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# Higher Order Field Equations for Coupled Oscillators

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### Introduction

The coupled oscillator model is used to explain many physical systems, for example, is the molecular structure of a crystal that can be represented as a set of oscillators coupled by springs that play the role of molecular interactions between the molecules of the crystal.

Given the importance of coupled oscillators, we derive higher order equations from the equations of motion of a one-dimensional system of $N$ particles of the same mass coupled by identical springs. Having obtained higher order equations, the main objective of this research is to limit the continuum to obtain a higher order Klein-Gordon equation and study its properties and characteristics.

### Main goals

1. To derive fourth order equations from the equations of motion of coupling oscillators.
2. To obtain the continuum limit of this equation and determine a higher order field equation.
3. To study properties and solutions of higher order field equations.
4. To study limiting cases that return in second order field equation.

## 1 Theoretical Model

Consider a one-dimensional system of $N$ particles of the same mass $m$ coupled by springs with the same spring constant $k$, as shown in Figure. 1:

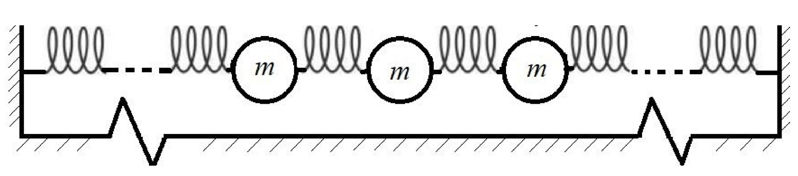


**Figure 1:** One-dimensional coupled oscillators.

The equations of motion for this model can be written as [3, 2]

$$\ddot{q}_i(t) = \omega^2[q_{i+1}(t) - 2q_i(t) + q_{i-1}(t)] \quad (i = 1, ...N), \tag{1}$$

where $q_i(t)$ and $\ddot{q}_i(t)$ is the position and acceleration of the ith particle at a given instant $t$ and $\omega = \sqrt{k/m}$ is the angular frequency. This equation is coupled since it depends on the positions of neighboring particles. One of the ways to decouple this equation is by increasing the order of the derivatives, but in compensation the number of spurious solutions increases. Deriving the equation (1) up to fourth order and with some algebraic manipulations we arrive at the following equation:

$$\ddddot{q}_i(t) + 4\omega^2\ddot{q}_i(t) = \omega^4[q_{i+2}(t) - 2q_i(t) + q_{i-2}(t)] \quad (i = 1, ...N), \tag{2}$$

still remains a coupled equation, the same order raising procedure can continue to be done until obtaining a higher order equation that will only depend on variables of the ith particle. But our main interest is not decoupling, but to obtain a Klein-Gordon equation from the equation (2).

The limit of the continuum of this model consists in assuming an infinite number of particles with the same mass $m \to 0$ coupled by springs with constant $k$ and separated by a distance $d \to 0$, such that the density linear $\lambda = m/d$ be a constant [1]. The position function $q_i(t)$ becomes a field function $\varphi(x, t)$.

Using finite difference approximations, algebraic manipulations and following procedures analogous to what is done in the case of a finite wire, from (2) we arrive at the following field research

$$\Box^2\varphi(x, t) = 0 \tag{3}$$

In addition to this equation, we obtained two others through different algebraic manipulations

$$\frac{1}{v^4}\frac{\partial^4\varphi}{\partial t^4} - \frac{\partial^4\varphi}{\partial x^4} = 0, \tag{4}$$

$$\frac{1}{4a^2}\frac{\partial^4\varphi}{\partial t^4} + \frac{1}{v^2}\frac{\partial^2\varphi}{\partial t^2} - \frac{\partial^2\varphi}{\partial x^2} = 0, \tag{5}$$

where $v$ is a velocity parameter and $a$ is an acceleration parameter.

## 2 Discussion

The equations (3), (4) and (5) are not equivalent, this shows us that the same higher order discrete equation can generate field equations that represent different physical systems. This raises some questions such as: is there a higher order field equation that is privileged? What types of physical systems can be described by these equations?

## 3 Conclusions

So far we have come to the conclusion that the same higher order discrete equation can generate field equations that are not equivalent and represent different physical systems. The next studies in this research consist of finding solutions to the field equations, understanding which physical systems are represented by these equations and analyzing limiting cases, for example the equation (5) which falls into the Klein-Gordon equation when $a \to \infty$. We hope that among the solutions we find there may be one that is a solution to the Klein-Gordon equation.

### References

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