Dynamical Attractors in a Full Transport Approach

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Outline

- Attractors in uRHICs
- Relativistic Boltzmann Transport Approach
- Boost-invariant systems
- Breaking boost-invariance
- Summary and outlook

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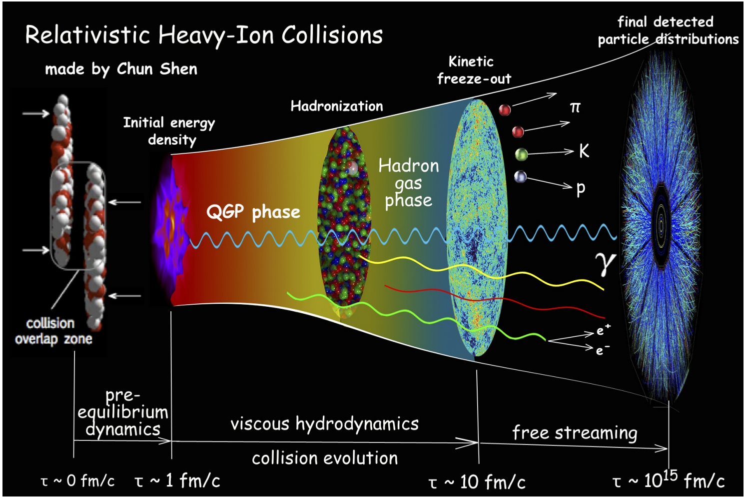
ultra-Relativistic Heavy-Ion Collisions (uRHICs)



QGP characterisation

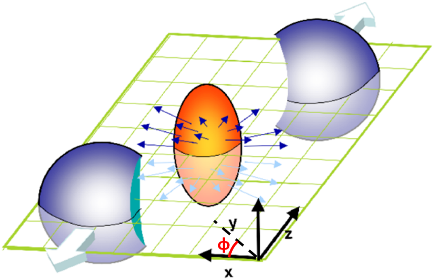
Non central collision $\implies$ eccentricity in coordinate phase $\implies$ azimuthal anisotropy in momentum space

Elliptic flow $v_2 \simeq \left\langle \frac{p_x^2 - p_y^2}{p_x^2 + p_y^2} \right\rangle$

Elliptic flow suggests QGP has small η/s (shear viscosity/entropy density) ratio.

- $\eta/s \to 0 \longrightarrow$ ideal hydrodynamics
- $\eta/s \to \infty \longrightarrow$ free streaming (**no hydro!**)

Hydrodynamics gives a satisfactory picture of nearly-thermalised QGP, with predictions at fixed η/s.



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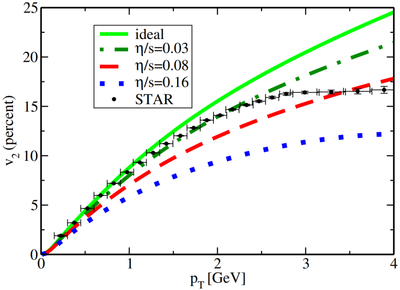

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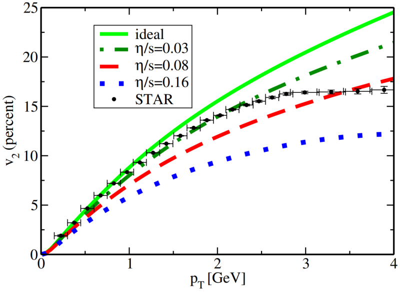

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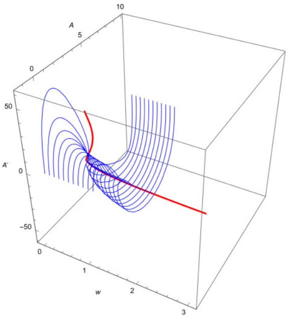
Attractors

What is an attractor?

Subset of the phase space to which all trajectories converge after a certain time.

Why do we look for attractors?

- Uncertainties in initial conditions affect final observables?
  Memory of initial conditions?
- Appearance of attractors and hydrodynamisation. The issue of small systems, as produced by p-p or p-A collisions.



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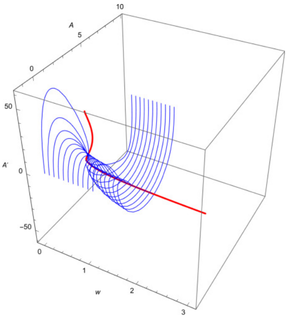

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Initial distribution function

Romatschke-Strickland Distribution Function

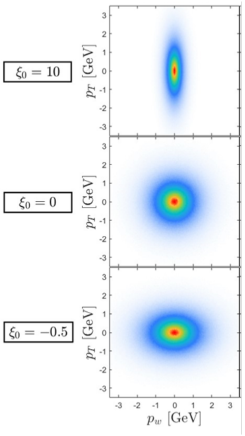
$$f_0(\mathrm{p};\gamma_0,\Lambda_0,\xi_0) = \gamma_0 \exp\left(-\frac{1}{\Lambda_0}\sqrt{p_\perp^2 + p_w^2(1+\xi_0)}\right),$$

where $p_\perp^2 = p_x^2 + p_y^2$ and $p_w = (p \cdot z)$.

- ξ_0 fixes initial pressure anisotropy P_L/P_T.
- Λ_0 and γ_0 fix initial energy density ε and particle density n.
- If $\xi_0 \to 0$ (isotropic distribution), $\Lambda_0 \to T_0$, $\gamma_0 \to \Gamma_0$.

Milne Coordinates (η_s, x, y, τ) : $\eta_s = \mathrm{atanh}(z/t)$, $\tau = \sqrt{t^2 - z^2}$.

