Parity-odd terms in the trace anomaly of Weyl fermions: elusive or delusive?

S. Franchino-Viñas

Genova, Giugno 2023

COST Action CA18108:

Quantum gravity phenomenology in the multi-messenger approach

https://qg-mm.unizar.es/

Quantum gravity phenomenology at the dawn of the multi-messenger era—A review, Prog.Part.Nucl.Phys. 125 (2022) 103948, arXiv:2111.05659

Is there an imaginary term in the trace anomaly of Weyl fermions id $D = 4$?

Outline

Consider a classical field theory. If the system under study possesses a continuous symmetry, then from Noether's theorem (in a classical, i.e. non-quantum theory)

$$\exists j^\mu, \partial_\mu j^\mu \overset{*}{=} 0. \tag{1}$$

If the system is quantized, $\partial_\mu \langle \hat{j}^\mu \rangle \overset{?}{=} 0$?

If $\neq 0$, the symmetry is said to be anomalous.

Why anomalies?

- Pion decay (Bell and Jackiw): $\pi \to \gamma\gamma$ (anomaly in J^μ_A).
- Applications in condensed matter/hydrodynamics (review Chernodub et al., 2021, experiment Gooth et al., 2017): $\overrightarrow{J}_\epsilon = \left(a_{F\tilde{F}} + a_{R\tilde{R}} T^2\right) \overrightarrow{B}$ (anomaly in J^μ_A).
- Kuzmin, Rubakov and Shaposhnikov (1985); still new proposals as QCD baryogenesis [1911.01432v2 [hep-ph]].
- Bouncing universes (instead of Big Bang) from anomalies, Fabris, Pelinson, Shapiro [gr-qc/9810032], Asorey et al. [2202.00154], Camargo and Shapiro [2206.02839] (trace anomaly).
- Axionic dark matter (Basilakos, Mavromatos, Solà [gr-qc/2001.03465]) (anomaly for a Kalb–Ramond field).
- BRST formalism and cohomological aspects.

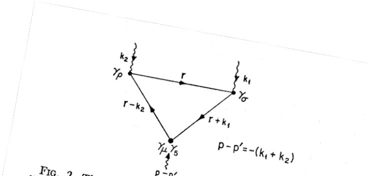
Anomalies

Consider a massless fermion, coupled to an *external* EM field A_μ

$$\mathcal{L} = i\bar{\psi}\gamma^\mu\partial_\mu\psi + eJ_V^\mu A_\mu, \quad J_V^\mu = \bar{\psi}\gamma^\mu\psi, \quad J_A^\mu = \bar{\psi}\gamma^\mu\gamma_5\psi \tag{2}$$

The system has two global symmetries: $\psi \to e^{i\alpha}\psi$, $e^{i\alpha\gamma_5}$, so that classically (on-shell) $\partial_\mu J_{A,V}^\mu = 0$.

$$\begin{aligned}\langle\partial_\mu J_A^\mu(x)\rangle &\sim \int \mathcal{D}\bar{\psi}\mathcal{D}\psi\partial_\mu J_A^\mu e^{i\int i\bar{\psi}\not\partial\psi + eJ_V^\mu A_\mu} \\ &= -\frac{e^2}{2}\partial_{x^\mu}\iint \mathrm{d}y\mathrm{d}z\langle J_A^\mu(x)J_V^\nu(y)J_V^\sigma(z)\rangle A_\nu(z)A_\sigma(y) + \cdots \\ &= -\frac{e^2}{16\pi^2}\epsilon^{\alpha\beta\sigma\lambda}F_{\alpha\beta}F_{\sigma\lambda}.\end{aligned} \tag{3}$$



STEPHEN L. ADI

term is ... appears

$$r\left\{\sum_{k=1}^{2n}\prod_{j=1}^{k-1}\left[\gamma^{(j)}\frac{1}{r+p_j-m_0}\right]\gamma^{(k)}\frac{1}{r+p_k-m_0}\gamma_\mu\gamma_5 \frac{1}{r+p_k+p'-p-m_0}\prod_{j=k+1}^{2n}\left[\gamma^{(j)}\frac{1}{r+p_j+p'-p-m_0}\right]\right\}\times(\cdots). \quad (12)$$

the in diverg linea oper Eq. T the of in

plying by $(p-p')^\mu$ and using Eq. (9) gives

$$\mathrm{Tr}\left\{\sum_{k=1}^{2n}\prod_{j=1}^{k-1}\left[\gamma^{(j)}\frac{1}{r+p_j-m_0}\right]\gamma^{(k)}\frac{1}{r+p_k-m_0}2m_0\gamma_5\frac{1}{r+p_k+p'-p-m_0}\prod_{j=k+1}^{2n}\left[\gamma^{(j)}\frac{1}{r+p_j+p'-p-m_0}\right]\right\}$$

$$(\cdots)+\int d^4r\,\mathrm{tr}\left\{\gamma_5\prod_{j=1}^{2n}\left[\gamma^{(j)}\frac{1}{r+p_j-m_0}\right] -\gamma_5\prod_{j=1}^{2n}\left[\gamma^{(j)}\frac{1}{r+p_j+p'-p-m_0}\right]\right\}(\cdots). \quad (13)$$

first term in Eq. (13) is the type-(b) contrib Λ^5 corresponding to Eq. (12), while makin nge of variable $r\to r+p'-p$ in the integra second term causes the second and third te ncel. This gives, when we sum over all t ntributions,

$$(p-p')^\mu\Lambda_\mu^{5(b)}(p,p')=2m_0\Lambda^{5(b)}(p,p').$$

he Ward identity of Eq. (7) is finally obt lding Eqs. (11) and (14).

Clearly, the only step of the above deriva not simply an algebraic rearrangement is f *integration variable* in the second term o This will be a valid operation provided that s at worst superficially logarithmically condition that is satisfied by loops with f photons, that is, loops with $n\geq 2$. Howev loop is a triangle graph with only two phot (See Fig. 2) we have $n=1$, and the integr

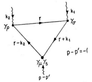


FIG. 2. The axial-vector triangle grap diagram, with the photon four-momenta a interchanged, which makes a contributio diagram pictured.

PHYSICAL REVIEW VOLUME 177, NUMBER 5 25 JANUARY 19

Axial-Vector Vertex in Spinor Electrodynamics

STEPHEN L. ADLER
Institute for Advanced Study, Princeton, New Jersey 08540

(Received 24 September 1968)

Working within the framework of perturbation theory, we show that the axial-vector vertex in spinor electrodynamics has anomalous properties which disagree with those found by the formal manipulation of field equations. Specifically, because of the presence of closed-loop "triangle diagrams," the divergence of the axial-vector current is not the usual expression calculated from the field equations, and the axial-vector current does not satisfy the usual Ward identity. One consequence is that, even after the external-line wave-function renormalizations are made, the axial-vector vertex is still divergent in fourth- (and higher-) order perturbation theory. A corollary is that the radiative corrections to ν_l elastic scattering in the local current-current theory diverge in fourth (and higher) order. A second consequence is that, in massless electrodynamics, despite the fact that the theory is invariant under γ_5 transformations, the axial-vector current is not conserved. In an Appendix we demonstrate the uniqueness of the triangle diagrams, and discuss a possible connection between our results and the $\pi^0\to 2\gamma$ and $\eta\to 2\gamma$ decays. In particular, we argue that as a result of triangle diagrams, the equations expressing partial conservation of axial-vector current (PCAC) for the neutral members of the axial-vector-current octet must be modified in a well-defined manner, which completely alters the PCAC predictions for the π^0 and the η two-photon decays.

INTRODUCTION

THE axial-vector vertex in spinor electrodynamics is of interest because of its connections (i) with radiative corrections to ν_l scattering and (ii) with the γ_5 invariance of massless electrodynamics. We will show in this paper, within the framework of perturbation theory, that the axial-vector vertex has anomalous properties which disagree with those found by the formal manipulation of field equations. In particular, because of the presence of closed-loop "triangle diagrams," the divergence of the axial-vector current is not the usual expression calculated from the field equations, and the axial-vector current does not satisfy the usual W...

well-defined manner, which completely alters the PCAC predictions for the π^0 and the η two-photon decays.

