Feynman Integral

Synergies Between Particle Physics and Gravitational Waves

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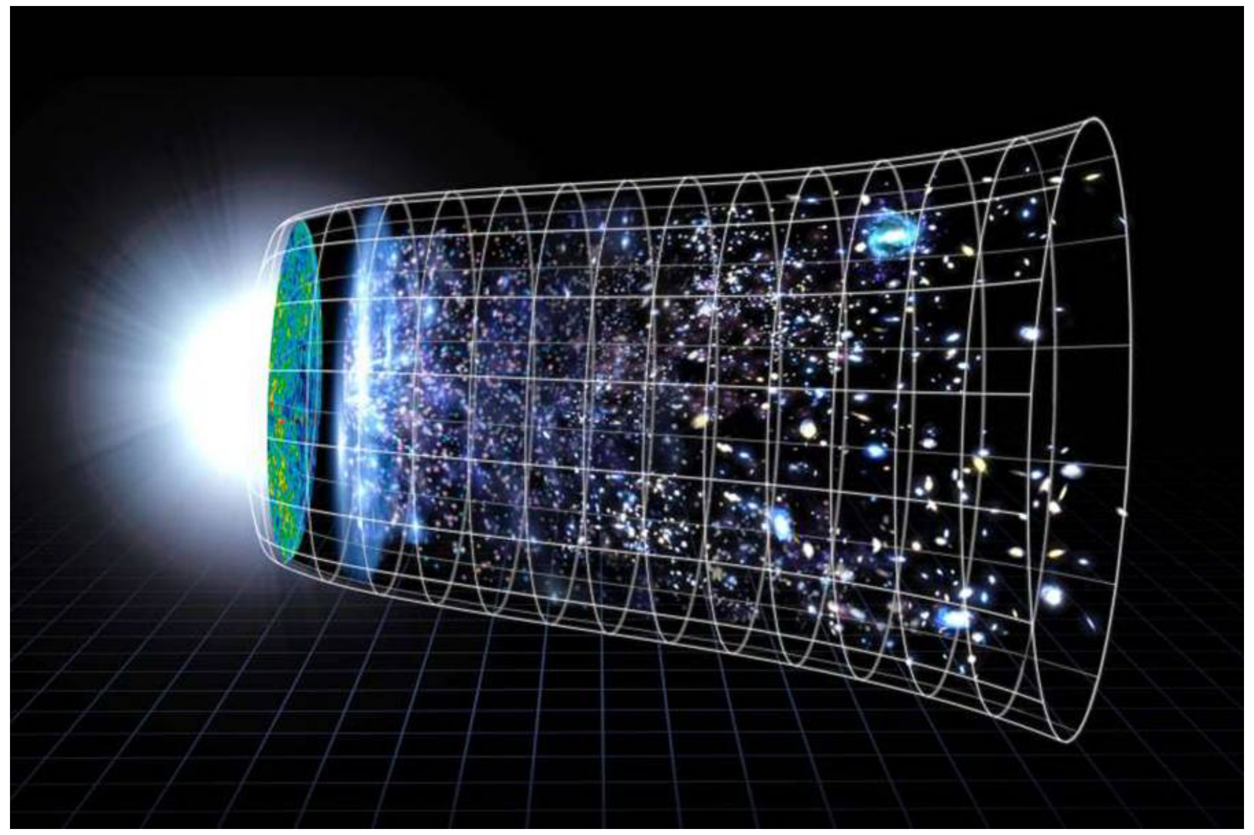
Istituto Nazionale di Fisica Nucleare

Scattering Amplitudes

Collider Phenomenology

Gravitational Waves



Cosmology

Precise Observables <—> Scattering Amplitudes

Scattering Amplitude: Connecting Theory and Experiment

Perturbative Expansion of Cross-Section

$$\sigma = \sigma^{(0)} + \alpha_s \sigma^{(1)} + \alpha_s^2 \sigma^{(2)} + \cdots$$

LO NLO NNLO

Cross-section Measured in Experiment

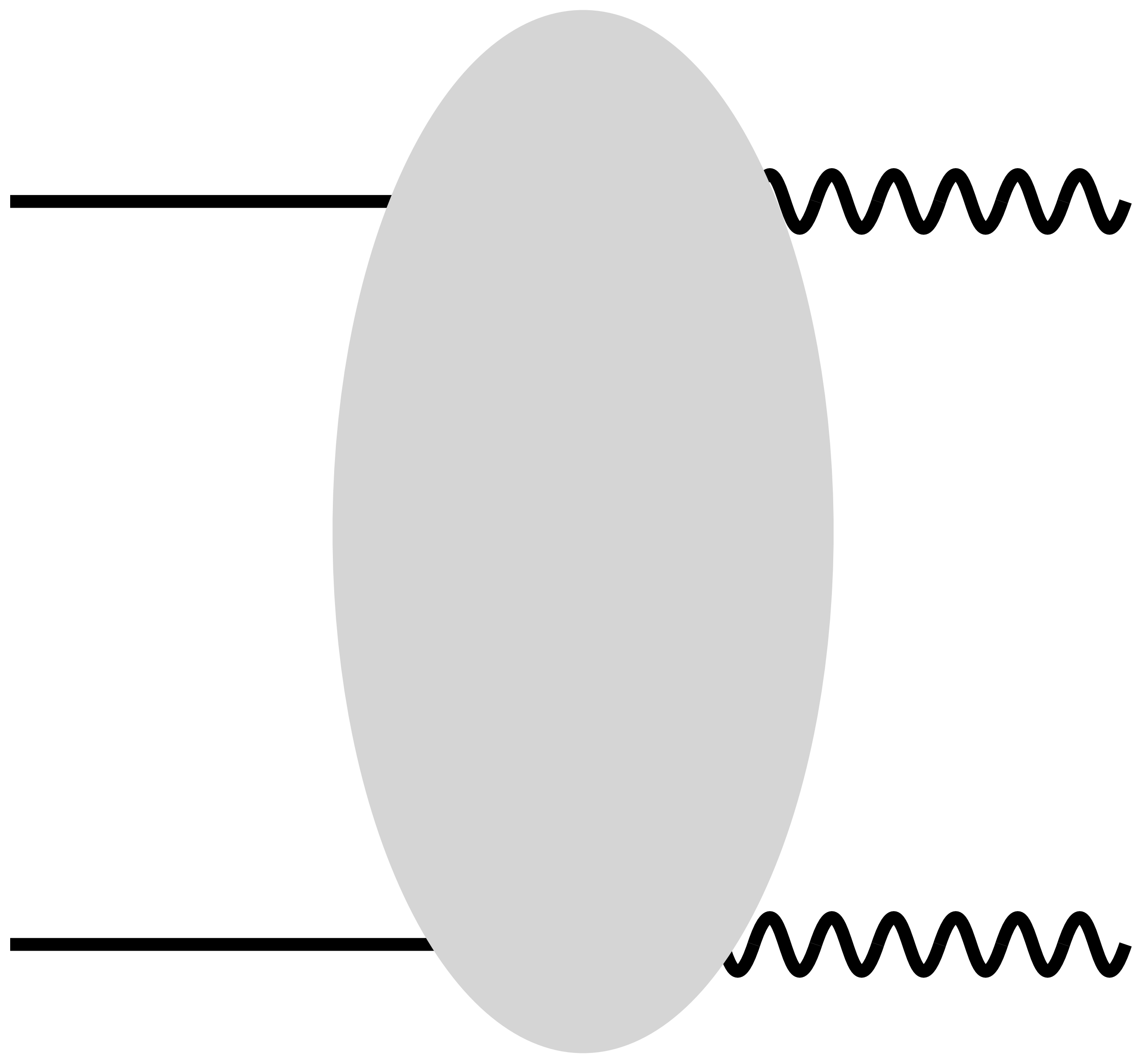
$$\sigma^0 \approx \int |\mathcal{M}_N^{(0)}|^2 d\Phi_N$$

Sum of Feynman Diagrams

Theory

Scattering Amplitudes

Scattering Amplitude



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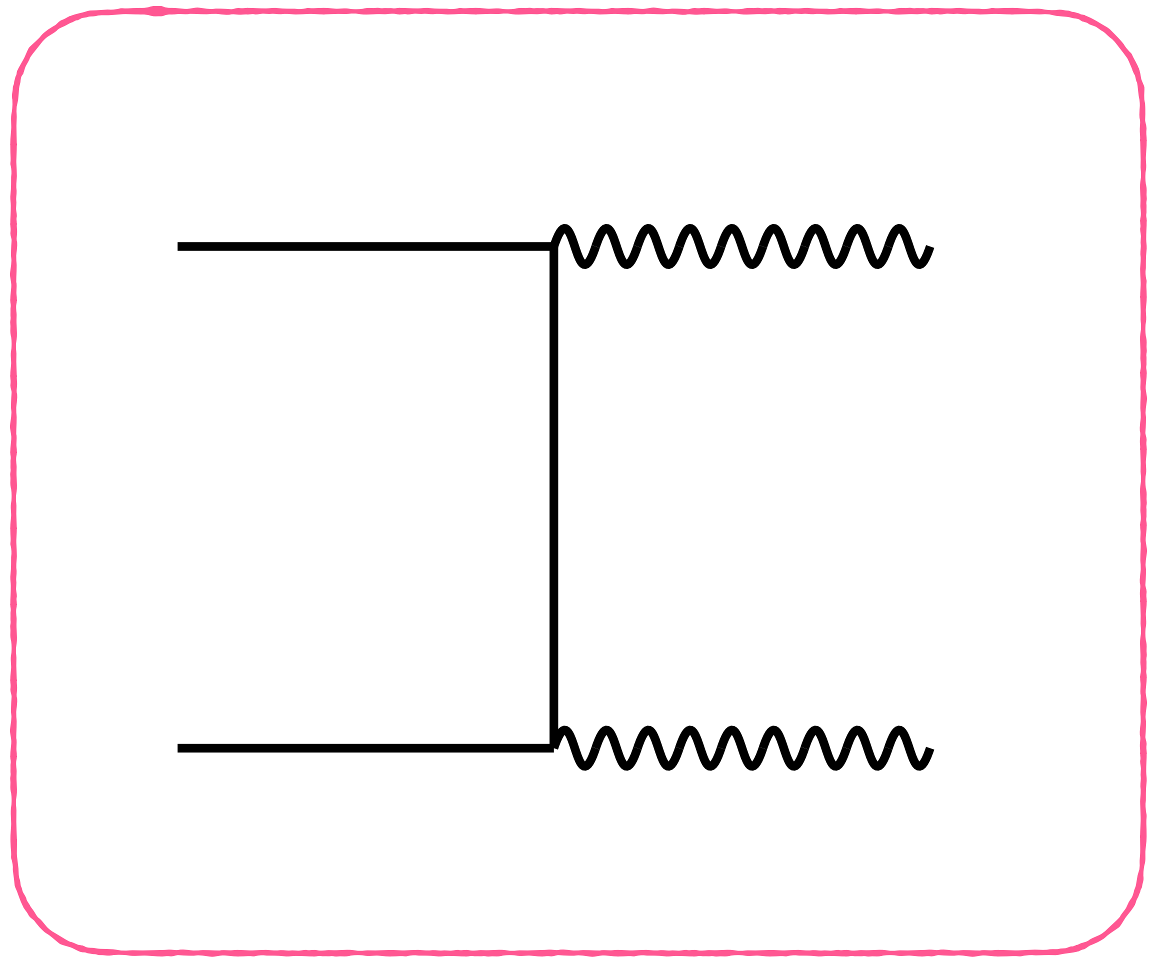
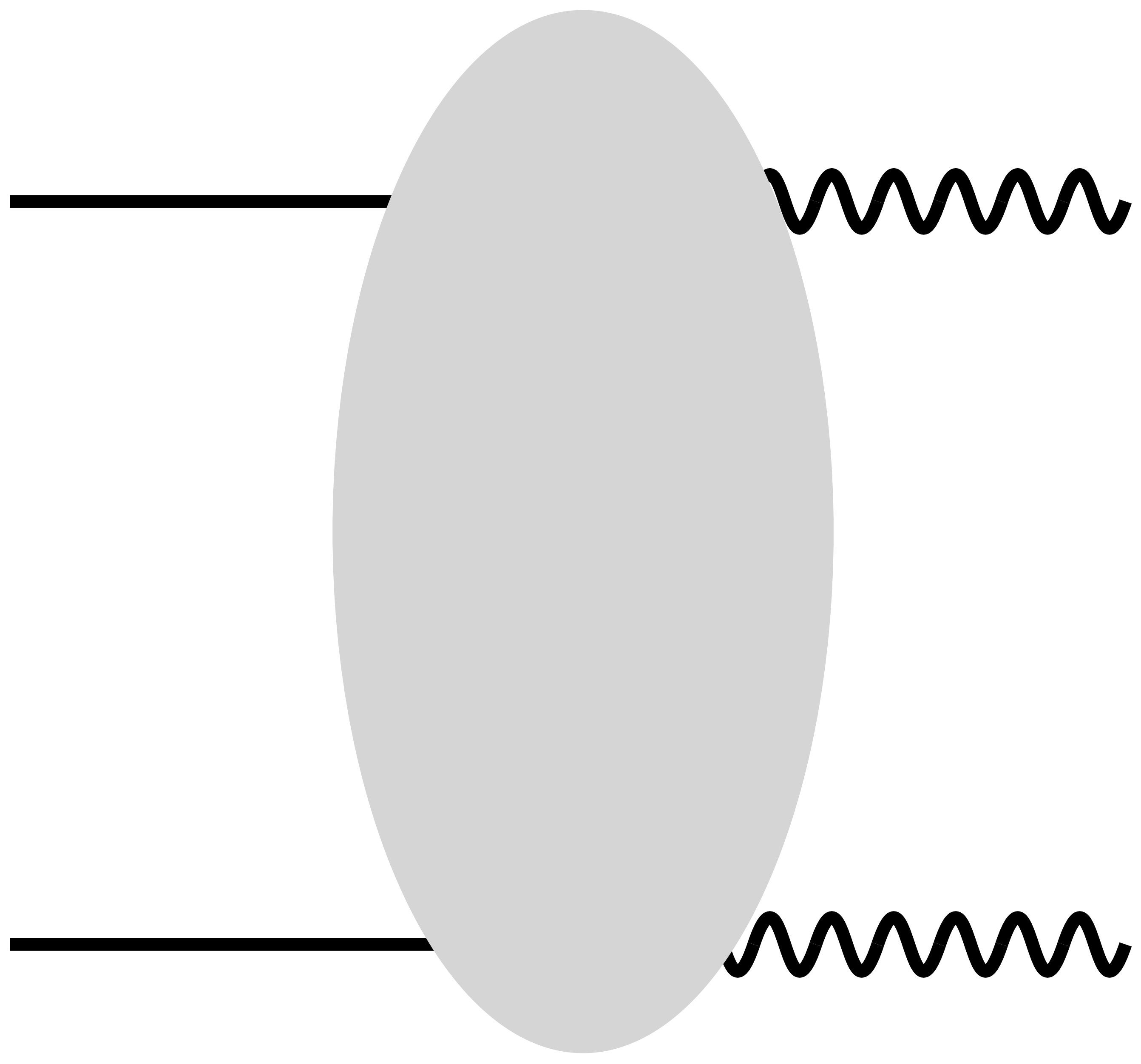
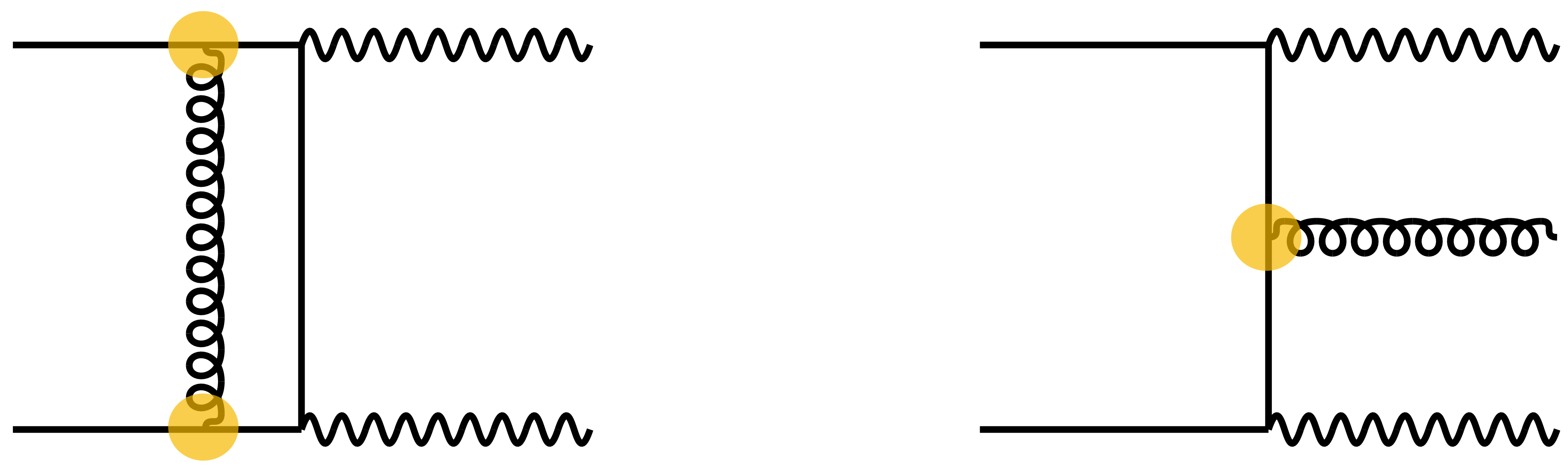