$f(p)$ in momentum space



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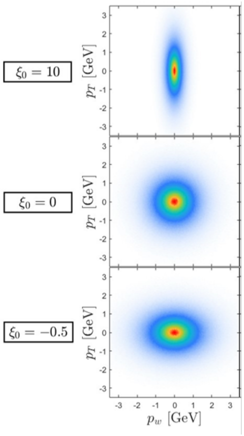

Normalized moments

Moments $M^{nm}[f]$ of the distribution function $f(p)$

$$M^{nm}[f] = \int \frac{d^3\vec{p}}{(2\pi)^3 p^0} (u \cdot p)^n (z \cdot p)^{2m} f(p)$$

They carry information about the $f(p)$ (M. Strickland *JHEP* 12, 128, (2010)) .

All moments $\Longleftrightarrow$ whole $f(p)$

Attractors spotted in the normalized moments

$$\overline{M}^{nm}[f] = \frac{M^{nm}[f]}{M^{nm}[f_{eq}(T_{eff}, \Gamma_{eff})]}$$

$f_{eq} = \Gamma_{eff} \exp(-p^0/T_{eff})$). Matching conditions imply: $M^{10} = n$, $M^{20} = \varepsilon$, $M^{01} = P_L$.
System equilibrates at large $\tau \implies \lim_{\tau\to\infty} \overline{M}^{nm}[f] = 1$.

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Boltzmann Equation

Solve the Relativistic Boltzmann Equation with the full collision integral:

$$p^{\mu}\partial_{\mu}f(x,p) = C\left[f(x,p)\right]_{\mathrm{p}} \, , \tag{1}$$

Only binary elastic $2 \leftrightarrow 2$ collisions:

$$C\left[f\right]_{\mathrm{p}} = \int \frac{\mathrm{d}^3 p_2}{2E_{\mathrm{p}_2}(2\pi)^3} \int \frac{\mathrm{d}^3 p_{1'}}{2E_{\mathrm{p}_{1'}}(2\pi)^3} \int \frac{\mathrm{d}^3 p_{2'}}{2E_{\mathrm{p}_{2'}}(2\pi)^3} \left(f_{1'}f_{2'} - f_1 f_2\right) \times |\mathcal{M}|^2 \, \delta^{(4)}\left(p_1 + p_2 - p_{1'} - p_{2'}\right) \tag{2}$$

$\mathcal{M}$: transition amplitude. $|\mathcal{M}|^2 = 16\pi \, s\,(s - 4m^2)\, d\sigma/dt$.

How to solve the Boltzmann Equation with the **full collision integral** $C[f]$?

Numerical solution with test particle method: simulation of propagating particles which collide with locally fixed cross-section σ_{22}.

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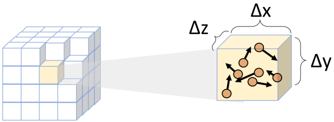
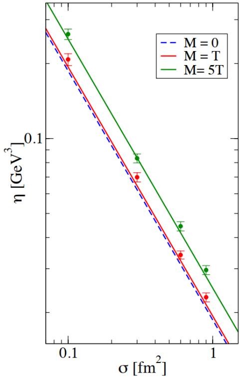
Relativistic Boltzmann Transport (RBT) Code

- C language: high performance (up to $3 \cdot 10^8 N_{\text{particles}}$)
- Stochastic Method to implement collisions (Xu, Greiner, *PRC 71* (2005), Ferini, Colonna, Di Toro, Greco, *PLB 670* (2009))
- Space discretisation: particles in the same cell can collide with probability $P_{22} \propto \sigma_{22}$
- $2 \leftrightarrow 2$ collisions $\Rightarrow$ Particle conservation: Fugacity $\Gamma \neq 1$
- Fix η/s by computing σ_{22} locally via the Chapman-Enskog formula (Plumari, Puglisi, Scardina, Greco, PRC 86 (2012)):

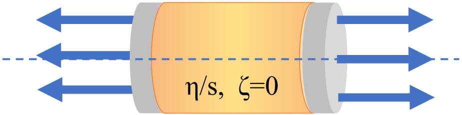
$$\eta = f(m/T)\frac{T}{\sigma_{22}} \overset{m=0}{\simeq} 1.2\frac{T}{\sigma_{22}}$$





Code setup for 1D boost-invariant systems

- Conformal system ($m = 0$)
- One-dimensional system
  Homogeneous distribution and periodic boundary conditions in the transverse plane.
- Boost-invariant system. No dependence on η!
  $dN/d\eta =$ const. in $[-\eta_{s\,max}, \eta_{s\,max}]$, $\eta_{s\,max}$ large enough to avoid propagation of information from boundaries.



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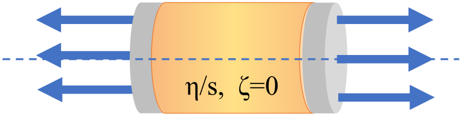

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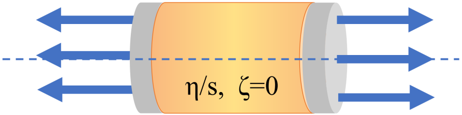

Forward Attractor

- Change initial anisotropy ξ_0 (and thus P_L/P_T).
- Fix anything else.

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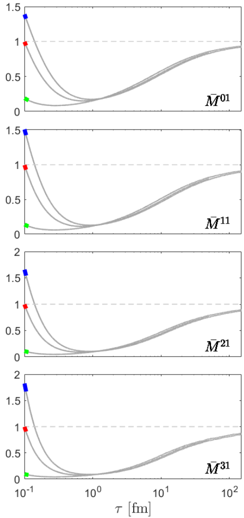
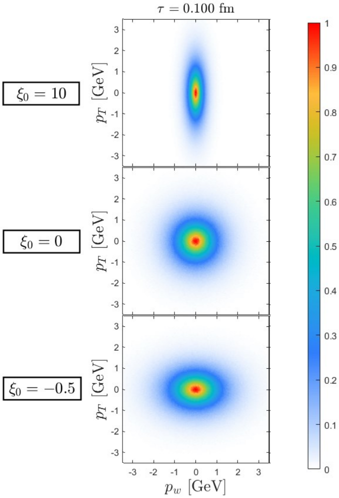
Distribution function evolution: Forward attractor vs τ, $\eta/s = 10/4\pi$.

