GAGRA = Gauge Theories and Gravity (without relying on supersymmetry)

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Main research in Roma1 Unit:
Large-N QCD
and QCD-like theories

Large-N QCD

Plan of the talk (in 20 minutes !)

History of large-N QCD

('t Hooft, Veneziano, Migdal, Witten; circa 1974-79)

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. . .

Present (GAGRA collaboration; circa 2023)

Computation in a closed form of the UV asymptotics of the generating functional of all the correlators of all the twist-2 operators with maximal spin components to the next-to-leading large-N order in Yang-Mills theory (by lowest-order perturbation theory, renormalization group (RG), a certain new understanding of operator mixing, plus some nonperturbative insight = logDet)…linked to some new idea for a nonperturbative (partial) solution

SU(N) QCD (with N_f massless quarks, for simplicity):

$$Z = \int \delta A \delta\psi \delta\bar{\psi} \exp(-\frac{N}{g^2} \int Tr F^2 + \sum_{N_f} \bar{\psi}_f \gamma_\alpha D_\alpha \psi_f)$$

is a theory of gluons and quarks massless to every order of perturbation theory, which are weakly coupled in the UV but strongly coupled in the infrared (IR) because of the asymptotic freedom

Its solution must unavoidably be nonperturbative.
Indeed, every physical mass scale of the theory must be proportional to the RG-invariant scale:

$$\Lambda_{RG} = const\, \Lambda\ \exp\left(-\frac{1}{2\beta_0 g^2}\right) (\beta_0 g^2)^{-\frac{\beta_1}{2\beta_0^2}} \left(1 + \sum_{n=1} c_n g^{2n}\right)$$

that vanishes to every order of perturbation theory

Thus, the solution for the physical mass spectrum -
and a fortiori for the S matrix -
is equivalent to solving a nonperturbative weak-coupling problem - for Λ_{RG} in terms of g -
of the finest asymptotic accuracy,
as g vanishes while the cutoff Lambda
diverges in order for Λ_{RG} to stay finite.

In relation to the problem above,
a considerable simplification occurs
in 't Hooft large-N limit with N_f fixed.

Indeed, the large-N 't Hooft limit of SU(N) QCD (with N_f massless quarks):

$$Z = \int \delta A \delta\psi \delta\bar{\psi} \exp(-\frac{N}{g^2} \int TrF^2 + \sum_{N_f} \bar{\psi}_f \gamma_\alpha D_\alpha \psi_f)$$

is a free theory of glueballs and mesons to leading 1/N order, which become weakly coupled at all energy scales to the next order, with couplings O(1/N) and O(1/sqrt N) respectively (G. 't Hooft 1974)

In the glueball sector:

$$< \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\cdots\mathcal{O}_n(x_n) >_{conn} \sim N^{2-n}$$

In the meson sector:

$$< \mathcal{M}_1(x_1)\mathcal{M}_2(x_2)\cdots\mathcal{M}_k(x_k) >_{conn} \sim N^{1-\frac{k}{2}}$$

In the meson/glueball sector:

$$< \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\cdots\mathcal{O}_n(x_n)\mathcal{M}_1(x_1)\mathcal{M}_2(x_2)\cdots\mathcal{M}_k(x_k) >_{conn} \sim N^{1-n-\frac{k}{2}}$$

In fact, to the leading 1/N order (i.e., 't Hooft planar theory), because of the vanishing of the interaction associated to 3 and multi-point correlators,

the connected two-point correlators, by assuming confinement, are an infinite sum of free propagators satisfying the the Kallen-Lehmann representation (Migdal, 1977):

$$\int \langle \mathcal{O}^{(s)}(x)\mathcal{O}^{(s)}(0)\rangle_{conn} e^{-ip\cdot x} d^4x = \sum_{n=1}^{\infty} P^{(s)}\left(\frac{p_\alpha}{m_n^{(s)}}\right) \frac{|<0|\mathcal{O}^{(s)}(0)|p,n,s>'|^2}{p^2+m_n^{(s)2}}$$

$$<0|\mathcal{O}^{(s)}(0)|p,n,s,j> = e_j^{(s)}\left(\frac{p_\alpha}{m}\right) <0|\mathcal{O}^{(s)}(0)|p,n,s>'$$

$$\sum_j e_j^{(s)}\left(\frac{p_\alpha}{m}\right)\overline{e_j^{(s)}\left(\frac{p_\alpha}{m}\right)} = P^{(s)}\left(\frac{p_\alpha}{m}\right)$$

Hence, the large-N nonperturbative solution would replace QCD

viewed as a theory of
gluons and quarks that is strongly coupled in the infrared in perturbation theory,

with a theory of glueballs and mesons
that is weakly coupled at all energy scales in the large-N expansion

Moreover, the large-N 't Hooft expansion may lead to a solution circumventing the problem of the nonperturbative nature of Λ_{RG}

Indeed, the large-N limit of SU(N) QCD (with massless quarks):

$$Z = \int \delta A \delta\psi \delta\bar{\psi} \exp(-\frac{N}{g^2}\int TrF^2 + \sum_{N_f} \bar{\psi}_f \gamma_\alpha D_\alpha \psi_f)$$

(G.'t Hooft 1974)

is conjectured to be solved by a yet-to-be-found string theory, of closed strings in the glueball sector and of open strings in the meson sector, with Λ_{RG} identified with the square root of the string tension !

The main evidence is the large-N counting of Feynman diagrams in 't Hooft double-line representation

In the glueball sector:

$$< \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\cdots\mathcal{O}_n(x_n) >_{conn} \sim N^{2-n}$$

In the meson sector:

$$< \mathcal{M}_1(x_1)\mathcal{M}_2(x_2)\cdots\mathcal{M}_k(x_k) >_{conn} \sim N^{1-\frac{k}{2}}$$

In the meson/glueball sector:

$$< \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\cdots\mathcal{O}_n(x_n)\mathcal{M}_1(x_1)\mathcal{M}_2(x_2)\cdots\mathcal{M}_k(x_k) >_{conn} \sim N^{1-n-\frac{k}{2}}$$

This is exactly the canonical counting that we would get from a string theory with string coupling: g_s=1/N

of closed strings in the glueball sector:

a sphere with n punctures

of open strings in the meson sector:

a disk with k punctures on the boundary

and of open/closed strings in the meson/glueball sector:

a disk with k punctures on the boundary and n in the interior

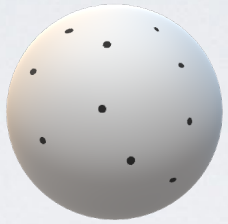
This is the 't Hooft planar theory, that describes tree amplitudes

Then, unitarization introduces higher-genus contributions, matching the topology of the 't Hooft expansion as well, that correct the planar theory by string diagrams with a weight that is 1/N to a power equal to minus the Euler characteristic

Perturbatively, topological interpretation of Feynman diagrams in 't Hooft double-line representation (1974):

χ is the Euler characteristic

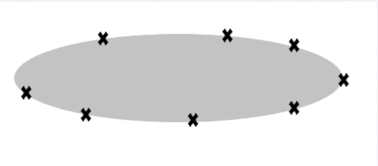
$$\langle \mathcal{G}_1 \ldots \mathcal{G}_n \rangle \sim N^{2-n}$$



