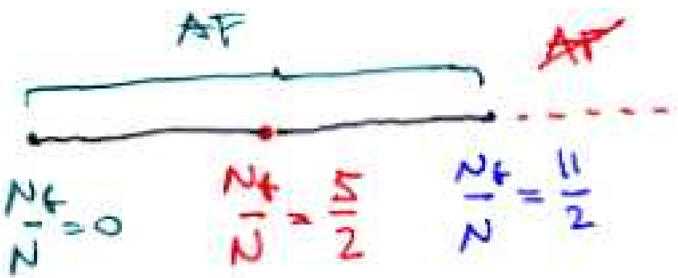
Conformal window in large N QCD

Result (easy to state): In large N QCD with N_f massless Dirac fermions $\frac{N_f}{N}$ = const. there is a phase transition from a phase with massive glueballs to a phase with no massive glueballs (Coulomb-like) precisely at $\frac{N_f}{N} = \frac{5}{2}$

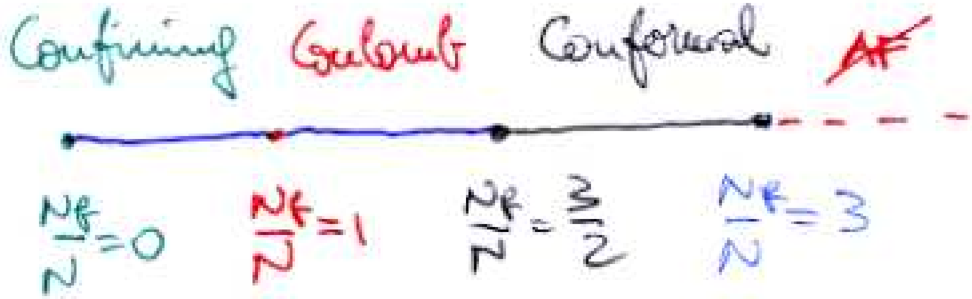


two loop β function $\rightarrow \frac{N_f}{N} = \frac{34}{13} \sim 2.61$

$N \to \infty$

Difficult to prove → Proof is the subject of this talk

Seiberg argued that in $\mathcal{N}=1$ SUSY QCD with N_f flavors of quarks in the N and $\bar{N}$ representation



Glueballs spectrum in large N Yang-Mills theory by localization on the fixed points of a semigroup contracting the functional measure

M.B. [hep-th/1011.1707] to appear in PoS Lattice 2010
[hep-th/0810.0776] PoS EPS-HEP 2009
[hep-th/0809.4662] JHEP 0905(2009)116

and to appear

Quantum Yang-Mills theory according to Clay Math. Inst.

by A. J. - E. W.

1) Prove the existence of the renormalized YM functional integral for every SU(N) gauge group

$$Z(\Lambda, g_{YM}) = \int e^{-\frac{1}{2g_{YM}^2}\sum_{\alpha\neq\beta}\int \mathrm{Tr}_f(F_{\alpha\beta}^2(A))} \prod_x DA_\alpha(x)$$

$\Lambda \to \infty$, $g_{YM} \to 0$ according to AF

2) Prove the existence of a mass gap

$$\left\langle \mathrm{Tr}\sum_{\alpha\neq\beta} F_{\alpha\beta}^2(x)\ \mathrm{Tr}\sum_{\alpha\neq\beta} F_{\alpha\beta}^2(y)\right\rangle \leq C(N)\, e^{-m(N)|x-y|}$$