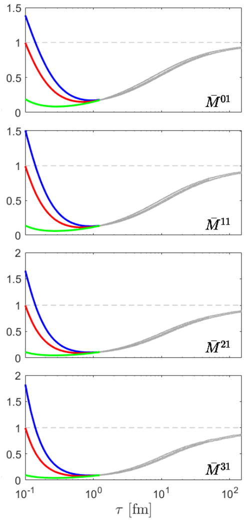
- At $\tau = \tau_0$, three different distributions in momentum space: oblate ($\xi_0 = 10$), spherical ($\xi_0 = 0$) and prolate ($\xi_0 = -0.5$).

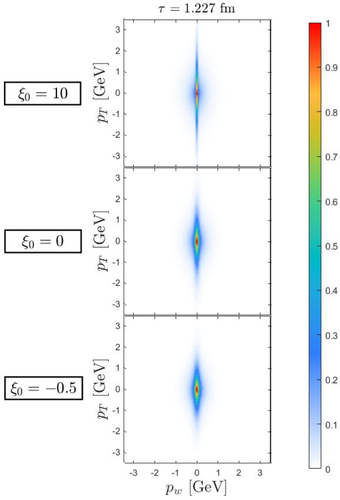




Distribution function evolution: Forward attractor vs τ, $\eta/s = 10/4\pi$.

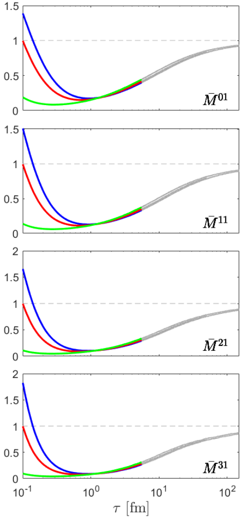
- Already at $\tau \sim 1$ fm, strong initial longitudinal expansion brings the system away from equilibrium
- Distribution functions have similar (but not identical) shape.

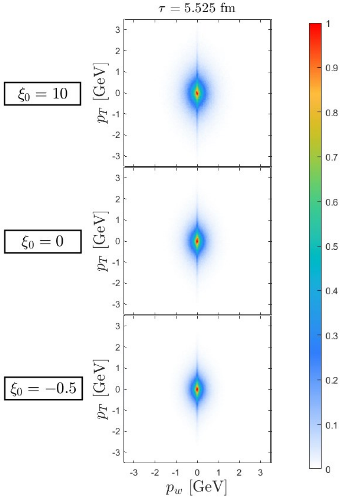




Distribution function evolution: Forward attractor vs τ, $\eta/s = 10/4\pi$.

- At $\tau \sim 5$ fm, clear universal behaviour also for the distribution functions.
- Two components: strongly peaked p_w distribution and a more isotropic one (Strickland, *JHEP* 12, 128)





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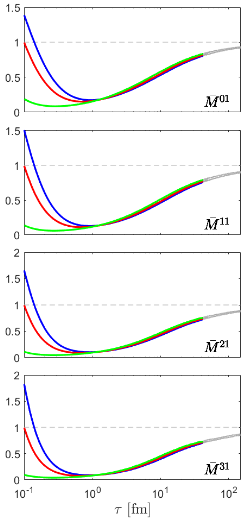
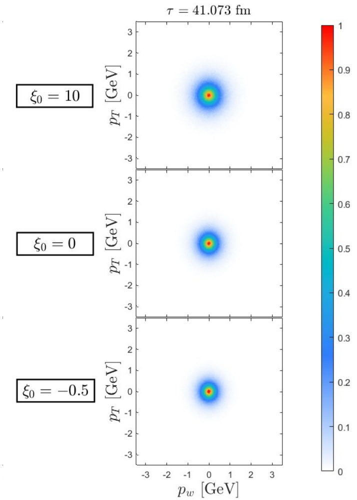
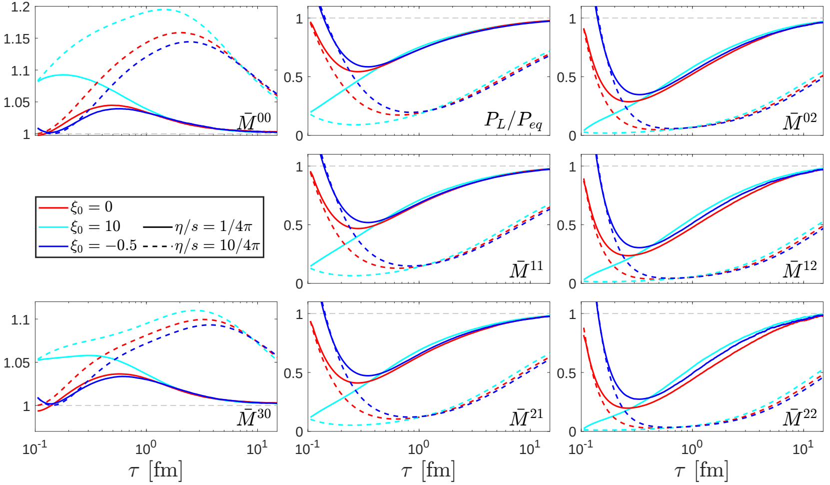
- For large τ the system is almost completely thermalized and isotropized.





