From the hierarchy problem to prime principle constraints on CFT_4

work in collaboration with V. Rychkov, E. Tonni and A. Vichi

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I. The Hierarchy Problem, Flavor and CFT's

II. Bounding operator dimensions in CFT_4

Λ_{UV}

Λ_{IR}

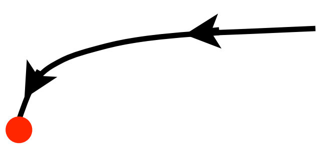
$\sim$ scale invariant dynamics

$\sim$ conformal invariance

- *stability* of $\Lambda_{IR} \ll \Lambda_{UV}$ characterized by dimensionality of *perturbations* at fixed point

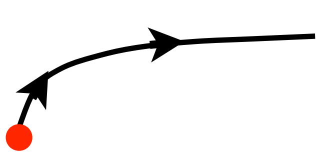
$$\Delta \mathcal{L} = \lambda \mathcal{O} \qquad \lambda(E) = \lambda(\Lambda_{UV}) \left(\frac{E}{\Lambda_{UV}} \right)^{d_{\mathcal{O}}-4}$$

$d_{\mathcal{O}} - 4 > 0$ irrelevant



$d_{\mathcal{O}} - 4 = 0$ marginal

$d_{\mathcal{O}} - 4 < 0$ relevant



Ex. scalar mass $\lambda(E) = \left(\frac{m}{E} \right)^2$

Natural Hierarchy

A. There exists no strongly relevant operator

most relevant $4 - d_{\mathcal{O}} = \epsilon \ll 1$ $\lambda(E) = \lambda_0 \left(\frac{\Lambda_{UV}}{E} \right)^{\epsilon}$

Ex: Yang-Mills theory $4 - d_{\mathcal{O}} = bg^2(E) > 0$

B. Strongly relevant operators exist, but can be controlled by a symmetry

Ex.

- ✦ quark mass in QCD $d_{\mathcal{O}} = 3$ controlled by chiral symmetry
- ✦ scalar masses in MSSM $d_{\mathcal{O}} = 2$ SUSY + chiral symm

Why we are interested in mass hierarchies

- $M_{\text{Planck}} \gg v_{\text{Fermi}}$

- To explain smallness of couplings (*features*) by power counting

Ex: Standard Model

$$\mathcal{L}_{SM} = \mathcal{L}_{d\leq 4} + \frac{1}{\Lambda_{\not{L}}}\mathcal{O}_{\not{L}} + \frac{1}{\Lambda^2_{\not{B}}}\mathcal{O}_{\not{B}} + \frac{\kappa}{\Lambda^2_{\not{F}}}\mathcal{O}_{\Delta S=2} + \ldots$$

$\Lambda_{\not{L}} \sim 10^{14}\,\text{GeV}$ $\Lambda_{\not{B}} \gtrsim 10^{15}\,\text{GeV}$

$\Lambda_{\not{F}} \gtrsim 10^{5}\,\text{TeV}$ $\kappa = 1$

$\Lambda_{\not{F}} \gtrsim 10\,\text{TeV}$ $\kappa = y_d y_s$

Smallness of B, L, F violation nicely explained if $\Lambda_{UV} \gg v_{\text{Fermi}}$

Standard Model ☺ $y_{ij}\, H \bar{F}_i \bar{F}_j$ (dim = 4) $\Lambda_{UV} \to \infty$

- y_{ij} unaffected
- extra unwanted Flavor effects decouple

$$\frac{1}{\Lambda_{UV}^2}\, \bar{q}_i q_j \bar{q}_k q_\ell$$

☹ very relevant operator $\Lambda_{UV}^2\, H^\dagger H$

makes $\Lambda_{UV} \to \infty$ problematic

Technicolor $H = \bar{\psi}\psi$ dimension ~ 3

Weinberg '79
Susskind '79

☺ no scalar singlet with dimension < 4

☹ Yukawas $\frac{y_{ij}}{\Lambda_{UV}^2} H \bar{F}_i F_j$ as relevant as $\frac{1}{\Lambda_{UV}^2}\, \bar{q}_i q_j \bar{q}_k q_\ell$

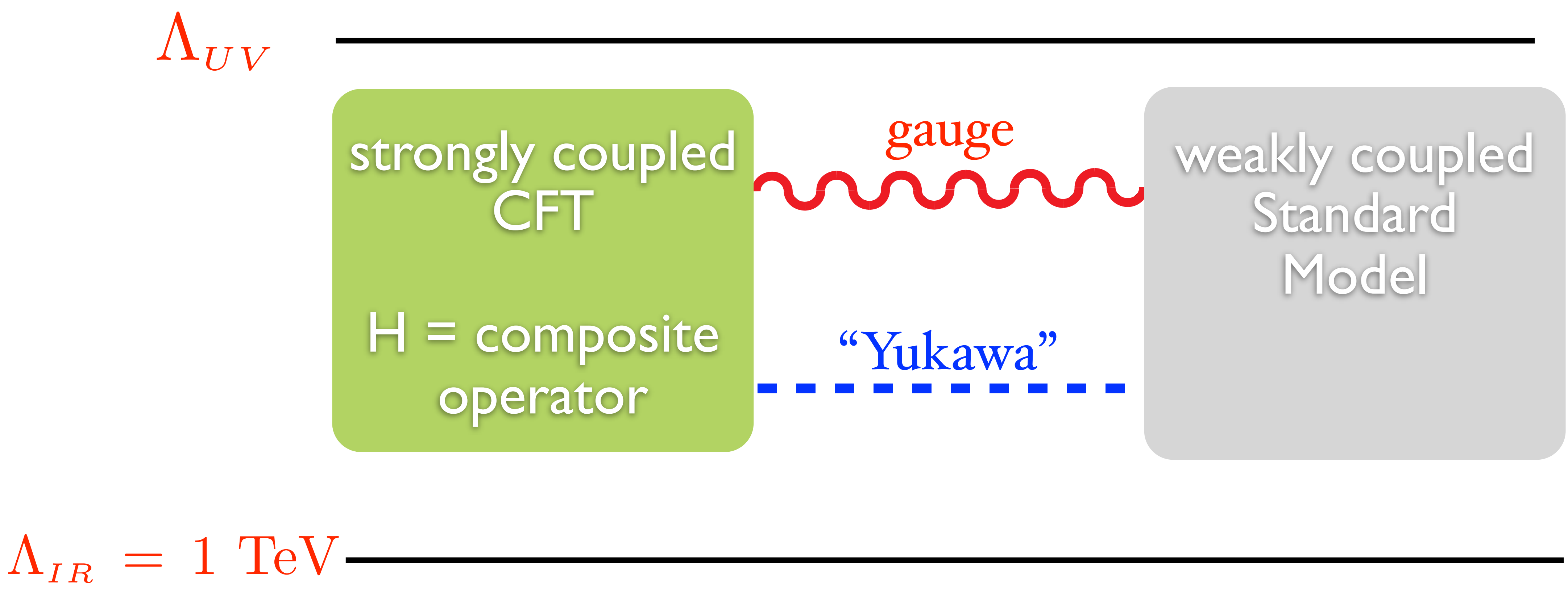
Ideally

- Flavor $d_H \to 1$
- Hierarchy $d_{H^\dagger H} \to 4$

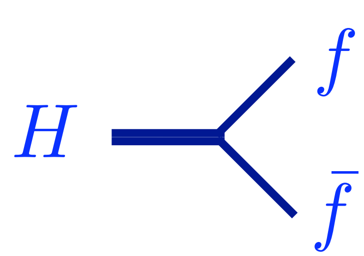
Conformal Technicolor

Luty-Okui 04

from walking TC
Holdom '86

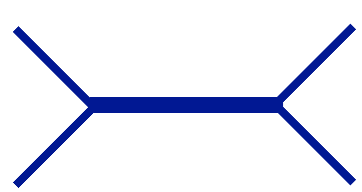


running Yukawas



$$y_f(\Lambda_{UV}) = y_f(\Lambda_{IR}) \left(\frac{\Lambda_{UV}}{\Lambda_{IR}}\right)^{d-1} \lesssim 4\pi$$