Punctured sphere

$$\chi = 2 - n$$

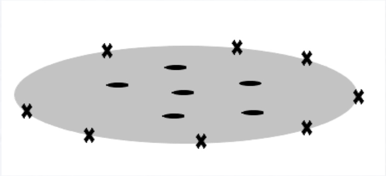
$$\langle \mathcal{M}_1 \ldots \mathcal{M}_m \rangle \sim N^{1-\frac{m}{2}}$$



Punctured disk on the boundary

$$\chi = 1 - \frac{m}{2}$$

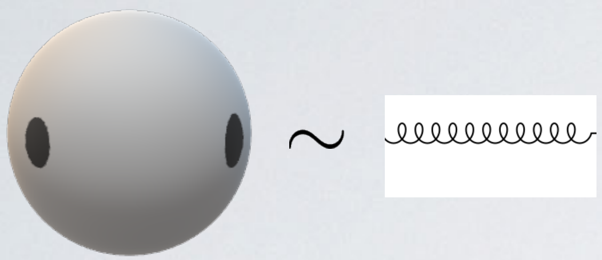
$$\langle \mathcal{G}_1 \ldots \mathcal{G}_n \mathcal{M}_1 \ldots \mathcal{M}_m \rangle \sim N^{1-n-\frac{m}{2}}$$

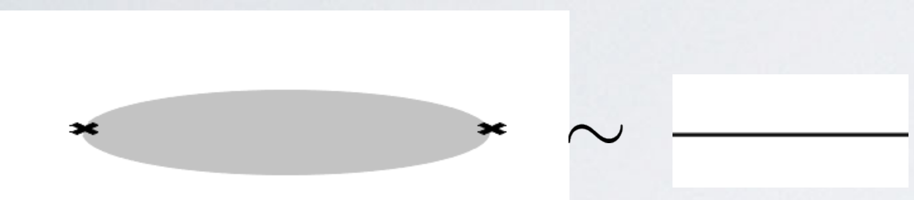


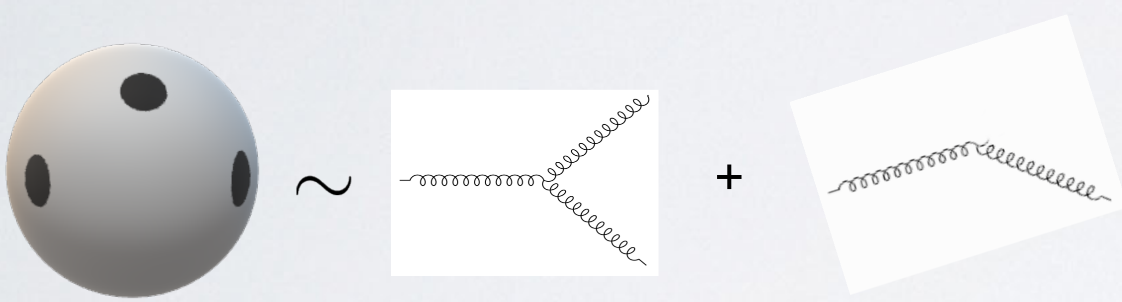
Punctured disk on the interior and the boundary

$$\chi = 1 - n - \frac{m}{2}$$

Nonperturbatively, in the effective theory of mesons and glueballs the planar theory corresponds to tree diagrams (Migdal 1977, Witten 1979):



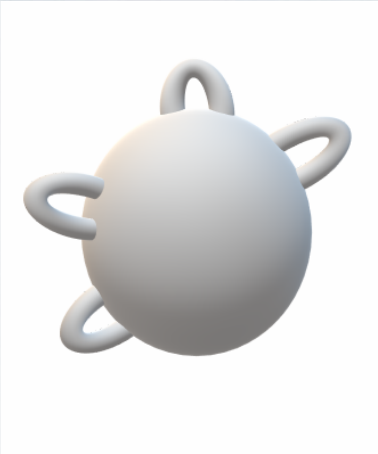




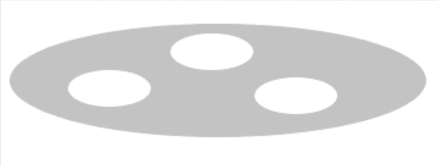
3-point vertices may have triple or double poles (Witten 1977)

The planar nonperturbative S matrix is not unitary, since it only contains tree diagrams: We need to include loops

In the large-N expansion loops are given by nonplanar diagrams



Glueball loops are given by handles



Meson loops are given by holes

In general, the N dependence of correlators is given by the Euler characteristic of a surface with genus g (handles) and h holes and n punctures in the interior and m punctures on the boundary:

$$\chi = 2 - 2g - h - n - \frac{m}{2}$$

$$\langle \mathcal{G}_1 \dots \mathcal{G}_n \mathcal{M}_1 \dots \mathcal{M}_m \rangle \sim N^\chi$$

Physically, this is the standard picture of confinement where

mesons are bound states of quarks linked by a chromo-electric flux tube and

glueballs are rings of chromo-electric flux

with the string world-sheet identified with the flux tube

Yet, this physical interpretation is not necessary in the canonical string framework. We may consider an arbitrary string background (higher dimensions, curvature, D-branes, RR sectors…) provided that it leads to a conformal-invariant string theory on the world sheet. The existence of the conformal structure is the new (arbitrary ?) assumption with respect to ’t Hooft topological classification !

Therefore, in the search for a (possibly stringy) solution of large-N YM or QCD, the following problems arise:

(1) Which nonperturbative UV features - that we may deduce from fundamental principles, i.e. asymptotic freedom and RG - should the solution, expressed only in terms of Λ_{RG}, of large-N YM and QCD satisfy?

(2) Is the canonical string ansatz (i.e., conformal symmetry on the world sheet + matching with 't Hooft large-N topological expansion) compatible with the above nonperturbative UV features ?

(3) Can we employ a detailed information on the above UV features as a guide to actually work out - perhaps by means of some new (noncanonical) idea - and verify or falsify candidates for a (partial) solution ?

Some years ago (circa 2013) we (M.B.) started the program of answering the above questions.

Eventually, this program merged into GAGRA (circa 2021)

We skip about 10 years to immediately discuss question (3) and GAGRA
(some key previous results about questions (1) and (2) are in the backup slides …)

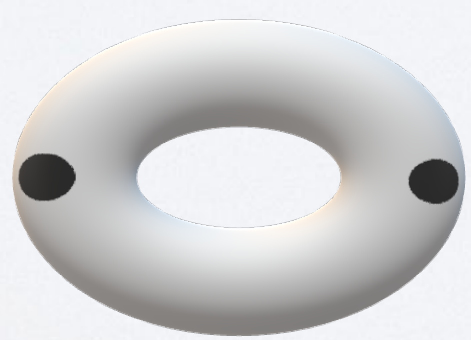
In relation to question (3), for deep reasons that we cannot explain in detail here, one of our aims is to get information on the sum of the glueball one-loop diagrams in large-N YM.