1) Is an ultraviolet problem

$$\frac{\partial g_{YM}}{\partial \log\Lambda} = \beta(g_{YM}) = -\beta_0 g_{YM}^3 - \beta_1 g_{YM}^5 + \dots$$

$$\beta_0 = \frac{1}{(4\pi)^2}\frac{11}{3}N$$

$$\beta_1 = \frac{1}{(4\pi)^4}\frac{34}{3}N^2$$

2) Is both an infrared and an ultraviolet problem

In YM there is no decoupling between the IR and the UV

$$\left[\frac{\partial}{\partial\log\Lambda} + \beta(g_{YM})\frac{\partial}{\partial g_{YM}}\right] m(\Lambda, g_{YM}) = 0 \qquad \text{(RG)}$$

$$m(\Lambda, g) = \text{const}\, \Lambda\, e^{-\frac{1}{2\beta_0 g_{YM}^2}} \left(\frac{1}{\beta_0 g_{YM}^2}\right)^{\frac{\beta_1}{\beta_0^2}} (1+\dots) = \text{const}\, \Lambda_{YM}$$

Consequences of the RG

1) Every physical mass scale is zero to every order of perturbation theory

2) The mass gap problem is a weak coupling problem that needs an amazing accuracy $o(e^{-\frac{1}{g^2_{YM}}})$ as $g_{YM} \to 0$ to be solved, not a strong coupling problem

3) The Clay Math. Inst. problem for YM in full generality is, in my opinion, hopeless

Thus we consider a simpler problem: SO(N) YM in the large N limit

$$Z = \int e^{-\frac{N}{2g^2}\int \sum_{\alpha\neq\beta} \mathrm{Tr}\, F_{\alpha\beta}^2\, d^4x}\, \delta A \;;\quad g^2 = g_{YM}^2 N = \text{const} \;;\quad N\to\infty$$

$$\text{(-)}\quad \left\langle \frac{1}{N}\mathrm{Tr}\, F_{\alpha\beta}^2(x_1)\cdots\frac{1}{N}\mathrm{Tr}\, F_{\alpha\beta}^2(x_k)\right\rangle \overset{N\to\infty}{=} \left\langle\frac{1}{N}\mathrm{Tr}\, F_{\alpha\beta}^2(x_1)\right\rangle\cdots\left\langle\frac{1}{N}\mathrm{Tr}\, F_{\alpha\beta}^2(x_k)\right\rangle$$

$$\text{(NL)}\quad \int \left\langle\frac{1}{N}\mathrm{Tr}\, F_{\alpha\beta}^2(x)\,\frac{1}{N}\mathrm{Tr}\, F_{\alpha\beta}^2(0)\right\rangle_{\text{conn}} e^{ip\cdot x}\, d^4x = \sum_{2}\frac{Z_2}{p^2+m_2^2} \sim$$

$$\sim g^4(p^2)\, p^4 \log\left(\frac{p^2}{\mu^2}\right) + \text{contact terms}$$

two loop perturbation theory

Long standing conjecture: large N YM is solved by a string theory

$$\left\langle \frac{1}{N} \mathrm{Tr} P e^{i\int_C A_\alpha dx^\alpha} \right\rangle = \int_{X^\alpha(0,\sigma)=C^\alpha} e^{-T S_{string}(X^\alpha,\varphi)} DX^\alpha(\tau,\sigma)\, D\varphi$$

C

string world sheet

$T \sim \Lambda_{YM}^2$

glueball propagator

Example: $\mathcal{N}=4$ SUSY YM (Maldacena 1997) computable explicitly for $g_{YM}\to\infty$; $\beta(g_{\mathcal{N}=4})=0$

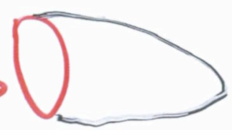
The string program for YM is very ambitious = solving the large N limit of the whole theory

Easier problem: solve the large N limit for special Wilson loops, called twistor Wilson loops.