Flavor breaking



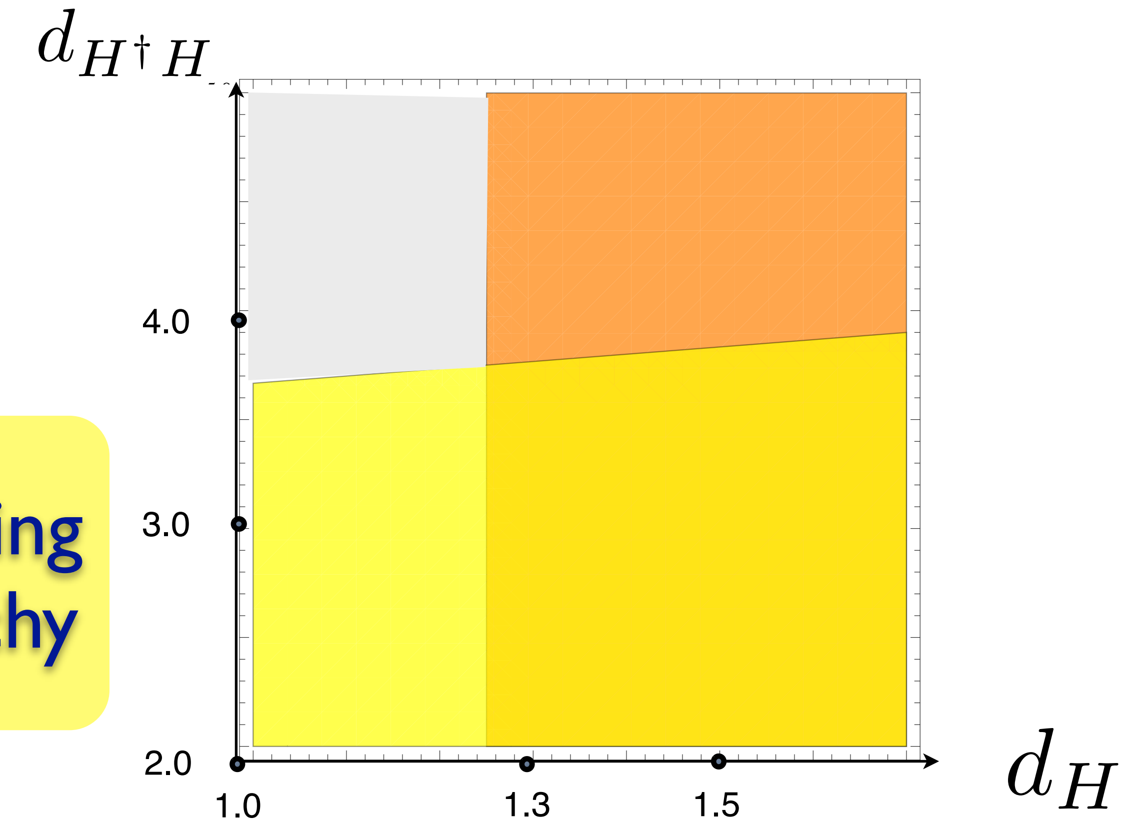
$$\frac{\kappa}{\Lambda_{IR}^2} \left(\frac{\Lambda_{UV}}{\Lambda_{IR}}\right)^{2(d-2)}$$

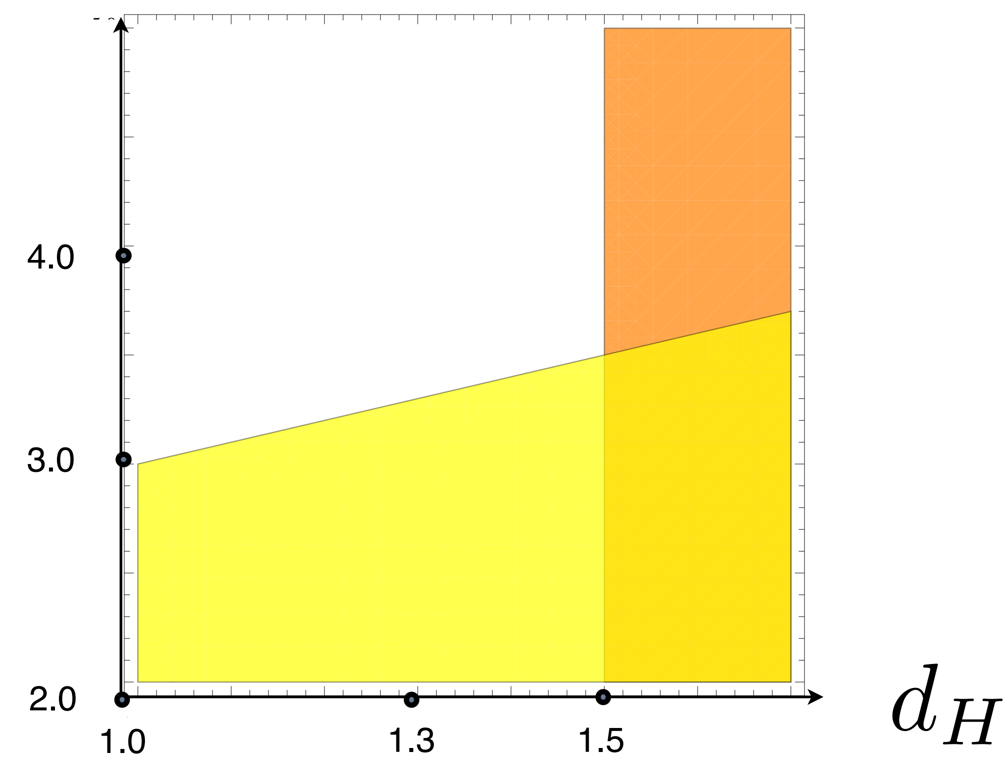
κ model dependent

$$\kappa \sim 10^{-2}$$
$$\Lambda_{UV} > 10^3\ \mathrm{TeV}$$

$$\kappa \sim y_d y_s \sim 10^{-8}$$
$$\Lambda_{UV} > 10\ \mathrm{TeV}$$

< 0.1 tuning in hierarchy

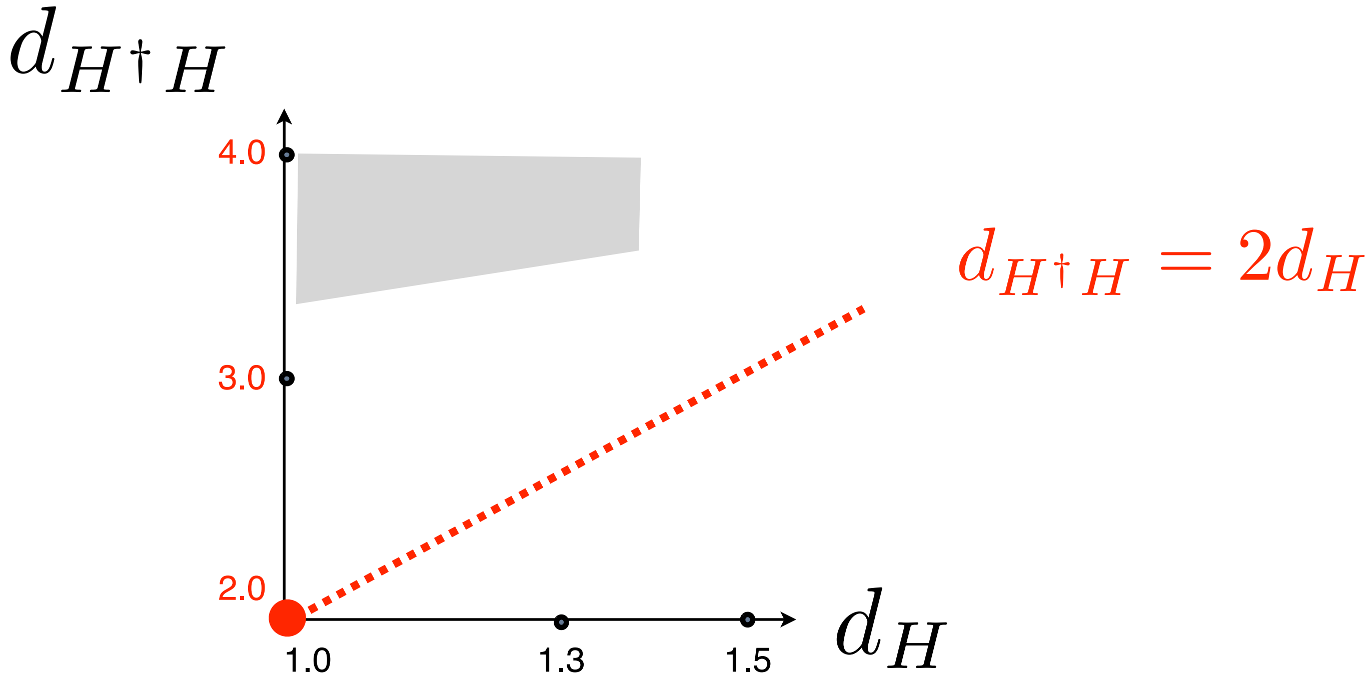




$\Delta\epsilon_k < \epsilon_K^{SM}$

and

$y_{top} < 4\pi$



- Interesting region is not attainable at weak coupling or large N

- Is it at all compatible with prime principles?