We pointed out (M.B. HADRON2015) that the leading non-planar glueball effective action should have the structure of the logarithm of a functional determinant that sums the glueball 1-loop diagrams:

$$\Gamma^{E}_{glueball\ 1-loop} = \frac{1}{2}\log\det\left(-\Delta + m^2 + \frac{1}{N} * \mathcal{G} * \right)$$

In the 't Hooft expansion (and in the string solution - if it exists) they are the leading-order nonplanar contribution, summing diagrams with the topology of a punctured torus:

$\sum$



$+\dots$

The above follows from the existence of an unknown glueball effective action. Schematically:

$$\Gamma^E_{\text{canonical}} = \frac{1}{2}\int d^4x \,\mathrm{Tr}\,\mathcal{G}\left(-\Delta + m^2\right)\mathcal{G} + \frac{1}{N}\int d^4x\, \mathcal{G} * \mathcal{G} * \mathcal{G} + \frac{1}{2}\log\det\left(-\Delta + m^2 + \frac{1}{N} * \mathcal{G} *\right) + \dots$$

In fact, what we may actually compute asymptotically in the UV is the generating functional of the connected correlators (essentially the inverse of the Legendre transform of the effective action):

$$\Gamma^E_{glueball\,1-loop} \longrightarrow W^E_{glueball\,1-loop}$$

Physically, the difference is that the effective action is the sum of the 1PI diagrams with amputated external lines, while the generating functional above involves connected correlators with non-amputated external lines.

Finally, the UV asymptotics of the aforementioned generating functional should inherit the very same structure of the log of a functional determinant as well.

We could test the above statement by computing, for example, the UV asymptotics of the analytic continuation to Euclidean space-time of the generating functional of all the correlators of all the twist-2 operators with maximal spin component along the p_+ direction, in order to verify that it has actually the predicted structure of the log of a functional determinant

In the light-cone gauge the above twist-2 operators read:

$$V_+ = \frac{V_0 + V_3}{\sqrt{2}} \qquad V_- = \frac{V_0 - V_3}{\sqrt{2}}$$
$$V = \frac{V_1 + iV_2}{\sqrt{2}} \qquad \bar{V} = \frac{V_1 - iV_2}{\sqrt{2}}$$

$$\mathbb{O}_s = \mathrm{Tr}\, \partial_+ \bar{A}(x)(i\overrightarrow{\partial}_+ + i\overleftarrow{\partial}_+)^{s-2} C^{\frac{5}{2}}_{s-2}\left(\frac{\overrightarrow{\partial}_+ - \overleftarrow{\partial}_+}{\overrightarrow{\partial}_+ + \overleftarrow{\partial}_+}\right) \partial_+ A(x)$$

$$\tilde{\mathbb{O}}_s = \mathrm{Tr}\, \partial_+ \bar{A}(x)(i\overrightarrow{\partial}_+ + i\overleftarrow{\partial}_+)^{s-2} C^{\frac{5}{2}}_{s-2}\left(\frac{\overrightarrow{\partial}_+ - \overleftarrow{\partial}_+}{\overrightarrow{\partial}_+ + \overleftarrow{\partial}_+}\right) \partial_+ A(x)$$

$$\mathbb{S}_s = \frac{1}{\sqrt{2}}\, \mathrm{Tr}\, \partial_+ \bar{A}(x)(i\overrightarrow{\partial}_+ + i\overleftarrow{\partial}_+)^{s-2} C^{\frac{5}{2}}_{s-2}\left(\frac{\overrightarrow{\partial}_+ - \overleftarrow{\partial}_+}{\overrightarrow{\partial}_+ + \overleftarrow{\partial}_+}\right) \partial_+ \bar{A}(x)$$

$$\bar{\mathbb{S}}_s = \frac{1}{\sqrt{2}}\, \mathrm{Tr}\, \partial_+ A(x)(i\overrightarrow{\partial}_+ + i\overleftarrow{\partial}_+)^{s-2} C^{\frac{5}{2}}_{s-2}\left(\frac{\overrightarrow{\partial}_+ - \overleftarrow{\partial}_+}{\overrightarrow{\partial}_+ + \overleftarrow{\partial}_+}\right) \partial_+ A(x)$$

with $C^{\frac{5}{2}}_{s-2}$ the Gegenbauer polynomials, which are a special case of the Jacobi polynomials

Typically, this computation looks quite difficult.
Therefore, we (M.B. and M.P.) thought it should be conveniently performed with the help of our PhD student at the time - Francesco Scardino.

The UV asymptotics of the generating functional is computed in 3 steps.

(1) Computation of correlators to the lowest order of perturbation theory + reconstruction of the (conformal) generating functional that turns out to have the structure of logDet

(2) Control of renormalization and operator mixing to all orders of perturbation theory

(3) Construction of the generating functional of asymptotic correlators that turns out to have the structure of logDet as well

3 steps in 3 papers

M. Bochicchio, M. Papinutto, F. Scardino,
n-point correlators of twist-2 operators in SU(*N*) Yang-Mills theory to the lowest perturbative order,
JHEP **08** (2021) 142

M. Bochicchio, On the geometry of operator mixing in massless QCD-like theories,
Eur. Phys. J. C **81**, 749 (2021)

M. Bochicchio, M. Papinutto, F. Scardino,
UV asymptotics of *n*-point correlators of twist-2 operators in SU(*N*) Yang-Mills theory,
arXiv:2208.14382 [hep-th].

Remarkably, the generating functional of the conformal correlators computed after step (1) admits the structure of the log of a functional determinant:

$$
\begin{aligned}
&W_{conf}[j_{\mathbb{O}}, j_{\tilde{\mathbb{O}}}, j_{\mathbb{S}}, j_{\bar{\mathbb{S}}}] \\
&= -\frac{N^2-1}{2} \log \mathrm{Det}\left[\mathbb{I} + \mathcal{D}^{-1} j_{\mathbb{O}} + \mathcal{D}^{-1} j_{\tilde{\mathbb{O}}}\right] \\
&\quad - \frac{N^2-1}{2} \log \mathrm{Det}\left[\mathbb{I} + \mathcal{D}^{-1} j_{\mathbb{O}} - \mathcal{D}^{-1} j_{\tilde{\mathbb{O}}}\right] \\
&\quad - \frac{N^2-1}{2} \log \mathrm{Det}\left[\mathbb{I} - 2\left(\mathbb{I} + \mathcal{D}^{-1} j_{\mathbb{O}} - \mathcal{D}^{-1} j_{\tilde{\mathbb{O}}}\right)^{-1} \mathcal{D}^{-1} j_{\bar{\mathbb{S}}} \left(\mathbb{I} + \mathcal{D}^{-1} j_{\mathbb{O}} + \mathcal{D}^{-1} j_{\tilde{\mathbb{O}}}\right)^{-1} \mathcal{D}^{-1} j_{\mathbb{S}}\right]
\end{aligned}
$$

where j are the sources dual to the corresponding operators, with:

$$
\begin{aligned}
\mathcal{D}^{-1}_{s_1 k_1, s_2 k_2}(x-y) &= \frac{i^{s_1+1}}{2} \frac{\Gamma(3)\Gamma(s_1+3)}{\Gamma(5)\Gamma(s_1+1)} \binom{s_1}{k_1} \binom{s_2}{k_2+2} (-\partial_+)^{s_1-k_1+k_2} \Box^{-1}(x-y) \\
&= \frac{i^{s_1}}{8\pi^2} \frac{\Gamma(3)\Gamma(s_1+3)}{\Gamma(5)\Gamma(s_1+1)} \binom{s_1}{k_1} \binom{s_2}{k_2+2} (-\partial_+)^{s_1-k_1+k_2} \frac{1}{|x-y|^2 - i\epsilon}
\end{aligned}
$$

