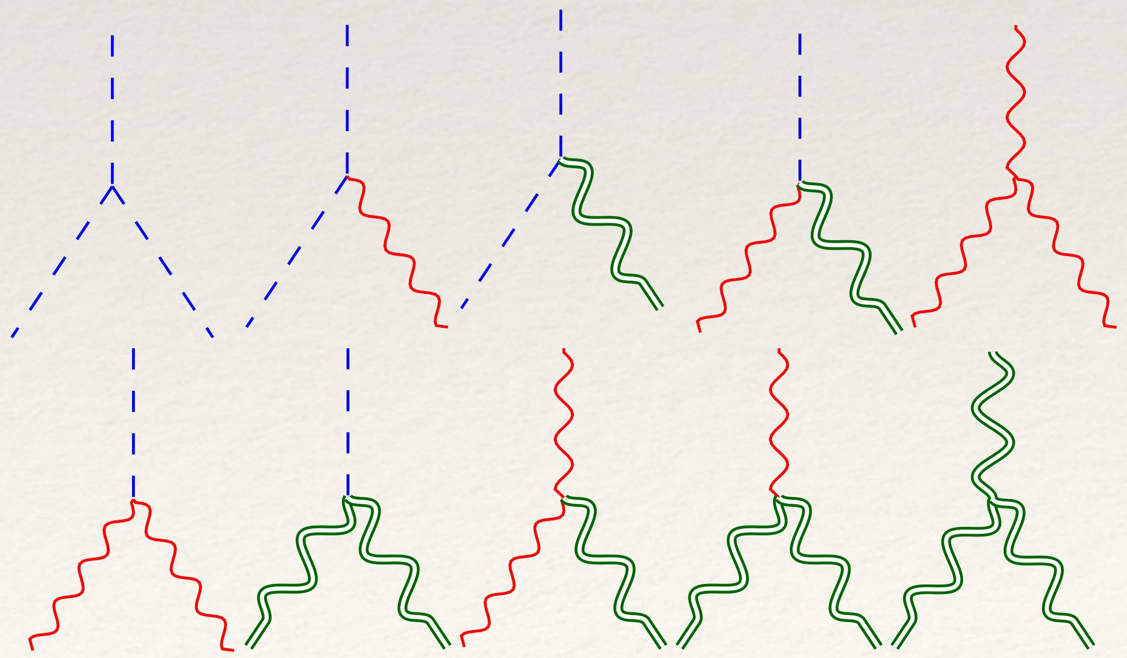
Source couplings:



$+ \ldots$

Spin dependence

Self-interactions:



$+ \ldots$

GREFT Diagrams & 2pt-QFT Integrals

Kim, Levi, Yin (2022)

Mandal, Patil, Steinhoff & P.M. (2022)

S^0

Order	Diagrams	Loops	Diagrams
0PN	1	0	1
1PN	4	1	1
		0	3
2PN	21	2	5
		1	10
		0	6
3PN	130	3	8
		2	75
		1	38
		0	9

(a) Non-spinning sector

S^1

Order	Diagrams	Loops	Diagrams
LO	2	0	2
NLO	13	1	8
		0	5
N^2LO	100	2	56
		1	36
		0	8
N^3LO	894	3	288
		2	495
		1	100
		0	11

(b) Spin-orbit sector

S^2

Order	Diagrams	Loops	Diagrams
LO	1	0	1
NLO	7	1	3
		0	4
N^2LO	58	2	27
		1	24
		0	7
N^3LO	553	3	125
		2	342
		1	76
		0	10

(a) Spin1-Spin2 and Spin1^2 (Spin2^2) sector

Order	Diagrams	Loops	Diagrams
LO	1	0	1
NLO	4	1	1
		0	3
N^2LO	25	2	7
		1	12
		0	6
N^3LO	168	3	15
		2	101
		1	43
		0	9

(b) ES^2 sector

Order	Loops	Diagrams
LO	1	1

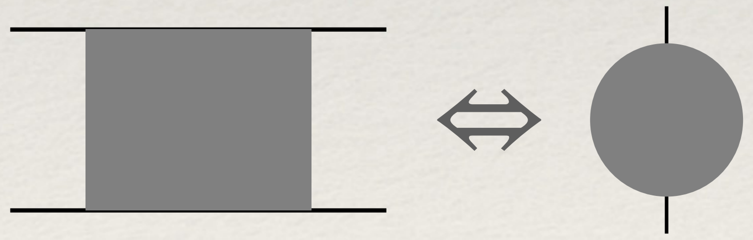
(c) E^2 sector

Order	Loops	Diagrams
LO	1	1

(d) E^2S^2 sector

▸ **Mapping to 2-point Functions**

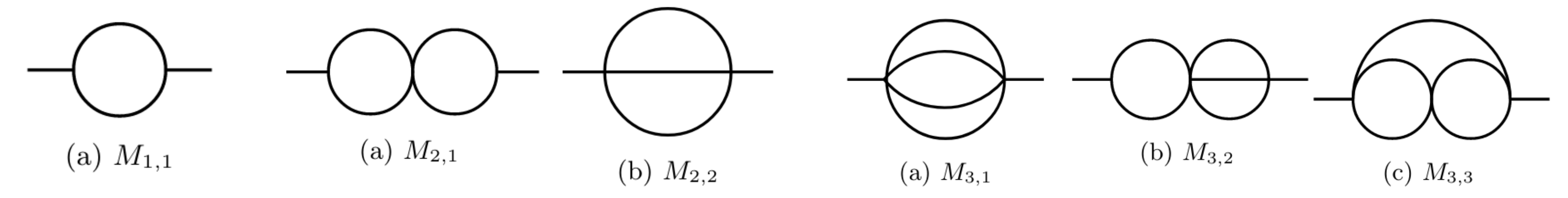
$$\mathscr{L}_{eff}[x_a, \dot{x}_a, \ddot{x}_a, \ldots, S_a, \dot{S}_a, \ldots] = -i \lim_{d\to 3} \int \frac{d^d\mathbf{p}}{(2\pi)^d} e^{i\mathbf{p}\cdot\mathbf{r}} \Big($$  $$\Big)$$



