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SHORTWAVE FORMALISM FOR GRAVITATIONAL WAVE PROPAGATION IN CURVED BACKGROUND

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Introduction

Nowadays it's possible to detect gravitational waves of black holes and neutron stars coalescences Abbott and et al. (2016). But these detections were made in a region not very different, where there is practically no curvature of spacetime. In other words, the properties of the wave may change if it passes close to black holes at the center of galaxies or due to the acceleration of the universe, experiencing deflection. Therefore, it is necessary to consider a curved background and no longer Minkowski, requiring a new theoretical framework.

Objectives

To find an equation that allows for calculations of the propagation of gravitational waves in a curved background

Methodology

For a general metric, was made a perturbation $h_{\mu\nu}$ on the background $g^{\mathbf{B}}_{\mu\nu}$

$$g_{\mu\nu} = g^{(\mathbf{B})}_{\mu\nu} + h_{\mu\nu}, \tag{1}$$

and the raising and lowering of indices was made with respect to the background metric $g^{(\mathbf{B})}$. As an analogy, one can think these perturbation as a roughness in a surface of an orange, that does not change its geometry in "large scale".

So one can define $\mathscr{R}$ the radius of curvature background, λ reduced wavelength $\left(\frac{\lambda}{2\pi}\right)$ and $\mathscr{A}$ amplitude of gravitational waves.

Thus, the proprieties are Misner et al. (2017):

1. The $\mathscr{A}$ is the amplitude of perturbation

$$h_{\mu\nu} \leq (\text{tipical value of} g^{(\mathbf{B})}_{\mu\nu})\mathscr{A}; \tag{2}$$

2. The scale on which $g^{(\mathbf{B})}_{\mu\nu}$ varies is $\geq \mathscr{R}$

$$\partial_\alpha g^{(\mathbf{B})}_{\mu\nu} \leq \frac{(\text{tipical value of } g^{(\mathbf{B})}_{\mu\nu})}{\mathscr{R}}; \tag{3}$$

3. The scale on which $h_{\mu\nu}$ varies is $\sim \lambda$

$$\partial_\alpha h_{\mu\nu} \sim \frac{(\text{tipical value of } g^{(\mathbf{B})}_{\mu\nu})}{\lambda}; \tag{4}$$

In order to use this new metric in the Einstein equations, which is given by Weinberg (1972):

$$R_{\mu\nu} = \chi\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right), \tag{5}$$

first one need to find the inverse metric in second order:

$$g^{\mu\nu} = g^{(\mathbf{B})\mu\nu} - h^{\mu\nu} + h^{\mu\alpha}h^{\nu}_{\alpha} - h^{\mu\alpha}h^{\beta}_{\alpha}h^{\nu}_{\beta} + O(h^3) \tag{6}$$

In this manner, one can find the Ricci tensor:

$$R_{\mu\nu} = R^{\mathbf{B}}_{\mu\nu} + R^{(1)}_{\mu\nu} + R^{(2)}_{\mu\nu} + ... \tag{7}$$

Since it is a metrics variation, the Ricci tensor also has variation, described by Eq (7). But to discriminate each one of its terms, the Ricci tensor:

$$R_{\mu\alpha\nu}{}^{\alpha} = R_{\mu\nu} = \partial_\mu\Gamma_{\alpha\nu}{}^{\alpha} - \partial_\alpha\Gamma^{\alpha}_{\mu\nu} + \Gamma_{\mu\lambda}{}^{\alpha}\Gamma_{\alpha\nu}{}^{\lambda} - \Gamma_{\alpha\lambda}{}^{\alpha}\Gamma_{\mu\nu}{}^{\lambda}, \tag{8}$$

and its variation is:

$$\delta R_{\mu\nu} = \partial_\mu\delta\Gamma_{\mu\nu}{}^{\alpha} - \partial_\alpha\delta\Gamma^{\alpha}_{\mu\nu} + \Gamma_{\mu\lambda}{}^{\alpha}\delta\Gamma_{\alpha\nu}{}^{\lambda} + \Gamma_{\alpha\nu}{}^{\lambda}\delta\Gamma_{\mu\lambda}{}^{\alpha} - \Gamma_{\alpha\lambda}{}^{\alpha}\delta\Gamma_{\mu\nu}{}^{\lambda} - \Gamma_{\mu\nu}{}^{\lambda}\delta\Gamma_{\alpha\lambda}{}^{\alpha} \tag{9}$$

and the connection variation $\delta\Gamma$:

$$\delta\Gamma_{\mu\nu}{}^{\alpha} = \frac{1}{2}g^{\lambda\alpha}\left[\nabla_\nu\delta g_{\lambda\nu} + \nabla_\nu\delta g_{\mu\lambda} - \nabla_\lambda\lambda g_{\mu\nu}\right] \tag{10}$$