Later F.S. has discovered that the calculation may be performed by functional-integral methods and provides directly the generating functional as a logDet, since in the light-cone gauge the above twist-2 operators are quadratic in the fundamental fields and the functional integral to the lowest order reduced to a Gaussian one

$$\mathcal{Z}_{conf}[J_{\mathbb{O}}, J_{\tilde{\mathbb{O}}}, J_{\mathbb{S}}, J_{\bar{\mathbb{S}}}] = \frac{1}{Z}\int \mathcal{D}A\mathcal{D}\bar{A}\, e^{-i\int d^4x\, \bar{A}^a \Box A^a} \exp\left(\int d^4x \sum_s J_{\mathbb{O}_s}\mathbb{O}_s + J_{\tilde{\mathbb{O}}_s}\tilde{\mathbb{O}}_s + J_{\mathbb{S}_s}\mathbb{S}_s + J_{\bar{\mathbb{S}}_s}\bar{\mathbb{S}}_s\right)$$

In order to perform the functional integral it is convenient to write more explicitly:

$$\mathbb{O}_s = \frac{1}{2}\bar{A}^a(x)\overleftarrow{\partial}_+(i\overrightarrow{\partial}_+ + i\overleftarrow{\partial}_+)^{s-2}C^{\frac{5}{2}}_{s-2}\left(\frac{\overrightarrow{\partial}_+ - \overleftarrow{\partial}_+}{\overrightarrow{\partial}_+ + \overleftarrow{\partial}_+}\right)\overrightarrow{\partial}_+A^a(x)$$

$$= \frac{1}{2}\bar{A}^a(x)\mathcal{Y}^{\frac{5}{2}}_{s-2}(\overrightarrow{\partial}_+, \overleftarrow{\partial}_+)A^a(x) \qquad s = 2, 4, 6, \ldots$$

$$\tilde{\mathbb{O}}_s = \frac{1}{2}\bar{A}^a(x)\overleftarrow{\partial}_+(i\overrightarrow{\partial}_+ + i\overleftarrow{\partial}_+)^{s-2}C^{\frac{5}{2}}_{s-2}\left(\frac{\overrightarrow{\partial}_+ - \overleftarrow{\partial}_+}{\overrightarrow{\partial}_+ + \overleftarrow{\partial}_+}\right)\overrightarrow{\partial}_+A^a(x)$$

$$= \frac{1}{2}\bar{A}^a(x)\mathcal{H}^{\frac{5}{2}}_{s-2}(\overrightarrow{\partial}_+, \overleftarrow{\partial}_+)A^a(x) \qquad s = 3, 5, 7, \ldots$$

$$\mathbb{S}_s = \frac{1}{2\sqrt{2}}\bar{A}^a(x)\overleftarrow{\partial}_+(i\overrightarrow{\partial}_+ + i\overleftarrow{\partial}_+)^{s-2}C^{\frac{5}{2}}_{s-2}\left(\frac{\overrightarrow{\partial}_+ - \overleftarrow{\partial}_+}{\overrightarrow{\partial}_+ + \overleftarrow{\partial}_+}\right)\overrightarrow{\partial}_+\bar{A}^a(x)$$

$$= \frac{1}{2\sqrt{2}}\bar{A}^a(x)\mathcal{Y}^{\frac{5}{2}}_{s-2}(\overrightarrow{\partial}_+, \overleftarrow{\partial}_+)\bar{A}^a(x) \qquad s = 2, 4, 6, \ldots$$

$$\bar{\mathbb{S}}_s = \frac{1}{2\sqrt{2}}A^a(x)\overleftarrow{\partial}_+(i\overrightarrow{\partial}_+ + i\overleftarrow{\partial}_+)^{s-2}C^{\frac{5}{2}}_{s-2}\left(\frac{\overrightarrow{\partial}_+ - \overleftarrow{\partial}_+}{\overrightarrow{\partial}_+ + \overleftarrow{\partial}_+}\right)\overrightarrow{\partial}_+A^a(x)$$

$$= \frac{1}{2\sqrt{2}}A^a(x)\mathcal{Y}^{\frac{5}{2}}_{s-2}(\overrightarrow{\partial}_+, \overleftarrow{\partial}_+)A^a(x) \qquad s = 2, 4, 6, \ldots$$

Even spin:

$$\mathcal{Y}^{\frac{5}{2}}_{s-2}(\overrightarrow{\partial}_+, \overleftarrow{\partial}_+) = \overleftarrow{\partial}_+(i\overrightarrow{\partial}_+ + i\overleftarrow{\partial}_+)^{s-2}C^{\frac{5}{2}}_{s-2}\left(\frac{\overrightarrow{\partial}_+ - \overleftarrow{\partial}_+}{\overrightarrow{\partial}_+ + \overleftarrow{\partial}_+}\right)\overrightarrow{\partial}_+$$

$$= \frac{\Gamma(3)\Gamma(s+3)}{\Gamma(5)\Gamma(s+1)}i^{s-2}\sum_{k=0}^{s-2}\binom{s}{k}\binom{s}{k+2}(-1)^{s-k}\overleftarrow{\partial}_+^{s-k-1}\overrightarrow{\partial}_+^{k+1}$$

Odd spin:

$$\mathcal{H}^{\frac{5}{2}}_{s-2}(\overrightarrow{\partial}_+, \overleftarrow{\partial}_+) = \overleftarrow{\partial}_+(i\overrightarrow{\partial}_+ + i\overleftarrow{\partial}_+)^{s-2}C^{\frac{5}{2}}_{s-2}\left(\frac{\overrightarrow{\partial}_+ - \overleftarrow{\partial}_+}{\overrightarrow{\partial}_+ + \overleftarrow{\partial}_+}\right)\overrightarrow{\partial}_+$$

$$= \frac{\Gamma(3)\Gamma(s+3)}{\Gamma(5)\Gamma(s+1)}i^{s-2}\sum_{k=0}^{s-2}\binom{s}{k}\binom{s}{k+2}(-1)^{s-k}\overleftarrow{\partial}_+^{s-k-1}\overrightarrow{\partial}_+^{k+1}$$

As a consequence the functional integral reads:

$$\mathcal{Z}_{conf}[J_{\mathbb{O}}, J_{\tilde{\mathbb{O}}}, J_{\mathbb{S}}, J_{\bar{\mathbb{S}}}] = \frac{1}{Z} \int \mathcal{D}A \mathcal{D}\bar{A}\, e^{-\frac{i}{2}\int d^4x \begin{pmatrix} A^a(x) & \bar{A}^a(x) \end{pmatrix} \mathcal{M}^{ab} \begin{pmatrix} \bar{A}^b(x) \\ A^b(x) \end{pmatrix}}$$

where the block matrix is valued in the operators:

$$\mathcal{M}^{ab} = \delta^{ab} \begin{pmatrix} \Box + \frac{i}{2}\sum_s J_{\mathbb{O}_s} \otimes \mathcal{Y}^{\frac{5}{2}}_{s-2} - \frac{i}{2}\sum_s J_{\tilde{\mathbb{O}}_s} \otimes \mathcal{H}^{\frac{5}{2}}_{s-2} & \frac{i}{\sqrt{2}}\sum_s J_{\bar{\mathbb{S}}_s} \otimes \mathcal{Y}^{\frac{5}{2}}_{s-2} \\ \frac{i}{\sqrt{2}}\sum_s J_{\mathbb{S}_s} \otimes \mathcal{Y}^{\frac{5}{2}}_{s-2} & \Box + \frac{i}{2}\sum_s J_{\mathbb{O}_s} \otimes \mathcal{Y}^{\frac{5}{2}}_{s-2} + \frac{i}{2}\sum_s J_{\tilde{\mathbb{O}}_s} \otimes \mathcal{H}^{\frac{5}{2}}_{s-2} \end{pmatrix}$$