The v.e.v of twistor Wilson loops is trivially 1 for $N \to \infty$ but they admit non-trivial $\frac{1}{N}$ corrections in their correlators $\to$ glueballs spectrum restricted to this sector

New approach: Localization without supersymmetry =

= in certain cases the saddle-point approximation turns out to be exact

twistor Wilson loop $\rightarrow$  $= 1$

 = non-trivial

Localization

D-H formula

$$\int_M \frac{e^{-\beta H}}{n!}\omega^n = \sum_p \frac{e^{-\beta H(p)}}{\beta^n}\left(\frac{\operatorname{Det}\omega}{\operatorname{Det}\frac{\delta^2 H}{\delta x^2}}\right)^{\frac{1}{2}}(p)$$

H = Hamiltonian for a torus $= U(1)\times \dots U(1)$ action on a compact symplectic manifold M

p are the isolated fixed points for the torus action

$g\cdot p = p$

D-H localization is a cohomology theory (A-B, B)

$$Z = \int_M e^{-\omega} \quad ; \quad d\omega = 0 \quad ; \quad M \text{ compact without boundary} \Rightarrow \int_M d\alpha = \int_{\partial M} \alpha = 0$$

$$Z(t) = \int_M e^{-\omega - t\, d\alpha} \quad ; \quad \left.\frac{dZ(t)}{dt}\right|_{t=0} = -\int_M d\alpha\, e^{-\omega} + \int_M \alpha \,(\underset{\overset{\|}{0}}{+d\omega})\, e^{-\omega}$$

$$= Z(0)$$

$$= -\int_M d(\alpha e^{-\omega}) = 0$$

Take the limit $t \to +\infty$

$Z(t)$ localizes on the critical points of $d\alpha$
and the saddle point approximation is exact

Localization in SUSY gauge theories (W)

$(\Omega, d, \int)$ $\quad$ $(\Omega, Q, \int)$ $\quad$ BRST

$d^2 = 0$ $\quad$ $Q^2 = 0$

$\int d\alpha = 0$ $\quad$ $\int Q\alpha = 0$

Q = twisted super-charge

$Q S_{SUSY} = 0 \; ; \quad Q\mathcal{O} = 0$

$$\langle \mathcal{O} \rangle = \int \mathcal{O}\, e^{-S_{SUSY}} = \int \mathcal{O}\, e^{-S_{SUSY} - tQ\alpha}$$

$t \to \infty$ $\langle \mathcal{O} \rangle$ localizes on the critical points of $Q\alpha$

$\langle \mathcal{O} \rangle = [\mathcal{O}]$ = [Cohomology class]

Example: localization of (the cohomology of) 1 in $\mathcal{N}=2$ SUSY YM → Nekrasov partition function → → Seiberg-Witten prepotential $\mathcal{F}$

$$\langle 1 \rangle_{\mathcal{N}=2} = \sum_{Q} e^{-\frac{16\pi^2}{2g_W^2}Q} \Lambda^{n_B(Q) - \frac{n_F(Q)}{2}} \int_{\mathcal{M}_Q} \omega$$

by cohomological localization

$$\int_{\overline{\mathcal{M}}_Q} \omega\, e^{-\varepsilon\mu} = \frac{1}{\varepsilon^n}\left(\frac{\operatorname{Det}\omega}{\operatorname{Det}\frac{\delta^2\mu}{\delta m^2}}\right)^{\frac{1}{2}}(P)$$

by the D-H formula

P = fixed point for the torus in $SU(N)\times SO(4)$

$$\langle 1 \rangle_{\mathcal{N}=2} = e^{\frac{1}{\varepsilon^2}\mathcal{F}}$$

The β-function for the Wilsonian coupling g_W is one-loop exact

Localization in $\mathcal{N}=1$ SUSY YM ?

No twist of SUSY

Nicolai map: change of variables from the gauge connection to ASD part of the curvature: $A_\alpha \to F^-_{\alpha\beta}$

$$F^-_{\alpha\beta} = F_{\alpha\beta} - \tilde{F}_{\alpha\beta}$$

$$\tilde{F}_{\alpha\beta} = \frac{1}{2}\varepsilon_{\alpha\beta\gamma\delta} F_{\gamma\delta}$$

