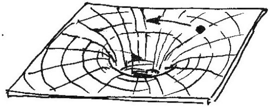
Orbital Motion in the Reference Frame of a Blackhole

Vladimir Yershov[1]

Co-authors A.A. Raikov[2] and E.A. Popova[3]



[1] formerly – Mullard Space Science Laboratory (UCL), London, U.K.
[2] Special Astrophysical Observatory, Russian Academy of Sciences, Nizhny Arkhyz, Russia
[3] Pulkovo Observatory, Russian Academy of Scinces, St.Petersburg, Russia

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Preliminaries

- ♢ Few people are aware of an existing issue with the processing scheme of gravitational-wave signals for estimating parameters of coalescent binary systems detected with the LIGO/Virgo/KAGRA interferometers
- ♢ The issue is related to the detector-to-source coordinate reference frame transformation, and it affects mainly the accuracy of the estimated blackhole masses (neutron stars are almost unaffected)
- ♢ The issue is subtle. So, in a brief presentation I will outline a simplified case of an equal-mass binary with circularised orbits, from which the root of the problem might be clearly seen

Contents

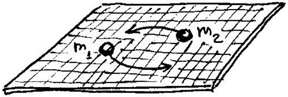
- Reformulation as an equivalent one-body problem
- GW150914: a convenient case of an almost equal-mass binary blackhole system
- Published proof of existence of an issue with GW-signal processing
- Source-to-detector coordinate transformation
- Reverse coordinate transformation
- Implications

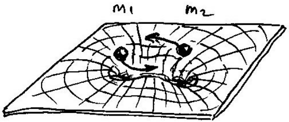
Reformulation as a one-body problem

- Gravitational-wave (GW) signals generated by coalescing binary blackhole systems are commonly processed by using the one-body formalism, mapping of the relative motion of two bodies to the motion of a reduced-mass (μ) particle around a fictitious central mass $M = m_1 + m_2$, the reduce mass being $\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2}$.
- Gravitational-wave waveform templates needed for processing GW signals are calculated then by using the Effective One-Body (EOB) formalism [Buonanno A., Damur T. (1999) Phys. Rev. D59, 084006], which is a post-Newtonian (PN) approximation [Peters P.C., Mathews J. (1963) Phys. Rev. 131, 435] for dealing with the linearised Schwarzschild metric deformed by the presence of a massive body.
- The EOB theory is validated by numerical-relativity (NR) simulations [Ossokine, S. et al. (2020) Phys. Rev. D 102, 044055] for the last circular orbits before the plunge stage. So, the GW waveform templates are regarded as almost exact and providing accurate information about blackhole masses, spins, orbit inclination and the luminosity distance to the merger.

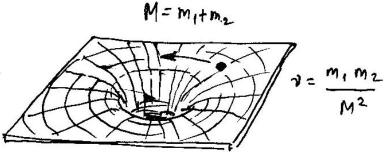
Reformulation as a one-body problem

PN-approximation (flat space-time): suitable for neutron stars at far-distances from each other





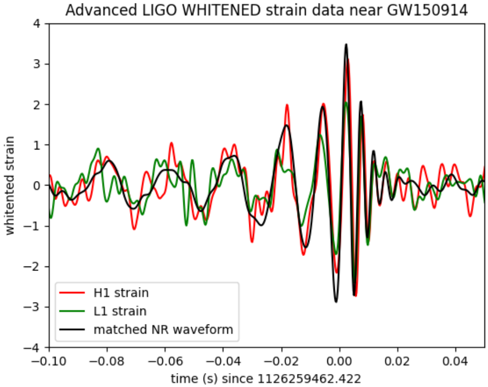
A real (non-linear) two-body case: massive blackholes at near-distances to each other



Effective-one-body reformulation of the real case using a PN-approximation for dealing with the linearised Schwarzschild metric of a fictitious central mass M, the metric being deformed (with the deformation parameter ν) by the presence of an orbiter of the reduced mass μ

Gravitational wave signal GW150914

A convenient case for exploring the coordinate-transformation issue is the case of the first-detected gravitational wave signal GW150914 [Abbott B.P. et al. (LVC), *Phys. Rev. Lett.*, **2016**, *vol.116*, 241102], which corresponds to a binary of nearly equal-mass blackholes ($\sim 35\,M_\odot$) with circularised orbits.