$$\sigma^0 \approx \int |\mathcal{M}_N^{(0)}|^2 d\Phi_N$$

Scattering Amplitude



NLO

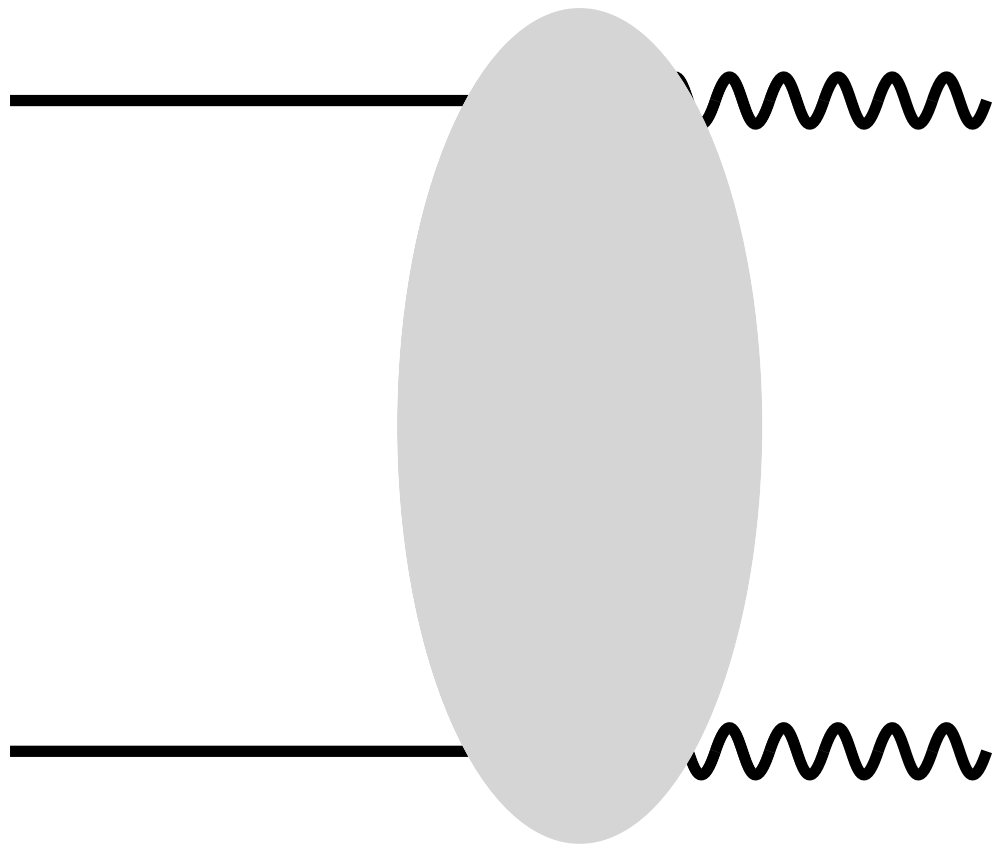


$$\sigma_N^{(1)} \approx \int 2\mathrm{Re}\left(\mathcal{M}_N^{(0)*}\mathcal{M}_N^{(1)}\right) d\Phi_N + \int |\mathcal{M}_{N+1}^{(0)}|^2 d\Phi_{N+1}$$

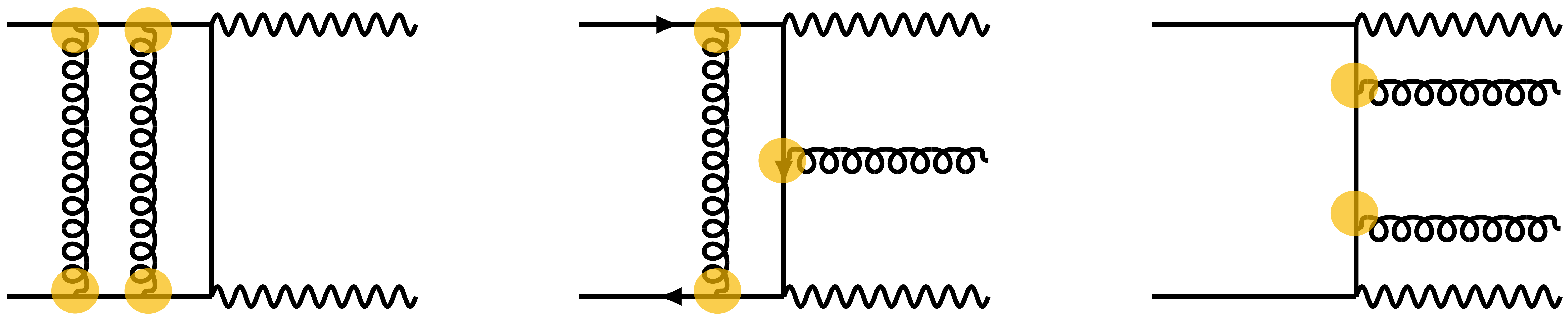
$$\int \left[\frac{V_2}{\epsilon^2} + \frac{V_1}{\epsilon^1} + V_0\right] d\phi_2$$

$$\int [R_0]\, d\phi_3$$

Scattering Amplitude



NNLO



$$\sigma_N^{(2)} \approx \int 2\mathrm{Re}\left(\mathcal{M}_N^{(0)*}\mathcal{M}_N^{(2)}\right) d\Phi_N + \int 2\mathrm{Re}\left(\mathcal{M}_{N+1}^{(0)*}\mathcal{M}_{N+1}^{(1)}\right) d\Phi_{N+1} + \int |\mathcal{M}_{N+2}^{(0)}|^2 d\Phi_{N+2}$$

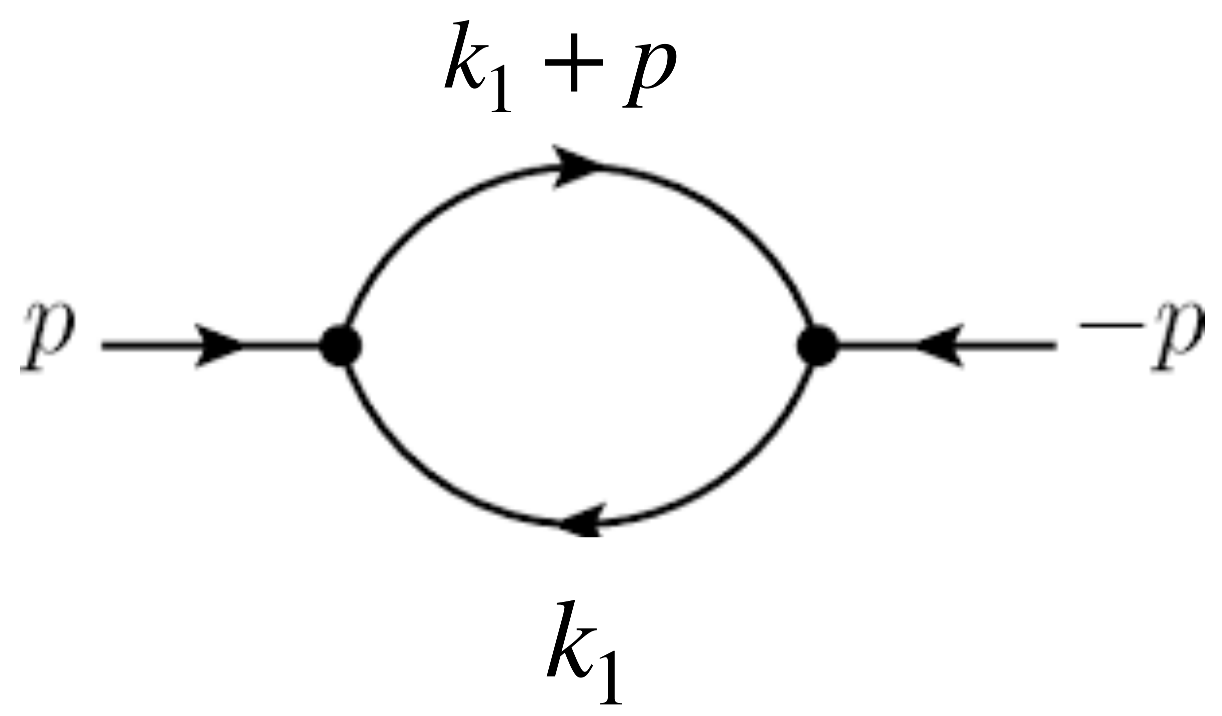
$$\int \left[\frac{VV_4}{\epsilon^4} + \frac{VV_3}{\epsilon^3} + \frac{VV_2}{\epsilon^2} + \frac{VV_1}{\epsilon^1} + VV_0\right] d\phi_2$$

$$\int \left[\frac{RV_2}{\epsilon^2} + \frac{RV_1}{\epsilon^1} + RV_0\right] d\phi_3$$

$$\int [RR_0]\, d\phi_4$$

Loop Integral: An example

One Loop Massless Bubble



$$I(a_1, a_2) = \int \frac{d^d k_1}{(k_1^2)^{a_1} ((k_1 + p)^2)^{a_2}}$$

$D_1 = k_1^2$

$D_2 = (k_1 + p)^2$

Notion of Loop Integral

Number of Loops

$$I\left(a_1 \cdots a_N\right) = \int \cdots \int d^d k_1 \cdots d^d k_l \frac{\mathcal{N}\left(\{k_i\},\{p_j\}\right)}{D_1^{a_1} \cdots D_N^{a_N}}$$

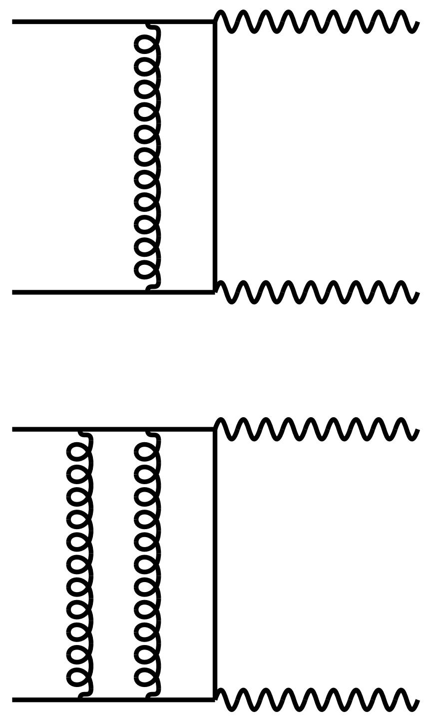
Loop Momenta

Number of Propagators

$$D_i = q_i^2 - m_i^2$$

$$q_i = \sum_j k_j + \sum_m p_m$$

Computation of the Loop Amplitude



Generation of the Diagrams via QGRAF

↓

Dirac algebra, Color sum, Trace in the numerators

↓

Reduction to scalar integrals

$$\mathcal{M} = \sum_i a_i I_i \qquad i = \mathcal{O}(10^5)$$

Integration-By-Parts Identity

Chetyrkin, Tkachov

Loop momenta

$$\int \prod_{\alpha=1}^{l} d^d k_\alpha \frac{\partial}{\partial k_{j,\mu}} \left(\frac{v^\mu}{D_1^{a_1} \cdots D_N^{a_N}} \right) = 0$$

Loop and external momenta

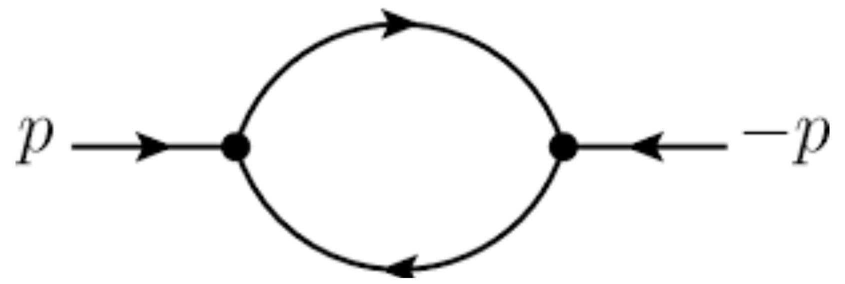
$$\int_{\alpha=1}^{l} \prod d^d k_\alpha \frac{\partial}{\partial k_{j,\mu}} \left(\frac{v^\mu}{D_1^{a_1} \cdots D_N^{a_N}} \right) = \int_{\alpha=1}^{l} \prod d^d k_\alpha \left[\frac{\partial v^\mu}{\partial k_{j,\mu}} \left(\frac{1}{D_1^{a_1} \cdots D_N^{a_N}} \right) - \sum_{j=1}^{N} \frac{a_j}{D_j} \frac{\partial D_j}{\partial k_{j,\mu}} \left(\frac{v^\mu}{D_1^{a_1} \cdots D_N^{a_N}} \right) \right]$$

$$C_1\, I(a_1, \cdots a_N - 1) + \cdots + C_r\, I(a_1 + 1, \cdots a_N) = 0$$

- ✺ Gives relations between different scalar integrals with different exponents
- ✺ l(l+E) number of equations
- ✺ Solve the system symbolically : Recursion relations — LiteRed
- ✺ Solve for specific integer value of the exponents : Laporta Algorithm — Fire, Reduze, Kira,..

Integration-By-Parts Identity (Example)

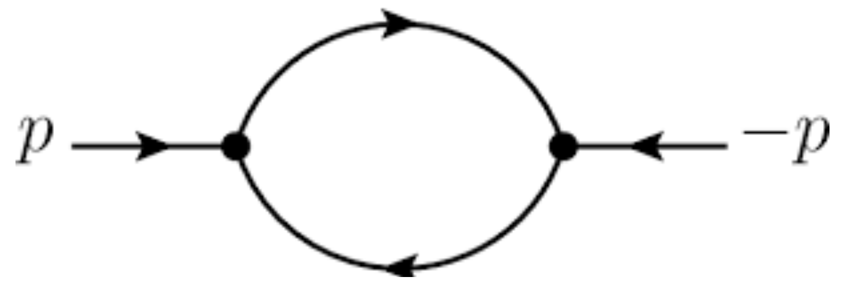
One Loop Massless Bubble



$$I(a_1, a_2) = \int \frac{d^d k_1}{(k_1^2)^{a_1} ((k_1 + p)^2)^{a_2}}$$

Integration-By-Parts Identity (Example)

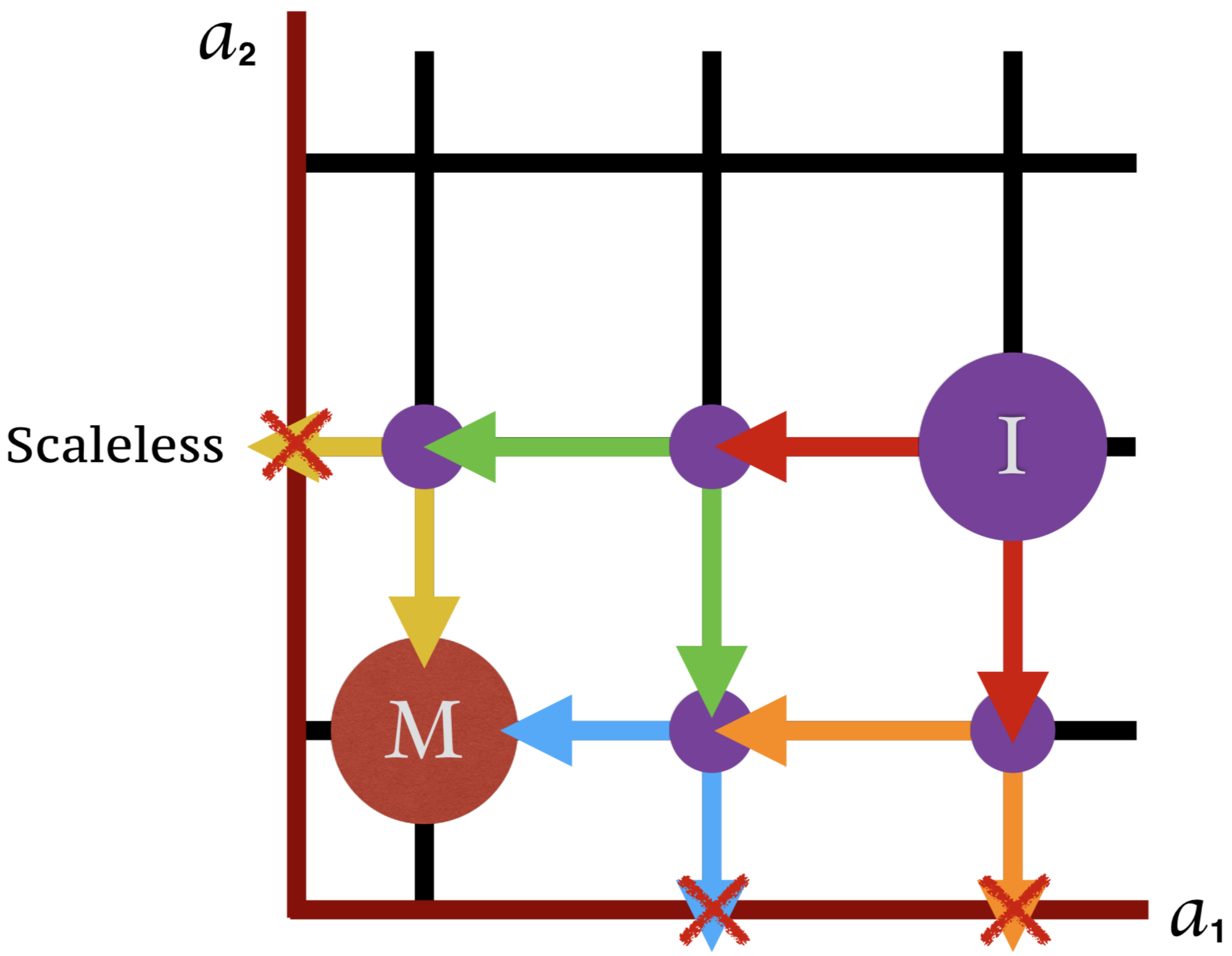
One Loop Massless Bubble



$$I(a_1, a_2) = \int \frac{d^d k_1}{(k_1^2)^{a_1} ((k_1+p)^2)^{a_2}}$$

IBP Identity

$$I(a_1, a_2) = \frac{a_1 + a_2 - d - 1}{p^2(a_2 - 1)} I(a_1, a_2 - 1) + \frac{1}{p^2} I(a_1 - 1, a_2)$$