$\mathcal{N}=1$ SUSY YM

$$Z_{\mathcal{N}=1} = \int e^{-\frac{1}{4g^2}\int \sum_{\alpha\beta} \mathrm{Tr}(F_{\alpha\beta}^2)} \,\mathrm{Det}\,\cancel{D}\; DA$$

$$= \int e^{-\frac{16\pi^2 QN}{2g^2} - \frac{1}{8g^2}\int \mathrm{Tr}(F_{\alpha\beta}^{-\,2})} \underbrace{\mathrm{Det}\,\cancel{D}\;\frac{DA}{DF^-}}_{1}\, DF^- \Big|_{A_+=0}$$

We have already seen that schematically

$$Z_{N=1} = \int e^{-Q - \int (F^-)^2} \frac{\delta F^-}{\delta A} DA \Big|_{A_+=0}$$

$$\frac{\delta F^-}{\delta A} = \int D\rho_{\alpha\beta}\, D\eta_\gamma \; e^{\rho_{\alpha\beta} \frac{\delta F^-_{\alpha\beta}}{\delta A_\gamma} \eta_\gamma} \Big|_{A_+=0}$$

$$Q A_\gamma = \eta_\gamma$$
$$Q \eta_\gamma = 0$$
$$Q \rho_{\alpha\beta} = F^-_{\alpha\beta}$$

$$\rightarrow \quad Z_{N=1} = \int e^{-\int \frac{H^{-2}}{2} + i F^- H + i\rho \frac{\delta F^-}{\delta A}\eta} \; DA\, DH\, D\rho\, D\eta$$

$$Q A_\gamma = \eta_\gamma$$
$$Q \eta_\gamma = 0$$
$$Q \rho_{\alpha\beta} = H_{\alpha\beta} \quad \rightarrow \quad Q^2 = 0$$
$$Q H_{\alpha\beta} = 0$$

Since $H = Q\rho$ also $-\frac{H^2}{2} = -\frac{Q(\rho Q \rho)}{2} = -\frac{Q\rho Q\rho}{2} - \frac{1}{2}\rho \underset{\overset{\shortparallel}{0}}{Q^2} \rho$

Thus H^2 is a coboundary

$$Z_{\sigma=1} = \int e^{i\int F^- H + \rho \frac{\delta F^-}{\delta A}\eta} \, DA\, DH\, D\rho\, D\eta$$

$$= \int \delta(F^-) \frac{\delta F^-}{\delta A} \, DA$$

$$= \int 1 \;\; \text{S Det } \omega$$

The $\mathcal{N}=1$ SUSY YM Nicolai map + tautological Parisi-Sourlas SUSY → Localization on instantons

Localization on instantons → exact NSVZ β-function

$$Z_{\mathcal{N}=1} = \int e^{-\frac{16\pi^2 Q N}{2 g_w^2}} \frac{\mathrm{Det}^{\frac{1}{2}} \langle \frac{\delta A}{\delta \mu} \frac{\delta A}{\delta \mu} \rangle}{\mathrm{Det}^{\frac{1}{2}} \langle \frac{\delta \psi}{\delta \eta} \frac{\delta \psi}{\delta \eta} \rangle} \delta\mu\, \delta\eta \; \Lambda^{n_B - \frac{1}{2} n_F}$$

$n_B = 4QN$

$n_F = 2QN$

$$-\frac{16\pi^2 QN}{2g_\omega^2(\Lambda)} + 3QN\log\left(\frac{\Lambda}{\mu}\right) = -\frac{16\pi^2 QN}{2g_\omega^2(\mu)}$$

$$\frac{1}{2g^2(\Lambda)} - \frac{3}{(4\pi)^2}\log\left(\frac{\Lambda}{\mu}\right) = \frac{1}{2g_\omega^2(\mu)} \rightarrow \frac{\partial g_\omega}{\partial \log\Lambda} = -\frac{3}{(4\pi)^2} g_\omega^3$$