Instrumental strains (10^{-21} units) of the GW150914 case from the LIGO detectors Hanford (H1, red curve) and Livingston (L1, green curve). The fitted fitted Numerical Relativity waveform template is shown as the black curve.

Published proof of existence of an issue

There are two publications of the same authors (the LIGO/Virgo Scientific Collaboration) that attest to the fact of existence of an issue with estimating the binary-blackhole system parameters from GW signals:

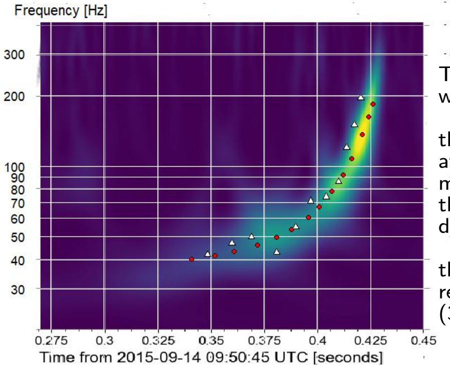
- The first publication (the GW discovery paper): [Abbott B.P. et al. (LVC) Observation of Gravitational Waves from a Binary Black Hole Merger, *Phys. Rev. Lett.*, **2016**, *vol.116*, 241102] presents the binary blackhole mass estimation based on Numerical Relativity (NR) waveforms accounting for the **full non-linearity of General Relativity**.

- The second publication [Abbott B.P. et al. (LVC) The basic physics of the binary black hole merger GW150914, *Ann. Phys. (Berlin)*, **2017**, *vol. 529*, 1600209] presents the estimation of the same binary mass based on the quadrupole post-Newtonian (PN) approximation, which is an **approximation of flat spacetime**.

The LVC authors highlight
that the estimated characteristic (chirp) masses in both cases **agree with each other**.

In our view, this agreement requires a very focused attention because the approximation of flat spacetime by design does not account for strong spacetime non-linearities when two merging blackholes are separated by just a few gravitational radii from each other. This coincidence of the mass estimations indicates that **there is an issue here**, and that the issue is likely to be with the estimation based on NR waveforms, because GR non-linearities should have produced a different result, as compared to the flat spacetime treatment.

GW signal GW150914



Time-frequency map for the H1 gravitational wave signal GW150914:

the white triangles indicate the frequencies at the successive zero-crossings of the instrumental strain time series used by the LVC for their PN-based calculation (34.5 $M_\odot$, noisy data points);

the red circles are the frequencies read directly from this diagram for our calculation ($30.1 \pm 2.4\,M_\odot$, filtered data).

Both PN-estimation from LVC-2017 estimation (34.5 $M_\odot$) based on the noisy data points from the successive zero-crossings of the instrumental strain, and our estimation here ($30.1 \pm 2.4\,M_\odot$) based on the filtered data points read directly from the time-frequency diagram (above), agree with the LVC-2016 result $30.4 \pm 2.4\,M_\odot$ [Abbott B.P. et al. (LVC) *Ann. Phys. (Berlin)*, **2017**, *vol. 529*, 1600209] based on numerical relativity.

Source-to-detector coordinate transformation

◇ Using Kepler's third law $r^3 = \frac{GM}{\omega^2}$ and its time derivative $\dot{r} = -\frac{2}{3}\frac{r\dot{\omega}}{\omega}$ we can calculate the orbital frequency growth due to the emission of gravitational waves:

$$\dot{\omega}_{\rm orb}^3 = \left(\frac{96}{5}\right)^3 \frac{\omega_{\rm orb}^{11}}{c^{15}} G^5 \mu^3 M^2 = \left(\frac{96}{5}\right)^3 \frac{\omega_{\rm orb}^{11}}{c^{15}} (G\mathcal{M})^5, \quad (1)$$

where

$$\mathcal{M} = (\mu^3 M^2)^{\frac{1}{5}} = \frac{(m_1 m_2)^{3/5}}{(m_1 + m_2)^{1/5}} \quad (2)$$

is the characteristic (chirp) mass of the system.