see MANDAL

$$\mathscr{M} = \sum_i c_i\, I_i^{MI}$$

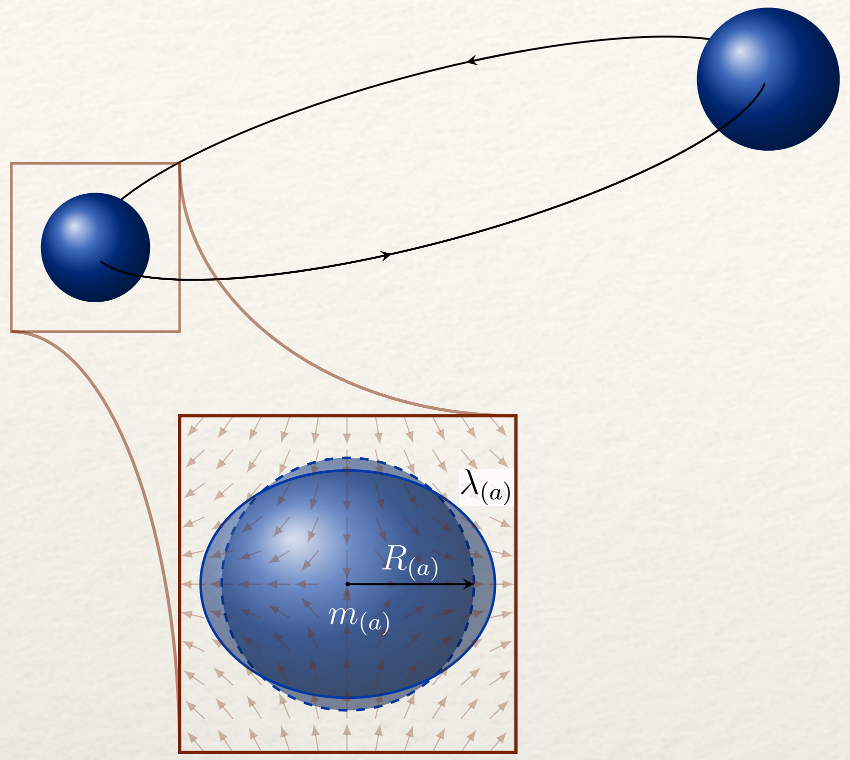
- **Dimensional Regularization** $d = 3 + \epsilon$
- **Integration-by-parts (IBP) decomposition**
- **Master Integrals evaluation**



Near Zone with Tidal Effects/PN Corrections

Bini, Damour, Faye (2012)
Steinhoff, Hinderer, Buonanno, Taracchini (2016)
Mandal, Patil, Silva, Steinhoff & P.M. (2023)

EFT Action for Tidal Effects



$$S_{pot}[x_a, g] = S_{GR}[g] + S_{m_a}[x_a, g] \qquad S_{m_a}[x_a, g] = S_{pp}[x_a, g] + \delta S_{m_a}[x_a, g]$$

$$S_{m_a}[x_a, g] \;=\; \sum_{a=1,2} \int \frac{\mathrm{d}\tau}{c} \left[-m_{(a)} z_{(a)} c^2 + \mathcal{L}_{\mathrm{FD}(a)} + \mathcal{L}_{\mathrm{MQ}(a)} + \mathcal{L}_{\mathrm{EQ(a)}} \right]$$

Tidal Effects

Order	Diagrams	Loops	Diagrams
0PN	1	0	1
1PN	4	0	3
		1	1
2PN	21	0	6
		1	10
		2	5

(a) Point particle sector

Order	Diagrams	Loops	Diagrams
0PN	1	0	1
1PN	4	0	3
		1	1
2PN	26	0	6
		1	12
		2	8

(b) EQ sector

Order	Diagrams	Loops	Diagrams
1PN	2	0	2
2PN	13	0	5
		1	8

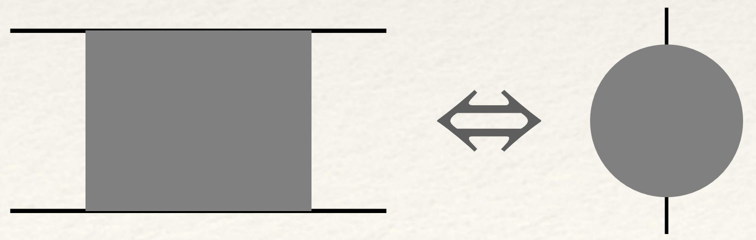
(c) FD sector

Order	Diagrams	Loops	Diagrams
1PN	1	0	1
2PN	4	0	3
		1	1

(d) MQ sector

$$\mathscr{L}_{eff}[x_a, \dot{x}_a, \ddot{x}_a, \ldots, S_a, \dot{S}_a, \ldots] = -\, i \lim_{d \to 3} \int \frac{d^d \mathbf{p}}{(2\pi)^d} e^{i \mathbf{p} \cdot \mathbf{r}} \Bigg(\quad$$  $$\Bigg)$$

see MANDAL



$$\mathscr{M} \;=\; \sum_i c_i \, I_i^{MI}$$

- Dimensional Regularization $d = 3 + \epsilon$
- Integration-by-parts (IBP) decomposition
- Master Integrals evaluation

Conservative Dynamics :: PM Corrections

Near Zone/PM Corrections

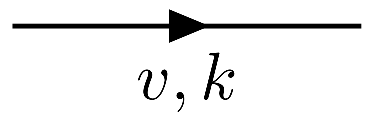
- **Heavy Quark EFT [HQET] in QCD**

Georgi
Manohar
Neubert, Becher
Brambilla, Vairo, Pineda
… … …

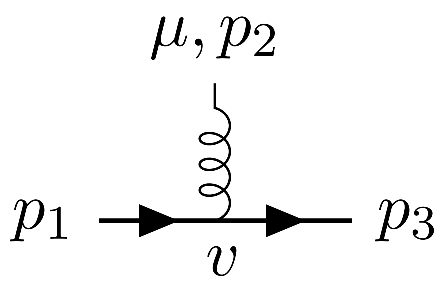
$$\mathcal{L}_{\rm QED} = \bar{\psi}\left(i\not{D} - m\right)\psi, \qquad \text{where} \quad D^\mu\psi \equiv (\partial^\mu + ieA^\mu)\psi.$$

$$p^\mu = mv^\mu \qquad v^2 = 1$$

$$\mathcal{L}_{\rm HQET} = \bar{Q}\left(iv\cdot D - \frac{D_\perp^2}{2m} + \frac{D_\perp^4}{8m^3} - \frac{e}{4m}\sigma_{\mu\nu}F^{\mu\nu} - \frac{e}{8m^2}v^\mu[D_\perp^\nu F_{\mu\nu}] + \frac{ie}{8m^2}v_\rho\sigma_{\mu\nu}\{D_\perp^\mu, F^{\rho\nu}\} + \frac{e}{16m^3}\{D_\perp^2, \sigma_{\mu\nu}F^{\mu\nu}\} + \frac{e^2}{16m^3}F_{\mu\nu}F^{\mu\nu}\right)Q + \mathcal{O}(m^{-4})$$