Canonical scheme

$$e^{-\frac{16\pi^2 N}{2g_\omega^2} Q(g_c A_c)} \int \mathrm{Det}^{\frac{1}{2}}(\omega(g_c A_c))$$

$$= e^{-\frac{16\pi^2 N}{2g_\omega^2} Q(g_c A_c)} \, g_c^{M_B - M_F} \int \mathrm{Det}^{\frac{1}{2}}(\omega(A_c))$$

$$\cong e^{-\frac{16\pi^2 N}{2g_c^2} Q(g_c A_c)} \int \mathrm{Det}^{\frac{1}{2}}(\omega(A_c))$$

$$\frac{1}{2g_c^2} = \frac{1}{2g_\omega^2} + \frac{2}{16\pi^2}\log g_c$$

$$\frac{\partial g_c}{\partial \log \Lambda} = \frac{-\frac{3}{(4\pi)^2} g_c^{\,3}}{1 - \frac{2}{(4\pi)^2} g_c^{\,2}}$$

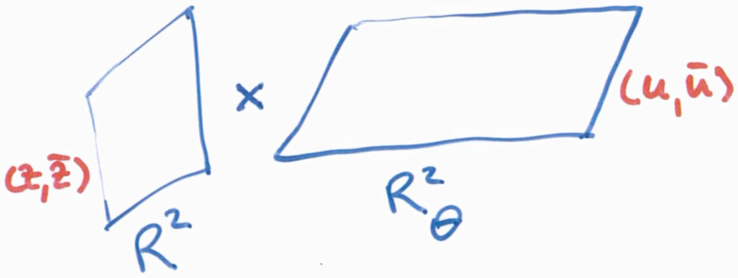
NSVZ

Localization in pure YM theory?

No cohomology

Localization in Large N YM

Twistor Wilson loops $\quad \mathrm{Tr}\,\Psi_\lambda(L_{ww}) = \mathrm{Tr}\,P\,e^{i\int_{L_{ww}} (A_z + \lambda D_u)\,dz + (A_{\bar z} + \lambda^{-1} D_{\bar u})\,d\bar z}$



$z = x_0 + i x_1 \; ; \quad u = x_2 + i x_3$

$[\partial_u, \partial_{\bar u}] = \theta^{-1}\mathbb{1} \quad [\bar u, u] = \theta\,\mathbb{1}$

$\theta \to \infty \sim N \to \infty$

$D_u = \partial_u + i A_u \; ; \quad D_{\bar u} = \partial_{\bar u} + i A_{\bar u}$

$\partial_u = [\theta^{-1}\bar u, \cdot\,] \; ; \quad \partial_{\bar u} = -[\theta^{-1} u, \cdot\,]$

$\mathrm{Tr} = \frac{1}{N}\mathrm{Tr}_{U(N)\times \text{Fock}}$

1) $\langle \mathrm{Tr}\,\Psi_\lambda(L_{ww})\rangle = \langle \mathrm{Tr}\,\Psi_1(L_{ww})\rangle$

2) $\lim_{\theta\to\infty} \langle \mathrm{Tr}\,\Psi_\lambda(L_{ww})\rangle = 1$ to all orders in g

i.e. twistor Wilson loops are in the "homology of 1" at large N

$$\lim_{\theta \to \infty} \langle \operatorname{Tr} \Psi_\lambda (L_{ww}) \rangle = 1$$

$$\left\langle \int_{L_{ww}} (A_z + \lambda D_u) dz + (A_{\bar z} + \lambda^{-1} D_{\bar u}) d\bar z \int_{L_{ww}} (A_z + \lambda D_u) dz + (A_{\bar z} + \lambda^{-1} D_{\bar u}) d\bar z \right\rangle$$

$$= 2 \int_{L_{ww}} dz \int_{L_{ww}} d\bar z \, \Big(\underbrace{\langle A_z A_{\bar z} \rangle + i^2 \langle A_u A_{\bar u} \rangle}_{\substack{\parallel \\ 0 \,;\, \theta \to \infty}} + o(\theta^{-1}) \Big)$$