Forward Attractor vs τ

Different initial anisotropies $\xi_0 = -0.5,\ 0,\ 10, \infty$, for $\eta/s = 1/4\pi$ and $\eta/s = 10/4\pi$.



- $\eta/s = 1/4\pi$: attractor at $\tau \sim 0.5$ fm
- $\eta/s = 10/4\pi$: attractor at $\tau \sim 1.0$ fm
- Not 10 times larger!
- Less collisions to reach the attractor?
- **Different attractors for different η/s?**

Definition of the relaxation time

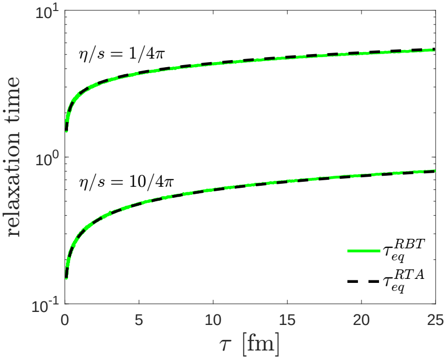
Hydrodynamics show attractors w.r.t scaled time τ/τ_{eq}. $\tau_{eq}^{\mathrm{RTA}} = 5(\eta/s)/T$ enters in their equations as the relaxation time (Denicol *et al.PRD* 83, 074019) .
In our approach no need for a τ_{eq}!

Natural time scale: **mean collision time per particle**

$$\tau_{coll} = \frac{1}{2}\left(\frac{1}{N_{\mathrm{part}}}\frac{\Delta N_{\mathrm{coll}}}{\Delta t}\right)^{-1}$$

It can be shown that

$$\tau_{eq}^{\mathrm{RTA}} = \tau_{tr} = \frac{3}{2}\tau_{coll} \equiv \tau_{eq}^{RBT}$$



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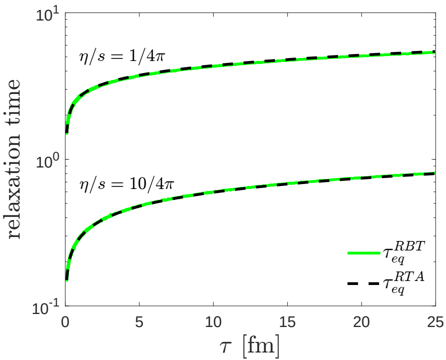

Forward Attractor vs τ/τ_{eq}

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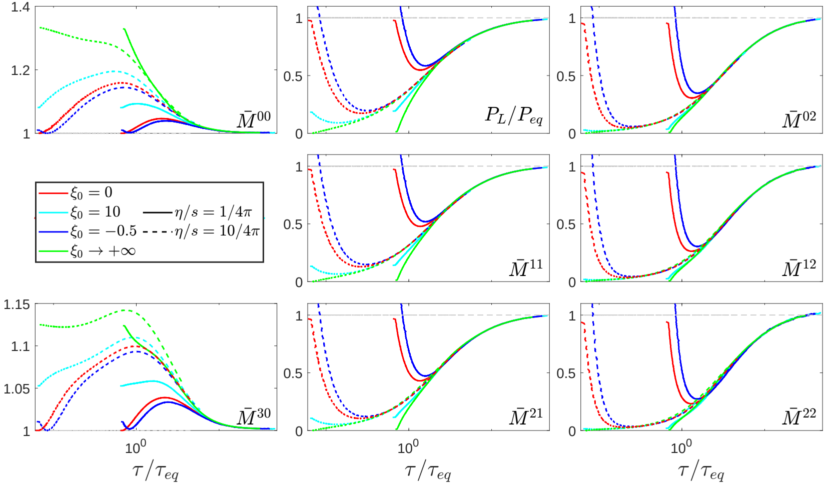

- $\eta/s = 1$: attractor at $\tau \sim 1.5\,\tau_{eq}$
- $\eta/s = 10$: attractor at $\tau \sim 0.2\,\tau_{eq}$
- Less collisions per particle to reach the attractor?
- **Unique attractor!**

Pull-back Attractor

- Fix the initial anisotropy ξ_0.
- Change initial scaled time $\tau_0 T_0/(\eta/s)$. (If ratio fixed, same curve!).

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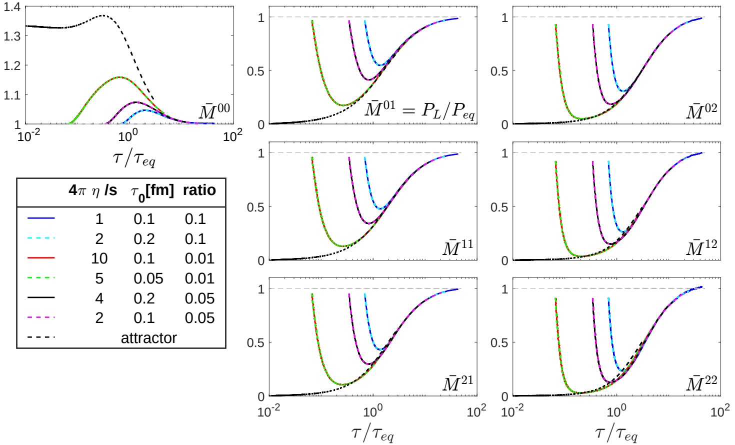
Fix ξ_0, change η/s and τ_0: three values for the ratio $\tau_0/(4\pi\eta/s)$: 0.1, 0.01, 0.05 fm.