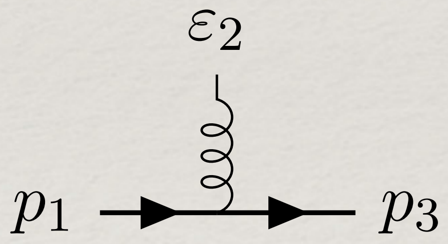
$$\frac{i}{v\cdot k + i\varepsilon}\frac{1+\not{v}}{2},$$



$$igT^a v_\mu \frac{1+\not{v}}{2}$$

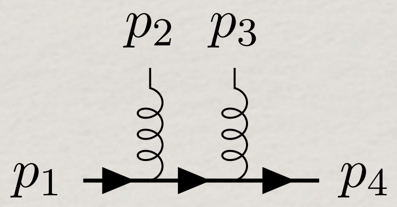
$$D_\perp^\mu \equiv D^\mu - v^\mu(v\cdot D)$$

$A_3^{\rm YM-M}(123) =$

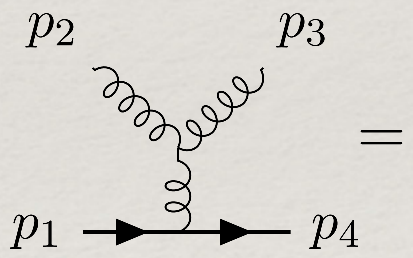


$= m\varepsilon_2\cdot v$

$A_4^{\rm YM-M}(1234) =$



$+$



$$= 2m\left(-\frac{\varepsilon_2\cdot p_3 v\cdot\varepsilon_3}{s_{23}} - \frac{\varepsilon_2\cdot\varepsilon_3 v\cdot p_2}{s_{23}} + \frac{\varepsilon_3\cdot p_2 v\cdot\varepsilon_2}{s_{23}} + \frac{v\cdot\varepsilon_2 v\cdot\varepsilon_3}{2v\cdot p_2}\right)$$

$$A_n^{\rm YM-M}(12\cdots n) = \sum_{\Gamma\in \text{ordered commutators } \{2,3,\cdots,n-1\}} \frac{\mathcal{N}_n(\Gamma, v)}{d_\Gamma},$$

Near Zone/PM Corrections

- Heavy Mass/Black-hole EFT [H(M/B)ET] in Gravity

Damgaard, Haddad, Helset
Brandhuber, Travaglini, Chen, Wen
+ Brown, De Angelis, Gowdy

$$\sqrt{-g}\mathcal{L}_m = \sqrt{-g}\left(\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}m^2\phi^2\right)$$

$$\sqrt{-g}\mathcal{L}_m = \sqrt{-g}\bar{\psi}\left(ie^\mu{}_a\gamma^a D_\mu - m\right)\psi$$

$$S_{\rm GR} = \frac{1}{16\pi G}\int d^4x\sqrt{-g}R$$

$$p^\mu = mv^\mu \qquad v^2 = 1,$$

$$A_n^{\rm YM-M}(12\cdots n) = \sum_{\Gamma\in\text{ordered commutators }\{2,3,\cdots,n-1\}} \frac{\mathcal{N}_n(\Gamma,v)}{d_\Gamma},$$

$$A_n^{\rm GR-M}(12\cdots n) = \sum_{\Gamma\in\text{non-ordered commutators}\{2,3,\cdots,n-1\}} \frac{\left[\mathcal{N}_n(\Gamma,v)\right]^2}{d_\Gamma}$$

see BRUNELLO

Conclusion

- GW Astronomy: a growing research field, where accuracy is not an option
- Compact objects evolution can benefit of the interplay between Cosmology, Astrophysics, and High-Energy Theoretical Physics
- Remarkable combination of traditional methods developed for the GR two-body problem and methods developed for elementary particle scattering to improve the GW waveforms modelling
- Scattering processes: a universal framework to investigate Nature at its most extreme conditions
- Under a diagrammatic viewpoint, Gravity is not so different from the other Fundamental Interactions

Conclusion

EFT - NRGR

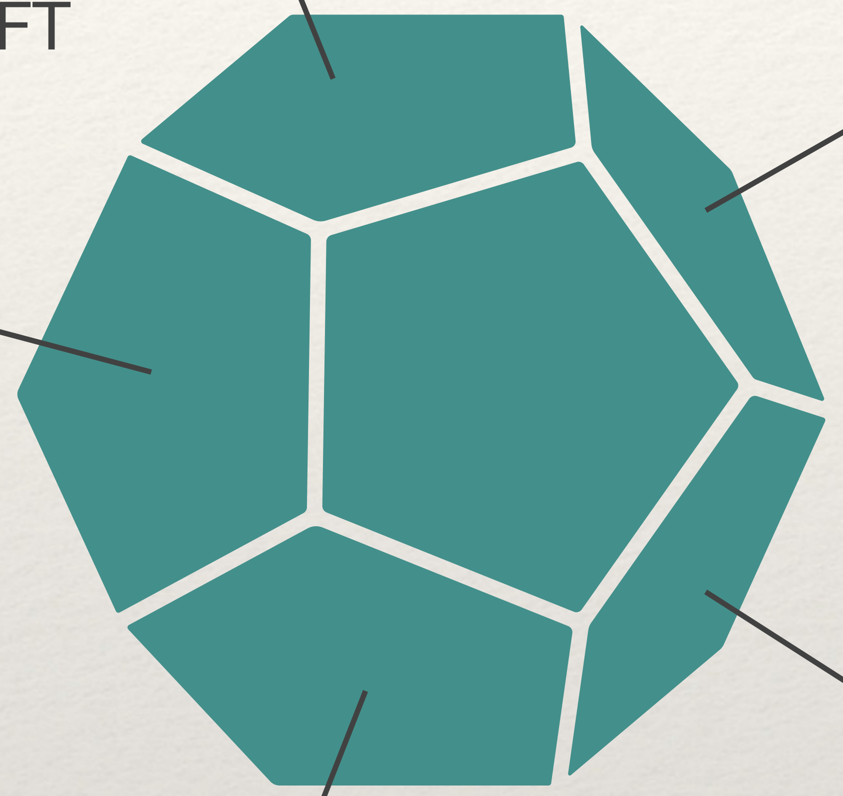
- PN & PM EFT
- In-in formalism
- NRQFT
- HQEFT
- H(P,B,M)EFT

GR-Techniques

- PN & PM
- Self-Force
- EOB
- Numerical Relativity
- Tutti Frutti

Amplitudes

- Unitarity-based methods
- Double-copy & BCJ relations
- Higher-spin
- Classical Scattering



Observable-based methods

- Eikonal approach
- Inclusive & differential formalisms
- Radial action
- S-matrix

Multiloop Techniques

- IBPs
- Difference & Differential Equations
- Theory of Special Functions
- High Precision arithmetics and Finite Fields
- Numerical Integration
- Asymptotic expansions

EXTRA

Definition. *Physics is a part of mathematics devoted to the calculation of integrals of the form $\int g(x)e^{f(x)}dx$. Different branches of physics are distinguished by the range of the variable x and by the names used for $f(x)$, $g(x)$ and for the integral.* [...]