I. AXIAL CURRENT DIVERGENCE AND WARD IDENTITY

We work in the usual spinor electrodynamics, described by the Lagrangian density[1]

$$\mathcal{L}(x)=\bar\psi(x)(i\gamma\cdot\Box-m_0)\psi(x)-\tfrac14 F_{\mu\nu}(x)F^{\mu\nu}(x) -:e_0\bar\psi(x)\gamma_\mu\psi(x)A^\mu(x):, \quad (1)$$

$\partial A_\mu(x) \quad \partial A_\nu(x)$

easily be calculated to be

$$\frac{\partial}{\partial x_\mu}j_\mu^5(x)=2im_0j^5(x). \quad (4)$$

From Eqs. (3) and (4), we obtain the usual axial-vector Ward identity

$$(p-p')^\mu\Gamma_\mu^5(p,p')=2m_0\Gamma^5(p,p') +S_F'(p)^{-1}\gamma_5+\gamma_5S_F'(p')^{-1}. \quad (5)$$

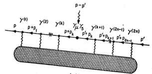
Our task in this section is to see whether Eqs. (4) and (5), which we have formally derived from the field equations, actually hold in perturbation theory.

To this end, let us rederive Eq. (5) in perturbation theory. It is convenient to write

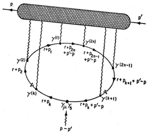
$$\Gamma_\mu^5=\gamma_\mu\gamma_5+\Lambda_\mu^5,\quad \Gamma^5=\gamma_5+\Lambda^5,\quad S_F'(p)^{-1}=p-m_0-\Sigma(p), \quad (6)$$

where the vertex corrections Λ_μ^5 and Λ^5 and the proper self-energy part $\Sigma(p)$ are calculated using $(p-m_0)^{-1}$ as the free propagator. (Use of the bare mass $m_0=m-\delta m$ in the free propagator automatically includes the mass-renormalization counter terms.) In terms of Λ_μ^5, Λ^5, and Σ, Eq. (5) becomes

$$(p-p')^\mu\Lambda_\mu^5(p,p')=2m_0\Lambda^5(p,p')-\Sigma(p)\gamma_5-\gamma_5\Sigma(p') \quad (7)$$



(a)



(b)

FIG. 1. Diagrams contributing to the axial-vector vertex. (a) The axial-vector vertex is attached to ... ning with ...

IL NUOVO CIMENTO VOL. LX A

A PCAC Puzzle: $\pi^0\to\gamma\gamma$ in the σ-Model.

J. S. BELL
CERN - Geneva

R. JACKIW (*)
CERN - Geneva

Jefferson Laboratory of Physics, Harvard University - Cambridge, Mass.

(ricevuto l'11 Settembre 1968)

Summary. — The effective coupling constant for $\pi^0\to\gamma\gamma$ should vanish for zero pion mass in theories with PCAC and gauge invariance. It does not so vanish in an explicit perturbation calculation in the σ-model. The resolution of the puzzle is effected by a modification of Pauli-Villars-Gupta regularization which respects both PCAC and gauge invariance.

1. – Introducti

The invar ization vecto

(1.1)

where p a their mas intereste above g vation. $= T^{\nu\mu}($

Since we are working to lowest order in electromagnetism F_1-F_3 possess a dynamical singularity at $k^2=0$, and therefore we recover th $T(k^2)=Ok^2$.

3. – Perturbation-theory argument.

The formal reasoning of the previous Section should be verifiable by plicit perturbation canculation in a model with PCAC and gauge invari We consider the σ-model interacting with the electromagnetic field a_μ, o ting the charged pion and the neutron fields, which are not necessary for problem. The Lagrange density is (8)

$$(3.1)\qquad \mathscr{L}=-\frac12\partial_\nu a_\mu\partial^\nu a^\mu+\bar\psi[i(\gamma\cdot\partial)+e(\gamma\cdot a)-m+g(\sigma+\varphi\gamma_5)]\psi+ +\frac12(\partial\varphi)^2+\frac12(\partial\sigma)^2-\frac12\mu^2\varphi^2-\frac12\left(\mu^2+\frac{2\lambda}{f}\right)\sigma^2-\lambda[(\sigma^2\ldots$$

Here $f=g/\ldots$

PHYSICAL REVIEW VOLUME 177, NUMBER 5 25 JANUARY 19

Axial-Vector Vertex in Spinor Electrodynamics

STEPHEN L. ADLER
Institute for Advanced Study, Princeton, New Jersey 08540
(Received 24 September 1968)

Working within the framework of perturbation theory, we show that the axial-vector vertex in spinor electrodynamics has anomalous properties which disagree with those found by the formal manipulation of field equations. Specifically, because of the presence of closed-loop "triangle diagrams," the divergence of the axial-vector current is not the usual expression calculated from the field equations, and the axial-vector current does not satisfy the usual Ward identity. One consequence is that, even after the external-line wave-function renormalizations are made, the axial-vector vertex is still divergent in fourth- (and higher-) order perturbation theory. A corollary is that the radiative corrections to $\nu_l l$ elastic scattering in the local current-current theory diverge in fourth (and higher) order. A second consequence is that, in massless electrodynamics, despite the fact that the theory is invariant under γ_5 transformations, the axial-vector current is not conserved. In an Appendix we demonstrate the uniqueness of the triangle diagrams, and discuss a possible connection between our results and the $\pi^0\to 2\gamma$ and $\eta\to 2\gamma$ decays. In particular, we argue that as a result of triangle diagrams, the equations expressing partial conservation of axial-vector current (PCAC) for the neutral members of the axial-vector-current octet must be modified in a well-defined manner, which completely alters the PCAC predictions for the π^0 and the η two-photon decays.

INTRODUCTION

The axial-vector vertex in spinor electrodynamics is of interest because of its connections (i) with radiative corrections to $\nu_l l$ scattering and (ii) with the γ_5 invariance of massless electrodynamics. We will show in this paper, within the framework of perturbation theory, that the axial-vector vertex has anomalous properties which disagree with those found by the formal manipulation of field equations. In particular, because of the presence of closed-loop "triangle diagrams," the divergence of the axial-vector current is not the usual expression calculated from the field equations, and the axial-vector current does not satisfy the usual W... well-defined manner, which completely alters the PCAC predictions for the π^0 and the η two-photon decays.

I. AXIAL CURRENT DIVERGENCE AND WARD IDENTITY

We work in the usual spinor electrodynamics, described by the Lagrangian density[1]

$$\mathcal{L}(x)=\bar\psi(x)(i\gamma\cdot\square-m_0)\psi(x)-\tfrac14 F_{\mu\nu}(x)F^{\mu\nu}(x)$$
$$-:e_0\bar\psi(x)\gamma_\mu\psi(x)A^\mu(x):,\quad(1)$$

...

...term is ... appears

$$\ldots\left\{\sum_{k=1}^{2n}\prod_{j=1}^{k-1}\left[\gamma^{(j)}\frac{1}{r+p_j-m_0}\right]\gamma^{(k)}\frac{1}{r+p_k-m_0}\gamma_\mu\gamma_5\ \frac{1}{\ldots-p_k+p'-p-m_0}\prod_{j=k+1}^{2n}\left[\gamma^{(j)}\frac{1}{r+p_j+p'-p-m_0}\right]\right\}\times(\cdots).\quad(12)$$

...plying by $(p-p')^\mu$ and using Eq. (9) gives

$$\mathrm{Tr}\left\{\sum_{k=1}^{2n}\prod_{j=1}^{k-1}\left[\gamma^{(j)}\frac{1}{r+p_j-m_0}\right]\gamma^{(k)}\frac{1}{r+p_k-m_0}2m_0\gamma_5\ \frac{1}{r+p_k+p'-p-m_0}\prod_{j=k+1}^{2n}\left[\gamma^{(j)}\frac{1}{r+p_j+p'-p-m_0}\right]\right\}$$
$$(\cdots)+\int d^4r\ \mathrm{tr}\left\{\gamma_5\prod_{j=1}^{2n}\left[\gamma^{(j)}\frac{1}{r+p_j-m_0}\right]-\gamma_5\prod_{j=1}^{2n}\left[\gamma^{(j)}\frac{1}{r+p_j+p'-p-m_0}\right]\right\}(\cdots).\quad(13)$$

...first term in Eq. (13) is the type-(b) contri... Λ^5 corresponding to Eq. (12), while maki... ...nge of variable $r\to r+p'-p$ in the integra... second term causes the second and third t... ...cel. This gives, when we sum over all t... ...ntributions,

$$(p-p')^\mu\Lambda_\mu^{5(b)}(p,p')=2m_0\Lambda^{5(b)}(p,p').$$

...he Ward identity of Eq. (7) is finally ob... ...ding Eqs. (11) and (14).

Clearly, the only step of the above deriva... not simply an algebraic rearrangement is ...f integration variable in the second term o... This will be a valid operation provided that ...s at worst superficially logarithmically ... condition that is satisfied by loops with f... photons, that is, loops with $n\geq 2$. Howev... loop is a triangle graph with only two phot... (See Fig. 2) we have $n=1$, and the integr...

the in... diverg... linear... oper... Eq. ... T... the ... of ... in...