Loop Amplitude

Reduction of scalar integrals to Master integrals

↓

Compute the Master Integrals

Number of Master Integrals

$$\mathcal{M} = \sum_i c_i J_i \qquad i = \mathcal{O}(10^2)$$

Integral Decomposition and Intersection Theory

Frellesvig, Gasparotto, **MKM**, Mastrolia, Mattiazzi, Mizera (2019)

Frellesvig, Gasparotto, Laporta, **MKM**, Mastrolia, Mattiazzi, Mizera (2019)

Frellesvig, Gasparotto, **MKM**, Mastrolia, Mattiazzi, Mizera (2020)

Chestnov, Frellesvig, Gasparotto, **MKM**, Mastrolia (2022)

Chestnov, Gasparotto, **MKM**, Mastrolia, Matsubara-Heo, Munch, Takayama (2022)

Decomposition of Feynman Integral

$$I = \sum_{i=1}^{\nu} c_i J_i$$

Decomposition of Feynman Integral

$$I = \sum_{i=1}^{\nu} c_i J_i$$

$$I \cdot J_i$$

$$J_i \cdot J_j = \delta_{ij}$$

$$I \cdot J_j \left(C^{-1}\right)_{ji}$$

$$J_i \cdot J_j = C_{ij} \neq \delta_{ij}$$

Intersection Theory and Feynman Integral

Mastrolia, Mizera (2018)

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Intersection Theory

Feynman Integral

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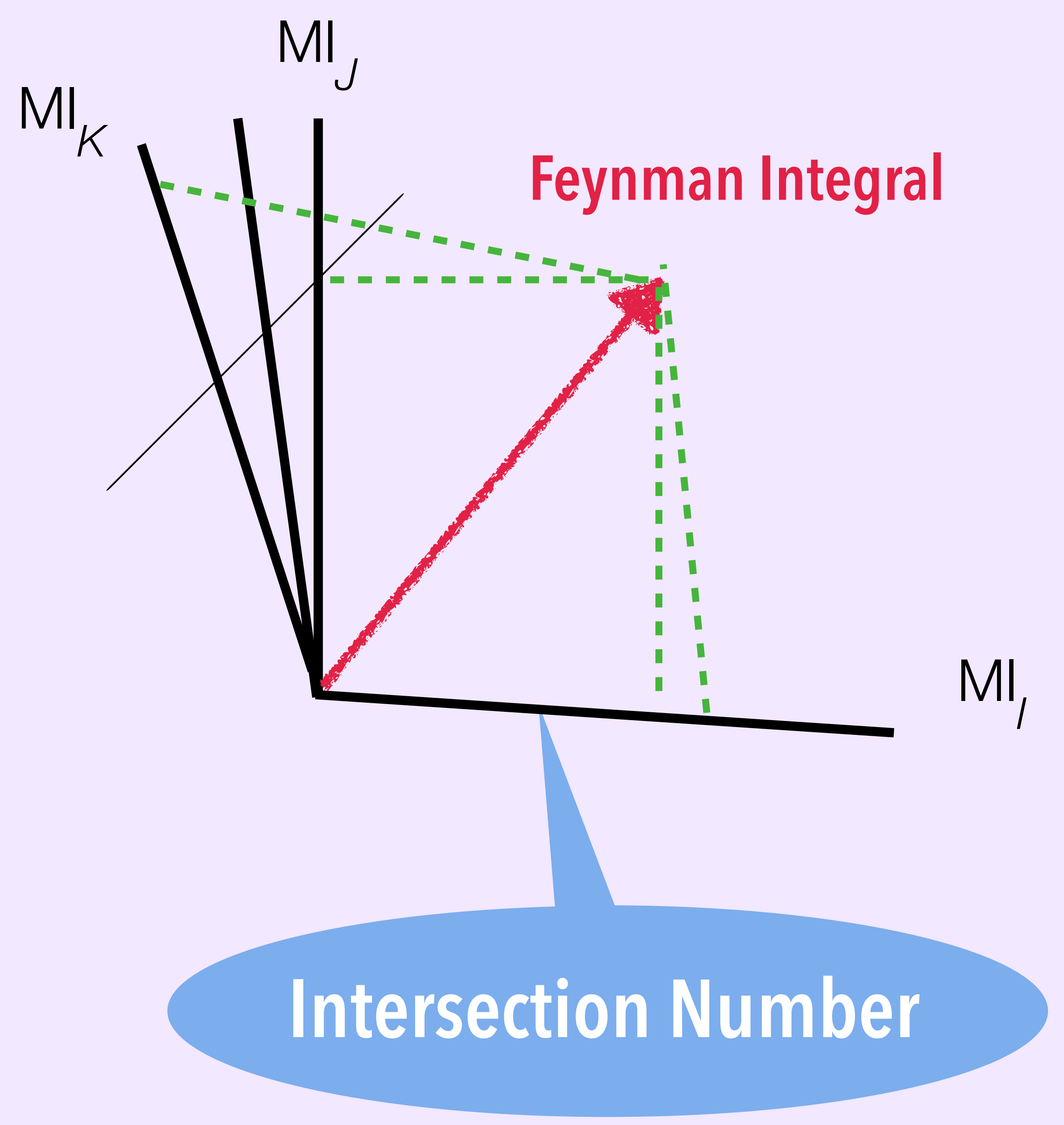
Intersection Theory — **Feynman Integral**

Feynman Integral decomposition



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Chestnov, Gasparotto, **MKM**, Mastrolia, Matsubara-Heo, Munch, Takayama (2022)

What is the Vector Space ?

How to define the scalar product ?

Intersection Theory

Aomoto, Gelfand, Kita, Cho, Matsumoto,
Mimachi, Mizera, Yoshida

Single valued differential form

$$I = \int_{\mathscr{C}} z^b (1-z)^{c-b} (1-tz)^{-a} \frac{dz}{z}$$

Multi-valued Function

Intersection Theory

Aomoto, Gelfand, Kita, Cho, Matsumoto,
Mimachi, Mizera, Yoshida

Single valued differential form

$$I = \int_{\mathscr{C}} z^b(1-z)^{c-b}(1-tz)^{-a}\frac{dz}{z}$$

Multi-valued Function

$$\langle \varphi | \mathcal{C} \otimes u]$$

Pairing

$$I = \int_{\mathcal{C}} u(\mathbf{z})\,\varphi(\mathbf{z})$$

Twisted Co-cycle

Twisted Cycle

$u(\mathbf{z})$ is a multi-valued function

$u(\mathbf{z})$ *vanishes on the boundaries* of $\mathcal{C}$, $u(\partial\mathcal{C}) = 0$

Basics of Intersection Theory

$$0 = \int_{\mathcal{C}} d\,(u\,\xi) = \int_{\mathcal{C}} (du \wedge \xi + u\,d\xi) = \int_{\mathcal{C}} u \left(\frac{du}{u} \wedge + d \right) \xi \equiv \int_{\mathcal{C}} u\, \nabla_{\omega} \xi .$$

$$\omega \equiv d \log u$$

$$\nabla_{\omega} \equiv d + \omega \wedge$$

Basics of Intersection Theory

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$\omega \equiv d \log u$

$\nabla_\omega \equiv d + \omega \wedge$

Equivalence Class $\quad {}_\omega\langle \varphi | : \; \varphi \sim \varphi + \nabla_\omega \xi$

$$\int_{\mathcal{C}} u\,\varphi = \int_{\mathcal{C}} u\,(\varphi + \nabla_\omega \xi)$$

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Vector Space of n-forms

$$H^n_\omega \equiv \{ n\text{-forms } \varphi_n \,|\, \nabla_\omega \varphi_n = 0 \} / \{ \nabla_\omega \varphi_{n-1} \}$$

Twisted Cohomology Group

Basics of Intersection Theory

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Equivalence Class $\quad {}_\omega\langle\varphi| : \; \varphi \sim \varphi + \nabla_\omega \xi \qquad \int_{\mathcal{C}} u\,\varphi = \int_{\mathcal{C}} u\,(\varphi + \nabla_\omega \xi)$

Vector Space of n-forms

$$H^n_\omega \equiv \{n\text{-forms } \varphi_n \,|\, \nabla_\omega \varphi_n = 0\} / \{\nabla_\omega \varphi_{n-1}\}$$

Twisted Cohomology Group

Dual space

$$H^n_{-\omega}, \qquad \nabla_{-\omega} = d - \omega\wedge$$

Dimension of the Vector Space: Number of MIs

$$\chi(X) = \sum_{k=0}^{2n} (-1)^k \dim H_\omega^k.$$

$H_\omega^{k\neq n}$ vanish.

Aomoto (1975)

$$\nu = (-1)^n \chi(X)$$

$$= (-1)^n \left(n+1 - \chi(\mathcal{P}_\omega)\right)$$

$$= \{\text{number of solutions of } \omega=0\}$$

Decomposition of differential forms

Number of Linearly independent forms (twisted co-cycle) is ν

Basis $\langle e_i|$ $i = 1, 2, \ldots, \nu$

Dual Basis $|h_j\rangle$ $j = 1, 2, \ldots, \nu$

Monomial Basis : $\langle e_i| = \langle \phi_i| \equiv z^{i-1} dz$

d-Log Basis : $\langle e_i| = \langle \varphi_i| \equiv \dfrac{dz}{z - z_i}$

Metric Matrix : $\mathbf{C}_{ij} = \langle e_i | h_j \rangle$

Decomposition of differential forms

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Basis	$\langle e_i \vert$	$i = 1, 2, \ldots, \nu$
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Metric Matrix : $\mathbf{C}_{ij} = \langle e_i | h_j \rangle$

$$\mathbf{M} = \begin{pmatrix} \langle \varphi|\psi\rangle & \langle \varphi|h_1\rangle & \langle \varphi|h_2\rangle & \ldots & \langle \varphi|h_\nu\rangle \\ \langle e_1|\psi\rangle & \langle e_1|h_1\rangle & \langle e_1|h_2\rangle & \ldots & \langle e_1|h_\nu\rangle \\ \langle e_2|\psi\rangle & \langle e_2|h_1\rangle & \langle e_2|h_2\rangle & \ldots & \langle e_2|h_\nu\rangle \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \langle e_\nu|\psi\rangle & \langle e_\nu|h_1\rangle & \langle e_\nu|h_2\rangle & \ldots & \langle e_\nu|h_\nu\rangle \end{pmatrix} \equiv \begin{pmatrix} \langle \varphi|\psi\rangle & \mathbf{A}^\intercal \\ \mathbf{B} & \mathbf{C} \end{pmatrix}$$

$$\det \mathbf{M} = \det \mathbf{C} \left(\langle \varphi|\psi\rangle - \mathbf{A}^\intercal \, \mathbf{C}^{-1} \, \mathbf{B} \right) = 0$$

$$\langle \varphi|\psi\rangle = \mathbf{A}^\intercal \, \mathbf{C}^{-1} \, \mathbf{B} = \sum_{i,j=1}^{\nu} \langle \varphi|h_j\rangle \, (\mathbf{C}^{-1})_{ji} \, \langle e_i|\psi\rangle$$

Master Decomposition Formula :

$$\langle \varphi| = \sum_{i,j=1}^{\nu} \langle \varphi|h_j\rangle \; \left(\mathbf{C}^{-1}\right)_{ji} \; \langle e_i|$$

Factorization of Identity

$$(i)\,(-i) = \mathbb{I}$$

Complex Number

$$\sum_n |n\rangle\langle n| = \mathbb{I}$$

Quantum Mechanics

$$\sum_{i,j=1}^{\nu} |e_j\rangle \left(C^{-1}\right)_{ij} \langle e_i| = \mathbb{I}_c$$

$$\sum_{i,j=1}^{\nu} |\mathcal{C}_j\rangle \left(H^{-1}\right)_{ij} \langle \mathcal{C}_i| = \mathbb{I}_h$$

Feynman Integral ?

Decomposition of Uni-variate Integral

Mastrolia, Mizera (2018)

Frellesvig, Gasparotto, MKM, Mastrolia, Mattiazzi, Mizera (2019)

Integrals

$$I = \int_{\mathcal{C}} u\, \varphi = \langle \varphi | \mathcal{C}]$$

Number of MIs

$$\omega \equiv d \log u$$

$$\nu = \{\text{the number of solutions of } \omega = 0\}$$

$$I = \sum_{i=1}^{\nu} c_i\, J_i \qquad J_i = \langle e_i | \mathcal{C}]$$

Matsumoto (1998)

Computation of Intersection Number

Uni-Variate

Choice of Bases / MIs

$$\langle e_i | = \langle \phi_i | \equiv z^{i-1} dz \qquad \langle e_i | = \langle \varphi_i | \equiv \frac{dz}{z - z_i}$$

Master Decomposition Formula

$$\langle \varphi | = \sum_{i,j=1}^{\nu} \langle \varphi | h_j \rangle \left(\mathbf{C}^{-1} \right)_{ji} \langle e_i |$$

Metric Matrix

$$\mathbf{C}_{ij} = \langle e_i | h_j \rangle$$

Decomposition of Multi-Variate Integral

Frellesvig, Gasparotto, MKM, Mastrolia, Mattiazzi, Mizera (2019)
Frellesvig, Gasparotto, Laporta, MKM, Mastrolia, Mattiazzi, Mizera (2020)

Integrals

$$I = \int_{\mathcal{C}} u\, \varphi = \langle \varphi | \mathcal{C}]$$

Number of MIs

$$\omega \equiv d\log u(\mathbf{z}) = \sum_{i=1}^{n} \hat{\omega}_i \, dz_i$$

$\nu =$ Number of solutions of the system of equations

$$\hat{\omega}_i \equiv \partial_{z_i} \log u(\mathbf{z}) = 0 \,, \qquad i = 1, \ldots, n$$

$$I = \sum_{i=1}^{\nu} c_i \, J_i \qquad J_i = \langle e_i | \mathcal{C}]$$

Choice of Bases

$$e_i(\mathbf{z}) \qquad h_i(\mathbf{z})$$

$$\mathbf{C}_{ij} = \langle e_i | h_j \rangle$$

Metric Matrix

$$\langle \varphi | = \sum_{i,j=1}^{\nu} \langle \varphi | h_j \rangle \left(\mathbf{C}^{-1} \right)_{ji} \langle e_i |$$

Master Decomposition Formula

Computation of Intersection Number

Multi-Variate

Computation of Intersection Number

Fibration Method

Matsumoto (1998)

Goto (2015)

Mizera (2019) Frellesvig, Gasparotto, Laporta, MKM, Mastrolia, Mattiazzi, Mizera (2019)

Wienzierl (2020) Frellesvig, Gasparotto, MKM, Mastrolia, Mattiazzi, Mizera (2020)

Caron-Huot, Pokraka (2021)

Secondary Equation

Matsubara-Heo (2019)

Chestnov, Gasparotto, MKM, Mastrolia, Matsubara-Heo, Munch, Takayama (2022)

Multivariate Differential Equation

Matsumoto (1998)

Chestnov, Frellesvig, Gasparotto, MKM, Mastrolia (2022)

Univariate Intersection Number

Matsumoto, Mizera

$$\langle \varphi_L | \varphi_R \rangle = \frac{1}{2\pi i} \int_X \varphi_L \wedge \varphi_R$$

Univariate Intersection Number

Matsumoto, Mizera

$$\langle \varphi_L | \varphi_R \rangle = \frac{1}{2\pi i} \int_X \varphi_L \wedge \varphi_R$$

$$\langle \varphi_L | \varphi_R \rangle_\omega = \sum_{p \in \mathcal{P}} \mathrm{Res}_{z=p} \Big(\psi_p \, \varphi_R \Big)$$

$$\nabla_{\omega_p} \psi_p = \varphi_{L,p}$$

First Order Differential Equation

Univariate Intersection Number

Matsumoto, Mizera

$$\langle \varphi_L | \varphi_R \rangle = \frac{1}{2\pi i} \int_X \varphi_L \wedge \varphi_R$$