- Unitarity + SO(4,2) : $d_\phi = 1$ ➡ $d_{\phi^2} = 2$ Mack '77

- Can one derive a theoretical upper bound on d_{ϕ^2} as a function of d_ϕ ?

- Standard proof for $d=1$ not extendable to $d = 1+ \varepsilon$

I. The Hierarchy Problem, Flavor and CFT's

II. A bound on operator dimensions in CFT_4

Formulation of the problem

OPE $\phi(x)\phi(0) = \frac{1}{x^{2d}} \left[\mathbb{I} + x^{\Delta_0}\phi^2(0) + \quad \ldots\ldots \quad \right]$

lowest dimension
scalar in $\phi \times \phi$

higher dimension
higher spin

What can one say on Δ_0 as a function of d ?

CFT redux

irrep of SO(4,2): (Δ, j_1, j_2) → primary operator

$$[K_\nu, \mathcal{O}(0)] = 0$$

$$[D, \mathcal{O}(0)] = i\Delta\mathcal{O}(0)$$

spin ℓ primaries: $(\Delta, \frac{\ell}{2}, \frac{\ell}{2})$ $\quad \mathcal{O}_{\mu_1 \ldots \mu_\ell}$ $\quad$ symmetric traceless

descendants $\quad \partial_{\nu_1} \ldots \partial_{\nu_n} \mathcal{O}_{\mu_1 \ldots \mu_\ell}$

Unitarity

$$\ell = 0 \qquad \Delta \geq 1$$

$$\ell > 0 \qquad \Delta \geq 2 + \ell$$

Mack '77

Method based on study of 4-point function

I. Conformal symmetry $\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle = \frac{1}{x_{12}^{2d}}\frac{1}{x_{34}^{2d}}\, g(u,v)$

$$x_{ij} = x_i - x_j \qquad u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2} \qquad v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}$$

$g(u,v)$ knows about operator content of the CFT

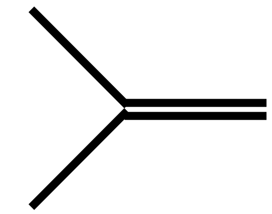
II. Conformal partial waves decomposition

basic idea derived using OPE

primary operator
of spin ℓ and dim Δ

descendands:

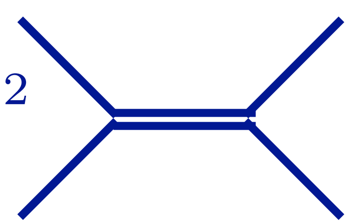
$$\phi(x_1)\phi(x_2) = \frac{1}{x_{12}^{2d}} \left\{ \sum_{\mathcal{O}} \lambda_{\mathcal{O}} \left[C_{d,\Delta,\ell}(x_{12})\mathcal{O} + \quad \cdots\cdots \right] \right\} = \sum_{\mathcal{O}} \lambda_{\mathcal{O}}$$



$\mathcal{O}$

- fully fixed by d, ℓ, Δ
- transforms like $\phi(x_1)\phi(x_2)$

$$\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4) \rangle = \sum_{\mathcal{O}} |\lambda_{\mathcal{O}}|^2$$



$$= \frac{1}{x_{12}^{2d} x_{34}^{2d}} \left(1 + \sum_{\mathcal{O}}{}' |\lambda_{\mathcal{O}}|^2 g_{\mathcal{O}}(u,v) \right)$$

- Unitarity → sum with positive weights $|\lambda_{\mathcal{O}}|^2$
- $g_{\mathcal{O}}(u,v) \equiv g_{\Delta,\ell}(u,v)$ = conformal bloks $\sim$ *spherical harmonics of conformal group*

Ferrara,Gatto,Grillo 1975

Casimir operator of conformal group

$$\mathbf{C} = \frac{1}{2} M_{\mu\nu} M_{\mu\nu} + D^2 - \frac{1}{2}(P_\mu K_\mu + K_\mu P_\mu) \equiv L_A L_A$$

$$c_{\Delta,\ell} = \ell(\ell+2) + \Delta(\Delta-4)$$

$$\mathbf{C} \cdot \mathcal{O}_{\Delta,\ell} = [L_A, [L_A, \mathcal{O}_{\Delta,\ell}]] = -c_{\Delta,\ell} \mathcal{O}_{\Delta,\ell}$$

- differential equation $$D_{u,v}\, g_{\Delta,\ell}(u,v) = c_{\Delta,\ell}\, g_{\Delta,\ell}(u,v)$$
- boundary condition provided by short distance behaviour

general solution: Dolan-Osborn 03

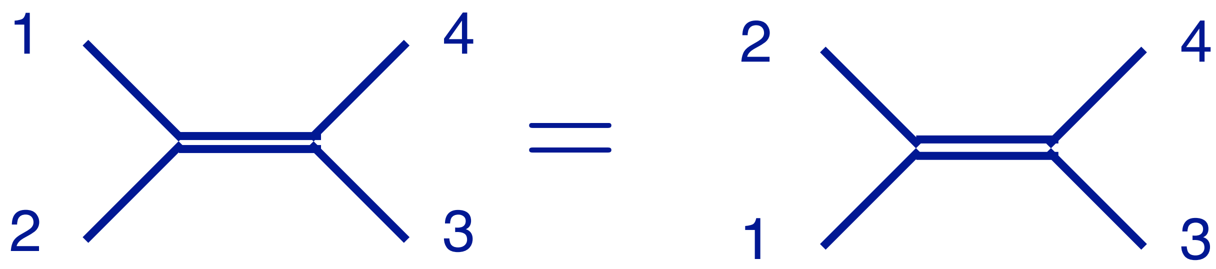
$$g_{\Delta,\ell}(u,v) = \frac{(-)^\ell}{2^\ell} \frac{z\bar{z}}{z-\bar{z}} \Big[f_{\Delta+\ell}(z) f_{\Delta-\ell-2}(\bar{z}) - (z \leftrightarrow \bar{z}) \Big]$$