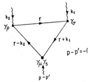


FIG. 2. The axial-vector triangle grap... diagram, with the photon four-momenta a... interchanged, which makes a contributio... diagram pictured.

easily be calculated to be

$$\frac{\partial}{\partial x_\mu}j_\mu^5(x)=2im_0j^5(x).\quad(4)$$

From Eqs. (3) and (4), we obtain the usual axial-vector Ward identity

$$(p-p')^\mu\Gamma_\mu^5(p,p')=2m_0\Gamma^5(p,p')+S_F'(p)^{-1}\gamma_5+\gamma_5S_F'(p')^{-1}.\quad(5)$$

Our task in this section is to see whether Eqs. (4) and (5), which we have formally derived from the field equations, actually hold in perturbation theory.

To this end, let us rederive Eq. (5) in perturbation theory. It is convenient to write

$$\Gamma_\mu^5=\gamma_\mu\gamma_5+\Lambda_\mu^5,$$
$$\Gamma^5=\gamma_5+\Lambda^5,$$
$$S_F'(p)^{-1}=p-m_0-\Sigma(p),\quad(6)$$

where the vertex corrections Λ_μ^5 and Λ^5 and the proper self-energy part $\Sigma(p)$ are calculated using $(p-m_0)^{-1}$ as the free propagator. (Use of the bare mass $m_0=m-\delta m$ in the free propagator automatically includes the mass-renormalization counter terms.) In terms of Λ_μ^5, Λ^5, and Σ, Eq. (5) becomes

$$(p-p')^\mu\Lambda_\mu^5(p,p')=2m_0\Lambda^5(p,p')-\Sigma(p)\gamma_5-\gamma_5\Sigma(\ldots)\quad(7)$$

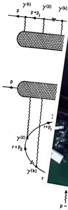
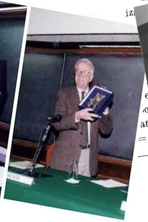


FIG. 1. Diagrams contributing to the axial-vector vertex. (a) The axial-vector vertex is attached ... ning with ...



IL NUOVO CIMENTO VOL. LX A, N. ...

A PCAC Puzzle: $\pi^0\to\gamma\gamma$ in the σ-Model.

J. S. BELL
CERN - Geneva

R. JACKIW (*)
CERN - Geneva
Jefferson Laboratory of Physics, Harvard University - Cambridge, Mass.

(ricevuto l'11 Settembre 1968)

Summary. — The effective coupling constant for $\pi^0\to\gamma\gamma$ should vanish for zero pion mass in theories with PCAC and gauge invariance. It does not so vanish in an explicit perturbation calculation in the σ-model. The resolution of the puzzle is effected by a modification of Pauli-Villars-Gupta regularization which respects both PCAC and gauge invariance.

1. – Introducti...

The invar... ...iza...

...ce we are working to lowest order in electromagnetism F_1-F_3 ... ...ess a dynamical singularity at $k^2=0$, and therefore we recover th... $=Ok^2$.

...turbation-theory argument.

...rmal reasoning of the previous Section should be verifiable b... ...rbation calculation in a model with PCAC and gauge invari... ...r the σ-model interacting with the electromagnetic field a_μ, o... ...rged pion and the neutron fields, which are not necessary for ...e Lagrange density is [8]

$$(3.1)\qquad \mathscr{L}=-\frac12\partial_\nu a_\mu\partial^\nu a^\mu+\bar\psi[i(\gamma\cdot\partial)+e(\gamma\cdot a)-m+g(\sigma+\varphi\gamma_5)]\psi+$$
$$+\frac12(\partial\varphi)^2+\frac12(\partial\sigma)^2-\frac12\mu^2\varphi^2-\frac12\left(\mu^2+\frac{2\lambda}{f}\right)\sigma^2-\lambda[(\sigma^2\ldots$$

Here $f=g/\ldots$

Outline

Dimensional dependent properties

Suppose we define $\gamma_*^{(4)} = -\frac{\mathrm{i}}{4!}\epsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma$ in $n = 4$ dimensions. Then

$$\begin{aligned} \mathrm{tr}\left(\gamma_*^{(4)}\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma\right) &= -\mathrm{i}\,\epsilon_{\mu\nu\rho\sigma}\mathrm{tr}\mathbb{1}, \\ \{\gamma_\mu, \gamma_*^{(4)}\} &= 0. \end{aligned} \tag{4}$$

Consider now γ_* in an arbitrary n with the same anticommutators + ciclicity of the trace:

$$n\,\mathrm{tr}\gamma_* = \mathrm{tr}\left(\gamma_*\gamma^\alpha\gamma_\alpha\right) = -\mathrm{tr}\left(\gamma^\alpha\gamma_*\gamma_\alpha\right) = -\mathrm{tr}\left(\gamma_*\gamma_\alpha\gamma^\alpha\right) = -n\,\mathrm{tr}\gamma_* \Longrightarrow n\,\mathrm{tr}\gamma_* = 0. \tag{5}$$

Mutatis mutandis...

$$\begin{aligned} n(n-2)\mathrm{tr}\left(\gamma_*\gamma_\mu\gamma_\nu\right) &= 0, \\ n(n-2)(n-4)\mathrm{tr}\left(\gamma_*\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma\right) &= 0, \\ &\vdots \end{aligned} \tag{6}$$

which contradicts formula (4)!

with

(18) $$A = -\frac{e^2}{(4\pi)^{\nu/2}}\cdot m^{\nu-3}\cdot\Gamma\left(2-\frac{\nu}{2}\right)\frac{\nu-1}{\nu-3},$$

(19) $$B = -\frac{e^2}{(4\pi)^{\nu/2}}\cdot m^{\nu-4}\cdot\Gamma\left(2-\frac{\nu}{2}\right)\cdot\frac{\nu-1}{\nu-3}$$

and

(20) $$\Sigma_f(p,\nu) = \frac{e^2}{(4\pi)^{\nu/2}}\cdot m^{\nu-4}\cdot\Gamma\left(2-\frac{\nu}{2}\right)\cdot$$

$$\cdot\left\{\left[(2-\nu)\frac{m-i\gamma\cdot p}{m^2\varrho}-4\frac{m-i\gamma\cdot p}{m^2\varrho^2}+\frac{2}{}\right].\ 2\ \ldots\right.$$

IL NUOVO CIMENTO VOL. 12 B, N. 1 11 Novembre 1972

Dimensional Renormalization: The Number of Dimensions as a Regularizing Parameter.

C. G. BOLLINI and J. J. GIAMBIAGI

Departamento de Física, Facultad de Ciencias Exactas
Universidad Nacional de La Plata
Consejo Nacional de Investigaciones Científicas y Técnicas - La Plata

(ricevuto l'8 Febbraio 1972)

Summary. — We perform an analytic extension of quantum electrodynamics matrix elements as (analytic) functions of the number of dimensions of space (ν). The usual divergences appear as poles for ν integer. The renormalization of those matrix elements (for ν arbitrary) leads to expressions which are free of ultraviolet divergences for ν equal to 4. This shows that ν can be used as an analytic regularizing parameter with advantages over the usual analytic regularization method. In particular, gauge invariance is mantained for any ν.

1. – Introduction.

In a previous paper (1) we pointed out the possibility of studying the struc- ... of the number of dimensions. We ...

Fig. 8.

$$\int d_n p\, \frac{\mathrm{Tr}\,[\gamma^\mu (i\gamma(p+k)+m)\gamma^\nu (i\gamma p + m)]}{((p+k)^2+m^2)(p^2+m^2)}.$$

The trace may be evaluated using (33), (34) and (35). Taking denominators together one obtains:

$$4\int_0^1 dx \int d_n p\, \frac{\delta_{\mu\nu}(m^2+p^2+pk) - 2p_\mu p_\nu - p_\mu k_\nu - k_\mu p_\nu}{(p^2+2pkx+k^2x+m^2)^2}.$$

Using the equations of appendix A:

$$= \frac{4i\pi^{\frac{1}{2}n}}{\Gamma(2)}\Gamma(2-\tfrac{1}{2}n)\int_0^1 dx\, \frac{2x(1-x)(k_\mu k_\nu - \delta_{\mu\nu}k^2)}{(m^2+k^2x(1-x))^{2-\frac{1}{2}n}}$$

which is manifestly gauge invariant.