Schwarz, Shapiro (2018)

Of course this is a joke, physics is not a part of mathematics. However, it is true that the main mathematical problem of physics is the calculation of integrals of the form

$$I(g) = \int g(x)e^{-f(x)}dx$$

[...] If f can be represented as $f_0 + \lambda V$ where f_0 is a negative quadratic form, then the integral $\int g(x)e^{f(x)}\,dx$ can be calculated in the framework of perturbation theory with respect to the formal parameter λ. We will fix f and consider the integral as a functional $I(g)$ taking values in $\mathbb{R}[[\lambda]]$. It is easy to derive from the relation

$$\int \partial_a(h(x)e^{f(x)})dx = 0$$

that the functional $I(g)$ vanishes in the case when g has the form

$$g = \partial_a h + (\partial_a f)h.$$

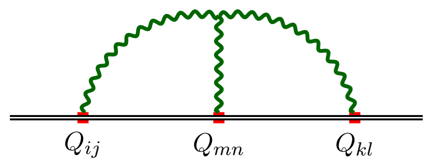
- Addressing a common math problem might be useful to make progress in different disciplines

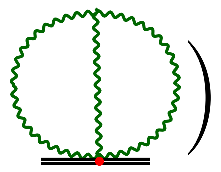
Far Zone/EFT Diagrammatic Approach

Foffa, Sturani
Blumlein, Maier, Marquard, Schaefer
Almeida, Foffa, Sturani
Brunello, Mandal, Patil & P.M. in progress

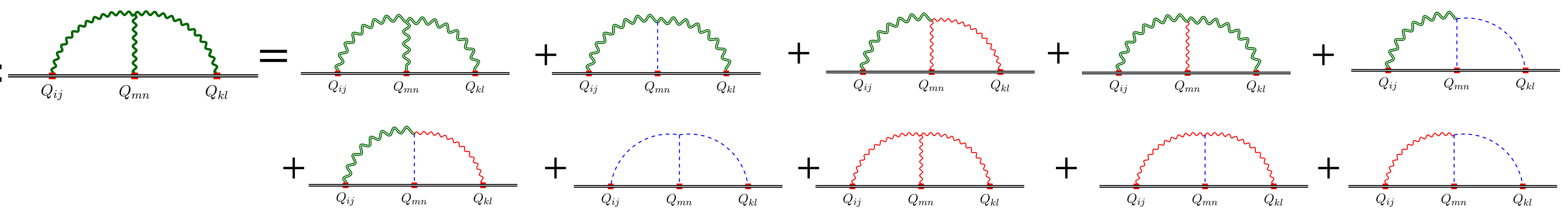
Memory Effect (5PN) within the *In-Out (causal)* formalism

Foffa, Sturani (2019)

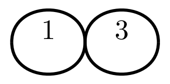
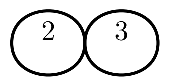
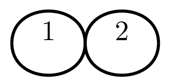
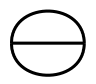
- Skeleton diagram: 

$= \int \frac{dk_0}{(2\pi)} \frac{dp_0}{(2\pi)} \Bigg($  $\Bigg) = \int \frac{dk_0}{(2\pi)} \frac{dp_0}{(2\pi)} \mathcal{M}^{Q^3}$

2 loop 1-point function

- Diagram Generation: (27 diagrams) 

- Tensor Decomposition: $\mathcal{M}^{Q^3} = Tr[\ Q(k_0)Q(p_0)Q(q_0)\ ]\ \tilde{\mathcal{M}}^{Q^3}$

- IBP Decomposition: $\tilde{\mathcal{M}}^{Q^3} = C_1$  $+ C_2$  $+ C_3$  $+ C_4$  $= 0$

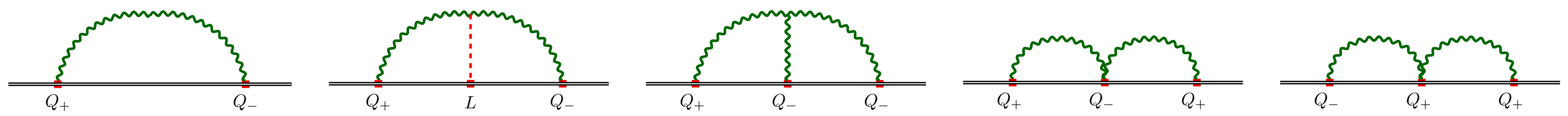
- Total result: $S^{Q^3}_{eff\ 5PN} = -i \int \frac{dk_0}{(2\pi)} \frac{dp_0}{(2\pi)} \tilde{\mathcal{M}}^{Q^3} = -\frac{G_N^2}{15} \int dt\ Tr\left[Q^{(4)}Q^{(4)}Q + \frac{4}{7}Q^{(3)}Q^{(3)}Q^{(2)}\right]$

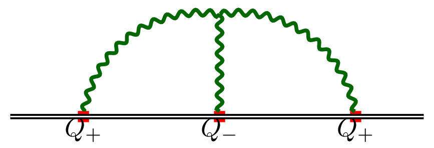
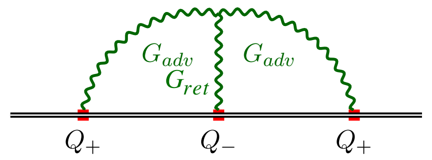
in agreement with
Foffa, Sturani (2019)

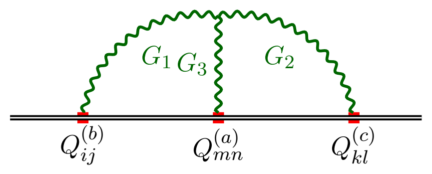
[credit: Brunello]

Memory Effect (5PN) within the *In-In* formalism

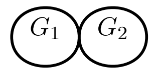
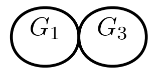
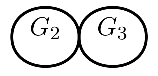
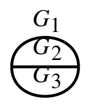
Blumlein, Maier, Marquard, Schaefer (2021)
Almeida, Foffa, Sturani (2022)



- Sample diagram:  $= < (V_{Qh})^{(+)}(V_{Qh})^{(-)}(V_{Qh})^{(+)}V_{h^3} > = 3$ 

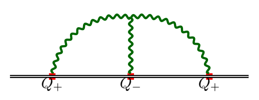
- Tensor Decomposition: 

$= Tr[\ Q^{(a)}Q^{(b)}Q^{(c)}\ ]\ \tilde{\mathcal{M}}^{Q^3}(G_1, G_2, G_3)$

- IBP Decomposition: $\tilde{\mathcal{M}}^{Q^3}(G_1, G_2, G_3) = C_1$  $+ C_2$  $+ C_3$  $+ C_4$  $= 0$