$$f_\beta(x) \equiv x^{\beta/2} {}_2F_1(\beta/2, \beta/2, \beta; x)$$

$$u = z\bar{z}, \quad v = (1-z)(1-\bar{z})$$

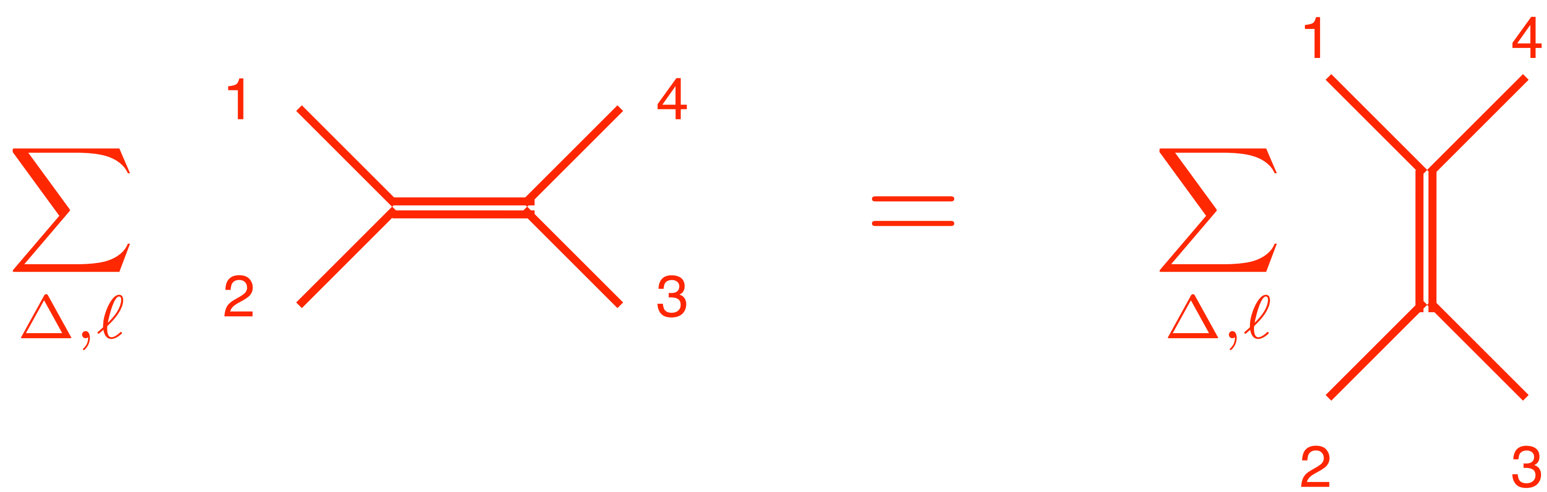
III. Crossing symmetry

A)



ℓ must be even

B)



$(u, v) \leftrightarrow (v, u)$

sum rule

$$1 = \sum_{\Delta,\ell}{}' |\lambda_{\Delta,\ell}|^2 F_{d,\Delta,\ell}(z,\bar{z})$$

non-trivial constraint on spectrum !!

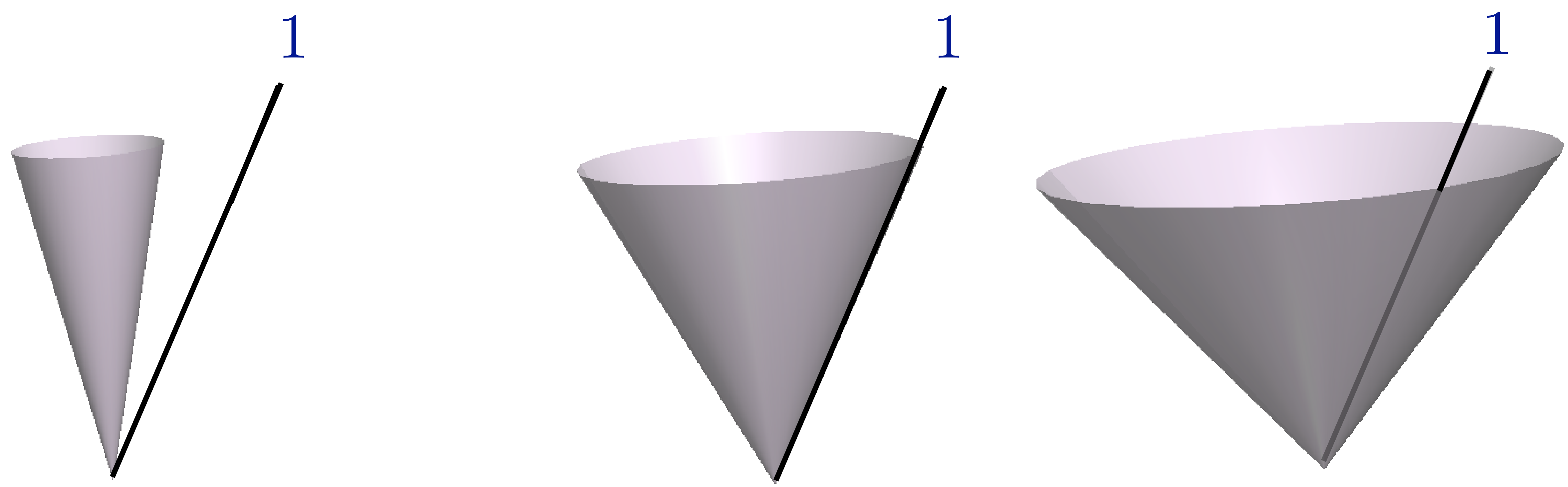
$$F_{d,\Delta,\ell}(z,\bar{z}) = \frac{v^d g_{\Delta,l}(u,v) - u^d g_{\Delta,l}(v,u)}{u^d - v^d}$$

vectors in function space

$$1 = \sum_{\Delta,\ell}{}' |\lambda_{\Delta,\ell}|^2 F_{d,\Delta,\ell}(z,\bar{z})$$

belongs to a **convex cone**

Given d, the broader the hypothetical spectrum $\{\Delta, \ \ell\}$ the wider the cone



$\Delta(\ell=0) \ > \ \infty$ no scalar composite with finite dimension	$\Delta(\ell=0) \ \geq \ \Delta_c$	$\Delta(\ell=0) \ \geq \ \Delta_{min}$ $\Delta_{min} \ < \ \Delta_c$
sum rule violated	critical case	sum rule satisfied

Function space $\{ F(z, \bar{z}) \}$ $\Longleftrightarrow$ $\{ F^{2n,2m} \}$ infinite vector space

$$1 = \sum_{\Delta,\ell} \lambda^2_{\Delta,\ell} F^{0,0}_{\Delta,\ell}$$

$$0 = \sum_{\Delta,\ell} \lambda^2_{\Delta,\ell} F^{2,0}_{\Delta,\ell}$$

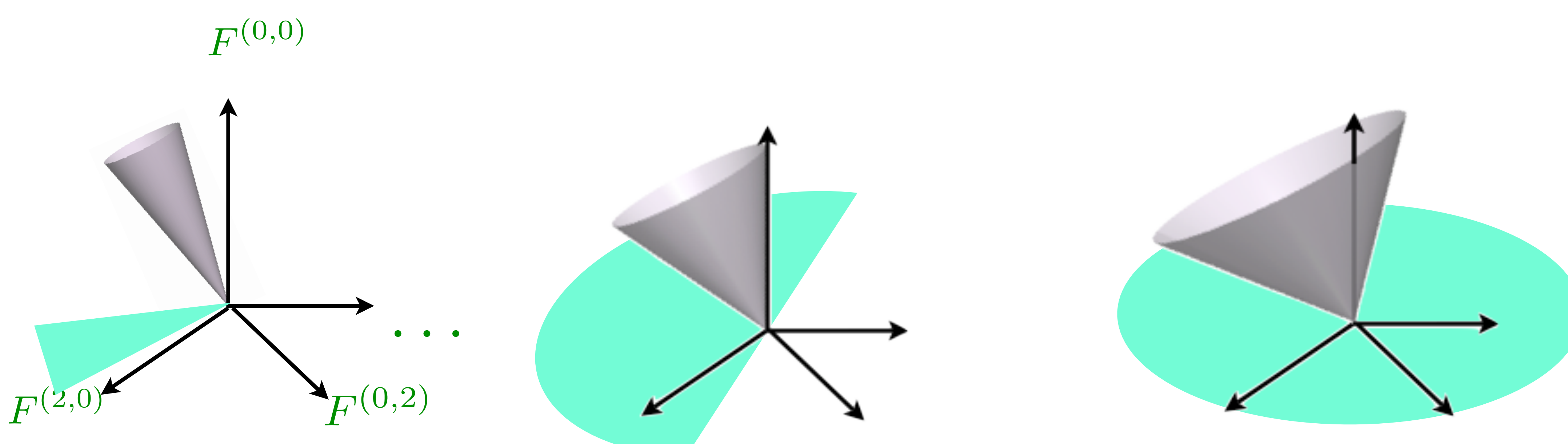
$$0 = \sum_{\Delta,\ell} \lambda^2_{\Delta,\ell} F^{0,2}_{\Delta,\ell}$$

$$\cdots\cdot$$

$$F^{2n,2m} \equiv \partial^{2n}_{z+\bar{z}} \partial^{2m}_{z-\bar{z}} F(z, \bar{z}) \Big|_{z=\bar{z}=\frac{1}{2}}$$