6. LIMITATIONS OF THE METHOD

The method fails if in the Ward identities there appear quantities that have the desired properties only in four dimensional space. An example is the completely antisymmetric tensor $\epsilon_{\mu\nu\alpha\beta}$. If the particular properties of this tensor are vital for the Ward identities to hold our method will fail because we cannot generalize $\epsilon_{\mu\nu\alpha\beta}$ to a tensor satisfying the required properties for non-integer n. Similarly for γ^5. One can write:

$$\gamma^5 = \frac{1}{4!}\epsilon_{\mu\nu\alpha\beta}\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta,$$

insert this whenever γ^5 occurs and take the ϵ-tensor outside of the expression to be generalized to non-integer n. However, if we are dealing with Ward identities that rely on

$$\{\gamma^5, \gamma^\alpha\} = 0 \qquad \text{for} \qquad \alpha = 1, \ldots, n$$

$$\mathrm{Tr}\,(\gamma^5\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta) = 4\epsilon_{\mu\nu\alpha\beta}$$

...ories of charged massive vector bosons. ... possible to formulate renormalizable ... particles, and in order to establish unitarity and causality ... The derived Feynman rules involv... tities are needed. The necessary combi... the treatment of ...

Chiral anomalies

Do we need to consider γ_*? The standard model is written in terms of chiral fermions.

We need a solution to this γ_* problem:

- give up the ciclicity of the trace (Kreimer+).
- give up $\{\gamma_\mu, \gamma_*\} = 0$
  - give up all (Thompson-Yu);
  - keep the first four (Breitenlohner-Maison);

BM scheme

- n-dimensional usual metric $\eta_{\mu\nu}$, γ_μ, p_μ, $\cdots$.
- decompose them into a four-dimensional part ($\bar{X}$) and an $(n-4)$-dimensional part ($\hat{X}$):

$$\eta_{\mu\nu} = \bar{\eta}_{\mu\nu} + \hat{\eta}_{\mu\nu}, \quad \gamma_\mu = \bar{\gamma}_\mu + \hat{\gamma}_\mu, \quad p_\mu = \bar{p}_\mu + \hat{p}_\mu, \ldots$$

$$\eta_\mu{}^\nu \hat{\eta}_{\nu\rho} = \hat{\eta}_{\mu\nu}\hat{\eta}^\nu{}_\rho = \hat{\eta}_{\mu\rho}, \quad \hat{\eta}_{\mu\nu} = \hat{\eta}_{\nu\mu},$$

$$\bar{\eta}^{\mu\nu}\hat{\eta}_{\nu\rho} = 0, \quad \bar{\eta}_\mu{}^\nu p_\nu = \bar{p}_\mu, \quad \hat{\eta}_\mu{}^\nu \gamma_\nu = \hat{\gamma}_\mu, \cdots$$

- $\epsilon_{\mu\nu\rho\sigma}$ is a purely four-dimensional object: $\epsilon_{\mu\nu\rho\sigma} = \bar{\epsilon}_{\mu\nu\rho\sigma}$, $\hat{\eta}^{\alpha\mu}\epsilon_{\mu\nu\rho\sigma} = 0$.
- γ_* the same as in four dimensions,

$$\gamma_* \equiv -\frac{\mathrm{i}}{4!}\epsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma.$$

Chiral anomalies

Do we need to consider γ_*? The standard model is written in terms of chiral fermions.

We need a solution to this γ_* problem:

- give up the ciclicity of the trace (Kreimer+).
- give up $\{\gamma_\mu, \gamma_*\} = 0$
  - give up all (Thompson-Yu);
  - keep the first four (Breitenlohner-Maison);

BM scheme

- n-dimensional usual metric $\eta_{\mu\nu}$, γ_μ, p_μ, $\cdots$.
- decompose them into a four-dimensional part ($\bar{X}$) and an $(n-4)$-dimensional part ($\hat{X}$):

$$\eta_{\mu\nu} = \bar{\eta}_{\mu\nu} + \hat{\eta}_{\mu\nu}, \quad \gamma_\mu = \bar{\gamma}_\mu + \hat{\gamma}_\mu, \quad p_\mu = \bar{p}_\mu + \hat{p}_\mu, \ldots$$

$$\eta_\mu{}^\nu \hat{\eta}_{\nu\rho} = \hat{\eta}_{\mu\nu}\hat{\eta}^\nu{}_\rho = \hat{\eta}_{\mu\rho}, \quad \hat{\eta}_{\mu\nu} = \hat{\eta}_{\nu\mu},$$

$$\bar{\eta}^{\mu\nu}\hat{\eta}_{\nu\rho} = 0, \quad \bar{\eta}_\mu{}^\nu p_\nu = \bar{p}_\mu, \quad \hat{\eta}_\mu{}^\nu \gamma_\nu = \hat{\gamma}_\mu, \cdots$$

- $\epsilon_{\mu\nu\rho\sigma}$ is a purely four-dimensional object: $\epsilon_{\mu\nu\rho\sigma} = \bar{\epsilon}_{\mu\nu\rho\sigma}$, $\hat{\eta}^{\alpha\mu}\epsilon_{\mu\nu\rho\sigma} = 0$.
- γ_* the same as in four dimensions,

$$\gamma_* \equiv -\frac{\mathrm{i}}{4!}\epsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma.$$

Chiral anomalies

Do we need to consider γ_*? The standard model is written in terms of chiral fermions.

We need a solution to this γ_* problem:

- give up the ciclicity of the trace (Kreimer+).
- give up $\{\gamma_\mu, \gamma_*\} = 0$
  - give up all (Thompson-Yu);
  - keep the first four (Breitenlohner-Maison);

BM scheme

- n-dimensional usual metric $\eta_{\mu\nu}$, γ_μ, p_μ, $\cdots$.
- decompose them into a four-dimensional part ($\bar{X}$) and an $(n-4)$-dimensional part ($\hat{X}$):

$$\eta_{\mu\nu} = \bar{\eta}_{\mu\nu} + \hat{\eta}_{\mu\nu}, \quad \gamma_\mu = \bar{\gamma}_\mu + \hat{\gamma}_\mu, \quad p_\mu = \bar{p}_\mu + \hat{p}_\mu, \ldots$$

$$\eta_\mu{}^\nu \hat{\eta}_{\nu\rho} = \hat{\eta}_{\mu\nu}\hat{\eta}^\nu{}_\rho = \hat{\eta}_{\mu\rho}, \quad \hat{\eta}_{\mu\nu} = \hat{\eta}_{\nu\mu},$$

$$\bar{\eta}^{\mu\nu}\hat{\eta}_{\nu\rho} = 0, \quad \bar{\eta}_\mu{}^\nu p_\nu = \bar{p}_\mu, \quad \hat{\eta}_\mu{}^\nu \gamma_\nu = \hat{\gamma}_\mu, \cdots$$

- $\epsilon_{\mu\nu\rho\sigma}$ is a purely four-dimensional object: $\epsilon_{\mu\nu\rho\sigma} = \bar{\epsilon}_{\mu\nu\rho\sigma}$, $\hat{\eta}^{\alpha\mu}\epsilon_{\mu\nu\rho\sigma} = 0$.
- γ_* the same as in four dimensions,

$$\gamma_* \equiv -\frac{\mathrm{i}}{4!}\epsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma.$$

Outline

The trace anomaly

Consider a field theory in curved space, and analyze the (local) symmetry under changes of scales

$$g_{\mu\nu} \to e^{-2\sigma} g_{\mu\nu}, \; \Phi_i \to e^{k_i \sigma} \Phi, \tag{7}$$

where the ($n = 4$) Weyl weights for matter fields are $k_i = (-1, -3/2, 0)$ for scalar, fermionic and vector fields.

$$\begin{aligned} \frac{\delta S}{\delta \sigma(x)} &= \int \mathrm{d}^4 y \frac{\delta g_{\mu\nu}(y)}{\delta \sigma(x)} \frac{\delta S}{\delta g_{\mu\nu}(y)} + \#\mathrm{EOM} \\ &= -2 g_{\mu\nu} \frac{\delta S}{\delta g_{\mu\nu}}(x) + \#\mathrm{EOM} \\ &= -\sqrt{-g} g_{\mu\nu} T^{\mu\nu} + \#\mathrm{EOM} \end{aligned}$$

$\Longrightarrow$ if scale invariant, $T^{\mu}{}_{\mu} \overset{*}{=} 0$.

Does it survive the quantization, i.e. $g_{\mu\nu} \langle T^{\mu\nu} \rangle = 0$?

IL NUOVO CIMENTO VOL. 23 A, N. 1 1 Settembre 1974

Trace Anomalies in Dimensional Regularization.