IBPs are not sensitive to the prescription

- Total result:

$S^{Q_+Q_+Q_-}_{eff\ 5PN} = -i \lim_{d\to 3}$  $= \frac{1}{35} \int dt\ Tr\left[8(\ddot{Q}_+)^2\ddot{Q}_- + 7(\dddot{Q}_+)^2 Q_- - 12\ddot{Q}_+\ddot{Q}_+\ddot{Q}_- - 14\dddot{Q}_+Q_+\dddot{Q}_-\right]$

agreement with:
Almeida Foffa, Sturani (2022)
Bluemlein, Maier, Marquard, Schaefer (2022)

[credit: Brunello]

- known known FarZone-GREFT with causal propagators not adequate to describe Radiation/Hereditary effects
- known unknown: FarZone-GREFT within Keldysh-Schwinger "*in-in*" formalism under scrutiny

Conservative Dynamics :: **Near Zone** with Spin

Kim, Levi, Yin (2022)

Mandal, Patil, Steinhoff & P.M. (2022)

- Elimination of higher-order time derivatives / equation of motion

borrowed from: Damour, Schafer, Barker, O'Connell

$$\mathbf{x}_{(a)} \to \mathbf{x}_{(a)} + \delta\mathbf{x}_{(a)}$$

$$\delta\mathcal{L} = \left(\frac{\delta\mathcal{L}}{\delta\mathbf{x}^i_{(a)}}\right)\delta\mathbf{x}^i_{(a)} + \frac{1}{2}\left(\frac{\delta^2\mathcal{L}}{\delta\mathbf{x}^i_{(a)}\delta\mathbf{x}^j_{(a)}}\right)\delta\mathbf{x}^i_{(a)}\delta\mathbf{x}^j_{(a)} + \mathcal{O}\left(\delta\mathbf{x}^3_{(a)}\right)$$

$$\mathbf{\Lambda}^{ij}_{(a)} \to \mathbf{\Lambda}^{ij}_{(a)} + \delta\mathbf{\Lambda}^{ij}_{(a)} \qquad \boldsymbol{S}^{ij}_{(a)} \to \boldsymbol{S}^{ij}_{(a)} + \delta\boldsymbol{S}^{ij}_{(a)} \qquad \delta\mathbf{\Lambda}^{ij}_{(a)} = \mathbf{\Lambda}^{ik}_{(a)}\boldsymbol{\omega}^{kj}_{(a)} + \mathcal{O}\left(\boldsymbol{\omega}^2_{(a)}\right) \qquad \delta\boldsymbol{S}^{ij}_{(a)} = 2\boldsymbol{S}^{k[i}_{(a)}\boldsymbol{\omega}^{j]k}_{(a)} + \mathcal{O}\left(\boldsymbol{\omega}^2_{(a)}\right)$$

$$\delta\mathcal{L}' = -\left(\frac{1}{c}\right)\frac{1}{2}\dot{\boldsymbol{S}}^{ij}_{(a)}\boldsymbol{\omega}^{ij}_{(a)} - \left(\frac{1}{c}\right)\frac{1}{2}\boldsymbol{S}^{ij}_{(a)}\dot{\boldsymbol{\omega}}^{ik}_{(a)}\boldsymbol{\omega}^{kj}_{(a)} - \left(\frac{\delta V}{\delta\boldsymbol{S}^{ij}_{(a)}}\right)\delta\boldsymbol{S}^{ij}_{(a)} + \mathcal{O}\left(\boldsymbol{\omega}^3_{(a)}, \delta\boldsymbol{S}^2_{(a)}\right)$$

➡ $\mathcal{L}'' = \mathcal{L} + \delta\mathcal{L} + \delta\mathcal{L}'$ free of higher-order time derivatives

- Elimination of 1/(d-3) divergences and spurious Logarithmic terms / canonical transformations

$$\mathcal{H}(\mathbf{x}, \mathbf{p}, \mathbf{S}) = \sum_{a=1,2} \mathbf{p}^i_{(a)}\dot{\mathbf{x}}^i_{(a)} - \mathcal{L}''(\mathbf{x}, \dot{\mathbf{x}}, \mathbf{S})$$

may contain divergences and spurious logarithmic term

➡ $\mathcal{H}' = \mathcal{H} + \{\mathcal{H}, \mathcal{G}\}$ → educated guess

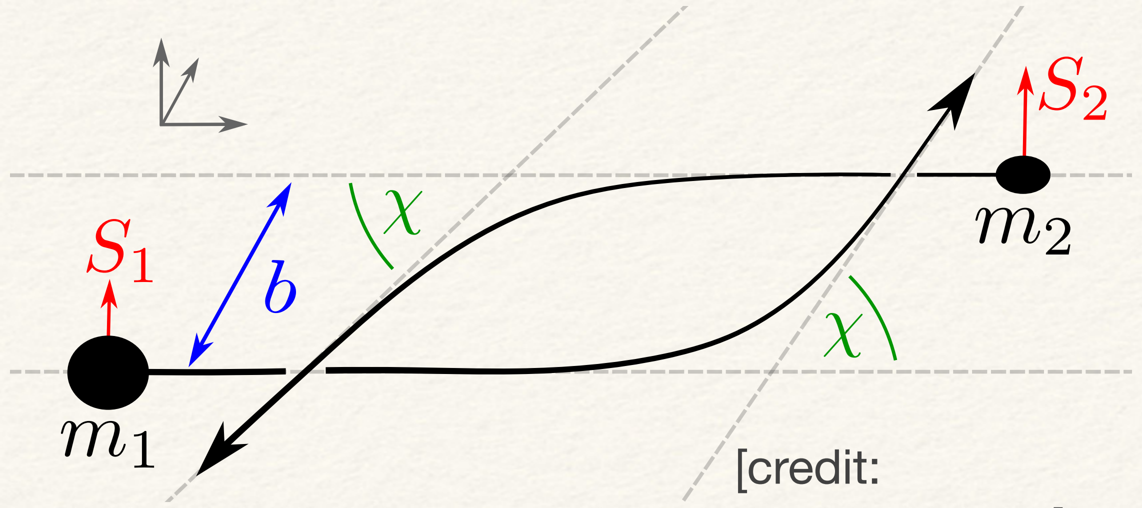
Effective Hamiltionian free of unphysical terms

Scattering Angle :: **Near Zone** with Spin

Mandal, Patil, Steinhoff & P.M. (2022)

- **Aligned spins**

$$\chi = -2\int_{r_{\min}}^{\infty} dr \frac{\partial p_r}{\partial L} - \pi \qquad\qquad \chi = \chi^{loc}$$



[credit: Antornelli et al.]