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First Order Differential Equation

$$\nabla_{\omega_p} \psi_p = \varphi_{L,p}$$

Laurent Expansion around the poles of ω

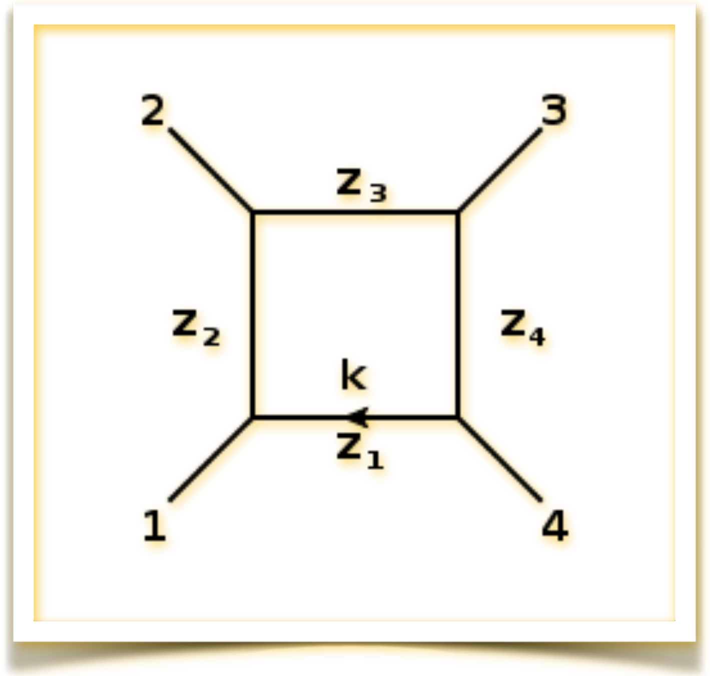
$$\tau \equiv z - p$$

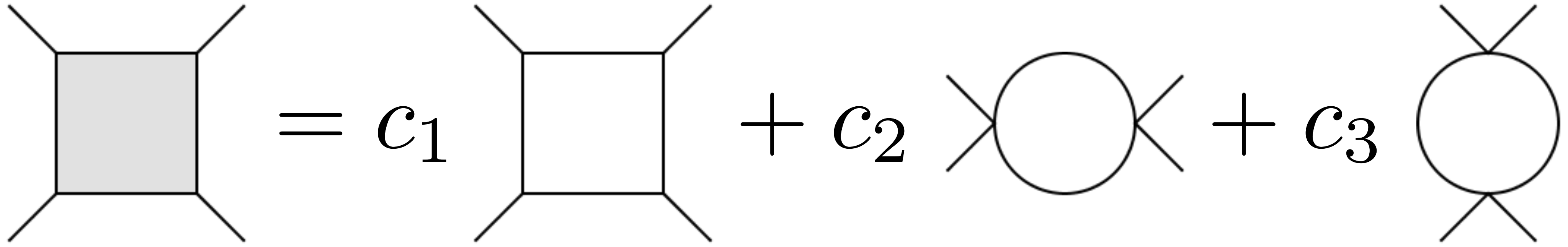
Known : $\varphi_{L,p}$

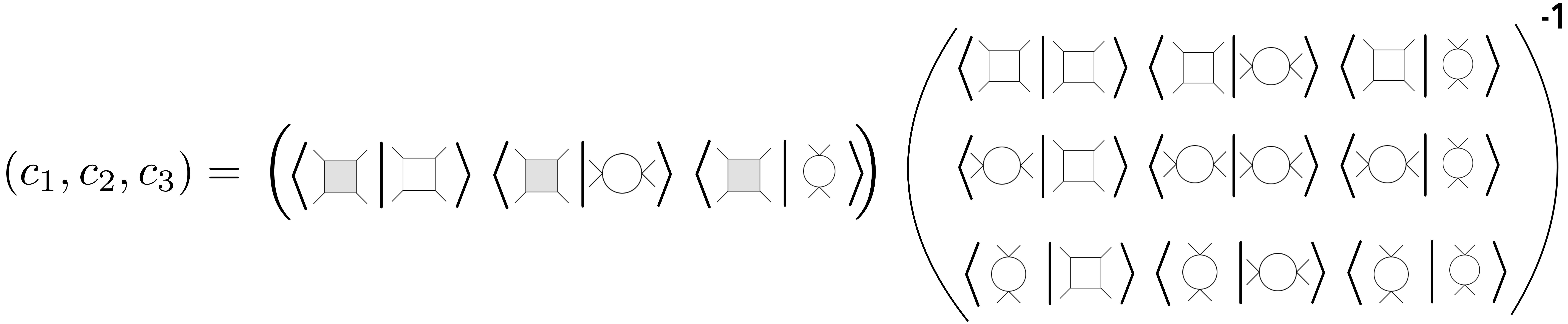
Ansatz : $\psi_p = \sum_{j=\min}^{\max} \psi_p^{(j)} \tau^j + \mathcal{O}\left(\tau^{\max+1}\right)$

The coefficients are obtained by solving the differential equation

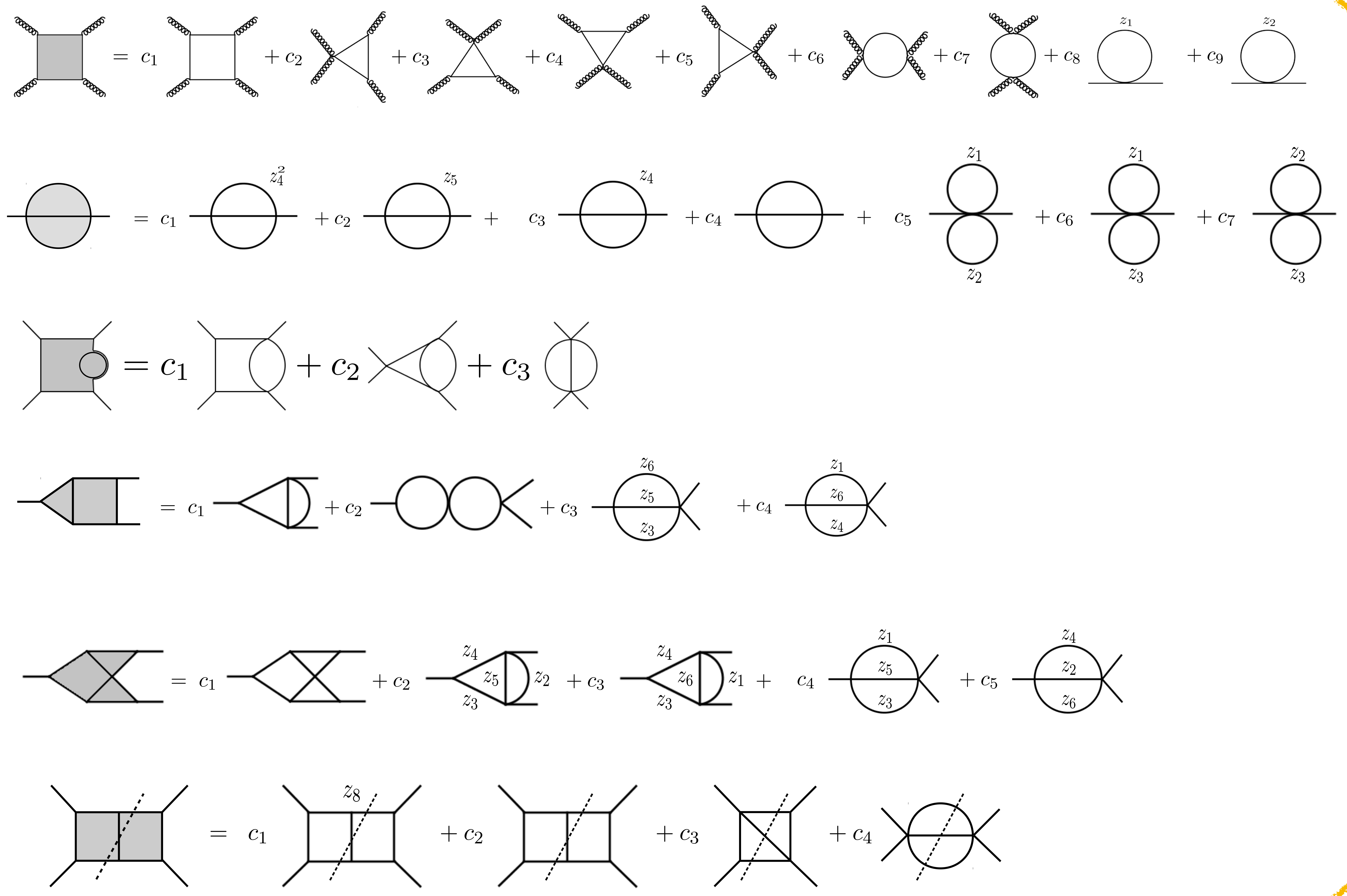
Examples of decomposition







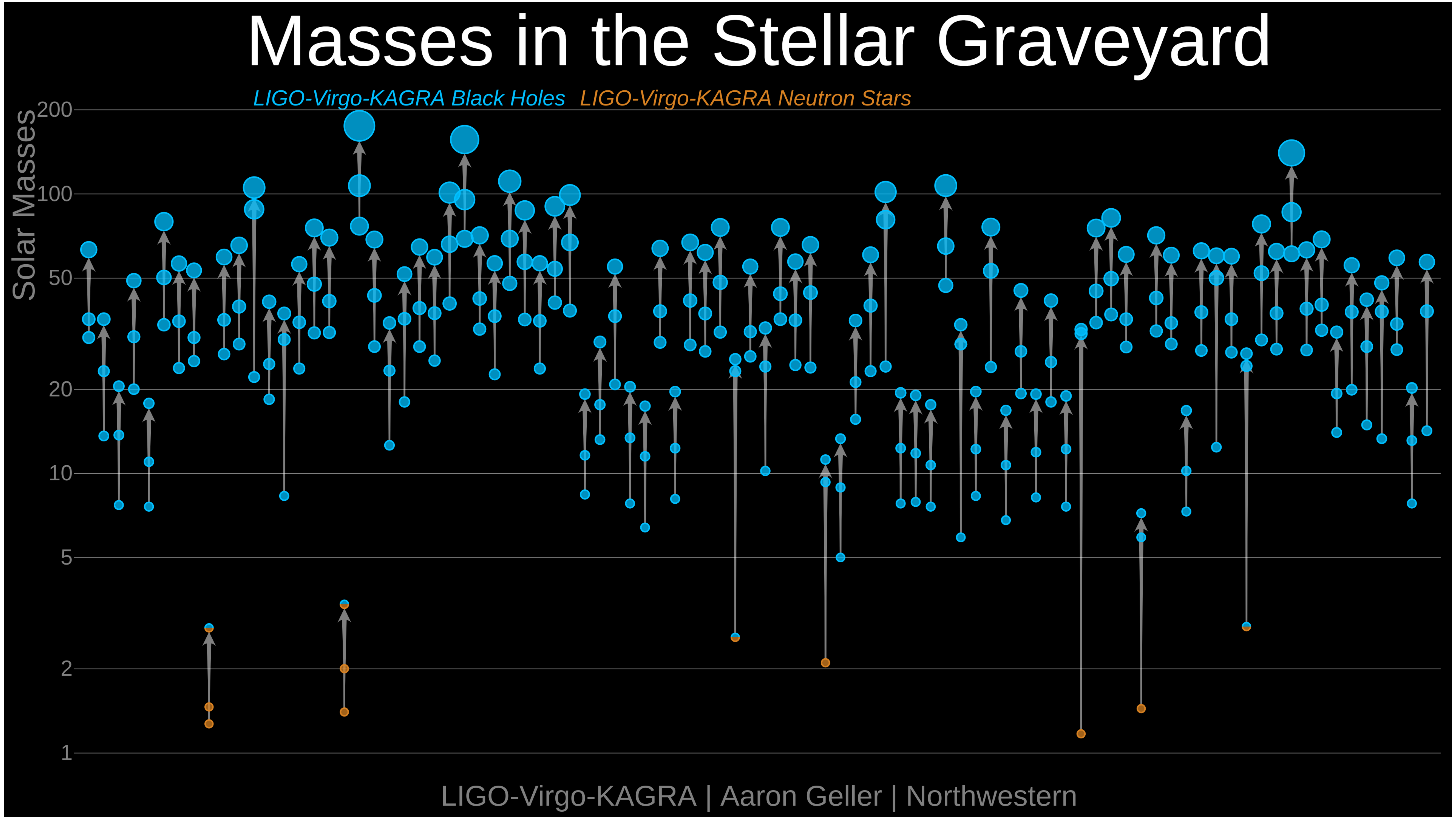
Further Applications



Gravitational Wave Observables

MKM, Mastrolia, Patil, Steinhoff (2022)
MKM, Mastrolia, Patil, Steinhoff (2022)
MKM, Mastrolia, O Silva, Patil, Steinhoff (2023)