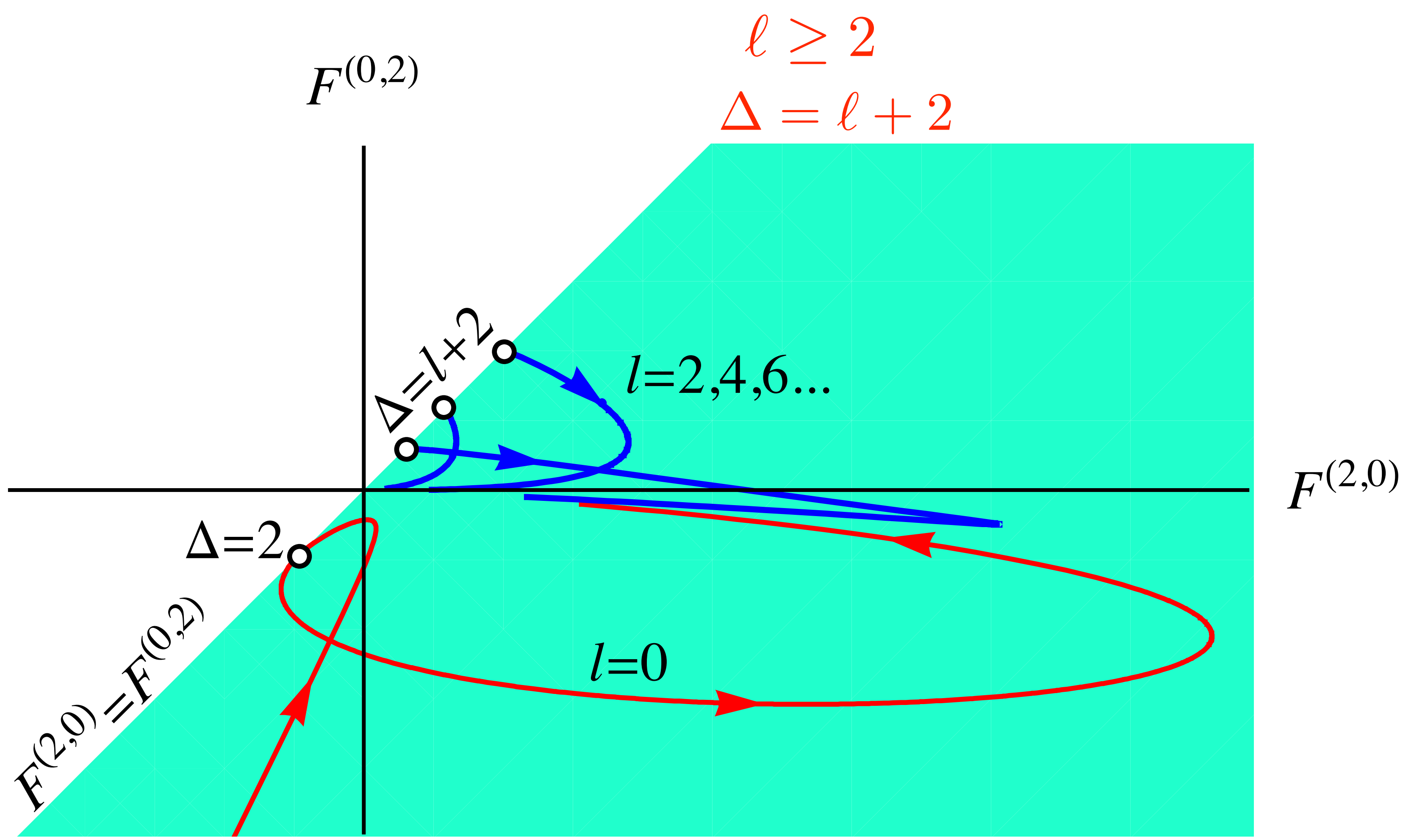
Projecting sum rule on subspaces: weaker but necessary constraint

Ex.: Cone projected on plane $F^{(0,0)} = 0$



Warm up exercise

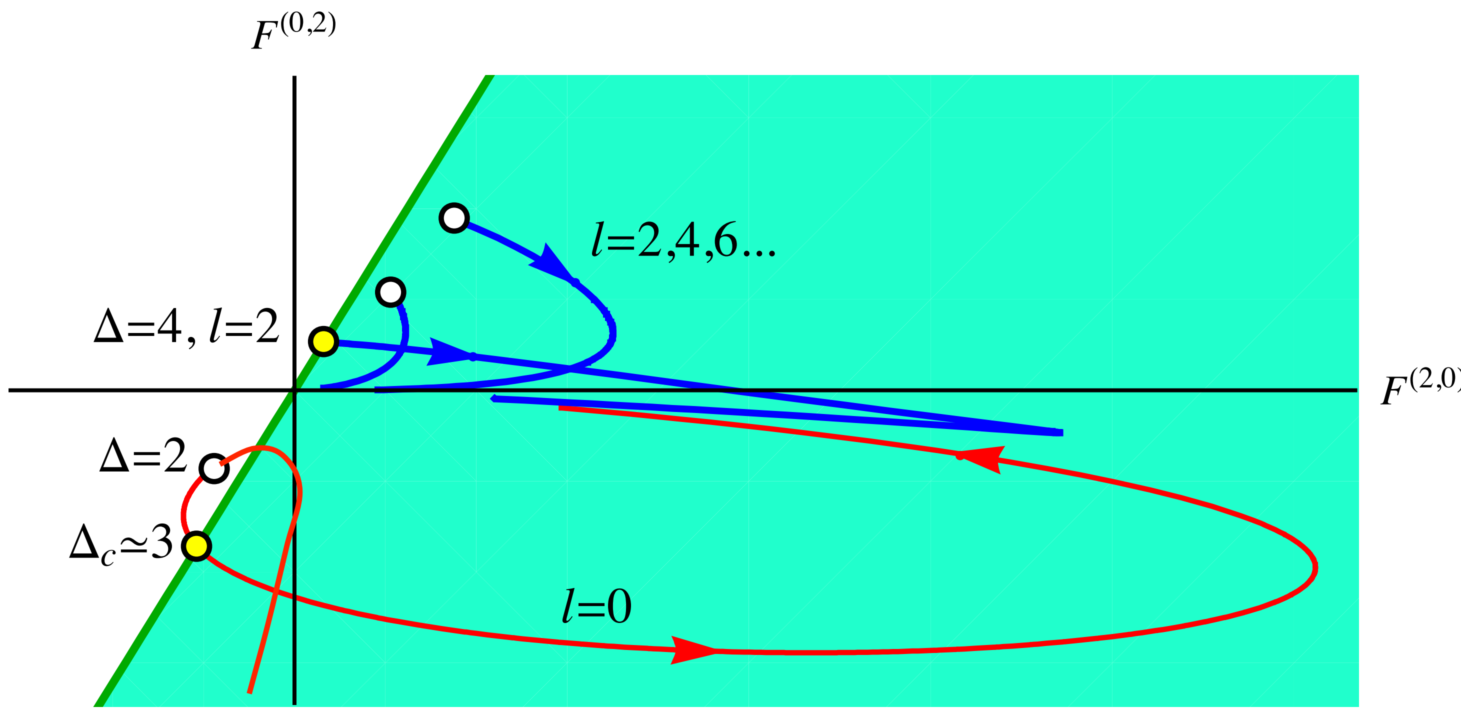
- $d = 1$
- project on subspace $(F^{2,0}, F^{0,2})$