$$\langle \mathrm{Tr}\,\Psi_\lambda \rangle = \langle \mathrm{Tr}\,\Psi_1 \rangle$$

$$= \int \mathrm{Tr}\, P\, e^{i\int (A+\lambda D)dz + (\bar{A}+\lambda^{-1}\bar{D})d\bar{z}}$$

$$\times\ e^{-\frac{N}{2g^2}\int \mathrm{Tr}\left(-i[D_\alpha D_\beta] - (\Theta^{-1})_{\alpha\beta}\mathbb{1}\right)^2} \quad \delta D_\alpha$$

$$= \int \mathrm{Tr}\,\Psi_\lambda\; e^{-\frac{N}{2g^2}\int \mathrm{Tr}\left((-i[D_\alpha D_\beta])^2 + (\Theta^{-1})^2_{\alpha\beta}\mathbb{1} + 2i\,[D_\alpha D_\beta](\Theta^{-1})_{\alpha\beta}\right)} \prod_\alpha \delta D_\alpha$$

$$= \int \mathrm{Tr}\, P\, e^{i\int (A+D')dz + (\bar{A}+\bar{D}')d\bar{z}}$$

$$\times\ e^{-\frac{N}{2g^2}\int \mathrm{Tr}\left((-i[D'_\alpha D'_\beta])^2 + (\Theta^{-1}_{\alpha\beta})^2\mathbb{1} + 2i\,[D'_\alpha D'_\beta](\Theta^{-1})_{\alpha\beta}\right)} \prod_\alpha \delta D'_\alpha$$

$D_z \to D'_z$

$D_{\bar{z}} \to D'_{\bar{z}}$

$D_u \to \lambda^{-1} D'_u$

$D_{\bar{u}} \to \lambda\, D'_{\bar{u}}$

$[D_u D_{\bar{u}}] \to [D'_u D'_{\bar{u}}]$

$[D_\alpha D_\beta]^2 \to [D'_\alpha D'_\beta]^2$

$(\Theta^{-1})_{u\bar{u}} \neq 0$

Non-supersymmetric analogue of the Nicolai map

$$1 = \int \delta(F^-_{\alpha\beta}(A) - \mu^-_{\alpha\beta})\, \delta\mu^-_{\alpha\beta}$$

$$Z_{YM} = \int e^{-\frac{N 8\pi^2}{g^2} Q - \frac{N}{4g^2}\sum_{\alpha\neq\beta}\int \mathrm{Tr}_f(\mu^-_{\alpha\beta})^2 d^4x}\, \delta(F^-_{\alpha\beta} - \mu^-_{\alpha\beta})\,\delta\mu^-_{\alpha\beta}\,\delta A_\alpha$$

$$\begin{cases} F^-_{01} = \mu^-_{01} \\ F^-_{02} = \mu^-_{02} \\ F^-_{03} = \mu^-_{03} \end{cases} \qquad \begin{cases} -iF_A + [D,\bar{D}] - \theta^{-1}\mathbb{1} = \mu^0 = \frac{1}{2}\mu^-_{01} \\ -i\partial_A \bar{D} = n = \frac{1}{4}(\mu^-_{02} + i\mu^-_{03}) \\ -i\bar{\partial}_A D = \bar{n} = \frac{1}{4}(\mu^-_{02} - i\mu^-_{03}) \end{cases} \qquad \begin{cases} -iF_{B_\rho} - \theta^{-1}\mathbb{1} = \mu_\rho \\ \qquad = \mu^0 + \rho^{-1} n - \rho\bar{n} \\ -i\partial_A \bar{D} = n \\ -i\bar{\partial}_A D = \bar{n} \end{cases}$$

$$B_\rho = (A_z + \rho D_u)\,dz + (A_{\bar{z}} + \rho^{-1} D_{\bar{u}})\,d\bar{z}$$