The tensor product symbol means that the differential operators do not act on the sources

Then:

$$\mathcal{W}_{conf}[J_{\mathbb{O}}, J_{\tilde{\mathbb{O}}}, J_{\mathbb{S}}, J_{\bar{\mathbb{S}}}] = \log \mathcal{Z}_{conf}[J_{\mathbb{O}}, J_{\tilde{\mathbb{O}}}, J_{\mathbb{S}}, J_{\bar{\mathbb{S}}}] = -\frac{1}{2} \log \mathrm{Det}(\mathcal{M})$$

The result is:

$$
\begin{aligned}
&\frac{1}{N^2-1}\ \mathcal{W}_{conf}[J_{\mathbb{O}'}, J_{\tilde{\mathbb{O}}'}, J_{\mathbb{S}'}, J_{\bar{\mathbb{S}}'}] = \\
&-\frac{1}{2}\log \mathrm{Det}\left(\mathcal{I} + \frac{1}{N}\sum_{k=0}^{s-2}\binom{s}{k}\binom{s}{k+2}(i\overrightarrow{\partial}_+)^{s-k-1} i\Box^{-1}(J_{\mathbb{O}'_s} + J_{\tilde{\mathbb{O}}'_s})(i\overrightarrow{\partial}_+)^{k+1}\right) \\
&-\frac{1}{2}\log \mathrm{Det}\left(\mathcal{I} + \frac{1}{N}\sum_{k=0}^{s-2}\binom{s}{k}\binom{s}{k+2}(i\overrightarrow{\partial}_+)^{s-k-1} i\Box^{-1}(J_{\mathbb{O}'_s} - J_{\tilde{\mathbb{O}}'_s})(i\overrightarrow{\partial}_+)^{k+1}\right) \\
&-\frac{1}{2}\log \mathrm{Det}\Bigg[\mathcal{I} - \frac{2}{N^2}\left(\mathcal{I} + \frac{1}{N}\sum_{k=0}^{s-2}\binom{s}{k}\binom{s}{k+2}(i\overrightarrow{\partial}_+)^{s-k-1} i\Box^{-1}(J_{\mathbb{O}'_s} - J_{\tilde{\mathbb{O}}'_s})(i\overrightarrow{\partial}_+)^{k+1}\right)^{-1} \\
&\quad \sum_{k_1=0}^{s_1-2}\binom{s_1}{k_1}\binom{s_1}{k_1+2}(i\overrightarrow{\partial}_+)^{s_1-k_1-1} i\Box^{-1} J_{\bar{\mathbb{S}}'_{s_1}}(i\overrightarrow{\partial}_+)^{k_1+1} \\
&\quad \left(\mathcal{I} + \frac{1}{N}\sum_{k_2=0}^{s_2-2}\binom{s_2}{k_2}\binom{s_2}{k_2+2}(i\overrightarrow{\partial}_+)^{s_2-k_2-1} i\Box^{-1}(J_{\mathbb{O}'_{s_2}} + J_{\tilde{\mathbb{O}}'_{s_2}})(i\overrightarrow{\partial}_+)^{k_2+1}\right)^{-1} \\
&\quad \sum_{k_3=0}^{s_3-2}\binom{s_3}{k_3}\binom{s_3}{k_3+2}(i\overrightarrow{\partial}_+)^{s_3-k_3-1} i\Box^{-1} J_{\mathbb{S}'_{s_3}}(i\overrightarrow{\partial}_+)^{k_3+1}\Bigg]
\end{aligned}
$$

We have verified (nontrivially) that this object coincides with our previously computed generating functional reconstructed form the correlators

Moreover, after rescaling the operators so that their 2-point correlators are of order 1 for large N, in a certain renormalization scheme where the twist-2 operators become multiplicatively renormalizable, the leading-order nonplanar generating functional of the Euclidean asymptotic correlators, as all the coordinates are rescaled by $\lambda \to 0$, after step (3) is:

$$
\begin{aligned}
&\mathcal{W}^E_{Torus\, asym}[J_{\mathbb{O}'^E}, J_{\tilde{\mathbb{O}}'^E}, J_{\mathbb{S}'^E}, J_{\bar{\mathbb{S}}'^E}, \lambda] = \\
&\frac{1}{2}\log\mathrm{Det}\left(I + \sum_{k=0}^{s-2}\binom{s}{k}\binom{s}{k+2}(-\overrightarrow{\partial}_z)^{s-k-1}\triangle^{-1}\frac{(Z_{\mathbb{O}'_s}(\lambda)J_{\mathbb{O}'^E_s} + Z_{\tilde{\mathbb{O}}'_s}(\lambda)J_{\tilde{\mathbb{O}}'^E_s})}{N\,\lambda^{2+s}}(-\overrightarrow{\partial}_z)^{k+1}\right) \\
&+\frac{1}{2}\log\mathrm{Det}\left(I + \sum_{k=0}^{s-2}\binom{s}{k}\binom{s}{k+2}(-\overrightarrow{\partial}_z)^{s-k-1}\triangle^{-1}\frac{(Z_{\mathbb{O}'_s}(\lambda)J_{\mathbb{O}'^E_s} - Z_{\tilde{\mathbb{O}}'_s}(\lambda)J_{\tilde{\mathbb{O}}'^E_s})}{N\,\lambda^{2+s}}(-\overrightarrow{\partial}_z)^{k+1}\right) \\
&+\frac{1}{2}\log\mathrm{Det}\Bigg[I - 2\left(I + \sum_{k=0}^{s-2}\binom{s}{k}\binom{s}{k+2}(-\overrightarrow{\partial}_z)^{s-k-1}\triangle^{-1}\frac{(Z_{\mathbb{O}'_s}(\lambda)J_{\mathbb{O}'^E_s} - Z_{\tilde{\mathbb{O}}'_s}(\lambda)J_{\tilde{\mathbb{O}}'^E_s})}{N\,\lambda^{2+s}}(-\overrightarrow{\partial}_z)^{k+1}\right)^{-1} \\
&\sum_{k_1=0}^{s_1-2}\binom{s_1}{k_1}\binom{s_1}{k_1+2}(-\overrightarrow{\partial}_z)^{s_1-k_1-1}\triangle^{-1}\frac{Z_{\bar{\mathbb{S}}'_{s_1}}(\lambda)}{N\,\lambda^{2+s_1}}J_{\bar{\mathbb{S}}'^E_{s_1}}(-\overrightarrow{\partial}_z)^{k_1+1} \\
&\left(I + \sum_{k_2=0}^{s_2-2}\binom{s_2}{k_2}\binom{s_2}{k_2+2}(-\overrightarrow{\partial}_z)^{s_2-k_2-1}\triangle^{-1}\frac{(Z_{\mathbb{O}'_{s_2}}(\lambda)J_{\mathbb{O}'^E_{s_2}} - Z_{\tilde{\mathbb{O}}'_{s_2}}(\lambda)J_{\tilde{\mathbb{O}}'^E_{s_2}})}{N\,\lambda^{2+s_2}}(-\overrightarrow{\partial}_z)^{k_2+1}\right)^{-1} \\
&\sum_{k_3=0}^{s_3-2}\binom{s_3}{k_3}\binom{s_3}{k_3+2}(-\overrightarrow{\partial}_z)^{s_3-k_3-1}\triangle^{-1}\frac{Z_{\mathbb{S}'_{s_3}}(\lambda)}{N\,\lambda^{2+s_3}}J_{\mathbb{S}'^E_{s_3}}(-\overrightarrow{\partial}_z)^{k_3+1}\Bigg]
\end{aligned}
$$