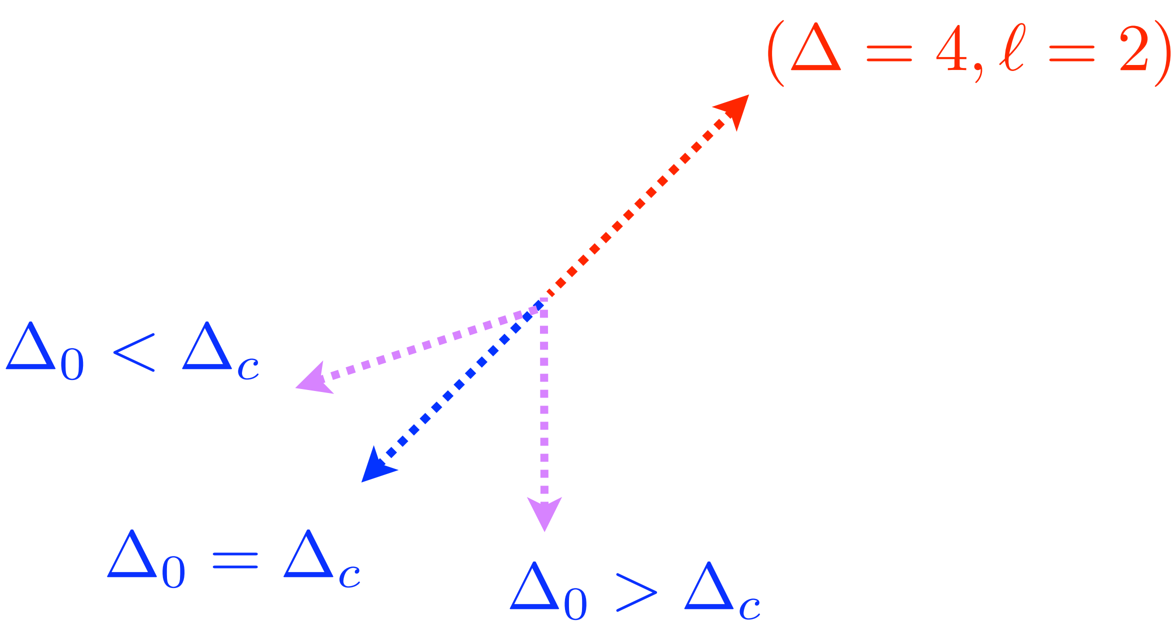
- projected sum rule implies that only twist 2 operators can appear in OPE !
- novel proof of known result that d=1 scalar is a free field

Less trivial exercise

- $d > 1$
- projecting on subspace $\left(F^{(2,0)}, F^{(0,2)}\right)$

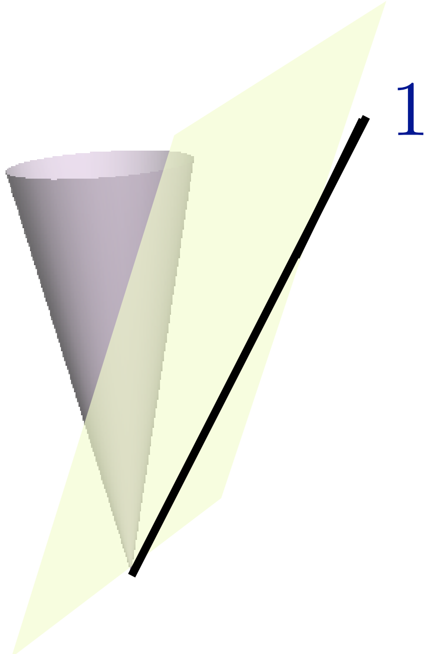


critical cone is determined by energy momentum tensor ray



$$\Delta_c \quad = 2 + \gamma\sqrt{d-1} + O(d-1)$$

$$\gamma \approx 2.929$$



analytically → ∃ linear functional Λ

$$\Lambda(F_{\Delta,\ell}) \geq 0$$

for (Δ, ℓ) in the spectrum

$$\Lambda(1) < 0$$

General Λ in derivative basis

$$\Lambda(f) = \sum_{n,m} a_{nm} \partial^{2n}_{z+\bar{z}} \partial^{2m}_{z-\bar{z}} f(z,\bar{z}) \Big|_{z=\bar{z}=\frac{1}{2}}$$

- ✦ given hypothetical spectrum, $\Lambda(F_{\Delta,\ell}) \geq 0$ defines convex subspace $\mathcal{P}$ of Λ space
- ✦ minimize Λ(1) on $\mathcal{P}$
- ✦ $\Lambda(1)_{min} < 0$ → sum rule cannot be satisfied

Linear Program !

In practice

✦ restrict to finite # of derivatives = minimize $\Lambda(1)$ on finite dimensional subspace of $\mathcal{P}$

necessary but
weaker constraint

✦ restrict to finite (Δ, ℓ) trial set

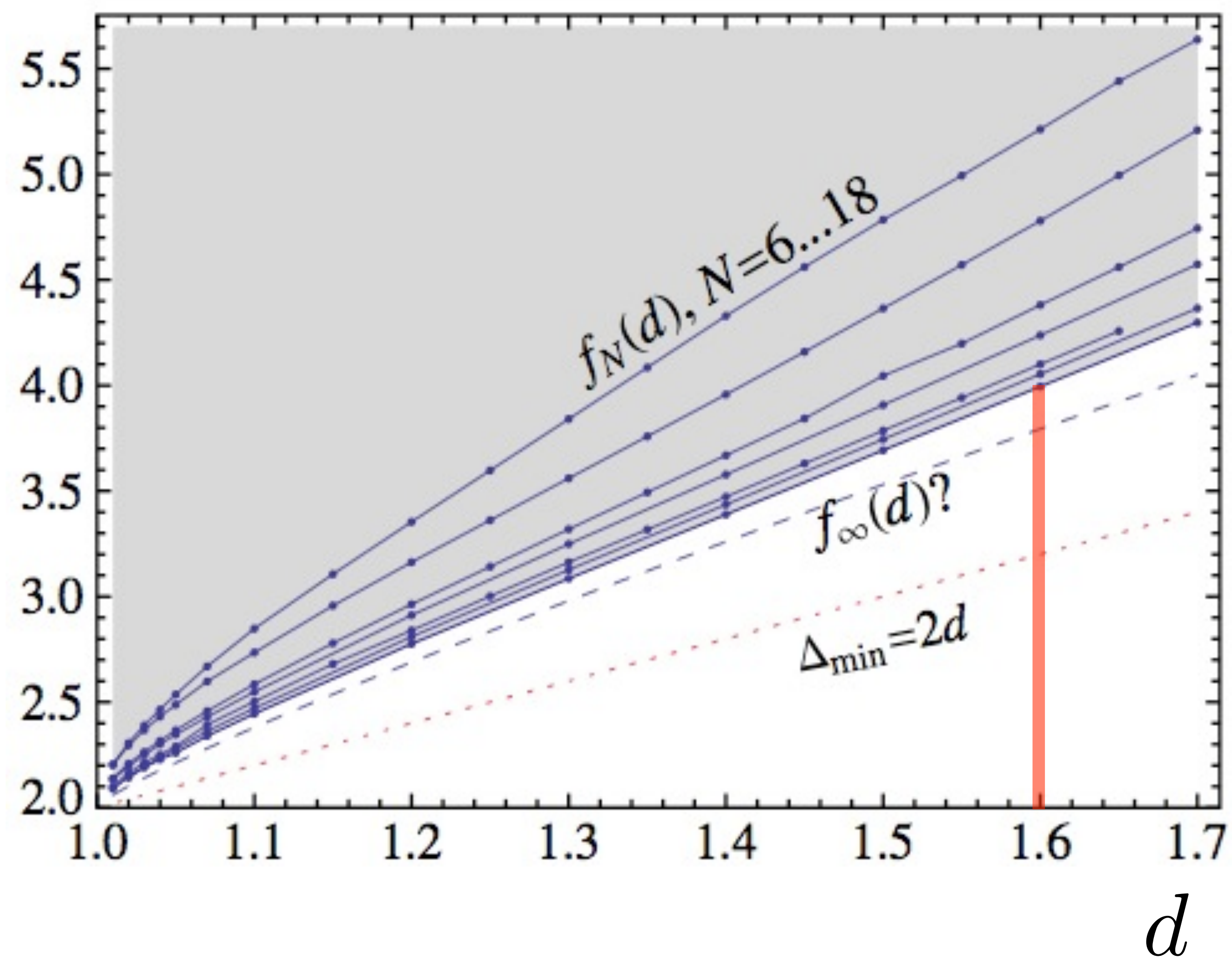
un-necessary
stronger constraint

- include spins up to ℓ_{max}
- include dimensions up to Δ_{max}
- discretize Δ
- to cover loose ends add to trial set the 'asymptotic ray' obtained by simple analytic formulae for derivatives at $\Delta, \ell \to \infty$