Applying the Eq. (10) and Eq. (6) in Eq. (9)

$$\delta R_{\mu\nu} = \frac{1}{2}\left[\nabla_\nu\nabla_\mu h + \nabla_\alpha\nabla_\nu h_\mu{}^{\alpha} - \nabla_\alpha\nabla_\nu h^{\alpha}{}_{\nu} + \Box h_{\mu\nu}\right] + O(h^2) \tag{11}$$

where the first term that has order h refers to $R^{(1)}_{\mu\nu}$. So, the first-order approximation and the propagation of vacuum space waves will be considered, such as:

$$R^{(1)}_{\mu\nu} = \frac{1}{2}\left[\nabla_\nu\nabla_\mu h + \nabla_\alpha\nabla_\nu h_\mu{}^{\alpha} - \nabla_\alpha\nabla_\nu h^{\alpha}{}_{\nu} + \Box h_{\mu\nu}\right] = 0 \tag{12}$$

In constructing the linearized theory, the reverse trace of $h_{\mu\nu}$ was employed, enabling its application here:

$$h_{\mu\nu} = \bar{h}_{\mu\nu} - \frac{1}{2}g^{(\mathbf{B})}_{\mu\nu}\bar{h} \tag{13}$$

Finally, it was identified the propriety of second order covariant derivative tensor:

$$\nabla_\alpha\nabla_\beta S^{\mu\nu} = \nabla_\beta\nabla_\alpha S^{\mu\nu} + R_{\alpha\rho\beta}{}^{\mu}S^{\rho\nu} + R_{\alpha\rho\beta}{}^{\nu}S^{\mu\rho} \tag{14}$$

Applying the Eq. (14) and Eq. (13) in Eq. (12):

$$\Box\bar{h}_{\mu\nu} - g^{(\mathbf{B})}_{\mu\nu}\nabla_\alpha\nabla_\beta\bar{h}^{\beta\alpha} - \nabla_\nu\nabla_\lambda\bar{h}_\mu{}^{\lambda} - \nabla_\mu\nabla_\lambda\bar{h}^{\lambda}{}_{\nu} + 2R_{\mu\lambda\nu\gamma}\bar{h}^{\gamma\lambda} - R_{\nu\gamma}\bar{h}_\mu{}^{\gamma} - R_{\mu\gamma}\bar{h}^{\gamma}{}_{\nu} = 0 \tag{15}$$

In a vacuum propagation scenario, the last two terms of Eq. (13) have vanished. Once again, retrieval from the linearized theory occurs through an appropriate choice of a quadrifunction, leading to the application of the Lorenz gauge, thus:

$$\nabla_\lambda\bar{h}^{\lambda\alpha} = 0 \tag{16}$$

Results

The equation to describe the gravitational waves propagation in a curved background is:

$$\Box\bar{h}_{\mu\nu} + 2R^{(\mathbf{B})}_{\mu\alpha\nu\beta}\bar{h}^{\alpha\beta} = 0 \tag{17}$$

It is possible to observe that Eq. (17) reduces to the equation for the propagation of gravitational waves in a flat background, as the Riemann tensor in Minkowski space is null. In other words, it provides a general form for calculating the propagation of gravitational waves in any background.

From the same equation, in the context of short-wave approximation, the eikonal approximation can be applied Lanczos (1986). Analogous to the electromagnetic case, where the conservation of photons is predicted, in the gravitational scenario, it leads to the conservation of gravitons, and propagate along a null geodesic Maggiore (2008).

Conclusion

The equation is now capable to describe the propagation in a curved background through a short-wave approximation, it can be used to explore a broader range of scenarios within our universe. This includes gravitational wave deflections by supermassive objects, the arrangement of massive objects generating gravitational lensing, or the propagation of gravitational waves in a primordial universe, simply by selecting the background metric to be utilized.

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