D. M. CAPPER
Department of Physics, Imperial College - London

M. J. DUFF
Department of Astrophysics, University of Oxford - Oxford

(ricevuto il 22 Gennaio 1974)

Summary. — An unusual perturbation theory anomaly is pointed out. If there exists a trace identity valid in an arbitrary number of dimensions, then employing dimensional regularization can result in an amplitude satisfying the identity in an arbitrary number of dimensions, but the finite part of the amplitude violating it in four dimensions. An example given here is the one-loop neutrino contribution to the graviton propagator. Anomalous behaviour, of a different origin, also occurs in the one-loop photon contribution. Both kinds of anomaly can be removed at the expense of introducing n-dimensional, rather than 4-dimensional, counterterms.

1. – Introduction.

Dimensional regularization [1-3], the scheme whereby the dimension of space-time is used as a regulating parameter in dealing with the divergences of quantum field theory, has gained a good deal of popularity. In gauge theories especially, where it is essential that the regularization scheme respect the Ward identities, this method seems particularly appealing.

In this paper, however, we wish to issue a word of warning. Dimensional regularization, at least in its conventional form, can and does give rise to anomalies. That is to say, symmetries present in the original Lagrangian are

(1) G. 'T HOOFT and M. VELTMAN: *Nucl. Phys.*, **44** B, 189 (1972).
(2) J. F. ASHMORE: *Lett. Nuovo Cimento*, **4**, 289 (1972); *Comm. Math. Phys.*, **29**, 177 (1973).
(3) C. G. BOLLINI and J. J. GIAMBIAGI: *Nuovo Cimento*, **12** B, 20 (1972).

be

$$Q_1:Q_2:Q_3:Q_4:Q_5 = 4:-2:3:2:-3\,.$$

gularization respect this conformal invariance? Once
. (15) that as far as the q_i are concerned, everything is
eed in this ratio, and $Q^{\gamma(\text{pole})}_{\mu\nu\varrho\sigma}$ is both transverse and traceless.
will not be in this ratio, since eq. (24) is a strictly 4-dimen-
eing no extra restriction on $q'_i(2)$. Consequently $Q^{\gamma(\text{finite})}_{\mu\nu\varrho\sigma}$,
is not traceless

$$Q^{\gamma(\text{finite})}_{\mu\mu\varrho\sigma} \neq 0$$

vith the requirements of conformal invariance.
hoton loop example, although we can patch up objection *B*)
Introduction, dimensional regularization inevitably «fails»
rdle via objection *A*). In our second example, the converse

mple. The neutrino loop.

onsider one-loop corrections to the graviton propagator, but this
neutrinos (8). The relevant action is

$$S_\nu[g,\psi,\bar\psi] = \int dx \left[\frac{i}{4}\sqrt{g}\,\bar\psi L^{\mu a}\gamma_a(1+\gamma^5)\overset{\leftrightarrow}{\nabla}_\mu\psi\right],$$

the covariant derivative and $L^{\mu a}(x)$ is the vierbein field

$$g^{\mu\nu} = L^{\mu a}L^\nu_a\,.$$

e not important in this context; we could equally well have chosen
ectrons instead of neutrinos. As in the photon case, the general
of $S_\nu[g,\bar\psi,\psi]$ means that

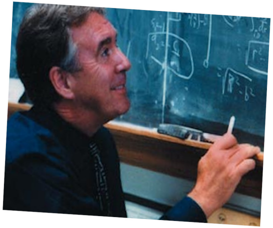
$$p_\mu Q^{(\nu)}_{\mu\nu\varrho\sigma}(p^2) = 0\,,$$

ilently

$$Q^\nu_1 + Q^\nu_4 + 2Q^\nu_5 = 0\,,$$
$$Q^\nu_2 + Q^\nu_4 = 0\,,$$
$$Q^\nu_3 + Q^\nu_5 = 0\,,$$

(8) M. CAPPER and M. J. DUFF: Imperial College preprint, ICTP/73/12.

1 it's wrong,



IL NUOVO CIMENTO Vol. 23 A, N. 1 1 Settembre 1974

Trace Anomalies in Dimensional Regularization.

D. M. Capper
Department of Physics, Imperial College - London

M. J. Duff
Department of Astrophysics, University of Oxford - Oxford

(ricevuto il 22 Gennaio 1974)

Summary. — An unusual perturbation theory anomaly is pointed out. If there exists a trace identity valid in an arbitrary number of dimensions, then employing dimensional regularization can result in an amplitude satisfying the identity in an arbitrary number of dimensions, but the finite part of the amplitude violating it in four dimensions. An example given here is the one-loop neutrino contribution to the graviton propagator. Anomalous behaviour, of a different origin, also occurs in the one-loop photon contribution. Both kinds of anomaly can be removed at the expense of introducing n-dimensional, rather than 4-dimensional, counterterms.

1. – Introduction.

Dimensional regularization ([1-3]), the scheme whereby the dimension of space-time is used as a regulating parameter in dealing with the divergences of quantum field theory, has gained a good deal of popularity. In gauge theories especially, where it is essential that the regularization scheme respect the Ward identities, this method seems particularly appealing.

In this paper, however, we wish to issue a word of warning. Dimensional regularization, at least in its conventional form, can and does give rise to anomalies. That is to say, symmetries present in the original Lagrangian are

(1) G. 'T Hooft and M. Veltman: *Nucl. Phys.*, **44** B, 189 (1972).
(2) J. F. Ashmore: *Lett. Nuovo Cimento*, **4**, 289 (1972); *Comm. Math. Phys.*, **29**, 177 (1973).
(3) C. G. Bollini and J. J. Giambiagi: *Nuovo Cimento*, **12** B, 20 (1972).

be

$$Q_1:Q_2:Q_3:Q_4:Q_5 = 4:-2:3:2:-3\,.$$

ularization respect this conformal invariance? Once
1. (15) that as far as the q_i are concerned, everything is
eed in this ratio, and $Q^{\gamma(\text{pole})}_{\mu\nu\varrho\sigma}$ is both transverse and traceless.
will not be in this ratio, since eq. (24) is a strictly 4-dimen-
eing no extra restriction on $q_i'(2)$. Consequently $Q^{\gamma(\text{finite})}_{\mu\nu\varrho\sigma}$,
is not traceless

$$Q^{\gamma(\text{finite})}_{\mu\mu\varrho\sigma} \neq 0$$

vith the requirements of conformal invariance.
hoton loop example, although we can patch up objection *B*)
Introduction, dimensional regularization inevitably «fails»
ırdle via objection *A*). In our second example, the converse

ample. The neutrino loop.

onsider one-loop corrections to the graviton propagator, but this
neutrinos ([8]). The relevant action is

$$S_\nu[g,\psi,\bar\psi] = \int dx\left[\frac{i}{4}\sqrt{g}\,\bar\psi L^{\mu a}\gamma_a(1+\gamma^5)\overleftrightarrow{\nabla}_\mu\psi\right],$$

the covariant derivative and $L^{\mu a}(x)$ is the vierbein field

$$g^{\mu\nu} = L^{\mu a} L^\nu_a\,.$$

e not important in this context; we could equally well have chosen
ectrons instead of neutrinos. As in the photon case, the general
of $S_\nu[g,\bar\psi,\psi]$ means that

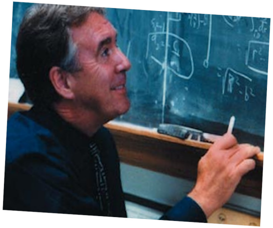
$$p_\mu Q^{(\nu)}_{\mu\nu\varrho\sigma}(p^2) = 0\,,$$

ılently

$$Q_1^\nu + Q_4^\nu + 2Q_5^\nu = 0\,,$$
$$Q_2^\nu + Q_4^\nu = 0\,,$$
$$Q_3^\nu + Q_5^\nu = 0\,,$$

(8) M. Capper and M. J. Duff: Imperial College preprint, ICTP/73/12.

1. it's wrong,
2. it's trivial



IL NUOVO CIMENTO VOL. 23 A, N. 1 1 Settembre 1974

Trace Anomalies in Dimensional Regularization.

D. M. CAPPER
Department of Physics, Imperial College - London

M. J. DUFF
Department of Astrophysics, University of Oxford - Oxford

(ricevuto il 22 Gennaio 1974)

Summary. — An unusual perturbation theory anomaly is pointed out. If there exists a trace identity valid in an arbitrary number of dimensions, then employing dimensional regularization can result in an amplitude satisfying the identity in an arbitrary number of dimensions, but the finite part of the amplitude violating it in four dimensions. An example given here is the one-loop neutrino contribution to the graviton propagator. Anomalous behaviour, of a different origin, also occurs in the one-loop photon contribution. Both kinds of anomaly can be removed at the expense of introducing n-dimensional, rather than 4-dimensional, counterterms.