	$4\pi\ \eta$ /s	τ_0[fm]	ratio
blue solid	1	0.1	0.1
cyan dashed	2	0.2	0.1
red solid	10	0.1	0.01
green dashed	5	0.05	0.01
black solid	4	0.2	0.05
magenta dashed	2	0.1	0.05
black dashed	attractor		

- Curves depend only on $\tau_0/(\eta/s)$ ratio
- Equilibration achieved at same τ/τ_{eq}
- Attractor reached at different τ/τ_{eq}
- Initial $\sim$ free streaming

Who is *the* attractor?

All curves scale to a universal behaviour. Which is the curve they converge to?

- Viscous (vHydro) and Anisotropic (aHydro) Hydrodynamics: analytical solution (M. Strickland *et al.PRD*, 97, 036020 (2018)) ;
- Relaxation Time Approximation (RTA) Boltzmann Equation (P. Romatschke *PRL* 120, 012301 (2018)) : $\tau_0 \ll 1$ and $\xi_0 \to \infty$ (in accordance with aHydro).

Infinitely oblate distribution $\xi_0 \to \infty$, initial scaled time $\tau_0 T_0/(\eta/s) \to 0$.

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- Relaxation Time Approximation (RTA) Boltzmann Equation (P. Romatschke *PRL* 120, 012301 (2018)) : $\tau_0 \ll 1$ and $\xi_0 \to \infty$ (in accordance with aHydro).

Infinitely oblate distribution $\xi_0 \to \infty$, initial scaled time $\tau_0 T_0/(\eta/s) \to 0$.

Is it the RBT attractor, too? It is.

The system initially is dominated by strong longitudinal expansion.

Who is *the* attractor?

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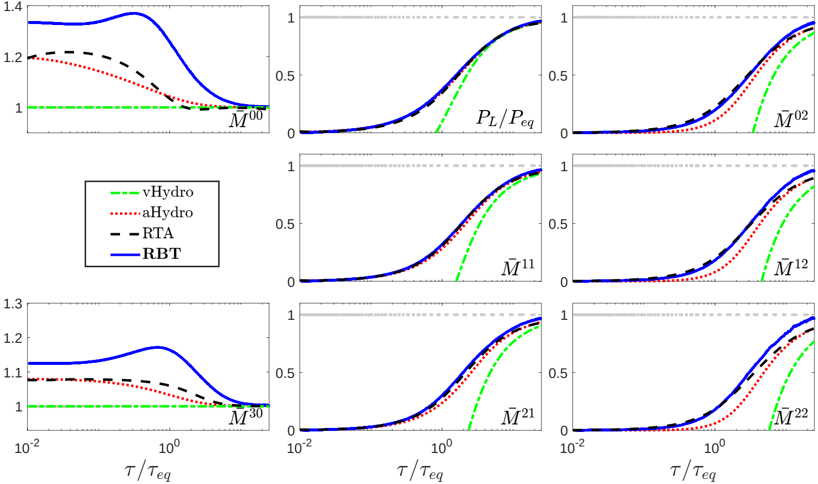
Attractors in different models



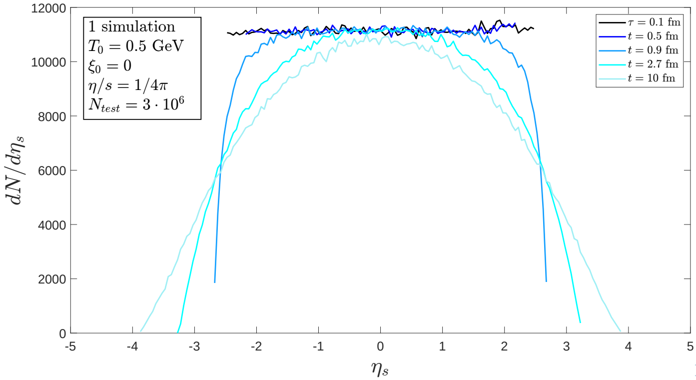
- $\overline{M}^{nm}$, $m > 0$: very good agreement
- Higher order moments $\rightarrow$ stronger departure between models
- **RBT** thermalizes earlier
- No agreement for M^{n0}

Are attractors due to boost-invariance?

Finite distribution in η

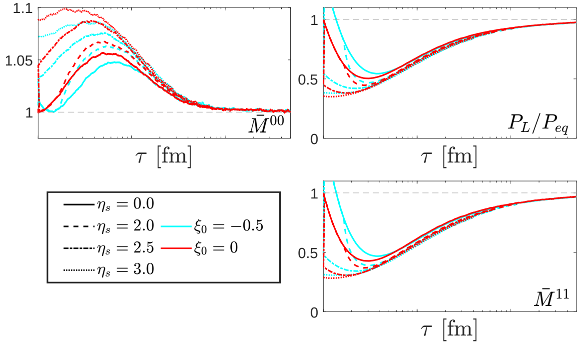
Breaking boost-invariance: $$\frac{dN}{d\eta_s}(\eta_s;\tau_0) = \begin{cases} \text{const.} & |\eta_s| < 2.5 \\ 0 & \text{elsewhere} \end{cases}$$

- Tails of the distribution function at $|\eta_s| > 1$
- Discontinuity in initial distribution $\rightarrow$ non-analyticity points in moments' evolution

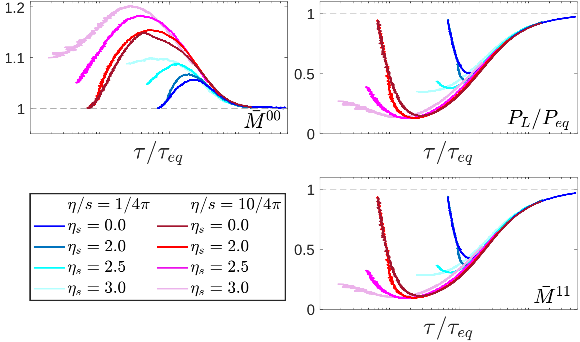


Attractors at finite rapidity

Forward attractor. Fixed $\eta/s = 1/4\pi$.