$$\chi(v, b, S_{(a)}) = \chi_{\rm pp}(v, b) + \chi_{\rm SO}(v, b, S_{(a)}) + \chi_{\rm SS}(v, b, S_{(a)})$$

$$\chi_{\rm SS}(v, b, S_{(a)}) = \chi_{\rm S1S2}(v, b, S_{(a)}) + \chi_{\rm S^2}(v, b, S_{(a)}) + \chi_{\rm ES^2}(v, b, S_{(a)}) + \chi_{\rm E^2S^2}(v, b, S_{(a)}) + \chi_{\rm E^2}(v, b, S_{(a)})$$

- **PM-expansion:**

$$\frac{1}{2}\chi(b, E) = \sum_n \chi_b^{(n)}(E)\left(\frac{GM}{b}\right)^n$$

- **PN-expansion:**

$$\chi_b^{(n)} = \sum_{k\geq 0} \chi_b^{(n,k)}\left(\frac{\mathrm{v}^2}{c^2}\right)^k$$

$$\begin{aligned}\frac{\chi_{\rm pp}}{\Gamma} = {} & \left(\frac{G_N M}{v^2 b}\right)\left\{2 + 2\left(\frac{v^2}{c^2}\right) + \mathcal{O}\left(\frac{v^8}{c^8}\right)\right\} \\ & + \pi\left(\frac{G_N M}{v^2 b}\right)^2\left\{3\left(\frac{v^2}{c^2}\right) + \frac{3}{4}\left(\frac{v^4}{c^4}\right) + \mathcal{O}\left(\frac{v^8}{c^8}\right)\right\} \\ & + \left(\frac{G_N M}{v^2 b}\right)^3\left\{-\frac{2}{3} + 2\frac{15-\nu}{3}\left(\frac{v^2}{c^2}\right) + \frac{60-13\nu}{2}\left(\frac{v^4}{c^4}\right) + \frac{40-277\nu}{12}\left(\frac{v^6}{c^6}\right) + \mathcal{O}\left(\frac{v^8}{c^8}\right)\right\} \\ & + \pi\left(\frac{G_N M}{v^2 b}\right)^4\left\{15\frac{7-2\nu}{3}\left(\frac{v^4}{c^4}\right) + \left(\frac{105}{4} - \frac{437}{8}\nu + \frac{123}{128}\pi^2\nu\right)\left(\frac{v^6}{c^6}\right) + \mathcal{O}\left(\frac{v^8}{c^8}\right)\right\}\end{aligned}$$

$$\begin{aligned}\frac{\chi_{\rm SO}}{\Gamma} = \frac{v}{bc}\left[a_{(+)} \quad \delta a_{(-)}\right]\cdot\Bigg(& \left(\frac{G_N M}{v^2 b}\right)\left\{\begin{bmatrix}-4\\0\end{bmatrix} + \mathcal{O}\left(\frac{v^8}{c^8}\right)\right\} \\ & + \pi\left(\frac{G_N M}{v^2 b}\right)^2\left\{-\frac{1}{2}\begin{bmatrix}7\\1\end{bmatrix} - \frac{3}{4}\begin{bmatrix}7\\1\end{bmatrix}\left(\frac{v^2}{c^2}\right) + \mathcal{O}\left(\frac{v^8}{c^8}\right)\right\} \\ & + \left(\frac{G_N M}{v^2 b}\right)^3\left\{-2\begin{bmatrix}5\\1\end{bmatrix} - 20\begin{bmatrix}5-\nu/2\\1\end{bmatrix}\left(\frac{v^2}{c^2}\right) - 10\begin{bmatrix}5-77\nu/20\\1\end{bmatrix}\left(\frac{v^4}{c^4}\right)\right. \\ & \left. \qquad + \frac{1}{4}\begin{bmatrix}177\nu\\0\end{bmatrix}\left(\frac{v^6}{c^6}\right) + \mathcal{O}\left(\frac{v^8}{c^8}\right)\right\} \\ & + \pi\left(\frac{G_N M}{v^2 b}\right)^4\left\{\frac{3}{4}\begin{bmatrix}-91+13\nu\\-21+\nu\end{bmatrix}\left(\frac{v^2}{c^2}\right) - \frac{1}{8}\begin{bmatrix}1365-777\nu\\315-45\nu\end{bmatrix}\left(\frac{v^4}{c^4}\right)\right. \\ & \left. \qquad - \frac{1}{32}\begin{bmatrix}1365 - \left(\frac{23717}{3} - \frac{733}{8}\pi^2\right)\nu\\315 - \left(\frac{257}{3} + \frac{251}{8}\pi^2\right)\nu\end{bmatrix}\left(\frac{v^6}{c^6}\right) + \mathcal{O}\left(\frac{v^8}{c^8}\right)\right\}\Bigg)\end{aligned}$$

$$\begin{aligned}\frac{\chi_{\rm S1S2} + \chi_{\rm S^2}}{\Gamma} = \frac{1}{b^2c^2}\left[a_{(+)}^2 \quad \delta a_{(+)}a_{(-)} \quad a_{(-)}^2\right]\cdot\Bigg(& \left(\frac{G_N M}{v^2 b}\right)\left\{\begin{bmatrix}1\\0\\-1\end{bmatrix} + \begin{bmatrix}1\\0\\-1\end{bmatrix}\left(\frac{v^2}{c^2}\right)\right\} \\ & + \pi\left(\frac{G_N M}{v^2 b}\right)^2\left\{\frac{3}{4}\begin{bmatrix}1\\0\\-1\end{bmatrix} + \frac{3}{16}\begin{bmatrix}41\\10\\-19\end{bmatrix}\left(\frac{v^2}{c^2}\right) + \frac{3}{128}\begin{bmatrix}55\\-10\\-41\end{bmatrix}\left(\frac{v^4}{c^4}\right)\right\} \\ & + \left(\frac{G_N M}{v^2 b}\right)^3\left\{2\begin{bmatrix}1\\0\\-1\end{bmatrix} + \begin{bmatrix}102-2\nu\\32\\-38-6\nu\end{bmatrix}\left(\frac{v^2}{c^2}\right) + \frac{1}{14}\begin{bmatrix}2332-499\nu\\688\\-748-37\nu\end{bmatrix}\left(\frac{v^4}{c^4}\right)\right. \\ & \left. \qquad + \frac{1}{140}\begin{bmatrix}1704-9925\nu\\-288\\-1096+5377\nu\end{bmatrix}\left(\frac{v^6}{c^6}\right)\right\} \\ & + \pi\left(\frac{G_N M}{v^2 b}\right)^4\left\{\frac{15}{16}\begin{bmatrix}63-2\nu\\22\\-21-6\nu\end{bmatrix}\left(\frac{v^2}{c^2}\right) - \frac{15}{448}\begin{bmatrix}-11063+2638\nu\\-4206+308\nu\\2657+790\nu\end{bmatrix}\left(\frac{v^4}{c^4}\right)\right. \\ & \left. \qquad - \frac{3}{229376}\begin{bmatrix}19102720 + (-29293696 + 135555\pi^2)\\-256(-24640+11507\nu)\\-4730880 + (8911744 - 139965\pi^2)\nu\end{bmatrix}\left(\frac{v^6}{c^6}\right)\right\}\Bigg)\end{aligned}$$