1. – Introduction.

Dimensional regularization [1-3], the scheme whereby the dimension of space-time is used as a regulating parameter in dealing with the divergences of quantum field theory, has gained a good deal of popularity. In gauge theories especially, where it is essential that the regularization scheme respect the Ward identities, this method seems particularly appealing.

In this paper, however, we wish to issue a word of warning. Dimensional regularization, at least in its conventional form, can and does give rise to anomalies. That is to say, symmetries present in the original Lagrangian are

(1) G. 'T HOOFT and M. VELTMAN: *Nucl. Phys.*, **44** B, 189 (1972).
(2) J. F. ASHMORE: *Lett. Nuovo Cimento*, **4**, 289 (1972); *Comm. Math. Phys.*, **29**, 177 (1973).
(3) C. G. BOLLINI and J. J. GIAMBIAGI: *Nuovo Cimento*, **12** B, 20 (1972).

be

$$Q_1:Q_2:Q_3:Q_4:Q_5 = 4:-2:3:2:-3\,.$$

gularization respect this conformal invariance? Once
4. (15) that as far as the q_i are concerned, everything is
eed in this ratio, and $Q^{\gamma(\text{pole})}_{\mu\nu\varrho\sigma}$ is both transverse and traceless.
will not be in this ratio, since eq. (24) is a strictly 4-dimen-
cing no extra restriction on $q_i'(2)$. Consequently $Q^{\gamma(\text{finite})}_{\mu\nu\varrho\sigma}$,
is not traceless

$$Q^{\gamma(\text{finite})}_{\mu\mu\varrho\sigma} \neq 0$$

with the requirements of conformal invariance.
hoton loop example, although we can patch up objection *B*)
Introduction, dimensional regularization inevitably « fails »
rdle via objection *A*). In our second example, the converse

ample. The neutrino loop.

onsider one-loop corrections to the graviton propagator, but this
neutrinos [8]. The relevant action is

$$S_\nu[g, \psi, \bar\psi] = \int \mathrm{d}x \left[\frac{i}{4}\sqrt{g}\,\bar\psi L^{\mu a}\gamma_a(1+\gamma^5)\overleftrightarrow{\nabla}_\mu\psi\right],$$

the covariant derivative and $L^{\mu a}(x)$ is the vierbein field

$$g^{\mu\nu} \equiv L^{\mu a} L^\nu_a\,.$$

e not important in this context; we could equally well have chosen
ectrons instead of neutrinos. As in the photon case, the general
of $S_\nu[g, \bar\psi, \psi]$ means that

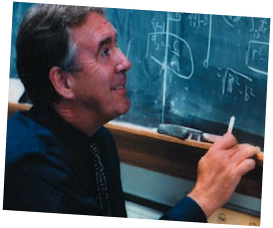
$$p_\mu Q^{(\nu)}_{\mu\nu\varrho\sigma}(p^2) = 0\,,$$

lently

$$Q_1^\nu + Q_4^\nu + 2Q_5^\nu = 0\,,$$
$$Q_2^\nu + Q_4^\nu = 0\,,$$
$$Q_3^\nu + Q_5^\nu = 0\,,$$

(8) M. CAPPER and M. J. DUFF: Imperial College preprint, ICTP/73/12.

1. it's wrong,
2. it's trivial
3. I thought it first.



Trace anomalies in one line

[1803:09764 [hep-th] Bruque, Cherchiglia and Pérez-Victoria]: consider a two-dimensional space in which the following integral appears in a perturbative expansion.

$$I_{\mu\nu} = \int \mathrm{d}^2 k \, \frac{k_\mu k_\nu}{(k^2+m^2)^2} \,. \tag{8}$$

We need to make sense of it by applying some regularization $R : I_{\mu\nu} \to [I_{\mu\nu}]^R$.

$$\begin{aligned}
[I_{\mu\nu}]^R &= \frac{1}{2}\left[\int \mathrm{d}^2 k \, \frac{\delta_{\mu\nu}}{k^2+m^2} - \int \mathrm{d}^2 k \, \partial_{k^\mu}\left(\frac{k_\nu}{k^2+m^2}\right)\right]^R \\
&= \frac{1}{2}\left[\int \mathrm{d}^2 k \, \frac{\delta_{\mu\nu}}{k^2+m^2}\right]^R \\
&= \frac{\delta_{\mu\nu}}{2}\left[\int \mathrm{d}^2 k \, \frac{k^2}{(k^2+m^2)^2} + \frac{m^2}{(k^2+m^2)^2}\right]^R \\
&= \frac{\delta_{\mu\nu}}{2}\left[I_{\alpha\alpha} + \pi\right]^R ,
\end{aligned}$$

$$\Longrightarrow \qquad \delta^{\mu\nu}[I_{\mu\nu}]^R = [\delta^{\mu\nu} I_{\mu\nu}]^R + \pi.$$

Anomaly induced action

In $n = 4$ spacetime dimensions, on dimensional grounds

$$g^{\mu\nu}\langle T_{\mu\nu}\rangle = wC^2 + bE_4 + c\Box R + \alpha R^2 + \beta F^2,$$

E_4: Euler characteristic,

C: Weyl tensor,

w, b, c, α, β: depend on the theory.

Consider semi-classical gravity: integrate out the (free) matter fields.

$$\Downarrow$$

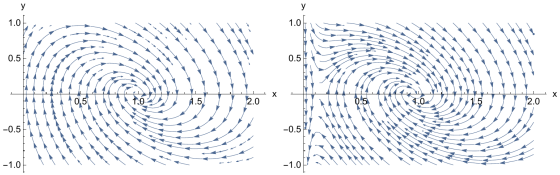
If the matter action is classically conformal invariant, the induced action for the (classical) gravitational field should satisfy the variational equation

$$\frac{\delta\Gamma_{\text{ind}}}{\delta\sigma(x)} = -\sqrt{-g}g_{\mu\nu}\langle T^{\mu\nu}\rangle.$$

We can integrate the anomaly to obtain Γ_{ind}!

Inflation, bouncing driven by quantum effects:

- Fabris, Pelinson, Shapiro [gr-qc/9810032]
- Hawking, Hertog, Reall [hep-th/0010232]
- ...



(©Shapiro+ [hep-th/2012.10554], $y \sim \dot{H}$, $x \sim H$)

Chiral trace anomaly

- Nakayama [1201.3428]: what if a Pontryagin term ($P = \tilde{R}R$) ...
- Bonora, Giaccari, Lima de Souza, +++ [arXiv:1403.2606, 1503.03326, 1703.10473, . . . , 2207.03279]:

  there is a P contribution to the chiral trace anomaly, which is purely imaginary ($\frac{\mathrm{i}}{180\,(4\pi)^2}\,\frac{15}{4}P$). Diagrammatic, heat-kernel (axial gravity).
- Bastianelli, Martelli, Broccoli [arXiv:1610.02304, 1911.02271, 2203.11668]: Pauli-Villars regularization, no P contribution.
- Abdallah, SF, Fröb [arXiv:2304.08939, arXiv:2101.11382]: diagrammatic, no P contribution.
- Duff [arXiv:2003.02688]: no computation.
- Discussion also on $F\tilde{F}$ contributions to trace anomaly.

Chiral trace anomaly

We consider a fermion in curved space,

$$S = -\int \bar{\psi}\gamma^{\mu}\nabla_{\mu}\psi\sqrt{-g}\,\mathrm{d}^n x, \qquad (9)$$

where as usual we introduce the vielbein

$$\gamma^{\mu} \equiv e^{\mu}{}_{b}\gamma^{b},$$

and the covariant derivative involves the spin connection

$$\omega_{\mu\rho\sigma} = \eta_{ab}\left(e_{\sigma}{}^{a}\partial_{[\mu}e_{\rho]}{}^{b} - e_{\rho}{}^{a}\partial_{[\mu}e_{\sigma]}{}^{b} + e_{\mu}{}^{a}\partial_{[\sigma}e_{\rho]}{}^{b}\right).$$

The fermion is chiral:

$$\psi = \mathcal{P}_{+}\psi \equiv \frac{1}{2}(\mathbb{1}+\gamma_{*})\psi, \qquad \bar{\psi} = \bar{\psi}\mathcal{P}_{-} \equiv \frac{1}{2}\bar{\psi}(\mathbb{1}-\gamma_{*}).$$

The EM tensor is given by

$$T^{\mu\nu} = \frac{1}{2}\bar{\psi}\mathcal{P}_{-}\gamma^{(\mu}\overleftrightarrow{\nabla}^{\nu)}\mathcal{P}_{+}\psi + \frac{1}{2}g^{\mu\nu}\bar{\psi}\mathcal{P}_{-}\gamma^{\rho}\overleftrightarrow{\nabla}_{\rho}\mathcal{P}_{+}\psi,$$

which is classically traceless (on-shell), since the action is scale invariant.