with $Z_{\mathcal{O}_i}(\lambda) = \left(\frac{g(\mu)}{g(\frac{\mu}{\lambda})}\right)^{\frac{\gamma_0 \mathcal{O}_i}{\beta_0}}$ and:

$$
\begin{aligned}
g^2(\frac{\mu}{\lambda}) &\sim \frac{1}{\beta_0 \log(\frac{\mu^2}{\lambda^2\Lambda^2_{YM}})}\left(1 - \frac{\beta_1}{\beta_0^2}\frac{\log\log(\frac{\mu^2}{\lambda^2\Lambda^2_{YM}})}{\log(\frac{\mu^2}{\lambda^2\Lambda^2_{YM}})}\right) \\
&\sim \frac{1}{\beta_0 \log(\frac{1}{\lambda^2})}\left(1 - \frac{\beta_1}{\beta_0^2}\frac{\log\log(\frac{1}{\lambda^2})}{\log(\frac{1}{\lambda^2})}\right)
\end{aligned}
$$

Conclusions

The generating functional of the RG-improved nonplanar correlators of twist-2 operators
has indeed the structure of the log of a functional determinant, as predicted on the basis of the existence of the glueball effective action (M.B. HADRON 2015)

Moreover, the above asymptotics provides us with very specific quantitative information on the large-N limit of Yang-Mills theory, and it is a constraint that any candidate nonperturbative solution has to satisfy.

Besides, it may be a powerful guide in the search for such a solution.

Indeed, in M.B. HADRON2015 it was introduced
a certain class of (noncanonical) string models
where the candidate glueball effective action
arises from the coupling to D-branes
and - again - has the structure of the
log of a functional determinant

Hence, now (for the first time in 50 years)
we have an ansatz to be compared with the
UV asymptotics of the real object
This is a problem for the future …

Backup slides

Step(1)

F.S. has computed all the n-point (conformal) correlators of all the twist-2 operators with maximal spin components along the p_+ direction (the restriction to maximal spin is a actually a simplification) to the lowest order of perturbation theory by Feynman diagrams

Some correlators are:

$$\langle \mathbb{O}_{s_1}(x_1)\dots\mathbb{O}_{s_n}(x_n)\tilde{\mathbb{O}}_{s_{n+1}}(x_{n+1})\dots\tilde{\mathbb{O}}_{s_{n+2m}}(x_{n+2m})\rangle_{conn}$$

$$= \frac{1}{(4\pi^2)^{n+2m}}\frac{N^2-1}{2^{n+2m}}2^{\sum_{l=1}^{n+2m}s_l}i^{\sum_{l=1}^{n+2m}s_l}\frac{\Gamma(3)\Gamma(s_1+3)}{\Gamma(5)\Gamma(s_1+1)}\dots\frac{\Gamma(3)\Gamma(s_{n+2m}+3)}{\Gamma(5)\Gamma(s_{n+2m}+1)}$$

$$\sum_{k_1=0}^{s_1-2}\dots\sum_{k_{n+2m}=0}^{s_{n+2m}-2}\binom{s_1}{k_1}\binom{s_1}{k_1+2}\dots\binom{s_{n+2m}}{k_{n+2m}}\binom{s_{n+2m}}{k_{n+2m}+2}$$

$$\frac{(-1)^{n+2m}}{n+2m}\sum_{\sigma\in P_{n+2m}}(s_{\sigma(1)}-k_{\sigma(1)}+k_{\sigma(2)})!\dots(s_{\sigma(n+2m)}-k_{\sigma(n+2m)}+k_{\sigma(1)})!$$

$$\frac{(x_{\sigma(1)}-x_{\sigma(2)})_+^{s_{\sigma(1)}-k_{\sigma(1)}+k_{\sigma(2)}}}{\left(|x_{\sigma(1)}-x_{\sigma(2)}|^2\right)^{s_{\sigma(1)}-k_{\sigma(1)}+k_{\sigma(2)}+1}}\dots\frac{(x_{\sigma(n+2m)}-x_{\sigma(1)})_+^{s_{\sigma(n+2m)}-k_{\sigma(n+2m)}+k_{\sigma(1)}}}{\left(|x_{\sigma(n+2m)}-x_{\sigma(1)}|^2\right)^{s_{\sigma(n+2m)}-k_{\sigma(n+2m)}+k_{\sigma(1)}+1}}$$

$$\langle \mathbb{S}_{s_1}(x_1)\dots\mathbb{S}_{s_n}(x_n)\bar{\mathbb{S}}_{s'_1}(y_1)\dots\bar{\mathbb{S}}_{s'_n}(y_n)\rangle = \frac{1}{(4\pi^2)^{2n}}\frac{N^2-1}{2^{2n}}2^{\sum_{l=1}^{n}s_l+s'_l}i^{\sum_{l=1}^{n}s_l+s'_l}$$

$$\frac{\Gamma(3)\Gamma(s_1+3)}{\Gamma(5)\Gamma(s_1+1)}\dots\frac{\Gamma(3)\Gamma(s_n+3)}{\Gamma(5)\Gamma(s_n+1)}\frac{\Gamma(3)\Gamma(s'_1+3)}{\Gamma(5)\Gamma(s'_1+1)}\dots\frac{\Gamma(3)\Gamma(s'_n+3)}{\Gamma(5)\Gamma(s'_n+1)}$$

$$\sum_{k_1=0}^{s_1-2}\dots\sum_{k_n=0}^{s_n-2}\binom{s_1}{k_1}\binom{s_1}{k_1+2}\dots\binom{s_n}{k_n}\binom{s_n}{k_n+2}$$

$$\sum_{k'_1=0}^{s'_1-2}\dots\sum_{k'_n=0}^{s'_n-2}\binom{s'_1}{k'_1}\binom{s'_1}{k'_1+2}\dots\binom{s'_n}{k'_n}\binom{s'_n}{k'_n+2}$$

$$\frac{2^{n-1}}{n}\sum_{\sigma\in P_n}\sum_{\rho\in P_n}(s_{\sigma(1)}-k_{\sigma(1)}+k'_{\rho(1)})!(s'_{\rho(1)}-k'_{\rho(1)}+k_{\sigma(2)})!$$

$$\dots(s_{\sigma(n)}-k_{\sigma(n)}+k'_{\rho(n)})!(s'_{\rho(n)}-k'_{\rho(n)}+k_{\sigma(1)})!$$