Pull-back attractor. Fixed $\xi_0 = 0$.



Universal behaviour even at $\eta_s = 3$, outside the initial distribution range!

Are attractors due to boost-invariance?

No.

Are attractors due to boost-invariance?

No.

Summary

- Attractors appear in the conformal boost-invariant case in the normalized moments of the distribution function and in the distribution function itself.
- RTA and aHydro attractors converge to the full Boltzmann ones: the larger the moments' order, the later the convergence.
- Non boost-invariant systems still show universal behaviour, also at quite large η_s.

Outlook

- Non-conformal simulation in progress
- Full 3+1D simulation in progress
- Realistic initial conditions
- Attractors in collective flows

Thank you for your attention.

LRF and matching conditions

Define the Landau Local Rest Frame (LRF) via the fluid four-velocity:

$$T^{\mu\nu} u_\nu = \varepsilon u^\mu, \\ n = n^\mu u_\mu$$

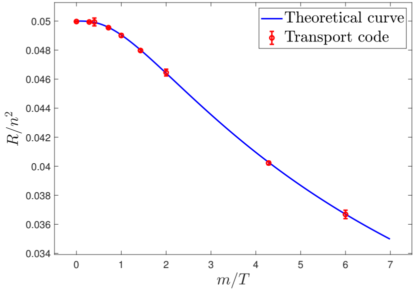
ε and n are the energy and particles density in the LRF.
Fluid is not in equilibrium $\implies$ define locally effective T and Γ via Landau matching conditions:

$$T = \frac{\varepsilon}{3\,n}, \qquad \Gamma = \frac{n}{d\,T^3/\pi^2},$$

d is the # of dofs, fixed $d = 1$.

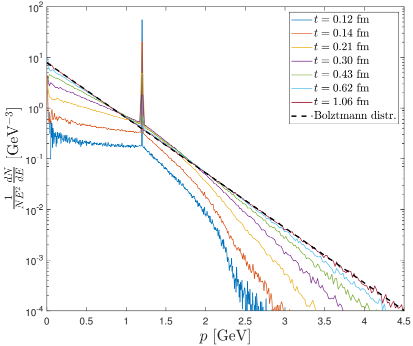
Transport code: consistency checks

Collision Rate



Expected and computed collision rate in unit of n^2 as a function of $z = m/T$. Theoretical value $R = \frac{1}{2}n^2\langle\sigma v\rangle$.

Thermalisation



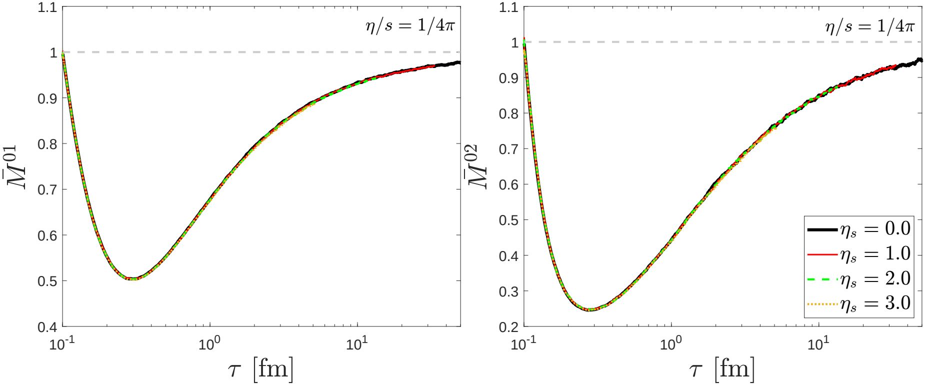
Particles initialised with momentum modulus $p = 1.2$ GeV. Within $t \sim 0.6$ fm the system thermalises; equilibrium temperature $T = 0.4$ GeV.

Code setup

- Cell: $\Delta x = \Delta y = 0.4$ fm, $\Delta\eta_s = 0.08$. Results taken in one-cell-thick slices in η_s.
- Test particles: from 10^7 up to $3 \cdot 10^8$.
- Time discretization: to avoid causality violation ($\sim 10^3$ time steps).
- Performance: 1 core-hour per 10^6 total particles in $2 \cdot 10^3$ time steps.
- Initial conditions: $T_0 = 0.5$ GeV, $\Gamma_0 = 1$, $\xi_0 = -0.5,\ 0,\ 10,\ +\infty$

Testing boost-invariance

Compute normalized moments at different η_s's within an interval $\Delta\eta_s = 0.04$.



No dependence on η! We look for them at midrapidity: $\eta \in [-0.02, 0.02]$

Boltzmann RTA Equation for number-conserving systems

Boltzmann equation in Relaxation Time Approximation (RTA) (Strickland, Tantary, JHEP10(2019) 069)

$$p^\mu \partial_\mu f_p = -\frac{p \cdot u}{\tau_{eq}}(f_{eq} - f_p).$$

Exactly solvable, by fixing number and energy conservation.
Two coupled integral equations for $\Gamma_{eff} \equiv \Gamma$ and $T_{eff} \equiv T$:

$$\Gamma(\tau) T^4(\tau) = D(\tau, \tau_0)\Gamma_0 T_0^4 \frac{\mathcal{H}(\alpha_0 \tau_0/\tau)}{\mathcal{H}(\alpha_0)} + \int_{\tau_0}^{\tau} \frac{d\tau'}{2\tau_{eq}(\tau')} D(\tau, \tau')\Gamma(\tau') T^4(\tau')\mathcal{H}\left(\frac{\tau'}{\tau}\right),$$

$$\Gamma(\tau) T^3(\tau) = \frac{1}{\tau}\left[D(\tau, \tau_0)\Gamma_0 T_0^3 \tau_0 + \int_{\tau_0}^{\tau} \frac{d\tau'}{\tau_{eq}(\tau')} D(\tau, \tau')\Gamma(\tau') T^3(\tau')\tau'\right].$$