How can we compute $\langle T^{\mu\nu}\rangle$?

Chiral trace anomaly

The simplest thing we know to do is to perform an expansion around a given metric

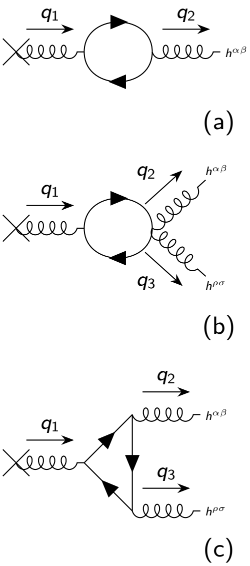
$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$$

$$\Downarrow$$

$$e^{\mu}{}_{a} = e^{(0)\,\rho}{}_{a}\left(\eta^{\mu}{}_{\rho} - \frac{1}{2}\kappa h^{\mu}{}_{\rho} + \frac{3}{8}\kappa^2 h^{\mu\sigma} h_{\sigma\rho}\right) + \mathcal{O}\left(\kappa^3\right),$$

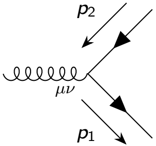
$$g^{\mu\nu} = \cdots,$$

$$\begin{aligned}\langle T^{\mu\nu}(x)\rangle_g &= \frac{\left\langle T^{\mu\nu}(x)\exp\left[\mathrm{i}\left(\kappa S_{(1)} + \kappa^2 S_{(2)}\right)\right]\right\rangle}{\left\langle \exp\left[\mathrm{i}\left(\kappa S_{(1)} + \kappa^2 S_{(2)}\right)\right]\right\rangle} + \mathcal{O}\left(\kappa^3\right) \\ &= \langle T^{\mu\nu}(x)\rangle_{(0)} + \kappa\,\langle T^{\mu\nu}(x)\rangle_{(1)} \\ &\quad + \kappa^2\,\langle T^{\mu\nu}(x)\rangle_{(2)} + \mathcal{O}\left(\kappa^3\right).\end{aligned}$$



Chiral trace anomaly

It is easy to write down the computations that should be done...

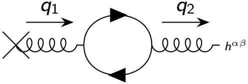


$$= \frac{\mathrm{i}}{8}\mathcal{P}_- \left[(p_1 + p_2)_\mu \gamma_\nu + (p_1 + p_2)_\nu \gamma_\mu\right] \mathcal{P}_+ \,,$$

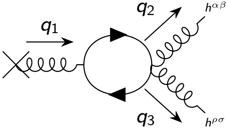
$$\mathcal{I}^{\mu_1\cdots\mu_m}(p) \equiv \int \frac{q^{\mu_1}\cdots q^{\mu_m}}{(q^2 - \mathrm{i}0)[(q+p)^2 - \mathrm{i}0]}\mathrm{d}^n q,$$

$$\mathrm{tr}\left[\gamma^\mu \mathcal{P}_+ \gamma^\tau \mathcal{P}_- \gamma^\sigma \mathcal{P}_+ \gamma^\delta \mathcal{P}_- \gamma^\alpha \mathcal{P}_+ \gamma^\lambda \mathcal{P}_-\right] ,$$

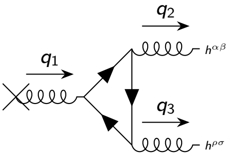
$$\mathcal{I}^{\mu_1\cdots\mu_m}(p,k) \equiv \int \frac{q^{\mu_1}\cdots q^{\mu_m}}{(q^2 - \mathrm{i}0)[(q-k)^2 - \mathrm{i}0][(q-p)^2 - \mathrm{i}0]}\mathrm{d}^n q.$$



(a)



(b)



(c)

$$
\begin{aligned}
\mathcal{I}^{\mu\nu\rho\sigma\alpha\beta}(p,k) = &\ \frac{1}{192}\frac{\mathrm{i}}{(4\pi)^2}\eta^{(\mu\nu}\eta^{\rho\sigma}\eta^{\alpha\beta)}\left[3(k^2)^2 - 6k^2\left(pk\right) + 4\left(pk\right)^2 + 5k^2p^2 - 6\left(pk\right)p^2 + 3(p^2)^2\right]\left(\mathcal{D} + \frac{3}{2}\right) \\
- &\ \frac{1}{16}\frac{\mathrm{i}}{(4\pi)^2}\eta^{(\mu\nu}\eta^{\rho\sigma}\left[p^{\alpha}p^{\beta)}\left[3p^2 + 2k^2 - 3\left(pk\right)\right] + k^{\alpha}k^{\beta)}\left[2p^2 + 3k^2 - 3\left(pk\right)\right] + p^{\alpha}k^{\beta)}[3p^2 + 3k^2 - 4(p\kappa)]\right](\mathcal{D}+1) \\
+ &\ \frac{1}{4}\frac{\mathrm{i}}{(4\pi)^2}\eta^{(\mu\nu}\left(p^{a}p^{\beta}p^{\rho}p^{\sigma)} + p^{\alpha}p^{\beta}p^{\rho}k^{\sigma)} + p^{\alpha}p^{\beta}k^{\rho}k^{\sigma)} + p^{\alpha}k^{\beta}k^{\rho}k^{\sigma)} + k^{\alpha}k^{\beta}k^{\rho}k^{\sigma)}\right)\mathcal{D} \\
- &\ \frac{15}{16}\frac{\mathrm{i}}{(4\pi)^2}\eta^{(\mu\nu}\eta^{\rho\sigma}\eta^{\alpha\beta)}\Big[(p^2)^2(G_{04}(p,k) + G_{02}(p,k) - 2G_{03}(p,k)) + \left(k^2\right)^2(G_{40}(p,k) + G_{20}(p,k) - 2G_{30}(p,k)) + 4p^2(pk)\Big(G_{13}(p,k) \\
&\ - G_{12}(p,k)\Big) + 4k^2(pk)(G_{31}(p,k) - G_{21}(p,k)) + 2p^2k^2\left(G_{11}(p,k) - G_{12}(p,k) + G_{22}(p,k) - G_{21}(p,k)\right)\Big] \\
+ &\ \frac{45}{2}\frac{\mathrm{i}}{(4\pi)^2}\eta^{(\mu\nu}\eta^{\rho\sigma}p^{\alpha}k^{\beta)}\left[p^2(G_{12}(p,k) - G_{13}(p,k)) + k^2\left(G_{21}(p,k) - G_{31}(p,k)\right) - 2(pk)G_{22}(p,k)\right] \\
+ &\ \frac{45}{4}\frac{\mathrm{i}}{(4\pi)^2}\eta^{(\mu\nu}\eta^{\rho\sigma}k^{\alpha}k^{\beta)}\left[p^2(G_{21}(p,k) - G_{22}(p,k)) + k^2\left(G_{30}(p,k) - G_{40}(p,k)\right) - 2(pk)G_{31}(p,k)\right] \\
+ &\ \frac{45}{4}\frac{\mathrm{i}}{(4\pi)^2}\eta^{(\mu\nu}\eta^{\rho\sigma}p^{\alpha}p^{\beta)}\left[p^2(G_{03}(p,k) - G_{04}(p,k)) + k^2\left(G_{12}(p,k) - G_{22}(p,k)\right) - 2(pk)G_{13}(p,k)\right] \\
- &\ \frac{15}{2}\frac{\mathrm{i}}{(4\pi)^2}\eta^{(\mu\nu}\left[p^{\rho}p^{\sigma}p^{\alpha}p^{\beta)}G_{04}(p,k) + 4p^{\rho}p^{\sigma}p^{\alpha}k^{\beta)}G_{13}(p,k) + 6p^{\rho}p^{\sigma}k^{\alpha}k^{\beta)}G_{22}(p,k) + 4p^{\rho}k^{\sigma}k^{\alpha}k^{\beta)}G_{31}(p,k) + k^{\rho}k^{\sigma}k^{\alpha}k^{\beta)}G_{40}(p,k)\right] \\
+ &\ \frac{\mathrm{i}}{(4\pi)^2}\Big[p^{\mu}p^{\nu}p^{\rho}p^{\sigma}p^{\alpha}p^{\beta}F_{06}(p,k) + 6p^{(\mu}p^{\nu}p^{\rho}p^{\sigma}p^{\alpha}k^{\beta)}F_{15}(p,k) + 15p^{(\mu}p^{\nu}p^{\rho}p^{\sigma}k^{\alpha}k^{\beta)}F_{24}(p,k) + 20p^{(\mu}p^{\nu}p^{\rho}k^{\sigma}k^{\alpha}k^{\beta)}F_{33}(p,k) \\
&\ + 15p^{(\mu}p^{\nu}k^{\rho}k^{\sigma}k^{\alpha}k^{\beta)}F_{42}(p,k) + 6p^{(\mu}k^{\nu}k^{\rho}k^{\sigma}k^{\alpha}k^{\beta)}F_{51}(p,k) + k^{\mu}k^{\nu}k^{\rho}k^{\sigma}k^{\alpha}k^{\beta}F_{60}(p,k)\Big],
\end{aligned}
$$

where F_{ij} and G_{ij} are functions of k and p which satisfy several nontrivial relations.