$$\begin{aligned}\frac{\chi_{\rm ES^2}}{\Gamma} = \frac{1}{b^2c^2}\left[a_{\rm ES^2(+)}^2 \quad \delta a_{\rm ES^2(-)}^2\right]\cdot\Bigg\{ & \left(\frac{G_N M}{v^2 b}\right)\left\{\begin{bmatrix}2\\0\end{bmatrix} + \begin{bmatrix}2\\0\end{bmatrix}\left(\frac{v^2}{c^2}\right)\right\} \\ & + \pi\left(\frac{G_N M}{v^2 b}\right)^2\left\{\frac{1}{2}\begin{bmatrix}3\\0\end{bmatrix} + \frac{1}{4}\begin{bmatrix}27\\3\end{bmatrix}\left(\frac{v^2}{c^2}\right) + \frac{1}{64}\begin{bmatrix}117\\57\end{bmatrix}\left(\frac{v^4}{c^4}\right)\right\} \\ & + \left(\frac{G_N M}{v^2 b}\right)^3\left\{\begin{bmatrix}4\\0\end{bmatrix} + \begin{bmatrix}76-4\nu\\8\end{bmatrix}\left(\frac{v^2}{c^2}\right) + \frac{1}{7}\begin{bmatrix}748-131\nu\\216\end{bmatrix}\left(\frac{v^4}{c^4}\right) + \frac{1}{70}\begin{bmatrix}1096-5013\nu\\704\end{bmatrix}\left(\frac{v^6}{c^6}\right)\right\} \\ & + \pi\left(\frac{G_N M}{v^2 b}\right)^4\left\{\frac{15}{4}\begin{bmatrix}11-\nu\\1\end{bmatrix}\left(\frac{v^2}{c^2}\right) + \frac{15}{224}\begin{bmatrix}2867-708\nu\\627-28\nu\end{bmatrix}\left(\frac{v^4}{c^4}\right)\right. \\ & \left. \qquad + \frac{15}{114688}\begin{bmatrix}1026816 - (2184832 + 27111\pi^2)\nu\\128(3262-395\nu)\end{bmatrix}\left(\frac{v^6}{c^6}\right)\right\}\Bigg\}\end{aligned}$$

$$\frac{\chi_{\rm E^2S^2}}{\Gamma} = \frac{1}{b^2c^2}\left[a_{\rm E^2S^2(+)}^2 \quad \delta a_{\rm E^2S^2(-)}^2\right]\cdot\left\{\pi\left(\frac{G_N M}{v^2 b}\right)^4\frac{15}{32}\nu\begin{bmatrix}1\\1\end{bmatrix}\left(\frac{v^6}{c^6}\right)\right\} + \mathcal{O}\left(G_N^5, \frac{v^8}{c^8}\right)$$

$$\frac{\chi_{\rm E^2}}{\Gamma} = \frac{1}{b^2c^2}\left[a_{\rm E^2(+)}^2 \quad \delta a_{\rm E^2(-)}^2\right]\cdot\left\{\pi\left(\frac{G_N M}{v^2 b}\right)^4\frac{45}{16}\nu\begin{bmatrix}1\\1\end{bmatrix}\left(\frac{v^6}{c^6}\right)\right\} + \mathcal{O}\left(G_N^5, \frac{v^8}{c^8}\right)$$

In agreement with:
Antonelli et al. (2020)
Kim et al. (2022)

Binding Energy :: **Near Zone** with Spin

Mandal, Patil, Steinhoff & P.M. (2022)

- **Circular Orbit and aligned spins**

$$E(x, \widetilde{S}_{(a)}) = E_{\rm pp}(x) + E_{\rm SO}(x, \widetilde{S}_{(a)}) + E_{\rm SS}(x, \widetilde{S}_{(a)})$$

$$E_{\rm SS}(x, \widetilde{S}_{(a)}) = E_{\rm S1S2}(x, \widetilde{S}_{(a)}) + E_{\rm S^2}(x, \widetilde{S}_{(a)}) + E_{\rm ES^2}(x, \widetilde{S}_{(a)}) + E_{\rm E^2S^2}(x, \widetilde{S}_{(a)}) + E_{\rm E^2}(x, \widetilde{S}_{(a)})$$

$$E_{\rm pp}(x) = -x\frac{1}{2} + x^2\left\{\frac{3}{8} + \frac{\nu}{24}\right\} + x^3\left\{\frac{27}{16} - \frac{19}{16}\nu + \frac{1}{48}\nu^2\right\} + x^4\left\{\frac{675}{128} + \left(-\frac{34445}{1152} + \frac{205\pi^2}{192}\right)\nu + \frac{155}{192}\nu^2 + \frac{35}{10368}\nu^3\right\}$$

$$E_{\rm SO}(x, \widetilde{S}) = x^{5/2}\left\{S^*(-\nu) + S\left(-\frac{4}{3}\nu\right)\right\} + x^{7/2}\left\{S^*\left(-\frac{3}{2}\nu + \frac{5}{3}\nu^2\right) + S\left(-4\nu + \frac{31}{18}\nu^2\right)\right\} + x^{9/2}\left\{S^*\left(-\frac{27}{8}\nu + \frac{39}{2}\nu^2 - \frac{5}{8}\nu^3\right) + S\left(-\frac{27}{2}\nu + \frac{211}{8}\nu^2 - \frac{7}{12}\nu^3\right)\right\} + x^{11/2}\left\{S^*\left(-\frac{135}{16}\nu + \frac{565}{8}\nu^2 - \frac{1109}{24}\nu^3 - \frac{25}{324}\nu^4\right) + S\left(-45\nu + \left(\frac{19679}{144} + \frac{29\pi^2}{24}\right)\nu^2 - \frac{1979}{36}\nu^3 - \frac{265}{3888}\nu^4\right)\right\},$$