$$\langle \Psi_\lambda(L_{ww}) \rangle = \int_{C_\rho} \delta\mu\, \delta\bar\mu\, \delta\mu_\rho \; e^{-\frac{N 8\pi^2}{g^2} Q - \frac{N4}{g^2}\int \mathrm{Tr}_f(\mu^0)^2 + 4\,\mathrm{Tr}_f(\mu\bar\mu)\, d^4x}$$

$$\times\ \mathrm{Tr}_f P\, e^{i\int_{L_{ww}} (A_z + \partial D_u) dz + (A_{\bar z} + \partial^{-1} D_{\bar u}) d\bar z}$$

$$\delta(-iF_{B_\rho} - \mu_\rho - \theta^{-1} 1)\ \delta(-i\partial_A \bar D - \mu)\ \delta(-i\bar\partial_A D - \mu)$$

$$\times\ \delta A\, \delta\bar A\, \delta D\, \delta\bar D = \int_{C_\rho} \delta\mu\, \delta\bar\mu\, \delta\mu_\rho\, e^{-\frac{N 8\pi^2}{g^2} Q - \frac{N4}{g^2}\int \mathrm{Tr}_f(\mu^0)^2 + 4\,\mathrm{Tr}_f(\mu\bar\mu)\, d^4x}$$

$\rho = \lambda \to 0$

$$\times\ \mathrm{Tr}_f P e^{i\int_{L_{ww}} (A_z + D'_u) dz + (A_{\bar z} + D'_{\bar u}) d\bar z}$$

$$\times\ \delta(\underbrace{-iF_A + [D', \bar D'] - \theta^{-1} 1 - \mu^0}_{0} - \underbrace{i\frac{\lambda}{\rho}\partial_A \bar D' + i\frac{\rho}{\lambda}\bar\partial_A D'}_{0} - \underbrace{\rho^{-1}\mu}_{0} + \underbrace{\rho\bar\mu}_{0})$$

$$\delta(-i\lambda\partial_A \bar D' - \mu)\ \delta(-i\lambda^{-1}\bar\partial_A D' - \mu)\ \delta A\, \delta\bar A\, \delta D'\, \delta\bar D'$$

$$\downarrow \lambda\to 0 \qquad\qquad \downarrow \lambda \to 0$$

$$\delta(\mu) \qquad\qquad \delta(\bar\partial_A D')$$

$$\uparrow$$

$$Z = \left[\int_{C_{0+}} \delta\mu'_{0+} \frac{\delta\mu_{0+}}{\delta\mu'_{0+}} e^{-\frac{N8\pi^2}{g^2}Q - \frac{N}{4g^2}\sum_{\alpha\neq\beta}\int \mathrm{Tr}_f(\mu^-_{\alpha\beta})^2 d^4x} \delta(F^-_{\alpha\beta} - \mu^-_{\alpha\beta}) \right]_{\mu=\bar{\mu}=0} \delta A_\alpha$$

$$= \left[\int_{C_{0+}} \delta\mu'_{0+} e^{-\frac{N8\pi^2}{g^2}Q - \frac{N}{4g^2}\sum_{\alpha\neq\beta}\int \mathrm{Tr}_f(\mu^-_{\alpha\beta})^2 d^4x} \times \right.$$

$$\left. \times \ \mathrm{Det}'^{-\frac{1}{2}}(-\Delta_A \delta_{\alpha\beta} + D_\alpha D_\beta + i\,\mathrm{ad}_{\mu^-_{\alpha\beta}}) \left(\frac{\Delta}{2\pi}\right)^{n_\mu} \mathrm{Det}^{\frac{1}{2}}\omega \frac{\delta\mu_{0+}}{\delta\mu'_{0+}} \right]_{\mu=\bar{\mu}=0}$$

$$\omega = \int |dz|^2 \mathrm{Tr}_f\left(\frac{\delta B_{1z}}{\delta m_i} \wedge \frac{\delta B_{1\bar{z}}}{\delta m_k} \delta m_k\right)$$