GW observations

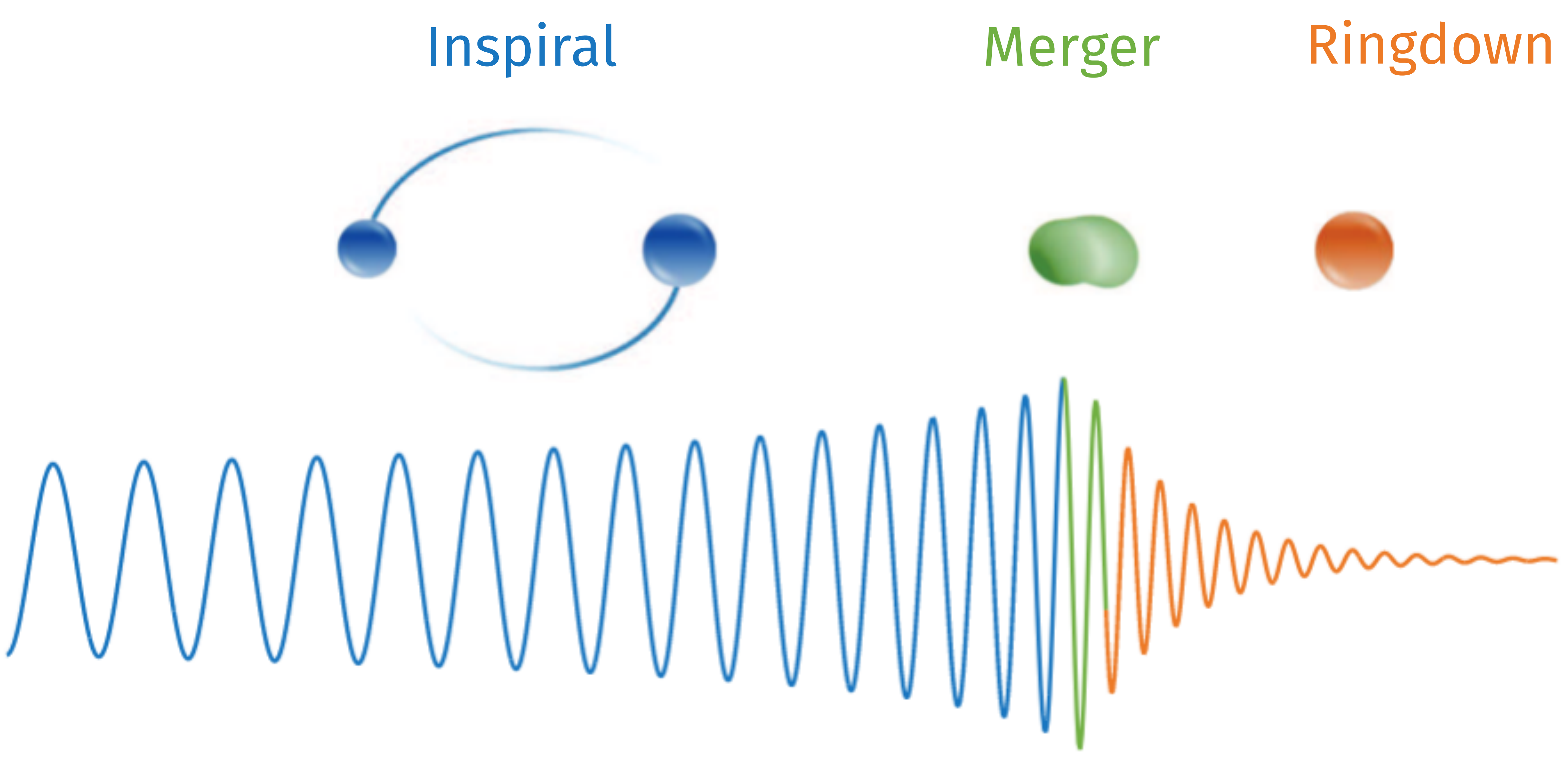


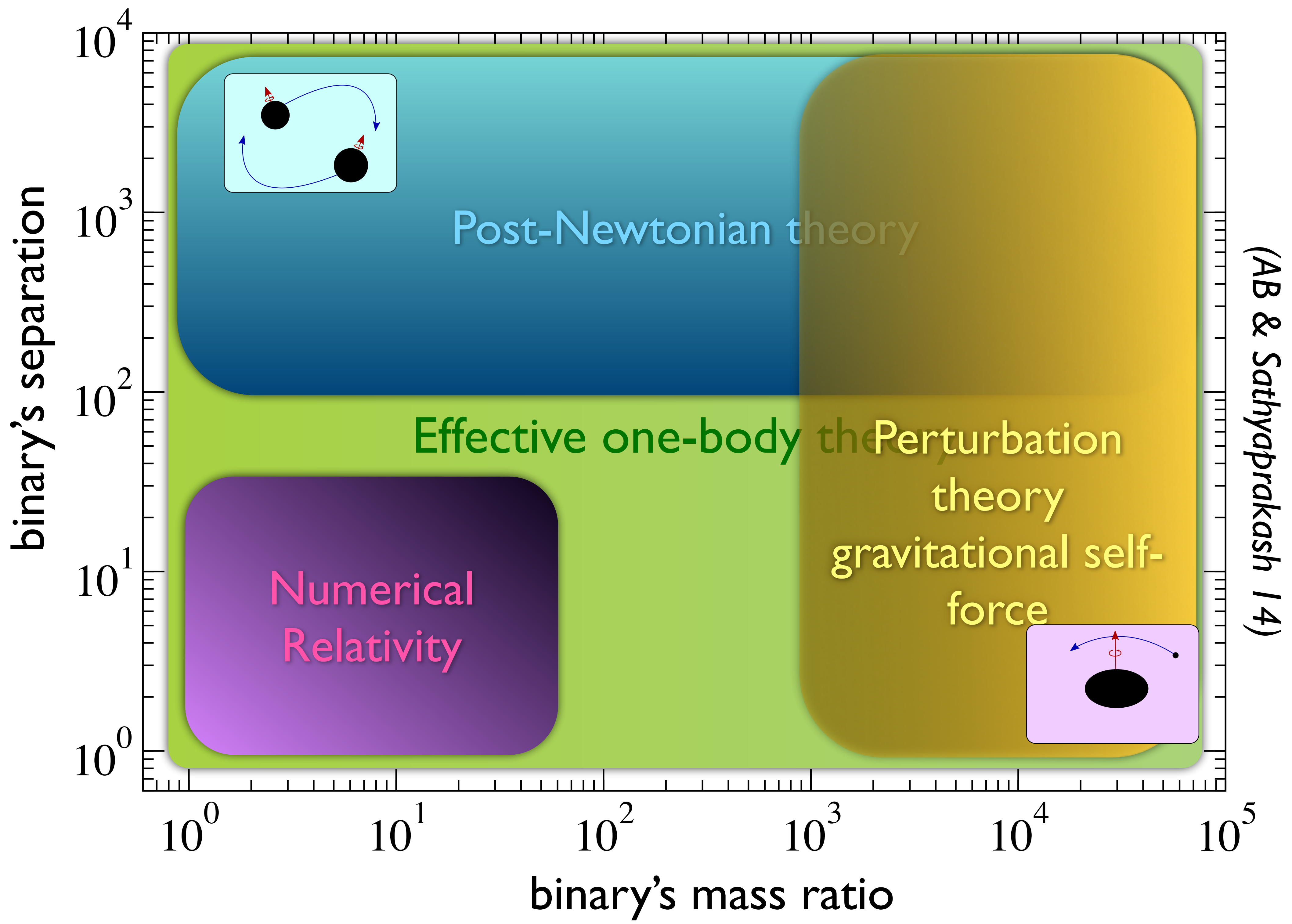
Tasks

- Supplement conventional Analysis
- Increase Theoretical Precision
- Perform Gravity phenomenology

Solving two-body problem in GR

Antelis, moreno (2016)





Post-Newtonian (PN)

Post-Minkowskian (PM)

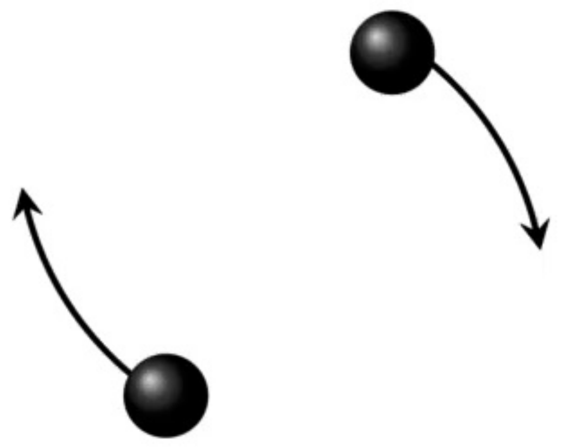
Numerical Relativity

Perturbation Theory

Analytical Approximation Methods

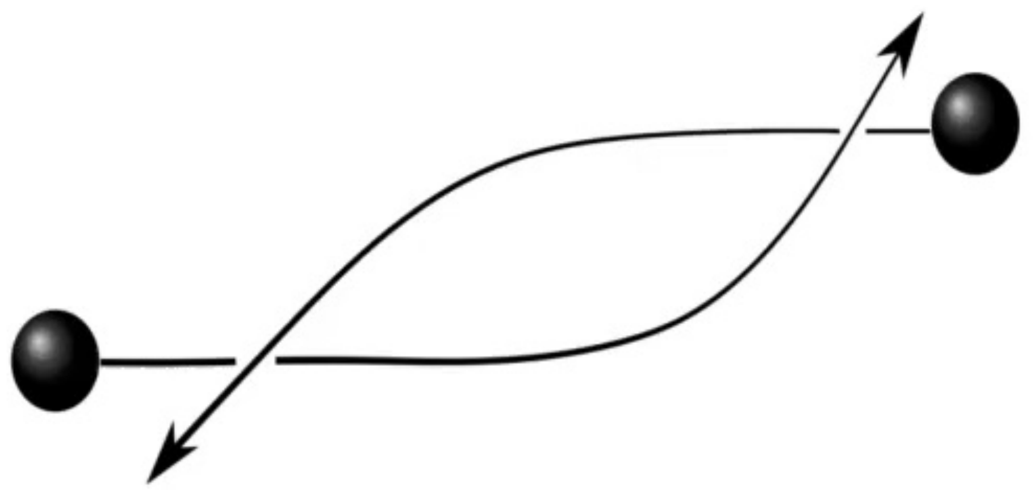
Post-Newtonian (PN)

$$\frac{v^2}{c^2} \sim \frac{GM}{rc^2} \ll 1$$



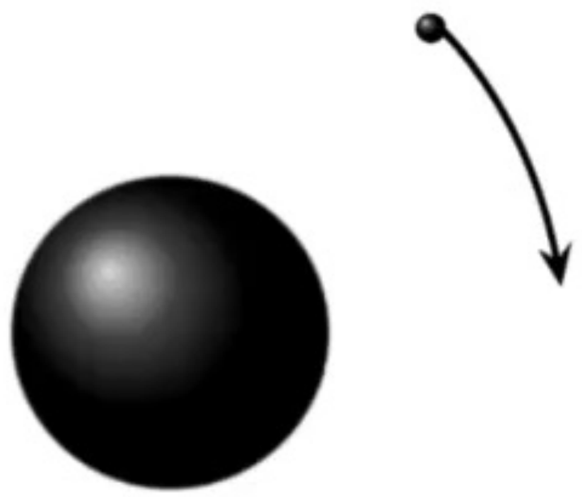
Post-Minkowskian (PM)

$$\frac{GM}{rc^2} \ll 1$$



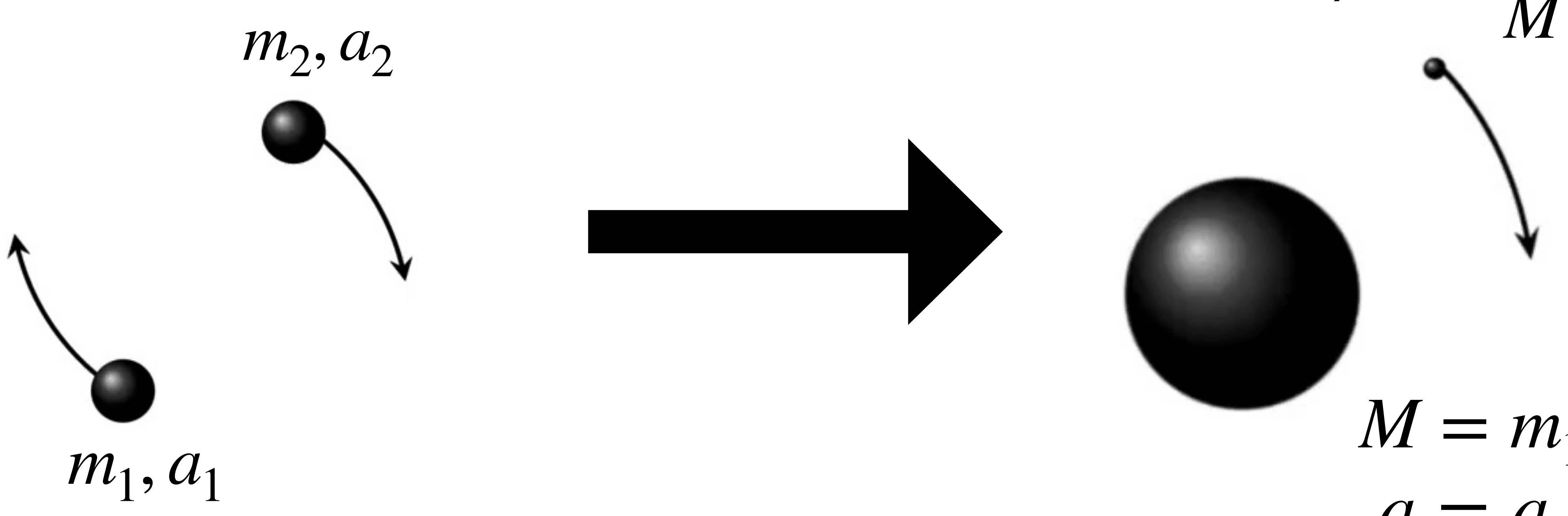
Self-Force (SF)

$$\frac{m_1}{m_2} \ll 1$$



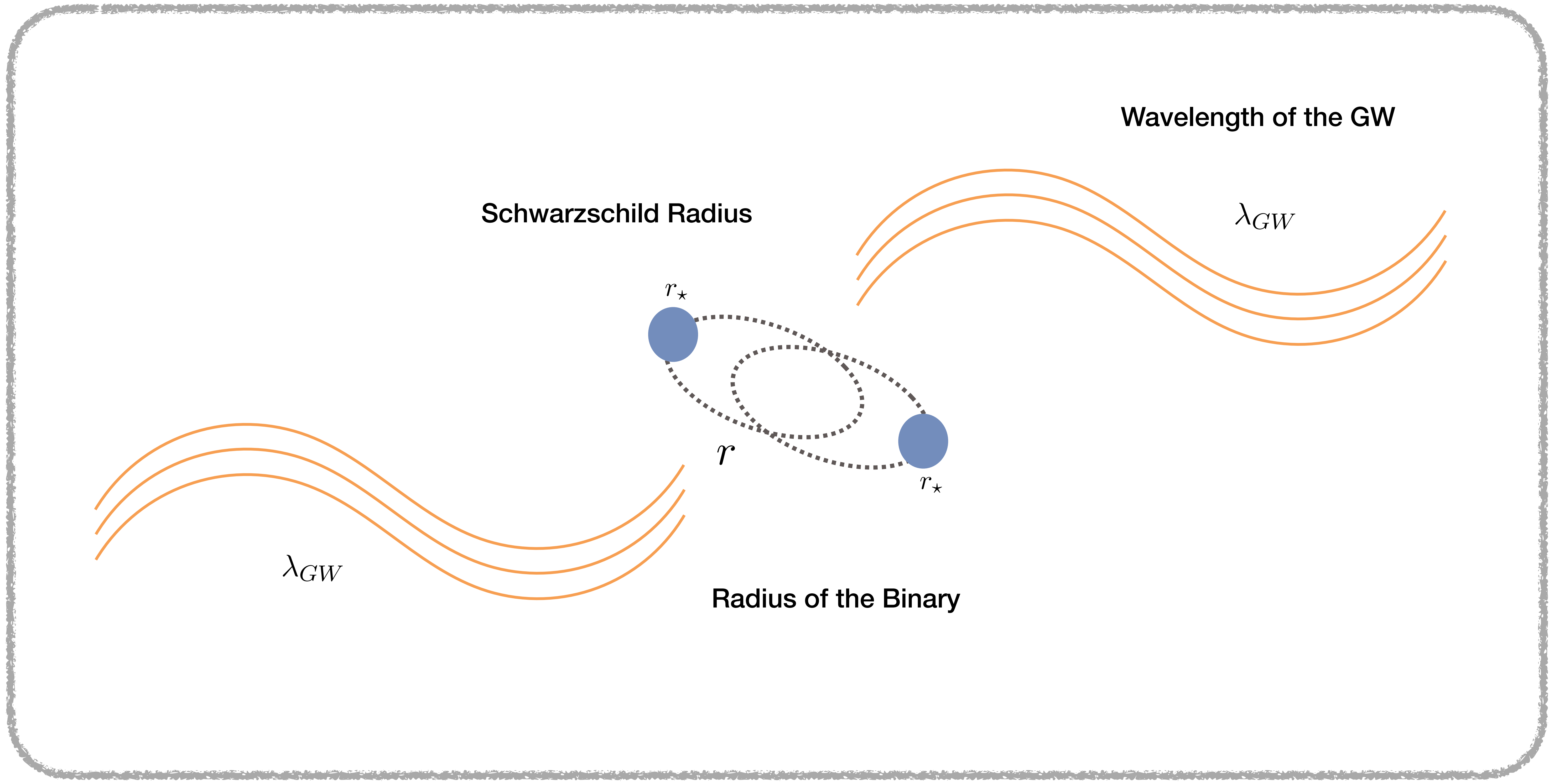
Effective One-Body (EOB)

$$\mu = \frac{m_1 m_2}{M}$$



$$M = m_1 + m_2$$
$$a = a_1 + a_2$$

Post-Newtonian Expansion EFT set up



Equations of Motion

$$\dot{r} = \frac{d\mathcal{H}}{dp_r} \qquad \dot{p}_r = -\frac{d\mathcal{H}}{dr} + \mathcal{F}_r$$

$$\dot{\phi} = \frac{d\mathcal{H}}{dp_\phi} \qquad \dot{p}_\phi = -\frac{d\mathcal{H}}{d\phi} + \mathcal{F}_\phi$$

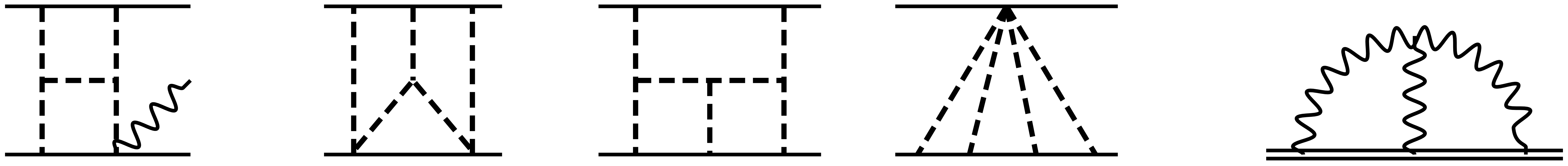
Need:

Hamiltonian $\mathcal{H}$

Radiation Reaction $\mathcal{F}$

Advantage of QFT techniques

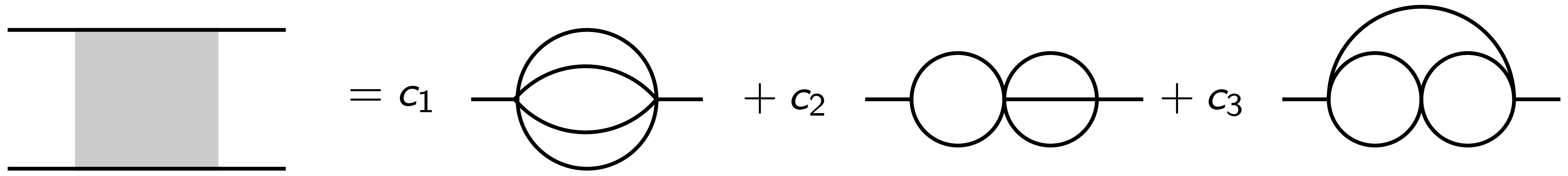
- Use of Feynman diagrams



- Dimensional regularization

Better to handle spurious divergences

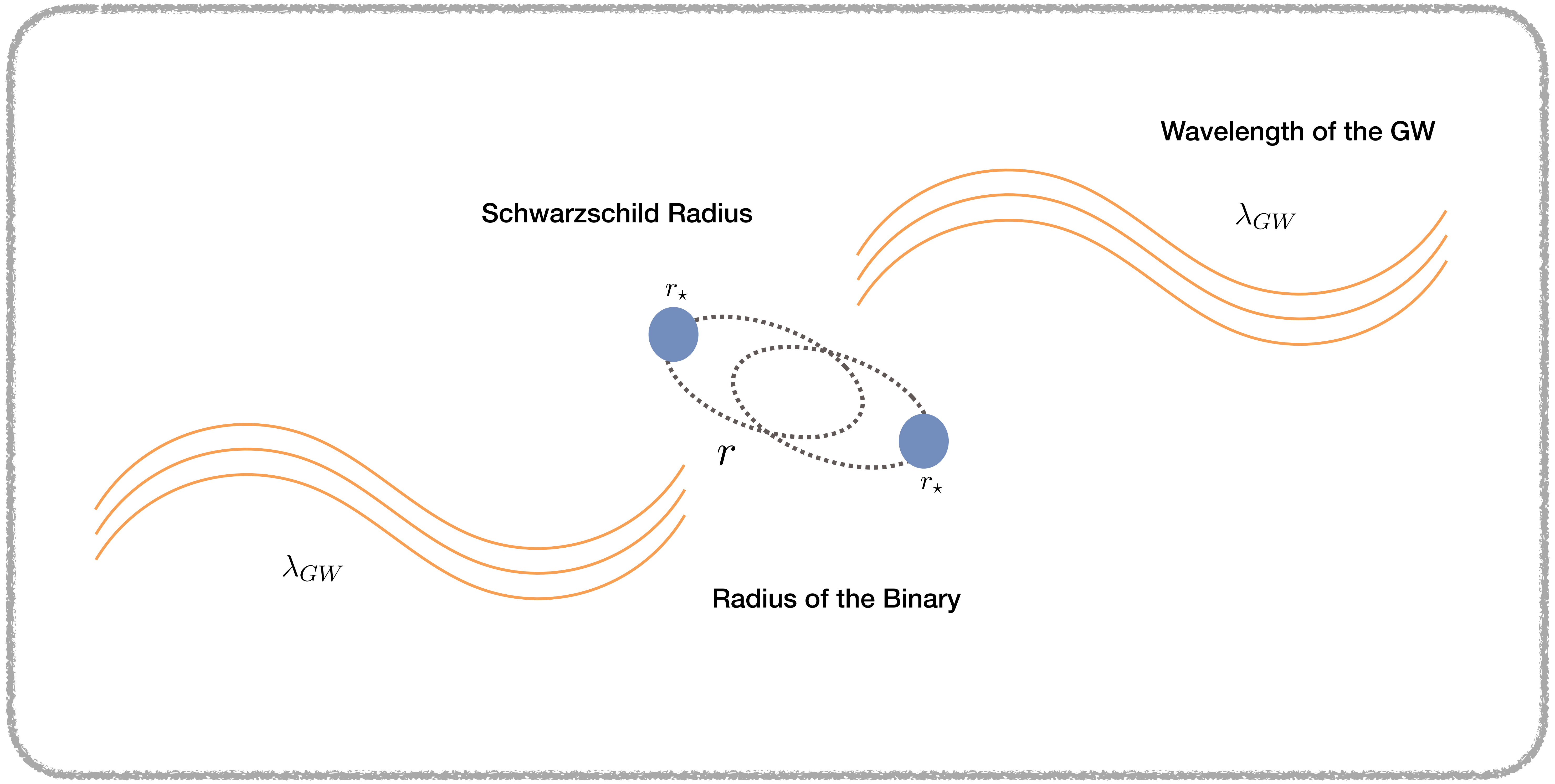
- Multi-loop Techniques



☑ IBP relations

☑ Differential Equations

Post-Newtonian Expansion EFT set up



Hierarchy of scales

$$r_\star << r << \lambda_{GW}$$

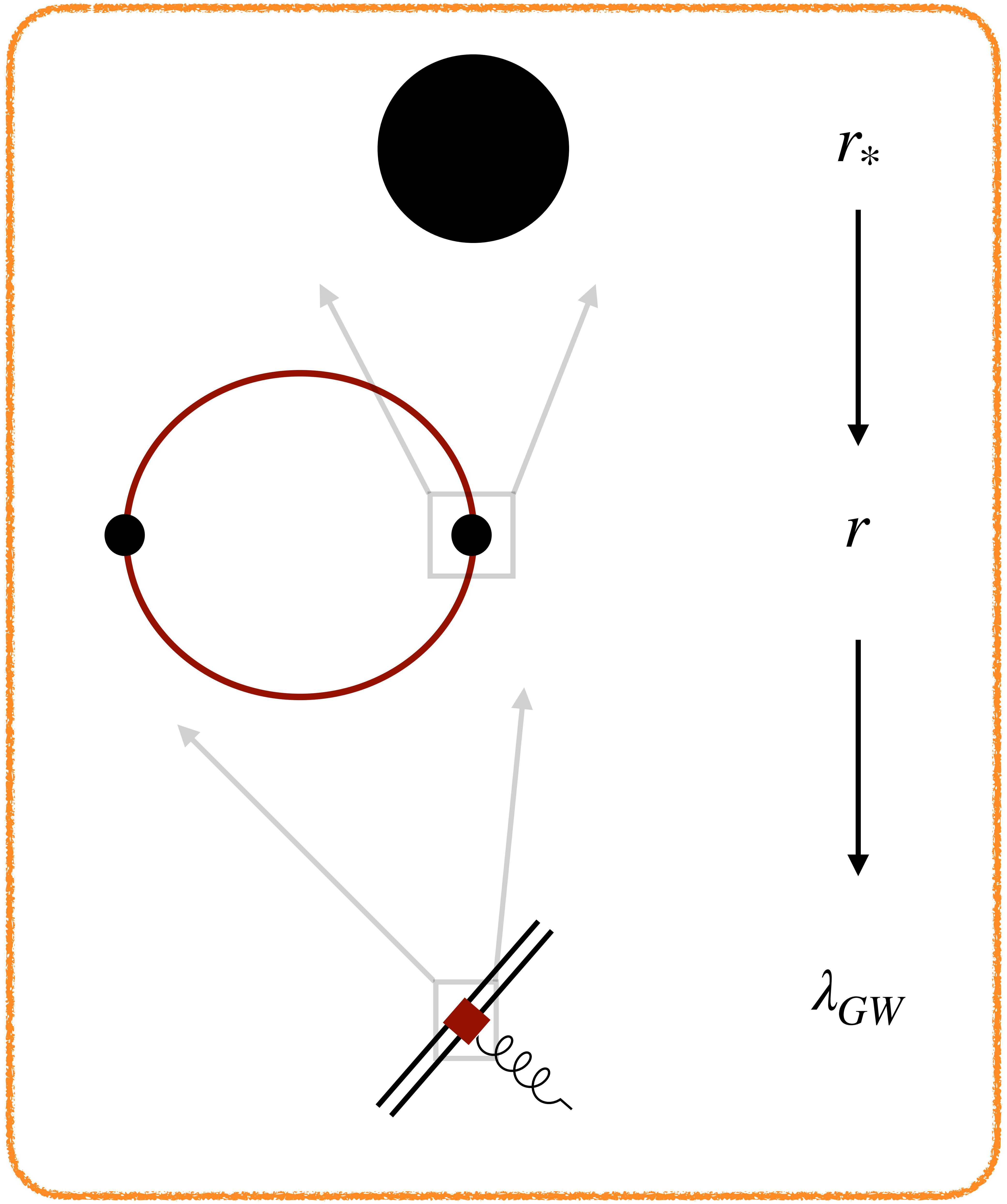
Post-Newtonian Expansion EFT set up

Hierarchy of scales

$$r_\star << r << \lambda_{GW}$$

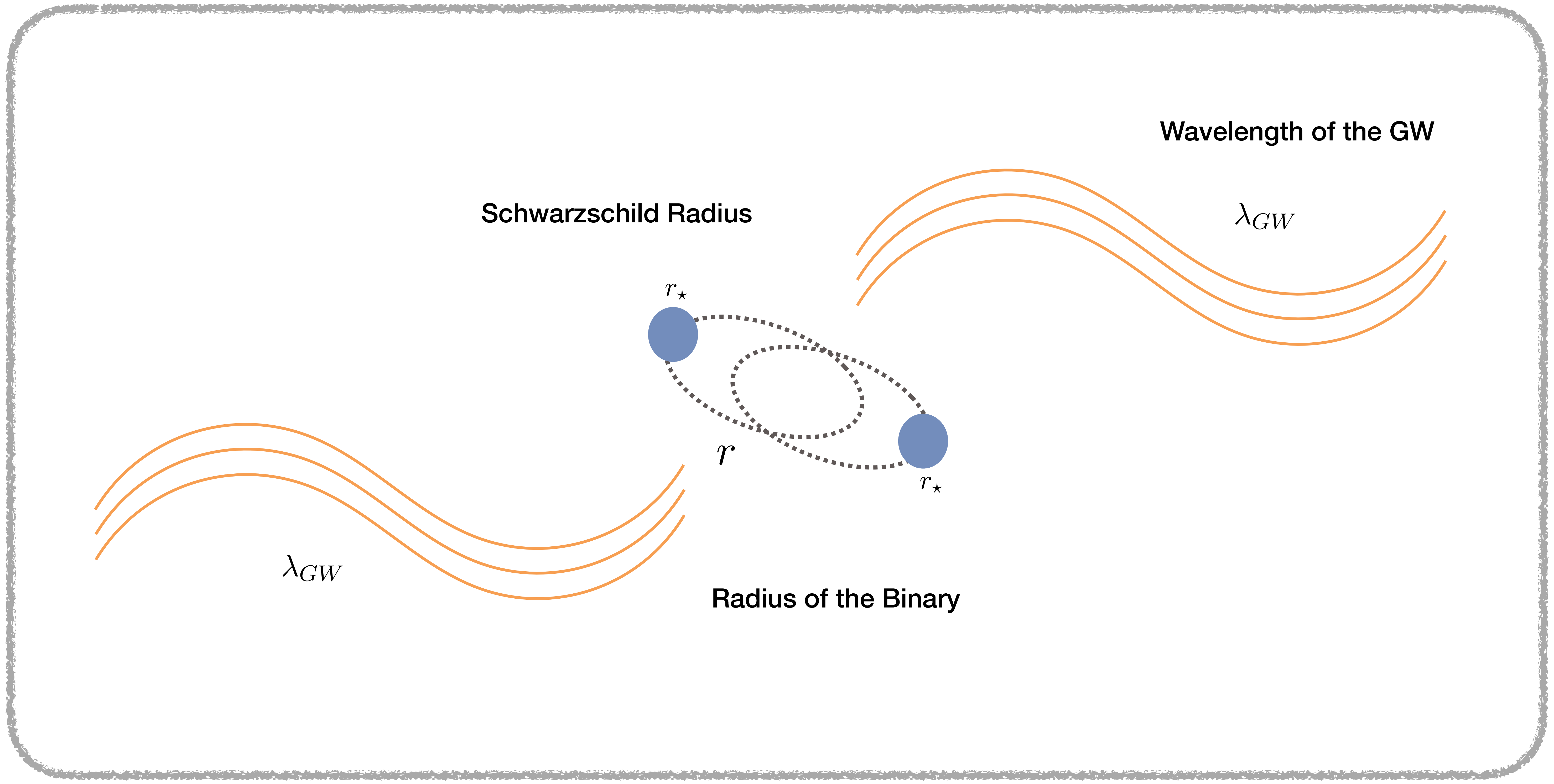
Tower of EFTs Goldberger, Rothstein

1. One-Particle EFT for Compact Object
2. EFT of Composite Particle for Binary
3. Effective Theory of Dynamical Multipoles



Post-Newtonian Expansion EFT set up

Goldberger, Rothstein



Hierarchy of scales

$$r_\star << r << \lambda_{GW}$$

Tower of EFTs

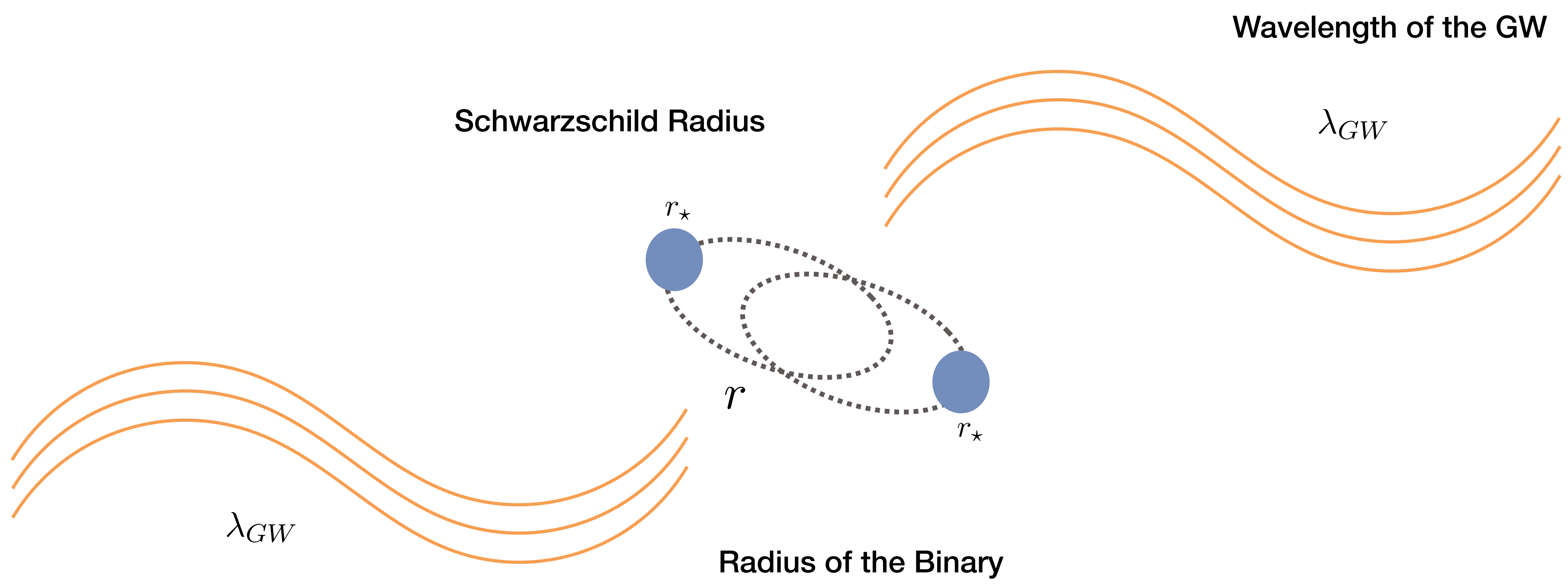
1. One-Particle EFT for Compact Object

$$S[g_{\mu\nu}] = -\frac{1}{16\pi G}\int d^4x\sqrt{g}R$$

$$S_{pp}[g_{\mu\nu}] = -m\int d\sigma\sqrt{u^2}$$

Post-Newtonian Expansion EFT set up

Goldberger, Rothstein



Hierarchy of scales

$$r_\star << r << \lambda_{GW}$$

Tower of EFTs

2. EFT of Composite Particle for Binary

$$S[g_{\mu\nu}] = -\frac{1}{16\pi G}\int d^4x\sqrt{g}R$$

$$g_{\mu\nu} = \eta_{\mu\nu} + H_{\mu\nu} + h_{\mu\nu}$$

$$S_{pp}[g_{\mu\nu}, x_K] = \sum_{K=1}^{2} - m_K \int d\sigma \sqrt{u_K^2}$$

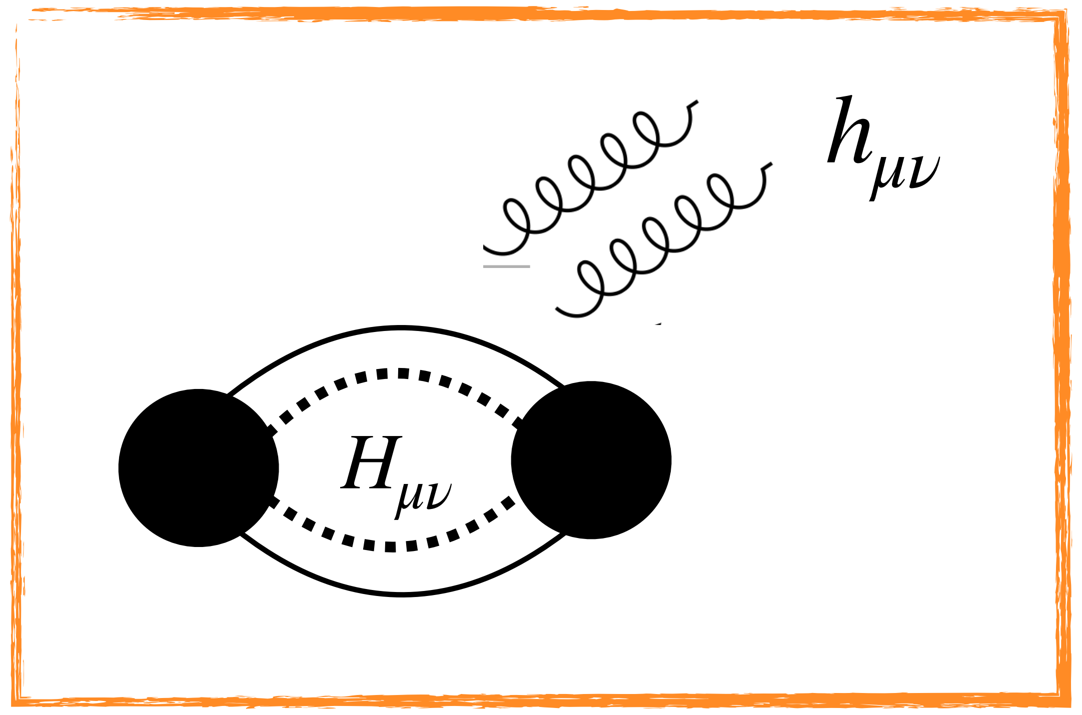
Method of Regions

potential gravitons $H_{\mu\nu}$ with scaling $(k_0, \mathbf{k}) \sim (v/r, 1/r)$

radiation gravitons $\bar{h}_{\mu\nu}$ with scaling $(k_0, \mathbf{k}) \sim (v/r, v/r)$