$$\frac{(x_{\sigma(1)}-y_{\rho(1)})_+^{s_{\sigma(1)}-k_{\sigma(1)}+k'_{\rho(1)}}}{\left(|x_{\sigma(1)}-y_{\rho(1)}|^2\right)^{s_{\sigma(1)}-k_{\sigma(1)}+k'_{\rho(1)}+1}}\frac{(y_{\rho(1)}-x_{\sigma(2)})_+^{s'_{\rho(1)}-k'_{\rho(1)}+k_{\sigma(2)}}}{\left(|y_{\rho(1)}-x_{\sigma(2)}|^2\right)^{s'_{\rho(1)}-k'_{\rho(1)}+k_{\sigma(2)}+1}}$$

$$\dots\frac{(x_{\sigma(n)}-y_{\rho(n)})_+^{s_{\sigma(n)}-k_{\sigma(n)}+k'_{\rho(n)}}}{\left(|x_{\sigma(n)}-y_{\rho(n)}|^2\right)^{s_{\sigma(n)}-k_{\sigma(n)}+k'_{\rho(n)}+1}}\frac{(y_{\rho(n)}-x_{\sigma(1)})_+^{s'_{\rho(n)}-k'_{\rho(n)}+k_{\sigma(1)}}}{\left(|y_{\rho(n)}-x_{\sigma(1)}|^2\right)^{s'_{\rho(n)}-k'_{\rho(n)}+k_{\sigma(1)}+1}}$$

Step (2)

The problem here is that twist-2 operators of higher spin mix with total derivatives of the same operators with lower spins.

However, we construct inductively order by order in perturbation theory a renormalization scheme where they are multiplicatively renormalizable, thanks to a geometric approach to operator mixing that involves the Poincare'-Dulac theorem

In case of mixing the Callan-Symanzik equation implies:

$$G^{(n)}_{k_1\ldots k_n}(\lambda x_1,\ldots,\lambda x_n;\mu,g(\mu)) = \sum_{j_1\cdots j_n} Z_{k_1 j_1}(\lambda)\ldots Z_{k_n j_n}(\lambda)\,\lambda^{-\sum_{i=1}^{n} D_{\mathcal{O}_i}} G^{(n)}_{j_1\ldots j_n}(x_1,\ldots,x_n;\mu,g\Big(\frac{\mu}{\lambda}\Big))$$

with:

$$\langle \mathcal{O}_{k_1}(x_1)\ldots\mathcal{O}_{k_n}(x_n)\rangle = G^{(n)}_{k_1\ldots k_n}(x_1,\ldots,x_n;\mu,g(\mu))$$

By the asymptotic freedom, RG-improved correlators must be asymptotic as $\lambda \to 0$ to the corresponding (presently unknown) nonperturbative correlators:

The corresponding UV asymptotics for $\lambda \to 0$ reads:

$$G^{(n)}_{k_1\ldots k_n}(\lambda x_1,\ldots,\lambda x_n;\mu,g(\mu)) \sim \sum_{j_1\cdots j_n} Z_{k_1 j_1}(\lambda)\ldots Z_{k_n j_n}(\lambda)\,\lambda^{-\sum_{i=1}^{n} D_{\mathcal{O}_i}} G^{(n)}_{conf\, j_1\ldots j_n}(x_1,\ldots,x_n)$$

provided that $G^{(n)}_{conf\, j_1\ldots j_n}(x_1,\ldots,x_n)$ — which can be computed to the lowest order of perturbation theory — does not vanish

$$Z(\lambda) = P\exp\Big(\int_{g(\mu)}^{g(\frac{\mu}{\lambda})} \frac{\gamma(g')}{\beta(g')}dg'\Big)$$

is the renormalized mixing matrix that in the general case involves the calculation of a path-ordered exponential

Hence, the evaluation of the asymptotic correlators involves the computation of sums of products of $G^{(n)}_{conf\, j_1 \dots j_n}(x_1, \dots, x_n)$ and $Z_{ij}(\lambda)$

These computations are technically challenging, even in the special case where the anomalous-dimension matrix is triangular, as it occurs for twist-2 operators (so that the expansion of the path-ordered exponential terminates to a finite order)

Therefore, it is of the outmost importance to establish whether a renormalization scheme exists where Z is diagonalizable to all perturbative orders

In such a scheme the operators would be multiplicatively renormalizable and only one term would contribute to the UV asymptotics:

$$G^{(n)}_{j_1 \dots j_n}(\lambda x_1, \dots, \lambda x_n; \mu, g(\mu)) \sim \frac{Z_{\mathcal{O}_{j_1}}(\lambda) \dots Z_{\mathcal{O}_{j_n}}(\lambda)}{\lambda^{D_{\mathcal{O}_1} + \dots + D_{\mathcal{O}_n}}} G^{(n)}_{conf\, j_1 \dots j_n}(x_1, \dots, x_n)$$

The existence of the above scheme may be decided as follows.

Renormalization may be interpreted in a differential-geometric setting, where a (finite) change of renormalization scheme is interpreted as a coupling-dependent change of the operator basis:

$$\mathcal{O}'(x) = S(g)\mathcal{O}(x)$$

The matrix: $$A(g) = -\frac{\gamma(g)}{\beta(g)} = \frac{1}{g}\left(\frac{\gamma_0}{\beta_0} + \sum_{n=1}^{\infty} C_n g^{2n}\right)$$

that occurs in the system of ODE defining the mixing matrix:

$$\left(\frac{\partial}{\partial g} + \frac{\gamma(g)}{\beta(g)}\right) Z = 0$$

is interpreted as a (formal) real-analytic connection, with a simple pole at g = 0,

that for the gauge transformation transforms as: $A'(g) = S(g)A(g)S^{-1}(g) + \frac{\partial S(g)}{\partial g} S^{-1}(g)$

Moreover, the operator: $\mathcal{D} = \frac{\partial}{\partial g} - A(g)$ is the covariant derivative that defines the linear system: $\mathcal{D}X = 0$

whose particular solution is Z : $Z(x,\mu) = \mathrm{P}\exp\left(\int_{g(x)}^{g(\mu)} A(g)\right)$

Therefore, Z is interpreted as a Wilson line that transforms under a gauge transformation as:

$$Z'(x,\mu) = S(g(\mu))Z(x,\mu)S^{-1}(g(x))$$

The Poincaré-Dulac theorem allows us to construct by induction order by order a formal analytic gauge transformation such that the gauge-transformed connection is one-loop exact:

$$A'(g) = \frac{1}{g}\frac{\gamma_0}{\beta_0}$$

provided that the eigenvalues of the matrix $\frac{\gamma_0}{\beta_0}$ in nonincreasing order do not differ by a positive even integer (nonresonant mixing):

$$\lambda_i - \lambda_j - 2k \neq 0$$

For twist-2 operators $\frac{\gamma_0}{\beta_0}$ is always diagonalizable and satisfies (numerically up to 10^4) the non-resonant condition
Hence, in the diagonal basis above the UV asymptotics of Z reduces essentially to the multiplicatively renormalizable case

$$Z_i(x,\mu) = \left(\frac{g(\mu)}{g(x)}\right)^{\lambda_i}$$

M.B. and S.P. Muscinelli, Ultraviolet asymptotics of glueball propagators, JHEP 08 (2013) 064

$$\int \langle \frac{\beta(g)}{gN} tr\left(\sum_{\alpha\beta} F^2_{\alpha\beta}(x)\right) \frac{\beta(g)}{gN} tr\left(\sum_{\alpha\beta} F^2_{\alpha\beta}(0)\right)\rangle_{conn} e^{ip\cdot x} d^4x$$

$$= C_S p^4 \left[\frac{1}{\beta_0 \log \frac{p^2}{\Lambda^2_{\overline{MS}}}} \left(1 - \frac{\beta_1}{\beta_0^2} \frac{\log \log \frac{p^2}{\Lambda^2_{\overline{MS}}}}{\log \frac{p^2}{\Lambda^2_{\overline{MS}}}} \right) + O\left(\frac{1}{\log^2 \frac{p^2}{\Lambda^2_{\overline{MS}}}} \right) \right]$$

Consequence: All the present proposals for string models of QCD-like theories based on the gauge/gravity duality disagree with the correct UV asymptotics above by positive powers of logs !