◇ Thus, by knowing two parameters, the frequency of a gravitational wave signal $f_{\rm gw} = 2\,f_{\rm orb} = \omega_{\rm orb}/\pi$ and its time derivative $\dot{f}_{\rm gw}$, we can find this mass according to the formula

$$\mathcal{M}(f_{\rm gw}, \dot{f}_{\rm gw}) = \frac{c^3}{G}\left[\frac{5}{96}\pi^{-8/3} f_{\rm gw}^{-11/3} \dot{f}_{\rm gw}\right]^{3/5}. \quad (3)$$

◇ These quadrupole-approximation formulae were used for calculating orbital paremeters of the famous binary pulsar PSR B1913+16 [Hulse R.A., Taylor J.H. (1975) ApJ 195 L51], whose orbital parameters were found to evolve in full agreement with the theoretical GW formulae. This agreement between theory and observations was interpreted as very strong, albeit indirect, evidence of the existence of gravitational waves.

Source-to-detector coordinate transformation

- ♢ The calculation of waveform templates for the detector reference frame envisages the coordinate transformation based on the Arnowitt-Deser-Misner (ADM) parametrisation [Arnowitt R., Deser S., Misner C.M. (1959) Phys. Rev. 116, 1322], which includes integration of the energy-momentum tensor over the entire space and the variation of the reduced action functional in order to obtain the detector-frame equations of motion.
- ♢ These equation of motion do not involve the lapse and shift functions, although they are fully GR-compliant. Therefore, the detector-frame GW waveforms are GR-accurate and correspond to the BH physical masses used for calculating these waveforms.
- ♢ **But this is just the one-way accuracy (source-to-detector)!** The source-to-detector coordinate transformation of GW waveforms uses the the physical masses, as required (which is OK). But the calculation of physical masses using the detector-frame GW signal **requires the knowledge of source-frame time derivatives of GW frequencies which are not available** at the detector reference frame.
- ♢ Therefore, by using GW signals as they are (taken from the detector reference frame), we obtain the detector-frame (non-physical) masses of coalescing binary blackholes. **Thus, a calculation based on the detector-frame frequencies f and $\dot{f}$ must be affected by a systematic error**.

Source-to-detector coordinate transformation

- A more clear picture of this problem can be seen in an example from classical textbooks for circular-orbits around a Schwarzschild blackhole with $r_g = 2\,\frac{GM}{c^2}$.
- In this case, a circular orbit corresponds to the following values of the specific angular momentum $\tilde{\mathcal{L}} = \frac{p_\varphi}{\mu} = u_\varphi$:

$$\tilde{\mathcal{L}} = \sqrt{\frac{rGM/c^2}{1 - 3GM/(rc^2)}} \tag{4}$$

and specific energy $\tilde{\mathcal{E}} = -\frac{p_t}{\mu} = -u_t$:

$$\tilde{\mathcal{E}} = \frac{1 - r_g/r}{\sqrt{1 - 3GM/(rc^2)}}\,. \tag{5}$$

- Consequently),

$$u^\varphi = \frac{\mathrm{d}\varphi}{\mathrm{d}\tau} = \frac{u_\varphi}{g_{\varphi\varphi}} = \frac{\tilde{\mathcal{L}}}{r^2}\,, \quad u^t = \frac{\mathrm{d}t}{\mathrm{d}\tau} = \frac{u_t}{g_{tt}} = \frac{\tilde{\mathcal{E}}}{1 - r_g/r}, \tag{6}$$

where g_{tt} and $g_{\varphi\varphi}$ are the spacetime metric coefficients for, respectively, time and the azimuthal angle in spherical coordinates, $\sqrt{g_{tt}}$ being the time lapse function.

- Then, by calculating $\mathrm{d}\varphi/\mathrm{d}t$ from (6), we get the angular frequency of orbital motion in the coordinate time t (i.e., in the detector reference frame)

$$\omega^{\mathrm{det}} = \frac{\mathrm{d}\varphi/\mathrm{d}\tau}{\mathrm{d}t/\mathrm{d}\tau} = \frac{u^\varphi}{u^t} = \sqrt{\frac{GM}{c^2 r^3}}, \tag{7}$$

which coincides with the Keplerian non-relativistic formula (but which is fully general-relativistic). So, the time-lapse function is lost on the way during the source-to-detector coordinate transformation (like in the ADM parametrisation.)

