Stability of the Photon Ring of a Charged Traversable Wormhole

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1 Introduction

A geometric procedure was adopted for deriving the photon sphere (circular photon orbit) and the radius of the throat/shadow of a charged wormhole This approach employs geodesic curvature and Gaussian curvature within the optical geometry of the charged wormhole spacetime to ascertain the the photon sphere radius and if it is a stable or unstable orbit.

2 Charged Wormholes

A wormhole is a object, with non-trivial topology, that connects two asymptotically flat spacetime regions [2]. The solution for a charged traversable wormhole was found by Kim and Lee [1]. The line element is given by:

$$ds^2 = -\left(1+\frac{Q^2}{r^2}\right)dt^2 + \frac{1}{1-\frac{b(r)}{r}+\frac{Q^2}{r^2}}dr^2 + r^2\left(d\theta^2 + sin^2(\theta)d\phi^2\right). \quad (1)$$

The corresponding components of the energy-momentum tensor are [1], .

$$\rho(r) = \frac{1}{8\pi}\left\{\frac{b'(r)}{r^2} - \frac{Q^2}{r^4}\right\}. \quad (2)$$

$$\sigma(r) = -\frac{1}{8\pi}\left\{\frac{b(r)}{r^3} - 2\left(1-\frac{b(r)}{r}\right)\frac{\Phi'(r)}{r} + \frac{Q^2}{r^4}\right\}. \quad (3)$$

$$p(r) = \frac{1}{8\pi}\left\{\left(1-\frac{b(r)}{r}\right)\left[\Phi''(r)+(\Phi'(r))^2 - \frac{b'(r)r-b(r)}{2r(r-b(r))}\Phi'(r) - \frac{b'(r)r-b(r)}{2r^2(r-b(r))} + \right.\right.$$
$$\left.\left. + \frac{\Phi'(r)}{r} - \frac{Q^2}{r^4}\right]\right\}. \quad (4)$$

Where $\rho(r)$ is the energy density, $\sigma(r)$ is the tension and $p(r)$ is the pressure. In the present work, we will focus on the shape function:

$$b(r) = \frac{b_0^2}{r}. \quad (5)$$

When $Q = 0$, the solution describes a neutral traversable wormhole, and when $b(r) = 0$ the spacetime geometry corresponds to the massless Reissner-Norström solution.

3 The Optical Metric Tensor, Geodesic and Gaussian Curvatures

The optical geometry is a tool to study the motion of null-like particles in curved spacetimes[4]. The optical geometry is obtained from directly from spacetime geometry. In terms of the line element, the optical geometry is a restriction to the metric tensor as, $d\tau^2 = -ds^2 = 0$, or, $ds^2 = g_{\mu\nu}dx^\mu dx^\nu = 0 \iff dt^2 = g_{ij}^{Optical}dx^i dx^j$. Moreover, due to the spherical symmetry, it is possible to investigate the behaviour of the light rays in $\theta = \frac{\pi}{2}$. Therefore, the general optical metric reads as:

$$dt^2 = -\frac{g_{rr}}{g_{tt}}dr^2 - \frac{g_{\phi\phi}}{g_{tt}}d\phi^2. \quad (6)$$

For the charged wormhole metric we have:

$$dt^2 = -\frac{1}{\left(1-\frac{b_0^2}{r^2}+\frac{Q^2}{r^2}\right)\left(-1-\frac{Q^2}{r^2}\right)}dr^2 - \frac{r^2}{-1-\frac{Q^2}{r^2}}d\phi^2. \quad (7)$$

Geodesic Curvature and Photon Radius

The main property explored in [4] was that the photon sphere of a black hole can be determined by the Geodesic curvature, when: $\kappa_g(r_{ph}) = 0$. Therefore, we propose the same approach but now for the wormhole. Explicitly, in this context of a static spherically symmetric spacetime, the whole equation is:

$$\kappa_g(r_{ph}) = \left[\frac{g_{tt}}{2g_{\phi\phi}}\sqrt{-\frac{g_{tt}}{g_{rr}}}\frac{\partial}{\partial r}\left(\frac{g_{\phi\phi}}{g_{tt}}\right)\right] = 0. \quad (8)$$

Gaussian Curvature and Stability of Photon Sphere

The stability of photon spheres in proximity to black holes and wormholes is entirely dictated by the Gaussian curvature within the optical geometry of the equatorial plane, and stems from an examination of bound photon orbits near these spheres, along with additional geometric and topological constraints—most notably, the Hadamard theorem in differential geometry and topology. Notably, the Hadamard theorem's argument restricts the Gaussian curvature without regard to the particular metric form [4].

Given a spherically symmetric black hole or Morris-Thorne wormhole, the signal of the gaussian curvature will dictate the conclusion for the stability of the photon sphere ($\kappa_g(r_{ph}) = 0 \implies r_{ph} = f(r)$). Explicitly, the Gausian curvature in this context is given by:

$$\mathcal{K} = \frac{g_{tt}}{\sqrt{g_{rr}g_{\phi\phi}}}\frac{\partial}{\partial r}\left[\frac{g_{tt}}{2\sqrt{g_{rr}g_{\phi\phi}}}\frac{\partial}{\partial r}\left(\frac{g_{\phi\phi}}{g_{tt}}\right)\right]. \quad (9)$$

The case when $\mathcal{K}(r_{ph}) > 0$, the photon sphere/radius is said to be stable. The case when $\mathcal{K}(r_{ph}) < 0$, the photon sphere/radius is said to be unstable.

4 Results

For the present case, we have, for the radius of the photon ring:

$$\kappa_g(r_{ph}) = \left[\frac{g_{tt}}{2g_{\phi\phi}}\sqrt{-\frac{g_{tt}}{g_{rr}}}\frac{\partial}{\partial r}\left(\frac{g_{\phi\phi}}{g_{tt}}\right)\right] = 0 \implies r_{ph} = \pm\sqrt{b_0^2 - Q^2}. \quad (10)$$

And for Gaussian Curvature:

$$\mathcal{K} = -\frac{(-b_0^2 + Q^2 + (r_{ph})^2)(2Q^2 + (r_{ph})^2)}{r_{ph}^5} < 0 \quad (11)$$

5 Conclusions and Further Developments

Given the charged traversable wormhole spacetime, we analized its photon rings and their stability. As we can see, the photon sphere radius is unstable, regardless the charge. Also, the photon rings in such spacetimes, are in fact the throat radius of the wormhole, since[3]:

$$1 - b_0^2/r^2 + Q^2/r^2 = 0 \implies r = \pm\sqrt{b_0^2 - Q^2}, \quad (12)$$

Therefore, massless particles can orbit such unstable circular orbits that occurs at the throat of the exotic object.

Referências

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