Chiral trace anomaly

You'd better use tensorial simplifications:

$$\mathcal{I}^{\mu_1\cdots\mu_m}(p,k) = 2\int_0^1\int_0^{1-y}\int \frac{(q+yp+xk)^{\mu_1}\cdots}{(q^2+M_E-\mathrm{i}0)^3}\mathrm{d}^n q\,\mathrm{d}x\,\mathrm{d}y.$$

$$M_E \equiv y(1-y)p^2 + x(1-x)k^2 - 2xy(p\cdot k)\,.$$

Because of symmetry, we can substitute

$$\begin{aligned} q^\mu q^\nu &\to \frac{1}{n}\eta^{\mu\nu}q^2, \\ q^\mu q^\nu q^\rho q^\sigma &\to \frac{3}{n(n+2)}\eta^{(\mu\nu}\eta^{\rho\sigma)}q^4, \\ &\vdots \end{aligned} \tag{10}$$

$\Downarrow$ the traceless part of $\eta^{(\mu\nu}\cdots$ vanishes

$$[\mathcal{I}^{\mu\cdots}(p,k)]_{\mathrm{trless}} = 2\int_0^1\int_0^{1-y}\int \frac{[(yp+xk)^{\mu}\cdots]_{\mathrm{trless}}}{(q^2+M_E-\mathrm{i}0)^3}\mathrm{d}^n q\,\mathrm{d}x\,\mathrm{d}y$$

$$
\begin{aligned}
\mathcal{I}^{\mu\nu\rho\sigma\alpha\beta}(p,k) = {} & \Big[\mathcal{I}^{\mu\nu\rho\sigma\alpha\beta}(p,k)\Big]_{\mathrm{trless}} \\
& + \frac{15\mathcal{I}(p-k)}{16(-1+n)(1+n)(8+n)}\eta^{(\rho\sigma} \\
& \Big[(2+n)(4+n)k^{\alpha}k^{\beta}k^{\mu}k^{\nu)} + 4(-4+n^2)k^{\alpha}k^{\beta}k^{\mu}p^{\nu)} + 6(-2+n)nk^{\alpha}k^{\beta}p^{\mu}p^{\nu)} \\
& + 4(-4+n^2)k^{\alpha}p^{\beta}p^{\mu}p^{\nu)} + (2+n)(4+n)p^{\alpha}p^{\beta}p^{\mu}p^{\nu)}\Big] \\
& - \frac{45\mathcal{I}(p-k)}{8(-1+n)(1+n)(6+n)(8+n)}\eta^{(\mu\nu}\eta^{\rho\sigma} \\
& \Big[2k^{\alpha}p^{\beta)}\Big(\Big(-2+n(4+n)\Big)k^2 - 10n(p\cdot k) + \Big(-2+n(4+n)\Big)p^2\Big) \\
& + k^{\alpha}k^{\beta)}\Big((2+n)\Big((6+n)k^2 - 10(p\cdot k)\Big) + \Big(8+n(4+n)\Big)p^2\Big) \\
& + p^{\alpha}p^{\beta)}\Big(\Big(8+n(4+n)\Big)k^2 + (2+n)\Big(-10(p\cdot k) + (6+n)p^2\Big)\Big)\Big] \\
& + \frac{15\mathcal{I}(p-k)}{16(-1+n)(1+n)(4+n)(6+n)(8+n)}\eta^{(\mu\nu}\eta^{\rho\sigma}\eta^{\alpha\beta)} \\
& \Big[3(4+n)(6+n)k^4 + 4\Big(48+n(4+n)\Big)(p\cdot k)^2 - 60(4+n)(p\cdot k)p^2 \\
& + 3(4+n)(6+n)p^4 + 2k^2\Big(-30(4+n)(p\cdot k) + \Big(64+n(22+3n)\Big)p^2\Big)\Big]
\end{aligned}
$$

A couple of contributions of this type...

- compute spinorial factors $(\gamma^{\mu}\cdots)$;
- renormalize;
- check divergence (diffeo anomaly);
- compute the trace;
- identify geometrical invariants (which are written as expansions in $h^{\mu\nu}$).

Chiral trace anomaly

Our results are exactly half of the trace anomaly for the Dirac spinor

$$(g_{\mu\nu}\langle T^{\mu\nu}\rangle)^{\text{ren}}_{(2)}(x) = \frac{1}{16\cdot 45(4\pi)^2}\left(-11\mathcal{E}_4 + 18C^{\mu\nu\rho\sigma}C_{\mu\nu\rho\sigma} + 12\nabla^2 R\right)_{(2)},$$
$$(\nabla_\mu\langle T^{\mu\nu}\rangle)^{\text{ren}}_{(2)}(x) = 0,$$

in terms of the Weyl tensor $C_{\mu\nu\rho\sigma}$ and the four-dimensional Euler density E_4, which in four dimensions satisfy

$$C^{\mu\nu\rho\sigma}C_{\mu\nu\rho\sigma} = R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} - 2R^{\mu\nu}R_{\mu\nu} + \frac{1}{3}R^2,$$
$$E_4 = R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} - 4R^{\mu\nu}R_{\mu\nu} + R^2.$$

What about the parity-odd contribution?

The parity-odd contribution

Two-point contributions, $\left\langle T_*^2 \right\rangle$:

$$\epsilon^{\delta\mu\sigma\tau} \times \text{symmetric in external momenta} = 0 \tag{11}$$

The $\langle T_* J \rangle$ contribution:

$$(2\epsilon^{\alpha\beta\gamma(\tau}\hat{\eta}^{\lambda)\mu} - \epsilon^{\alpha\beta\gamma\mu}\hat{\eta}^{\lambda\tau}) \times \delta^{\nu}{}_{[\tau} p_{\lambda]} p^2 = 0 \tag{12}$$

The three-point contribution, $\left\langle T_*^3 \right\rangle$:

$$\epsilon^{[\delta\mu\sigma\tau}\hat{\eta}^{\alpha]\lambda} \times \text{something} = 0 \tag{13}$$

Dimensionally dependent identities [arXiv:gr-qc/0105066v1, Edgar & Höglund]. The Cayley-Hamilton theorem for an $n \times n$ matrix ("*every square matrix over a commutative ring satisfies its own characteristic equation*"):

$$M^{c_1}{}_{[c_1} M^{c_2}{}_{c_2} \cdots M^{c_n}{}_{c_n} \delta^{b}{}_{a]} = 0 \tag{14}$$

Some current discussions

- The folklore says that the definition of the conformal anomaly for nonconformal theories is

$$\mathcal{A} = g^{\mu\nu}\langle T_{\mu\nu}\rangle - \langle g^{\mu\nu} T_{\mu\nu}\rangle. \tag{15}$$

  Why should we employ it for a conformal theory?

- For a CFT in $n = 4$, the correlator of three stress-energy tensors is necessarily parity-even (Stanev, arXiv: 1206.5639)

$$\langle T_{\mu\nu}(x_1) T_{\rho\sigma}(x_2) T_{\alpha\beta}(x_3)\rangle = (\text{parity-even term}) + 0 \times \epsilon_{\bullet\bullet\bullet\bullet}, \tag{16}$$

  but this is for the "regular" contribution. One may still have contributions at coincident points (that should be seen in our computation).

Outline

Outlook

- We have obtained no parity-odd contribution to the chiral trace anomaly (BM scheme, Feynman diagrams).
- Cosmology (inflation, baryogenesis, ...)
- SMEFT and anomaly cancellations [arXiv:2011.09976, Cata, Kilian and Kreher], [arXiv:2205.10381, 2012.13989, Feruglio], [arXiv:2104.13569, Passarino], [2205.02248 Quevillon et al.].
- Supersymmetric anomalies [arXiv:2104.13391, Minasian, Papadimitriou, Yi], [arXiv:2103.10048, Nakagawa and Nakayama].