Here $\alpha = (1 + \xi)^{-1/2}$. System solvable by iteration.

vHydro equations

Second-order dissipative viscous hydrodynamics equations according to DNMR derivation, starting from kinetic theory (G. S. Denicol *et al.*, *PRL*105, 162501 (2010)) :

$$\partial_\tau \varepsilon = -\frac{1}{\tau}(\varepsilon + P - \pi),$$
$$\partial_\tau \pi = -\frac{\pi}{\tau_\pi} + \frac{4}{3}\frac{\eta}{\tau_\pi \tau} - \beta_\pi \frac{\pi}{\tau},$$

where $\tau_\pi = 5(\eta/s)/T$ and $\beta_\pi = 124/63$.
Solved with a Runge-Kutta-4 algorithm.

aHydro for number-conserving systems

Formulation of dissipative anisotropic hydrodynamics with number-conserving kernel (Almaalol, Alqahtani, Strickland, PRC 99, 2019).
System of three coupled ODEs:

$$\partial_\tau \log\gamma + 3\partial_\tau \log\Lambda - \frac{1}{2}\frac{\partial_\tau \xi}{1+\xi} + \frac{1}{\tau} = 0;$$

$$\partial_\tau \log\gamma + 4\partial_\tau \log\Lambda + \frac{\mathcal{R}'(\xi)}{\mathcal{R}(\xi)}\partial_\tau \xi = \frac{1}{\tau}\left[\frac{1}{\xi(1+\xi)\mathcal{R}(\xi)} - \frac{1}{\xi} - 1\right];$$

$$\partial_\tau \xi - \frac{2(1+\xi)}{\tau} + \frac{\xi(1+\xi)^2\mathcal{R}^2(\xi)}{\tau_{eq}} = 0.$$

Solved with a Runge-Kutta-4 algorithm.

Auxiliary functions

$$D(\tau_2, \tau_1) = \exp\left[-\int_{\tau_1}^{\tau_2} \frac{d\tau}{\tau_{eq}\tau}\right];$$

$$\mathcal{H}^{nm}(y) = \frac{2y^{2m+1}}{2m+1} {}_2F_1\left(\frac{1}{2}+m, \frac{1-n}{2}; \frac{3}{2}+m; 1-y^2\right).$$

Computation of moments in other models

- RTA:

$$M^{nm}(\tau) = \frac{(n+2m+1)!}{(2\pi)^2}\left[D(\tau,\tau_0)\alpha_0^{n+2m-2}T_0^{n+2m+2}\Gamma_0\frac{\mathcal{H}^{nm}(\alpha\tau_0/\tau)}{[\mathcal{H}^{20}(\alpha_0)/2]^{n+2m-1}}+\right.$$
$$\left.+\int_{\tau_0}^{\tau}\frac{d\tau'}{\tau_{eq}(\tau')}D(\tau',\tau')\Gamma(\tau')T^{n+2m+2}(\tau')\mathcal{H}^{nm}\left(\frac{\tau'}{\tau}\right)\right];$$

- DNMR:

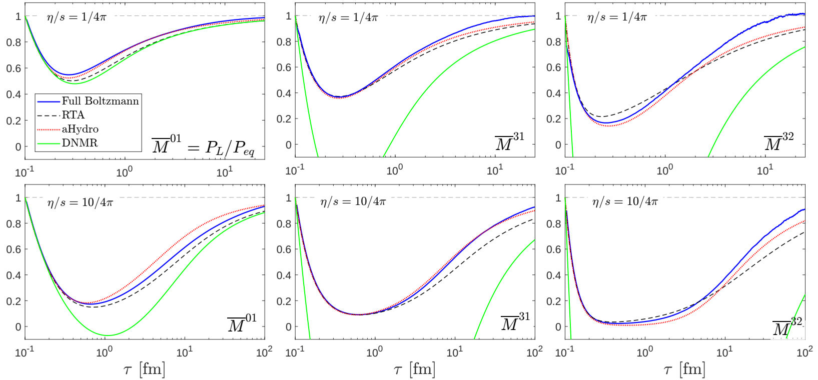
$$\overline{M}^{nm}_{\mathrm{DNMR}} = 1 - \frac{3m(n+2m+2)(n+2m+3)}{4(2m+3)}\frac{\pi}{\varepsilon};$$

- aHydro:

$$\overline{M}^{nm}_{\mathrm{aHydro}}(\tau) = (2m+1)(2\alpha)^{n+2m-2}\frac{\mathcal{H}^{nm}(\alpha)}{[\mathcal{H}^{20}(\alpha)]^{n+2m-1}};$$

Comparison with other models

Compute normalized moments with DNMR, anisotropic hydrodynamics (aHydro) and Relaxation Time Approximation (RTA) Boltzmann Equation.