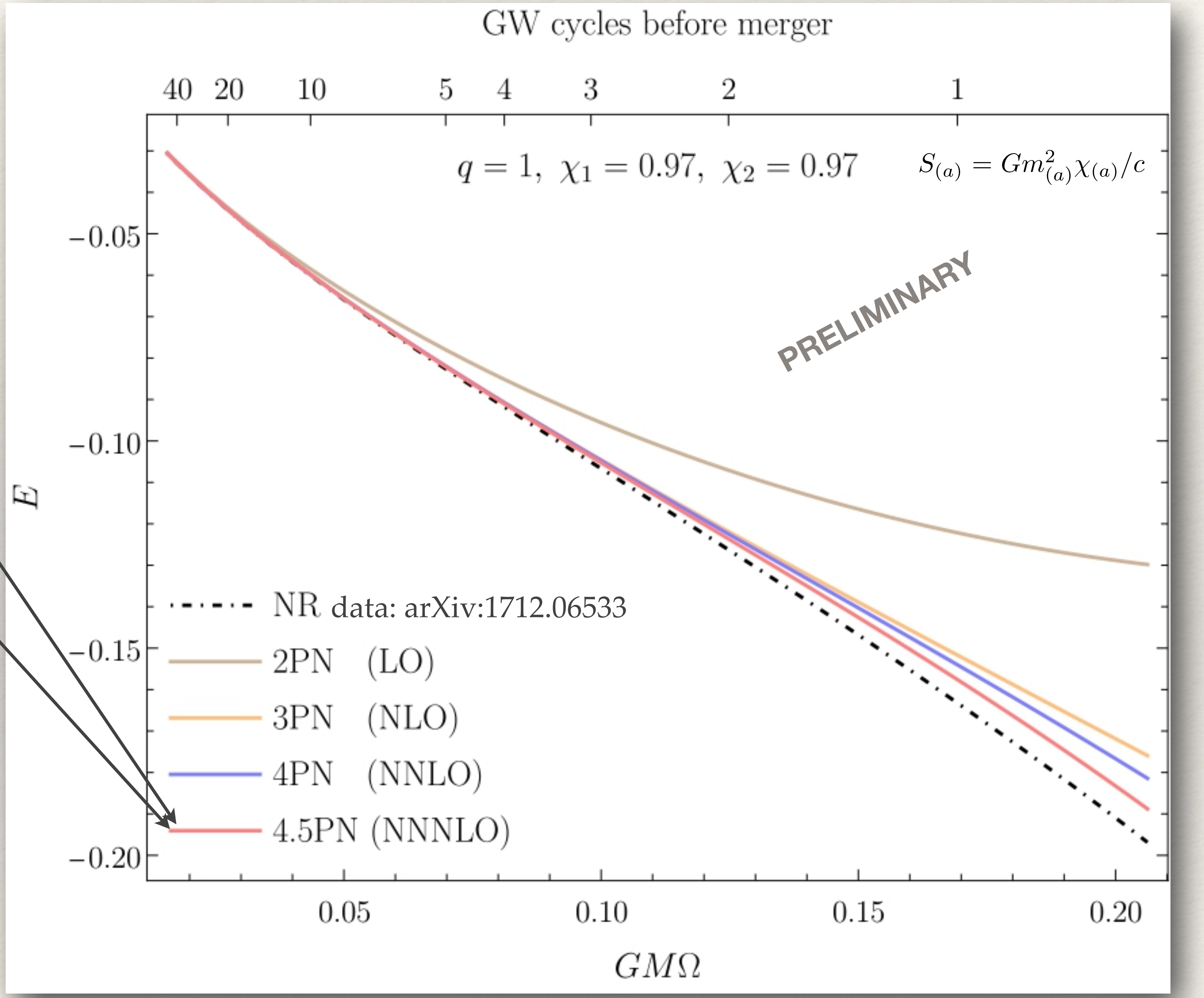
$$E_{\rm S1S2}(x, \widetilde{S}) = \widetilde{S}_{(1)}\widetilde{S}_{(2)}\left\{x^3\left\{\nu\right\} + x^4\left\{\frac{5}{6}\nu + \frac{5}{18}\nu^2\right\} + x^5\left\{\frac{35}{8}\nu - \frac{1001}{72}\nu^2 - \frac{371}{216}\nu^3\right\} + x^6\left\{\frac{243}{16}\nu - \left(\frac{2107}{16} - \frac{123}{32}\pi^2\right)\nu^2 + \frac{147}{8}\nu^3 + \frac{13}{16}\nu^4\right\}\right\},$$

$$E_{\rm S^2}(x, \widetilde{S}) = \widetilde{S}^2_{(1)}\left\{x^4\left\{\frac{25}{18}\nu^2 + \frac{1}{q}\left(-\frac{5}{2}\nu + \frac{5}{6}\nu^2\right)\right\} + x^5\left\{\frac{10}{3}\nu^2 - \frac{749}{108}\nu^3 + \frac{1}{q}\left(-\frac{21}{4}\nu - \frac{7}{6}\nu^2 - \frac{217}{36}\nu^3\right)\right\} + x^6\left\{\frac{1947}{112}\nu^2 - \frac{48357}{560}\nu^3 + \frac{159}{16}\nu^4 + \frac{1}{q}\left(-\frac{243}{16}\nu + \left(\frac{747}{16} - \frac{189\pi^2}{2048}\right)\nu^2 - \frac{13731}{280}\nu^3 + \frac{153}{16}\nu^4\right)\right\}\right\} + (1 \leftrightarrow 2),$$

$$E_{\rm ES^2}(x, \widetilde{S}) = \left(C^{(0)}_{\rm ES^2}\right)_{(1)}\widetilde{S}^2_{(1)}\left\{x^3\left\{\frac{1}{q}\frac{\nu}{2}\right\} + x^4\left\{\frac{5}{3}\nu^2 + \frac{1}{q}\left(\frac{5}{4}\nu + \frac{5}{4}\nu^2\right)\right\} + x^5\left\{\frac{31}{4}\nu^2 - \frac{35}{18}\nu^3 + \frac{1}{q}\left(\frac{63}{16}\nu + \frac{77}{48}\nu^2 - \frac{91}{48}\nu^3\right)\right\} + x^6\left\{\frac{789}{28}\nu^2 - \frac{156}{7}\nu^3 + \frac{5}{8}\nu^4 + \frac{1}{q}\left(\frac{405}{32}\nu + \left(\frac{3747\pi^2}{2048} - \frac{2389}{32}\right)\nu^2 - \frac{555}{56}\nu^3 + \frac{21}{32}\nu^4\right)\right\}\right\} + (1 \leftrightarrow 2),$$

$$E_{\rm E^2}(x, \widetilde{S}) = \left(C^{(2)}_{\rm E^2}\right)_{(1)}\widetilde{S}^2_{(1)}x^6\left\{9\nu^3\left(1 + \frac{1}{q}\right)\right\} + (1 \leftrightarrow 2),$$

$$E_{\rm E^2S^2}(x, \widetilde{S}) = \left(C^{(0)}_{\rm E^2S^2}\right)_{(1)}\widetilde{S}^2_{(1)}x^6\left\{\frac{3\nu^3}{2}\left(1 + \frac{1}{q}\right)\right\} + (1 \leftrightarrow 2).$$



[credit: Patil]

In agreement with:
Antonelli et al. (2020)
Kim et al. (2022)