Source-to-detector coordinate transformation

- In accordance with formula (7), the orbital period $T^{\rm det} = 2\pi/\omega^{\rm det}$ reads

$$T^{\rm det} = \frac{2\pi\, c\, r^{3/2}}{\sqrt{GM}}\,. \tag{8}$$

- For example, by knowing $r_{\rm ISCO} = 6\,\frac{GM}{c^2}$ we can calculate the ISCO orbital period for a test mass μ as measured in the detector reference frame:

$$T^{\rm det}_{\rm ISCO} = \frac{12\pi}{\sqrt{2/3}}\,\frac{2GM}{c^3}\,. \tag{9}$$

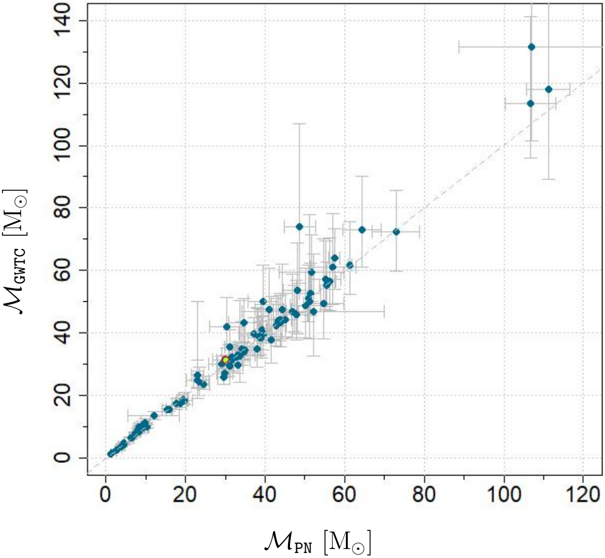
- Seemingly, everything looks fine, as the formulae (3) and (7) give the source-frame (physical) mass $M = m_1 + m_2$ of our binary blackhole as derived from the detector-frame quantities $\omega^{\rm det}$, $f_{\rm gw}$ and $\dot{f}_{\rm gw}$, whereas the motion of masses m_1 and m_2 is considered in the source reference frame.
- However, the quadrupole moments and changes in the orbital-motion parameters needed for obtaining $\dot{\omega}_{\rm orb}$ and the corresponding $\dot{f}_{\rm gw}$ are calculated within the quadrupole formalism framework, using local coordinates and proper time τ of the source.
- But sometimes, when calculating the quadrupole moments, this proper time is denoted as t; i.e. the distinction between τ and t is dropped. This is explicit in the LVC PN-analysis [Abbott B.P. et al. (LVC) *Ann. Phys. (Berlin)* 2017, 529, 1600209] when, for example, these authors derive formula (3). In that derivation, the authors, indeed, do not make any distinction between the proper and coordinate time.

Reverse coordinate transformation

Chirp masses $\mathcal{M}_i$ (GW150914) estimated for the moments of times taken from the time-frequency diagram for the instrumental strain of GW150914 t_i are the successive times; $f_{\rm gw}$ and $\dot{f}_{\rm gw}$ are the observed GW frequencies and their time derivatives for each time interval; r_i are the Keplerian distances between two objects at each time interval; $f_{\rm Kep}$ are the corresponding Keplerian orbital frequencies ($f_{\rm Kep} = 0.5\, f_{\rm gw}$); v_i are the orbital velocities for the total mass of the system $M = 67\,\mathrm{M}_\odot$; $\alpha_i^{\rm Sch}$ and $\alpha_i^{\rm Kerr}$ are the time-lapse factors for, respectively, the Schwarzschild and Kerr metrics; $f_{\rm gw}^{\rm src}$ is the GW frequency translated to the source reference frame by using $\alpha^{\rm Kerr}$.