Asymptotic theorem that determines the residues of the poles in the spectral representation of large-N 2-point correlators asymptotically for
large masses, in terms of the known anomalous dimension of the operators and the unknown spectral density:

M.B., Glueball and meson propagators of any spin in large-*N QCD, Nucl. Phys. B (875)2013*

Asymptotic Theorem:

$$\int \langle \mathcal{O}^{(s)}(x)\mathcal{O}^{(s)}(0)\rangle_{conn}\, e^{-ip\cdot x} d^4x \sim \sum_{n=1}^{\infty} P^{(s)}\Big(\frac{p_\alpha}{m_n^{(s)}}\Big) \frac{m_n^{(s)2D-4} Z_n^{(s)2} \rho_s^{-1}(m_n^{(s)2})}{p^2 + m_n^{(s)2}}$$

$$= P^{(s)}\Big(\frac{p_\alpha}{p}\Big)\, p^{2D-4} \sum_{n=1}^{\infty} \frac{Z_n^{(s)2} \rho_s^{-1}(m_n^{(s)2})}{p^2 + m_n^{(s)2}} + \cdots$$

$$\sim P^{(s)}\Big(\frac{p_\alpha}{p}\Big)\, p^{2D-4} \int_{m_1^{(s)2}}^{\infty} \frac{Z^{(s)2}(m)}{p^2+m^2} dm^2 + \cdots$$

$$\sim P^{(s)}\Big(\frac{p_\alpha}{p}\Big)\, p^{2D-4} \Bigg[\frac{1}{\beta_0 \log(\frac{p^2}{\Lambda^2_{QCD}})}\Bigg(1 - \frac{\beta_1}{\beta_0^2} \frac{\log\log(\frac{p^2}{\Lambda^2_{QCD}})}{\log(\frac{p^2}{\Lambda^2_{QCD}})} + O\Big(\frac{1}{\log(\frac{p^2}{\Lambda^2_{QCD}})}\Big)\Bigg)\Bigg]^{\frac{\gamma_0}{\beta_0}-1}$$

$$\sum_{n=1}^{\infty} f(m_n^{(s)2}) \sim \int_1^{\infty} f(m_n^{(s)2}) dn = \int_{m_1^{(s)2}}^{\infty} f(m^2)\rho_s(m^2) dm^2$$

$$Z_n^{(s)} \equiv Z^{(s)}(m_n^{(s)}) = \exp \int_{g(\mu)}^{g(m_n^{(s)})} \frac{\gamma_{\mathcal{O}^{(s)}}(g)}{\beta(g)} dg$$

$$Z_n^{(s)2} \sim \Bigg[\frac{1}{\beta_0 \log \frac{m_n^{(s)2}}{\Lambda^2_{QCD}}}\Bigg(1 - \frac{\beta_1}{\beta_0^2} \frac{\log\log \frac{m_n^{(s)2}}{\Lambda^2_{QCD}}}{\log \frac{m_n^{(s)2}}{\Lambda^2_{QCD}}} + O\Big(\frac{1}{\log \frac{m_n^{(s)2}}{\Lambda^2_{QCD}}}\Big)\Bigg)\Bigg]^{\frac{\gamma_0}{\beta_0}}$$

Nonperturbative renormalization of the large-N S matrix:

M. B., The large-*N* Yang–Mills S matrix is ultraviolet finite, but the large-*N* QCD S matrix is only renormalizable, Phys. Rev. D 95 (2017) 054010

In large-N YM the first-two coefficients of the beta function are only planar without 1/N corrections

$$\beta_0 = \beta_0^P = \frac{1}{(4\pi)^2}\frac{11}{3}$$

$$\beta_1 = \beta_1^P = \frac{1}{(4\pi)^4}\frac{34}{3}$$

as a consequence the expansion

$$\Lambda_{YM} \sim const\, \Lambda_{YM}^P \left(1 + \sum_{n=1} c_n\, O(\frac{1}{\log^n(\frac{\Lambda}{\Lambda_{YM}^P})})\right)$$

is finite, and so it is the large-N expansion of the YM S matrix

Instead, in large-N QCD the first-two coefficients of the beta function get corrections to the order N_f/N

$$\beta_0 = \beta_0^P + \beta_0^{NP} = \frac{1}{(4\pi)^2}\frac{11}{3} - \frac{1}{(4\pi)^2}\frac{2}{3}\frac{N_f}{N}$$

$$\beta_1 = \beta_1^P + \beta_1^{NP} = \frac{1}{(4\pi)^4}\frac{34}{3} - \frac{1}{(4\pi)^4}\left(\frac{13}{3} - \frac{1}{N^2}\right)\frac{N_f}{N}$$

As a consequence:

$$\Lambda_{QCD} \sim \Lambda_{QCD}^P \left[1 + \frac{\beta_0^{NP}}{\beta_0^P}\log\left(\frac{\Lambda}{\Lambda_{QCD}^P}\right) + \frac{\beta_1^P}{2{\beta_0^P}^2}\log\log\left(\frac{\Lambda}{\Lambda_{QCD}^P}\right)\left(\frac{\beta_1^{NP}}{\beta_1^P} - \frac{\beta_0^{NP}}{\beta_0^P}\right) + \ldots\right]$$

is divergent (though renormalizable), and so it is the nonperturbative large-N expansion of the QCD S matrix around the planar theory

No-go theorem for the canonical QCD string:
M. B., Renormalization in large-N QCD is incompatible with open/closed string duality, Phys. Lett. B 783 (2018) 341 M.

The UV finiteness of the large-N YM theory, due to AF and RG, that we have just mentioned is

compatible

with the universally believed UV finiteness of (consistent) closed-string theories (due to the underlying modular invariance on the closed-string side)

Thus, a canonical string solution of the pure large-N YM theory may exist

But, contrary to the universal belief,
the aforementioned renormalization properties in large-N QCD
+ the existence of the glueball mass gap at the lowest 1/N order, i.e. in the planar theory
+ UV finiteness of closed string trees

are incompatible with the open/closed duality of a would-be canonical string solution (canonical means that matches topologically 't Hooft expansion)
that therefore does not actually exist.
In fact, there is a stronger version of the no-go theorem, based on a low-energy theorem in large-N QCD that does not assume neither the mass gap nor the UV finiteness of closed string trees

The proof is in this picture of open/closed string duality that is a consequence of the conformal invariance of the string world sheet