EFT at the orbital scale: Conservative Dynamics

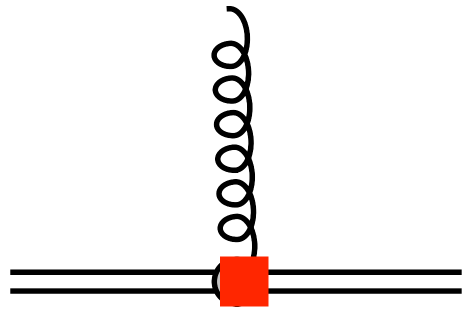
$$e^{i S_{eff}[x_K]} = \int \mathcal{D}\bar{h}_{\mu\nu} \int \mathcal{D}H_{\mu\nu} \, \exp\left\{ iS[\eta + \bar{h} + H] + i \sum_{K=1}^{2} S_{pp}[x_K(t), \eta + \bar{h} + H] \right\}$$



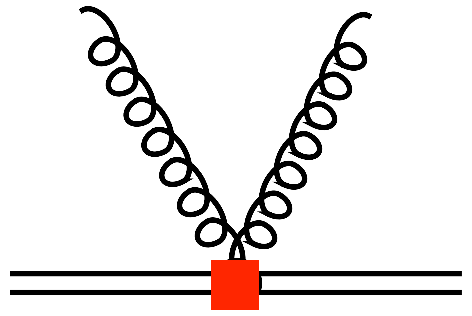
Effective Action for Dynamical Multipoles

$$e^{i S_{eff}[x_K]} = \int \mathcal{D}\bar{h}_{\mu\nu} \, \exp\left\{ iS[\eta + \bar{h}] + \right.$$

$+$



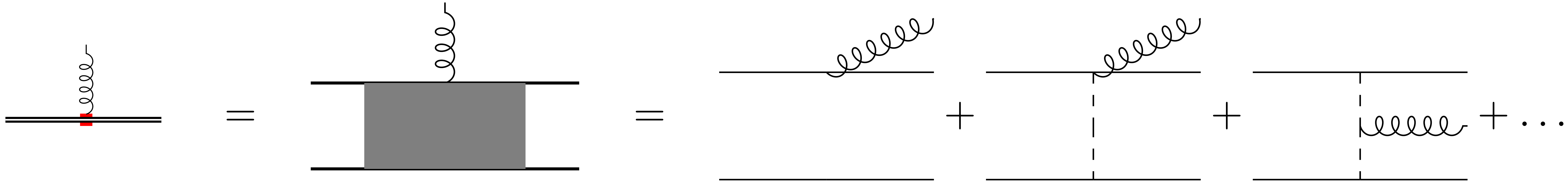
$+$



$$\left. + \cdots \right\}$$

Conservative Dynamics

$$\int \mathcal{D}H \, \exp\left\{ i\, S\,[\eta + H, h = 0] + i\, S_{pp}\,[x_K, \eta + H, h = 0] \right\} = e^{i\, S_{eff}[h=0, x_K]} = e^{i \int dt \, \mathcal{L}_{eff}}$$

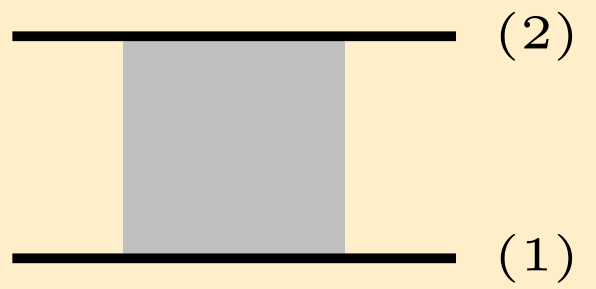


Potential for the 2-body system

Goldberger, Rothstein, Porto, Levi, …

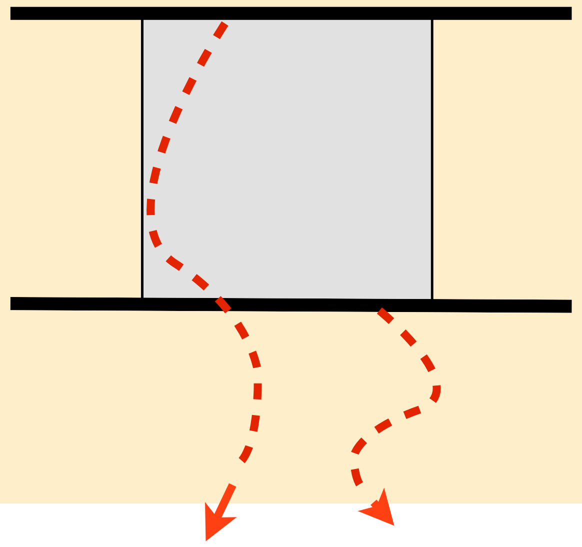
Foffa, Sturani, Sturm, Mastrolia (2016)

$$\mathcal{V}_{\rm eff} = \mathbf{i} \lim_{d\to 3} \int_{\mathbf{p}} e^{\mathbf{i}\mathbf{p}\cdot(\mathbf{x}_{(1)}-\mathbf{x}_{(2)})}$$



Key Observation

Diagrams in GR



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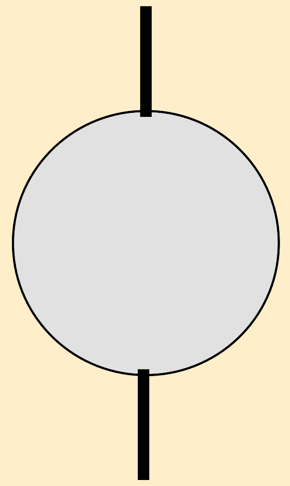
Diagrams in QFT



$$p^2 \equiv s \neq 0$$

Static Source :: Not Propagating

Status of PN Results

PN order	0	1	1.5	2	2.5	3	3.5	4	4.5	5	5.5	6	6.5	need up to
no spin	N	1PN		2PN		3PN		4PN		5PN		6PN		1PM
spin-orbit			LO SO		NLO SO		N2LO SO		N3LO SO		N4LO SO			2PM
spin^2				LO S2		NLO S2		N2LO S2		N3LO S2				3PM
spin^3							LO S3		NLO S3		N2LO S3			4PM
spin^4								LO S4		NLO S4				5PM
spin^5											LO S5		NLO S5	6PM
spin^6												LO S6		7PM

credit: Justin Vines

Levi, McLeod, Steinhoff, Teng, Von Hippel,..

Kim, Levi, Yin (2021)

Kim, Levi, Yin (2022)

MKM, Mastrolia, Patil, Steinhoff (2022)

Levi, Yin (2022)

MKM, Mastrolia, Patil, Steinhoff (2022)

1PN [Einstein, Infeld, Hoffman '38].

2PN [Ohta *et al.*, '73].

3PN [Jaranowski, Schaefer, '97; Damour, Jaranowski, Schaefer, '97; Blanchet, Faye, '00; Damour, Jaranowski, Schaefer, '01]

4PN [Damour, Jaranowski, Schäfer, Bernard, Blanchet, Bohe, Faye, Marsat, Marchand, Foffa, Sturani, Mastrolia, Sturm, Porto, Rothstein...]

5PN [Foffa, Mastrolia, Sturani, Sturm, Bodabilla, '19; Blümlein, Maier, Marquard, '19; Bini, Damour, Geralico, '19; Blümlein, Maier, Marquard, '19; Almeida, Foffa, Sturani, '22;]

Status of PN Results

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no spin	N	1PN		2PN		3PN		4PN		5PN		6PN	
spin-orbit			LO SO		NLO SO		N2LO SO		N3LO SO		N4LO SO		
spin^2				LO S2		NLO S2		N2LO S2		N3LO S2			
spin^3							LO S3		NLO S3		N2LO S3		
spin^4								LO S4		NLO S4			
spin^5											LO S5		NLO S5
spin^6												LO S6	

need up to

1PM
2PM
3PM
4PM
5PM
6PM
7PM

credit: Justin Vines

Levi, McLeod, Steinhoff, Teng, Von Hippel,..

Kim, Levi, Yin (2021)

Kim, Levi, Yin (2022)

MKM, Mastrolia, Patil, Steinhoff (2022)

Levi, Yin (2022)

MKM, Mastrolia, Patil, Steinhoff (2022)

Brunello, MKM, Mastrolia, Patil (W.I.P)

1PN [Einstein, Infeld, Hoffman '38].

2PN [Ohta *et al.*, '73].

3PN [Jaranowski, Schaefer, '97; Damour, Jaranowski, Schaefer, '97; Blanchet, Faye, '00; Damour, Jaranowski, Schaefer, '01]

4PN [Damour, Jaranowski, Schäfer, Bernard, Blanchet, Bohe, Faye, Marsat, Marchand, Foffa, Sturani, Mastrolia, Sturm, Porto, Rothstein...]

5PN [Foffa, Mastrolia, Sturani, Sturm, Bodabilla, '19; Blümlein, Maier, Marquard, '19; Bini, Damour, Geralico, '19; Blümlein, Maier, Marquard, '19; Almeida, Foffa, Sturani, '22;]

EFT of Spinning Objects

Levi, Steinhoff (2015)

$$S_{\rm EH} = -\frac{c^4}{16\pi G_N}\int d^4x\sqrt{g}\;R[g_{\mu\nu}] + \frac{c^4}{32\pi G_N}\int d^4x\sqrt{g}\;g_{\mu\nu}\Gamma^\mu\Gamma^\nu$$

$$\Omega^{\mu\nu}_{(a)} = \Lambda^\mu_{(a)A}\frac{d\Lambda^{A\nu}_{(a)}}{d\tau}$$

$$S_{\rm pp} = \sum_{a=1,2}\int d\tau\left(-m_{(a)}c\sqrt{u^2_{(a)}} - \frac{1}{2}S_{(a)\mu\nu}\Omega^{\mu\nu}_{(a)} - \frac{S_{(a)\mu\nu}u^\nu_{(a)}}{u^2_{(a)}}\frac{du^\mu_{(a)}}{d\tau} + \mathcal{L}^{(R)}_{(a)} + \mathcal{L}^{(R^2)}_{(a)} + \ldots\right)$$

$$S_{(a)\mu\nu} = -2\frac{\partial L_{\rm pp}}{\partial\Omega^{\mu\nu}_{(a)}}$$

$$\mathcal{L}^{(R)}_{(a)} = -\frac{1}{2m_{(a)}c}\left(C^{(0)}_{\rm ES^2}\right)_{(a)}\frac{E_{\mu\nu}}{u_{(a)}}\left[S^\mu_{(a)}S^\nu_{(a)}\right]_{\rm STF} + \ldots.$$

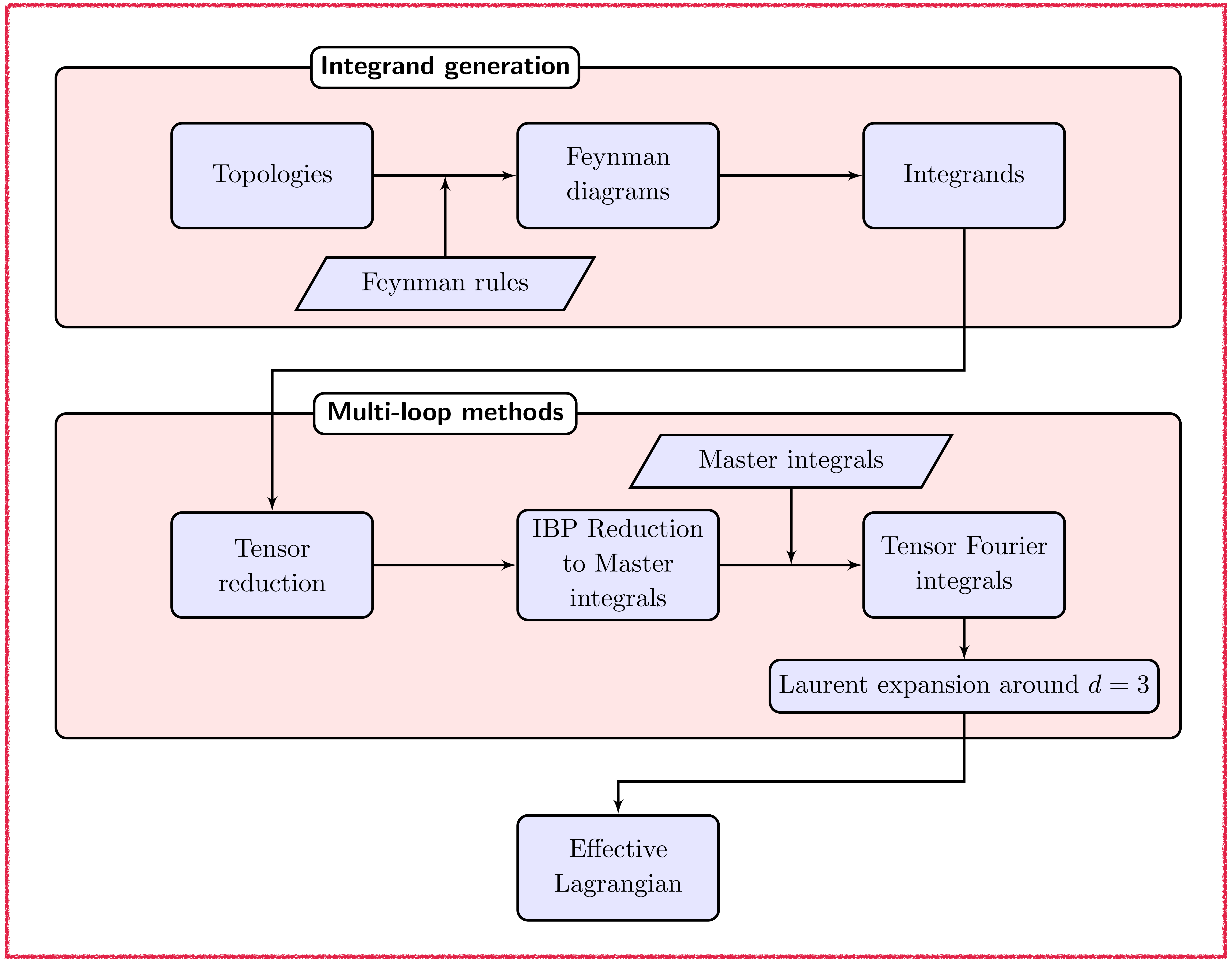
$$\mathcal{L}^{(R^2,S^0)}_{(a)} = \frac{1}{2}\left(C^{(2)}_{\rm E^2}\right)_{(a)}\frac{G_N^2 m_{(a)}}{c^5}\frac{E_{\mu\nu}E^{\mu\nu}}{u^3_{(a)}}S^2_{(a)} + \ldots.$$

$$\mathcal{L}^{(R^2,S^2)}_{(a)} = \frac{1}{2}\left(C^{(0)}_{\rm E^2S^2}\right)_{(a)}\frac{G_N^2 m_{(a)}}{c^5}\frac{E_{\mu\alpha}E_\nu{}^\alpha}{u^3_{(a)}}\left[S^\mu_{(a)}S^\nu_{(a)}\right]_{\rm STF} + \ldots.$$

Computational Algorithm : Towards Automation

MKM, Mastrolia, Patil, Steinhoff (2022)

MKM, Mastrolia, Patil, Steinhoff (2022)



- ☑ Automated in-house codes

- Aim to publish the code in future

Diagrams for Spinning Binaries

MKM, Mastrolia, Patil, Steinhoff (2022)

MKM, Mastrolia, Patil, Steinhoff (2022)

S^0

Order	Diagrams	Loops	Diagrams
0PN	1	0	1
1PN	4	1	1
		0	3
2PN	21	2	5
		1	10
		0	6
3PN	130	3	8
		2	75
		1	38
		0	9

(a) Non-spinning sector

S^1

Order	Diagrams	Loops	Diagrams
LO	2	0	2
NLO	13	1	8
		0	5
N^2LO	100	2	56
		1	36
		0	8
N^3LO	894	3	288
		2	495
		1	100
		0	11

(b) Spin-orbit sector

S^2

Order	Diagrams	Loops	Diagrams
LO	1	0	1
NLO	7	1	3
		0	4
N^2LO	58	2	27
		1	24
		0	7
N^3LO	553	3	125
		2	342
		1	76
		0	10

(a) Spin1-Spin2 and Spin1^2 (Spin2^2) sector

Order	Diagrams	Loops	Diagrams
LO	1	0	1
NLO	4	1	1
		0	3
N^2LO	25	2	7
		1	12
		0	6
N^3LO	168	3	15
		2	101
		1	43
		0	9