Finite dimensional Linear Programming: use routine in Mathematica

Best bound to date

$\ell_{max} = 20$
$\Delta_{max} = 200$
Δ step = 0.01

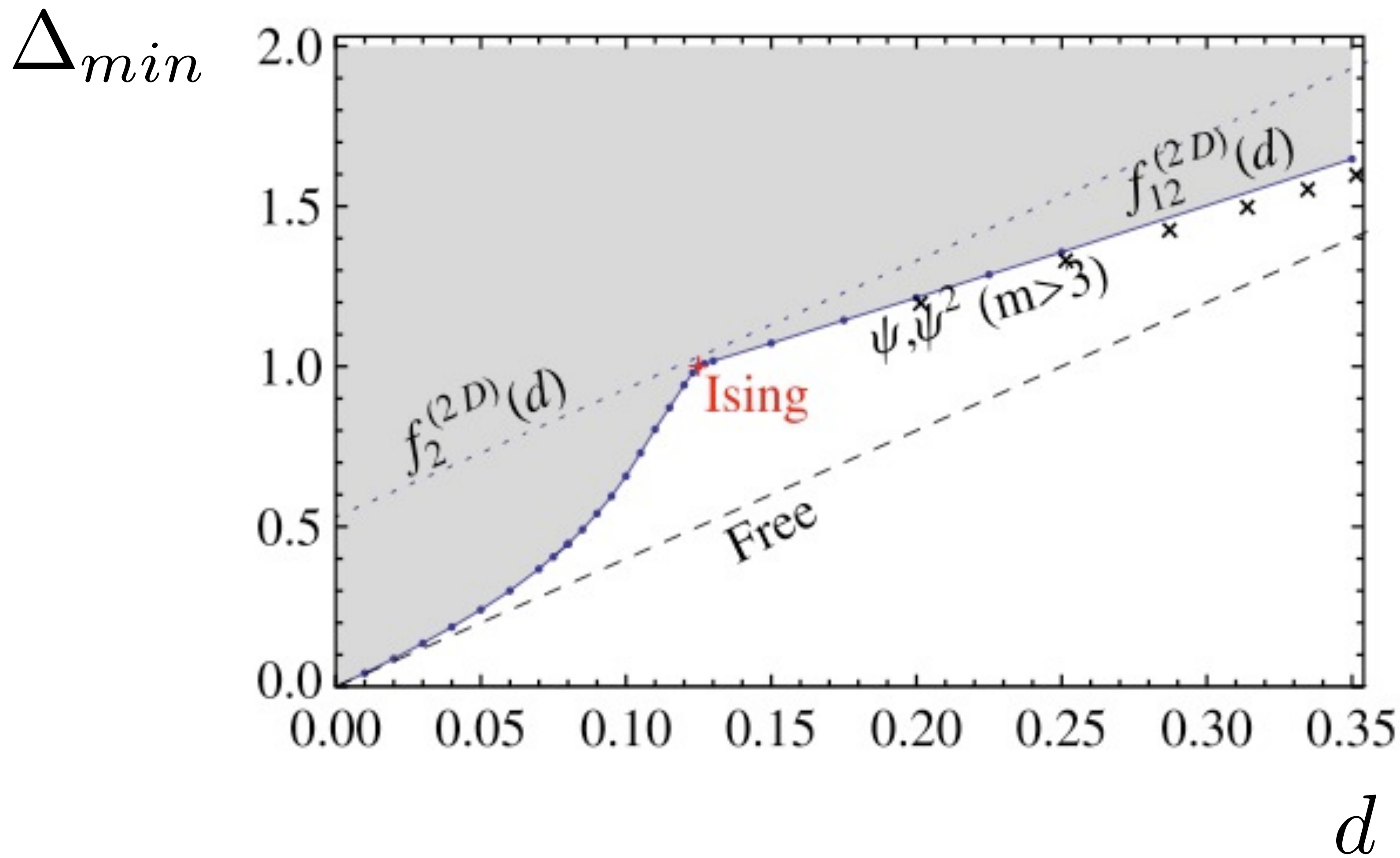


Rychkov, Vichi 09

$$\Delta < 2 + 0.7\sqrt{d-1} + 2.1(d-1) + 0.43(d-1)^{\frac{3}{2}}$$

Bound is trivially satisfied in known 4D CFTs (supersymmetry, large N)

Same bound in 2-dimensional CFT

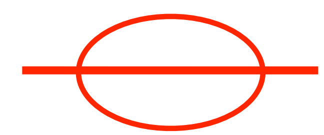


Crossing + Unitarity constraint seem to capture the relevant physics !

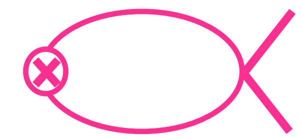
$D = 4 - \epsilon$

Wilson-Fisher O(N) model

$$d_\phi = \left(1 - \tfrac{\epsilon}{2}\right) + \tfrac{N+2}{4(N+8)^2}\epsilon^2$$



$$\Delta_{\phi^2} = (2 - \epsilon) + \tfrac{2}{N+8}\epsilon$$



- square root behaviour!
- numerical coefficient slightly 'violates' bound for N=1,2

not clear that we should worry

- bound strictly apply only to D=4
- not clear how to extend it to 4-ϵ

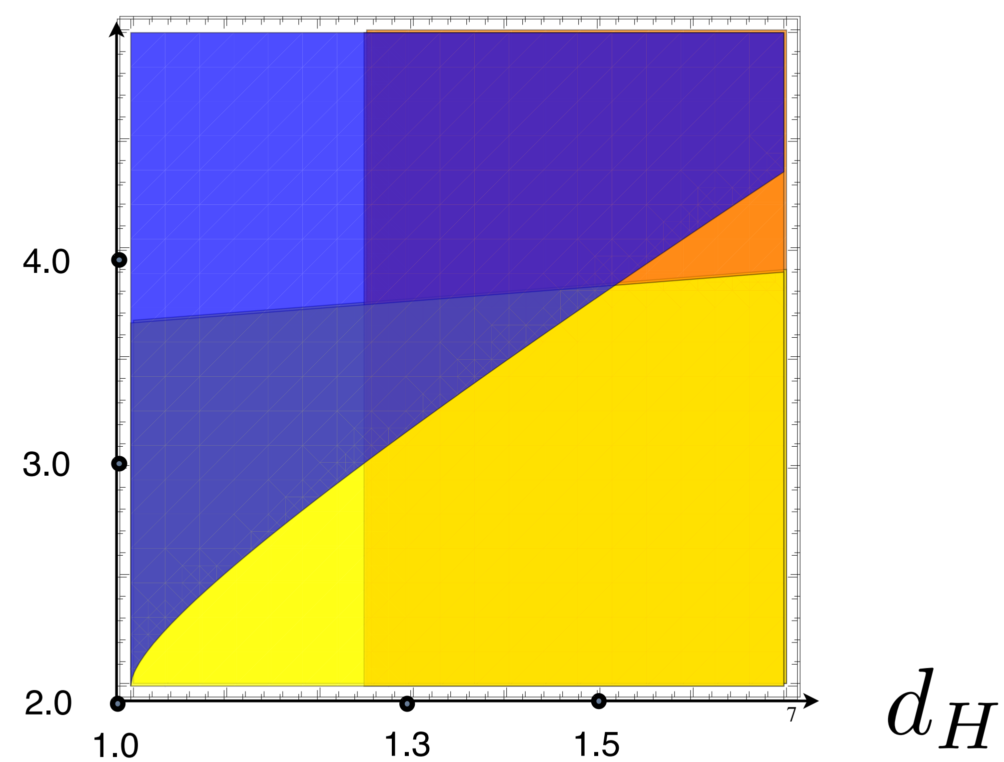
Back to Higgs doublet

$$H_i^\dagger \times H_j = S\,\delta_{ij} + T_A\,\tau^A_{ij} \equiv (\text{singlet}) + (\text{triplet})$$