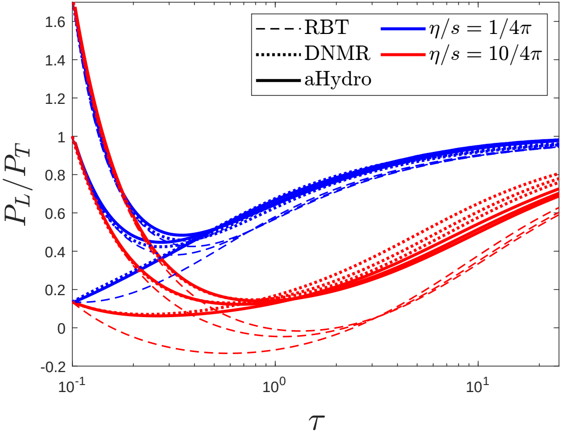


- Better agreement with RTA and aHydro for lower order moments
- Better agreement with DNMR for lower η/s

(V. Ambrus *et al.*, PRD 104.9 (2021))

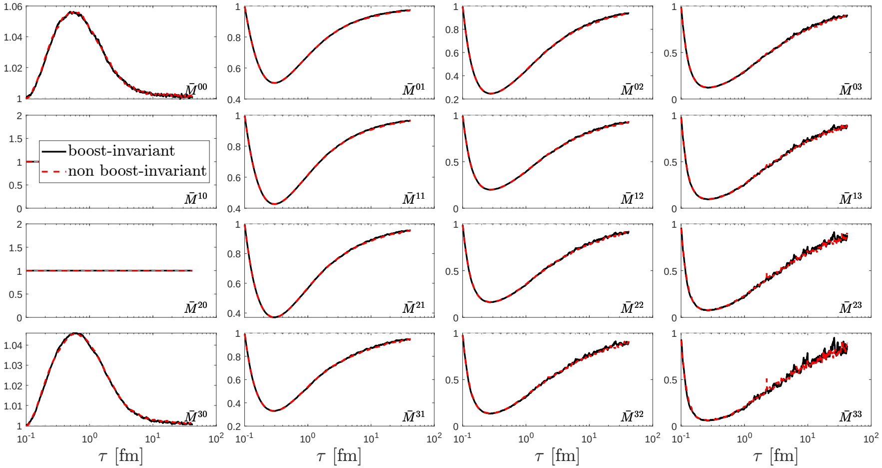
Pressure anisotropy in different frameworks

For $\eta/s = 1/4\pi$ and $\eta/s = 10/4\pi$, compute P_L/P_T from three different initial anisotropies: $\xi_0 = -0.5,\ 0,\ 10$.



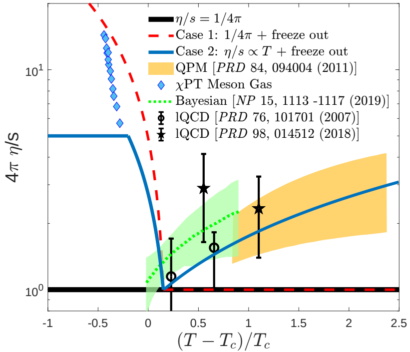
- RTA (not showed) really similar to aHydro
- aHydro attractor reached $\sim$ time than RBT
- vHydro attractor reached at later time, especially for larger η/s

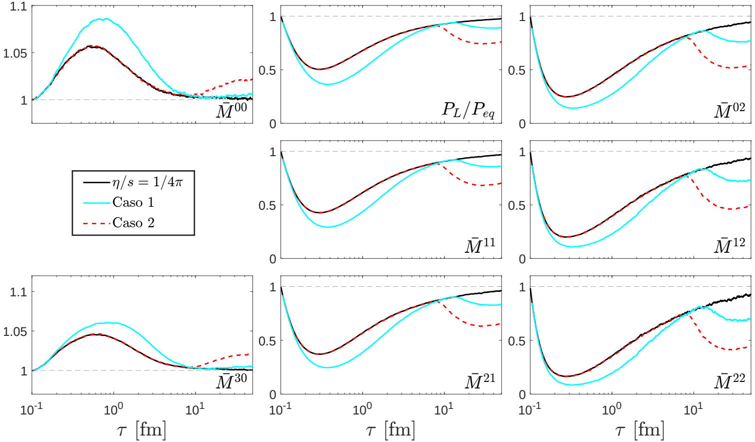
Midrapidity



At midrapidity no difference w.r.t. the boost invariant case.

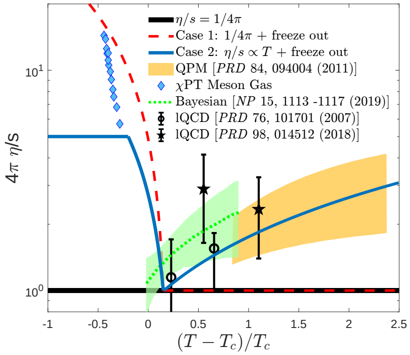
T-dependent η/s: Plot with respect to τ

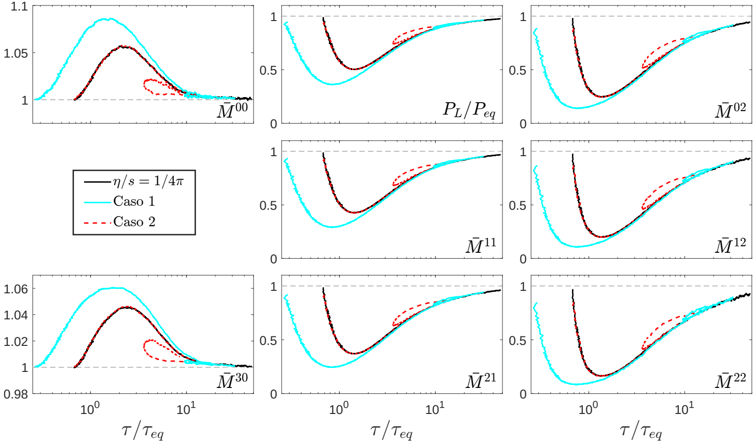




Universal behaviour lost at different τ (depend on local T)

T-dependent η/s: Plot with respect to τ/τ_{eq}





Universal behaviour restored after 'loops'.

Non-monotonic τ/τ_{eq} for Case 1

Loops when τ/τ_{eq} is no more a monotonic function: $\tau_{eq} \propto \eta/s(T)/T$ grows faster than τ.