(b) ES^2 sector

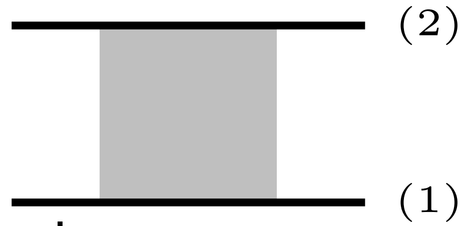
Order	Loops	Diagrams
LO	1	1

(c) E^2 sector

Order	Loops	Diagrams
LO	1	1

(d) E^2S^2 sector

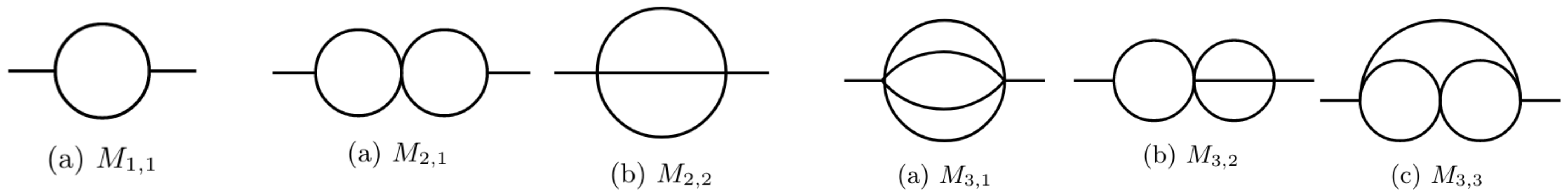
$$\mathcal{L}(x_a, \dot{x}_a, \ddot{x}_a, \ldots S_a, \dot{S}_a, \ddot{S}_a, \ldots) = -\,\mathbf{i} \lim_{d\to 3} \int_{\mathbf{p}} e^{\mathbf{i}\mathbf{p}\cdot(\mathbf{x}_{(1)} - \mathbf{x}_{(2)})}$$



$$= -\,\mathbf{i} \lim_{d\to 3} \int_{\mathbf{p}} e^{\mathbf{i}\mathbf{p}\cdot(\mathbf{x}_{(1)} - \mathbf{x}_{(2)})}$$

- Dimensional Regularization d= 3+ϵ
- IBP Decomposition

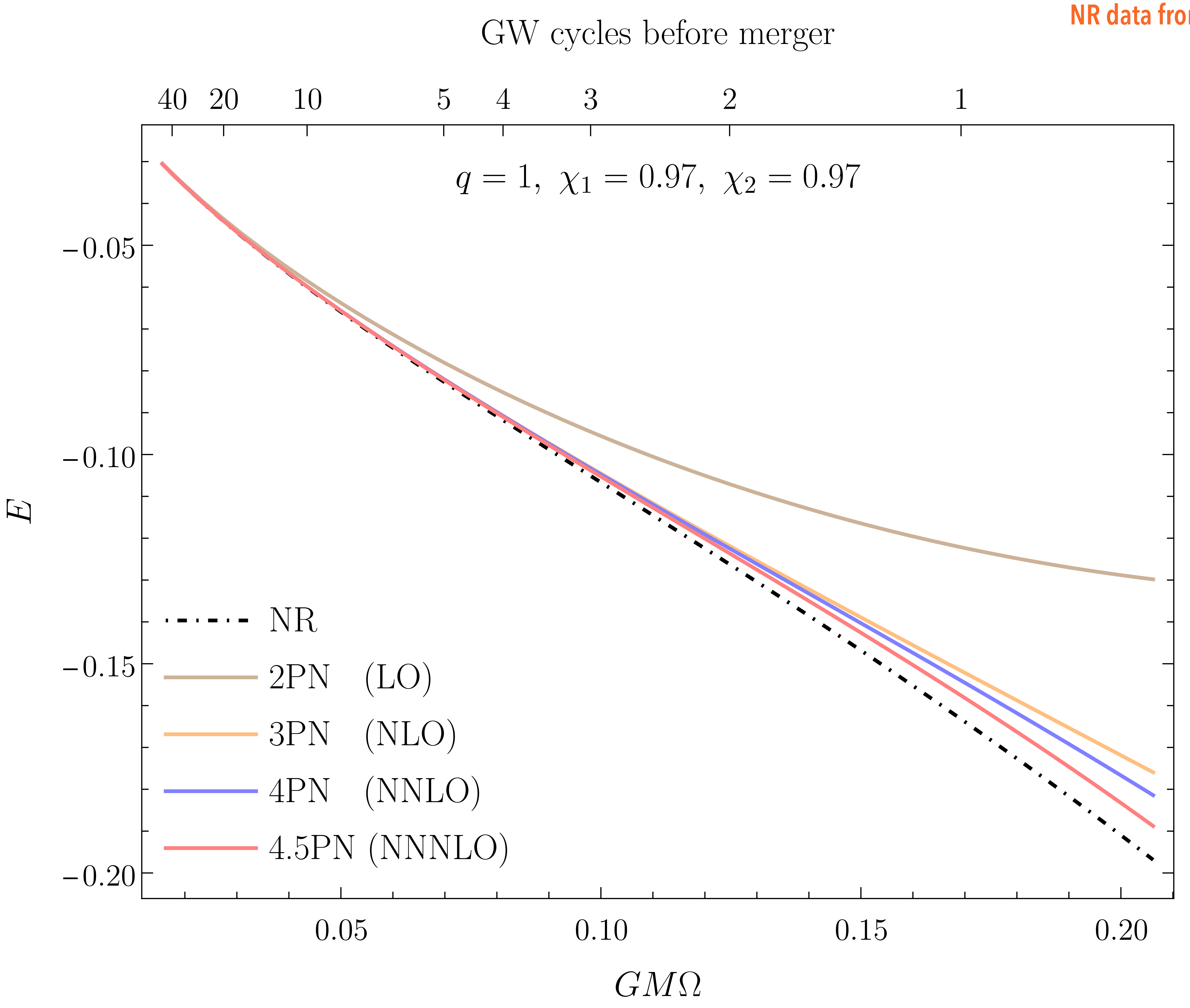
- Three Loop MIs



Binding Energy for Spin-Orbit Coupling

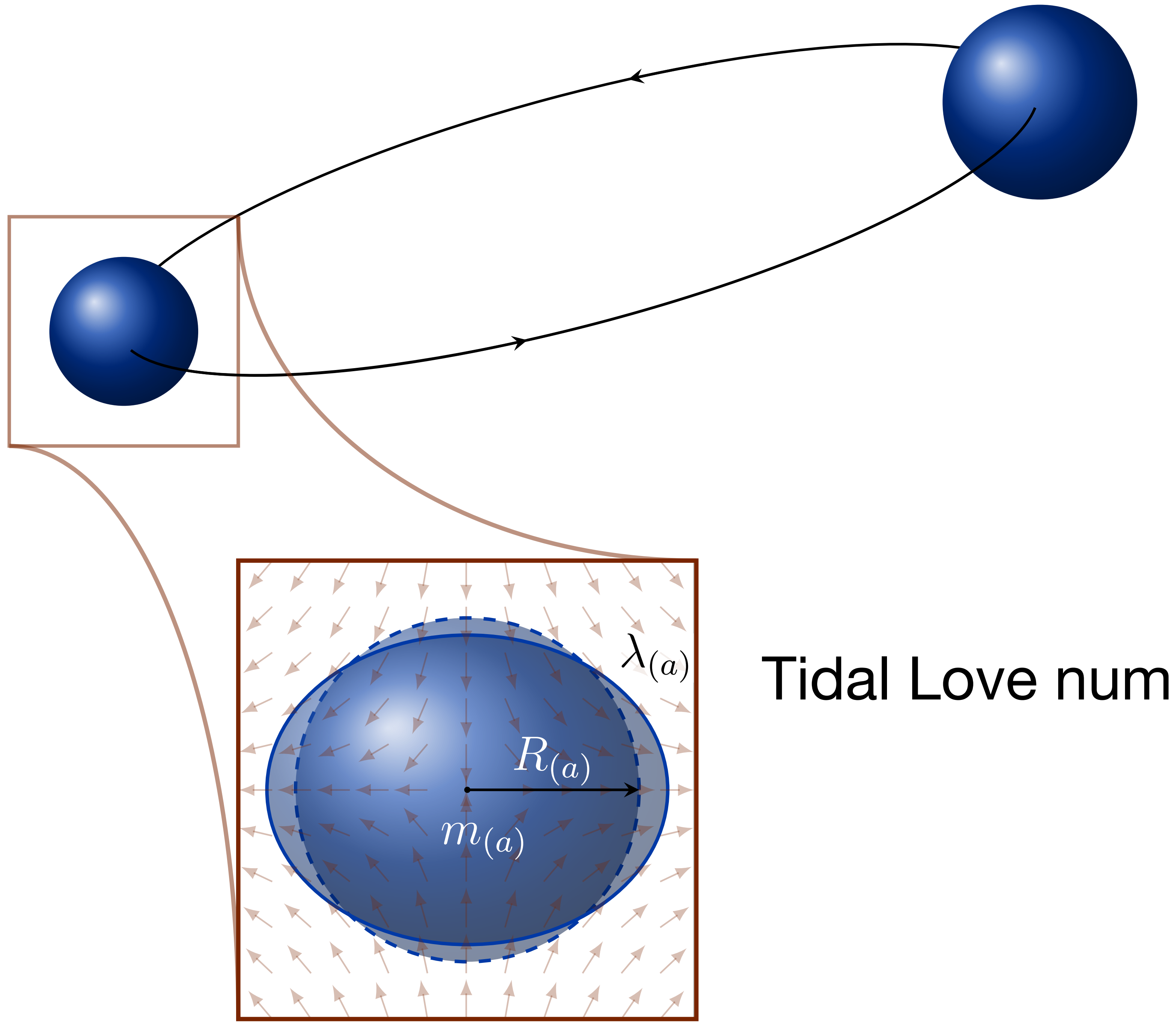
MKM, Mastrolia, Patil, Steinhoff (2022)

NR data from Ossokine, Dietrich, Foley, Katebi, Lovelace (2018)



Tidal Effects

MKM, Mastrolia, O Silva, Patil, Steinhoff (2023)



Tidal Love number

- NS features a number of oscillation modes
- The dominant mode is known as *f-mode*, which is the lowest frequency surface gravity waves
- The frequency depend only on the mean density of the star and not on the Equation of State of the NS
- The f-modes dynamical tides are important as it significantly affect the inference of the equations of state of NS

Pratten, Schmidt, Williams (2022)

Dynamical Electric Tides at 2 PN

MKM, Mastrolia, O Silva, Patil, Steinhoff (2023)

$$\mathcal{L}_{\rm DT} = \frac{z}{4\lambda\omega_f^2}\left[\frac{c^2}{z^2}\frac{{\rm d}Q_{\mu\nu}}{{\rm d}\tau}\frac{{\rm d}Q^{\mu\nu}}{{\rm d}\tau} - \omega_f^2 Q_{\mu\nu}Q^{\mu\nu}\right] - \frac{z}{2}E_{\mu\nu}Q^{\mu\nu}$$

Adiabatic limit: $\omega_f \to \infty$

$$Q_{\mu\nu} = -\lambda E_{\mu\nu}$$

Tidal deformability

Binding Energy

$$E_{\rm AT}(x,\widetilde{\lambda}_{(a)}) = -x^6(9\widetilde{\lambda}_{(+)}) + x^7\left[\left(\frac{33}{4}\nu - \frac{121}{8}\right)\widetilde{\lambda}_{(+)} - \left(\frac{55}{8}\right)\delta\widetilde{\lambda}_{(-)}\right]$$
$$+ x^8\left[\left(-\frac{91}{16}\nu^2 + \frac{2717}{42}\nu - \frac{20865}{224}\right)\widetilde{\lambda}_{(+)} + \left(\frac{715}{48}\nu - \frac{11583}{224}\right)\delta\widetilde{\lambda}_{(-)}\right]$$

Scattering Angle

$$\frac{\chi_{\rm AT}}{\Gamma} = \frac{1}{Mb^4}\left[\lambda_{(+)} \quad \delta\lambda_{(-)}\right]\cdot\left\{\pi\left(\frac{G_N M}{v^2 b}\right)^2\begin{bmatrix}1\\0\end{bmatrix}\left\{\frac{45}{16} + \frac{135}{32}\left(\frac{v^2}{c^2}\right) + \frac{1575}{256}\left(\frac{v^4}{c^4}\right)\right\}\right.$$
$$+\left(\frac{G_N M}{v^2 b}\right)^3\left\{48\begin{bmatrix}1\\0\end{bmatrix} + \begin{bmatrix}732/5\\12\end{bmatrix}\left(\frac{v^2}{c^2}\right) + \frac{3}{35}\begin{bmatrix}3073\\593\end{bmatrix}\left(\frac{v^4}{c^4}\right)\right\}$$
$$\left.+\pi\left(\frac{G_N M}{v^2 b}\right)^4\left\{\frac{315}{8}\begin{bmatrix}1\\0\end{bmatrix} + \frac{315}{64}\begin{bmatrix}51-2\nu\\5\end{bmatrix}\left(\frac{v^2}{c^2}\right) + \frac{15}{128}\begin{bmatrix}5331-274\nu\\1383\end{bmatrix}\left(\frac{v^4}{c^4}\right)\right\}\right\}$$

Conclusion

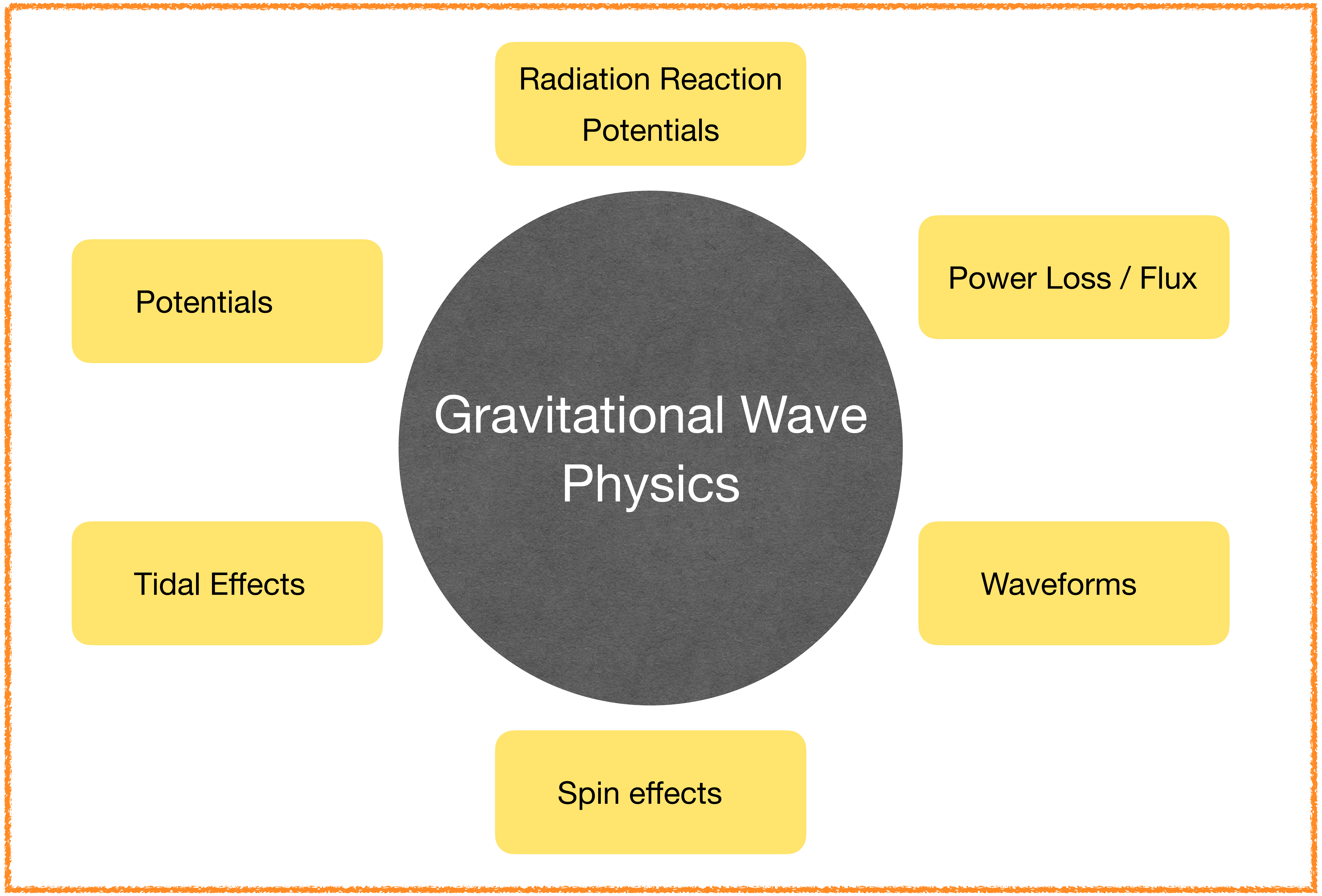
- ☑ Novel Algebraic Property Unveiled
  - ☑ The algebra of Feynman Integrals is controlled by intersection numbers
  - ☑ Intersection Numbers : Scalar Product/Projection between Feynman Integrals
  - ☑ Useful for both Physics and Mathematics

- ☑ Applications to GW phenomenology
  - ☑ progress in understanding spin effects / tidal effects for the compact binaries
  - ☑ A number of observables e.g binding energy, scattering angle has been computed to high precision

Outlook


Feynman Integral
Quantum Field Theory
Graph Polynomials
Quadratic Relations
Theoretical Physics
Master Integrals
Differential Equations
Vector Space Structure
IBP Identities
Intersection Theory
For
Twisted de-Rham Co-homology
Aomoto-Gel'fand Integral
Differential and Algebraic Geometry
Multivalued Functions
Twisted Reimann Period Relations
Mathematics
Independent Forms And Contours
Differential Equations
Cohomology Group
Contiguity Relation

Outlook


Radiation Reaction Potentials
Potentials
Power Loss / Flux
Gravitational Wave Physics
Tidal Effects
Waveforms
Spin effects

Thank You