i	t_i [s]	$f_{\rm gw}$ [s^{-1}]	$\dot{f}_{\rm gw}$ [s^{-2}]	$\mathcal{M}_i$ [$M_\odot$]	r_i [km]	$f_{\rm Kep}$ [s^{-1}]	v_i/c	$\alpha_i^{\rm Sch}$	$\alpha_i^{\rm Kerr}$	$f_{\rm gw}^{\rm src}$ [s^{-1}]	$\mathcal{M}_i^{\rm src}$ [$M_\odot$]
1	2	3	4	5	6	7	8	9	10	11	12
1	0.352	41.7	134.9	28.5	811.2	20.9	0.177	0.865	0.806	51.7	–
2	0.361	43.3	178.6	31.1	791.0	21.7	0.180	0.861	0.802	54.0	27.2
3	0.372	46.1	256.4	33.6	758.4	23.1	0.183	0.855	0.793	58.2	29.1
4	0.381	49.7	391.6	36.8	722.1	24.8	0.188	0.847	0.783	63.4	31.7
5	0.388	53.7	577.4	39.2	685.4	26.9	0.193	0.838	0.771	69.6	33.4
6	0.396	60.7	871.5	38.3	631.9	30.3	0.201	0.823	0.753	80.6	32.0
7	0.401	67.8	1279.6	38.7	591.0	33.5	0.208	0.809	0.736	91.2	31.9
8	0.407	78.1	1843.1	34.4	533.9	39.1	0.219	0.786	0.709	110.2	27.6
9	0.412	91.8	2735.8	30.6	479.4	45.9	0.231	0.758	0.678	135.4	23.7
10	0.416	107.4	3898.6	26.8	431.8	53.7	0.243	0.726	0.647	166.0	20.0
11	0.421	136.4	5790.6	20.1	368.4	68.2	0.263	0.668	0.599	227.7	13.9
12	0.424	161.7	8431.4	17.3	328.8	80.8	0.278	0.616	0.567	285.0	11.4
13	0.426	183.4	10887.5	15.3	302.2	91.7	0.290	0.570	0.548	334.9	9.6
		Average $\mathcal{M}$:		$30.1\pm2.4\,M_\odot$				Average $\mathcal{M}^{\rm src}$:		$24.3\pm2.6\,M_\odot$	

Other GW observations:



← The Pearson correlation test gives the following result:

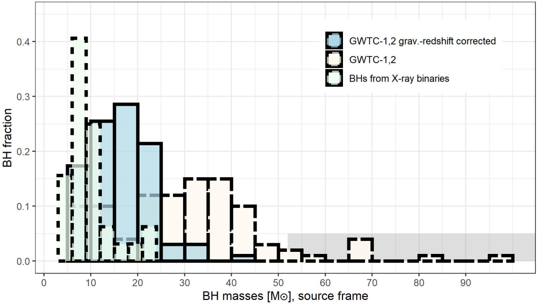
the correlation coefficient is: $0.985^{+0,004}_{-0.008}$

p-value $< 2.2 \cdot 10^{-16}$,

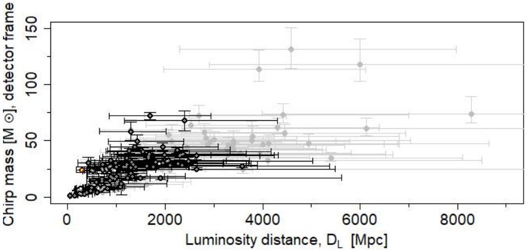
which indicates the practically perfect diagonality of the plot and demonstrates the full one-to-one correspondence between the masses $\mathcal{M}$ computed by two different methods.

Diagonal plot showing the correspondence between the 97 chirp masses $\mathcal{M}_{\mathrm{GWTC}}$ (detector-frame) from the Gravitational-Wave Transient Catalogue (vertical axis) and those calculated by using the PN-flat-spacetime formula (3) – the horizontal axis.

Recalculated massed and distances



← Unusually large BH masses, some of which being within the pulsational pair-instability gap (grey rectangle). After correction, the histogram of masses is closer to the histogram of previously known blackholes.



← Unusual correlation between BH masses and distances, which cannot be fully explained by the observational selection effect. After corrections, it is reduced.

Conclusions

- Here we have presented a proof of existence of a systematic error in the estimation of the characteristic (chirp) masses of coalescing binary systems discovered via gravitational wave detections. An important implication of this result is our conclusion that the estimated luminosity distances to the discovered gravitational sources are largely overestimated, especially in the cases with large chirp masses.

- Since the locations on the sky of gravitational sources are currently determined with large uncertainties, the search for any possible electromagnetic transient counterparts is performed within the locations of host-galaxy candidates pre-selected on the basis of information about the likely gravitational source distances. Therefore, more accurate knowledge of this distances is crucial for increasing the probability of finding electromagnetic counterparts of gravitational wave sources. So far, only one such counterpart was found.

A more detailed description can be found at:
Physica Scripta **98** (2023) 075019
`https://iopscience.iop.org/article/10.1088/1402-4896/ace00c`