- We did not use information about global quantum numbers
- The obtained bound is on $\Delta \equiv \min(\Delta_S, \Delta_T)$
- The 'Higgs mass' operator relevant to hierarchy is however S
- Analogy with O(N) Wilson-Fisher fixed point suggests $\Delta_S > \Delta_T$,

  so that actual bound on Δ_S may be weaker

Anyway let us pretend
the bound applies to Δ_S

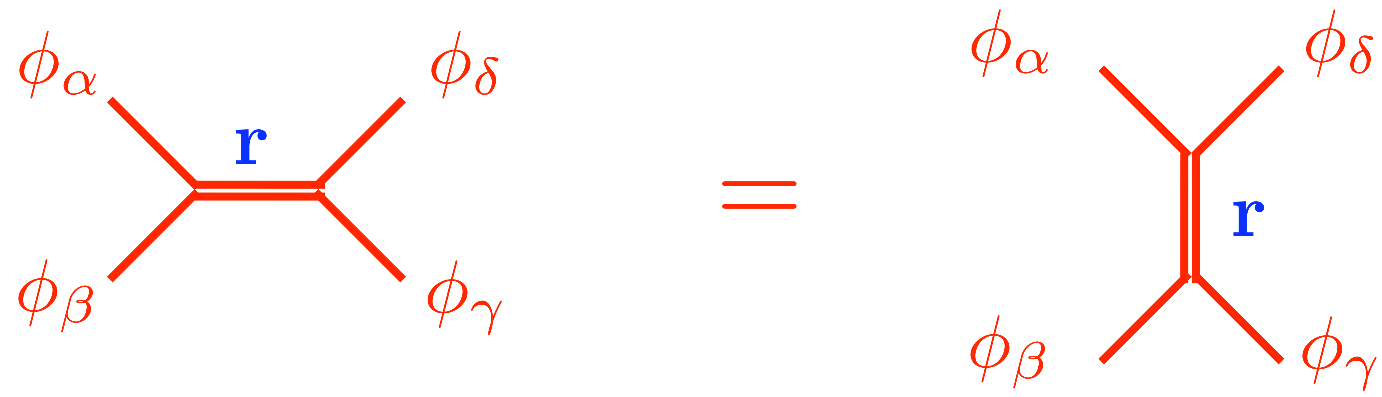


Voilà !

Adding ‘flavor’ to the CFT (global group G)

arXiv:1009.5985

Ex: ϕ real irrep



$$\frac{1}{u^d}\sum_{\mathbf{r}} T^{\mathbf{r}}_{\alpha\beta\gamma\delta}\, G_{\mathbf{r}}(u,v) \;=\; \frac{1}{v^d}\sum_{\mathbf{r}} T^{\mathbf{r}}_{\alpha\delta\gamma\beta}\, G_{\mathbf{r}}(v,u)$$

Fierz

$$T^{\mathbf{r}}_{\alpha\beta\gamma\delta} = \sum_{\mathbf{r}'} C^{\mathbf{r}\mathbf{r}'}\, T^{\mathbf{r}'}_{\alpha\delta\gamma\beta}$$

\# independent sum rules = # of G × parity channels

$G = SO(N)$

$\phi_i = \mathbf{N}$

$$\phi_i \times \phi_j \;=\; S_{ij} \oplus T_{ij} \oplus A_{ij}$$

ℓ even $\quad\quad$ ℓ odd

3 sum rules

- Can derive upper bound $\Delta_s < \Delta_s^{min}$ for d_ϕ close to 1
- Δ_s^{min} grows with d_ϕ
- $\Delta_s^{min} \to 2$ smoothly when $d_\phi \to 1$
- SO(N) (3 sum rules) × (3 channels) ➡ 9 times more complex
- 'Numerical instability' when trying to refine bound

G	$U(1) \equiv SO(2)$	$SO(3)$	$SO(4)$	$SU(2)$	$SU(3)$
d_*	1.063 $(k=2)$ 1.12 $(k=4)$	1.032 $(k=2)$ 1.08 $(k=4)$	1.017 $(k=2)$ 1.06 $(k=4)$	1.016 $(k=2)$	1.003 $(k=2)$

d_* = value of d_ϕ at which Δ_s^{min} crosses 4

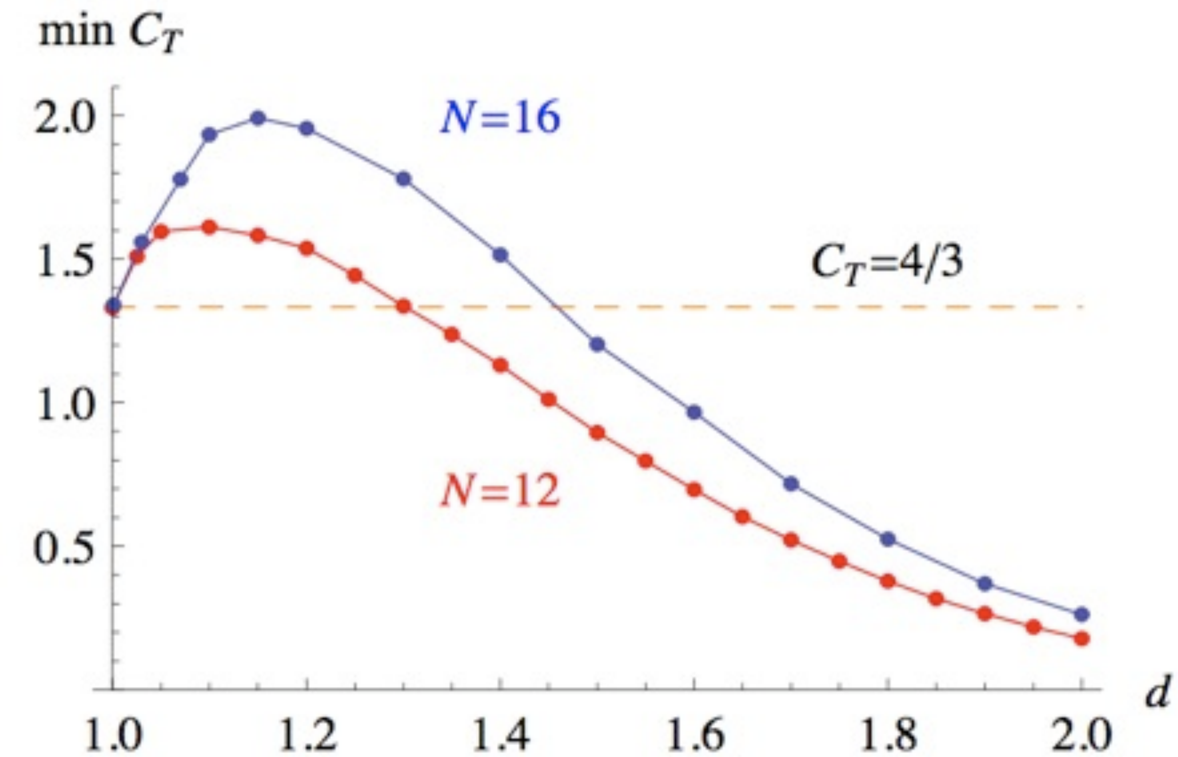
Summary

Conformal blocks + Unitarity + Crossing

$$\vec{\mathcal{V}} = \sum_{\mathcal{O}}{}' |\lambda_{\mathcal{O}}|^2 \vec{\mathcal{G}}_{\mathcal{O}}$$

- ✦ powerful constraint on spectrum of scalar operators (motivated by pheno)
- ✦ more widely applicable to constrain whole operator spectrum & couplings $\lambda_{\mathcal{O}}$

Ex: lower bound on central charge $C_T \propto \langle T_{\mu\nu} T_{\rho\sigma} \rangle$ $C_T = \frac{1}{|\lambda_{T_{\mu\nu}}|^2}$



Poland, Simmons-Duffin '10
Rattazzi, Rychkov, Vichi '10

...or current $C_J \propto \langle J_\mu J_\nu \rangle$

possible future directions

- try to stregthen bound on Δ_s by correlating it with sensible constraints on central charges (like suggested by exp bound on S-parameter)
- think of more efficient algorithm, taking into account the continuity of the constraints
- or think of alternative way to package the information in the sum rule, try and use analyticity of $g_{\Delta,\ell}$
- 3D CFTs and make contact with condensed matter systems: (watch: closed form of conformal blocks unknown in odd D !)