$$B'_\rho = (A_z + \rho D'_u)\,dz + (A_{\bar z} + \rho^{-1} D'_{\bar u})\,d\bar z$$

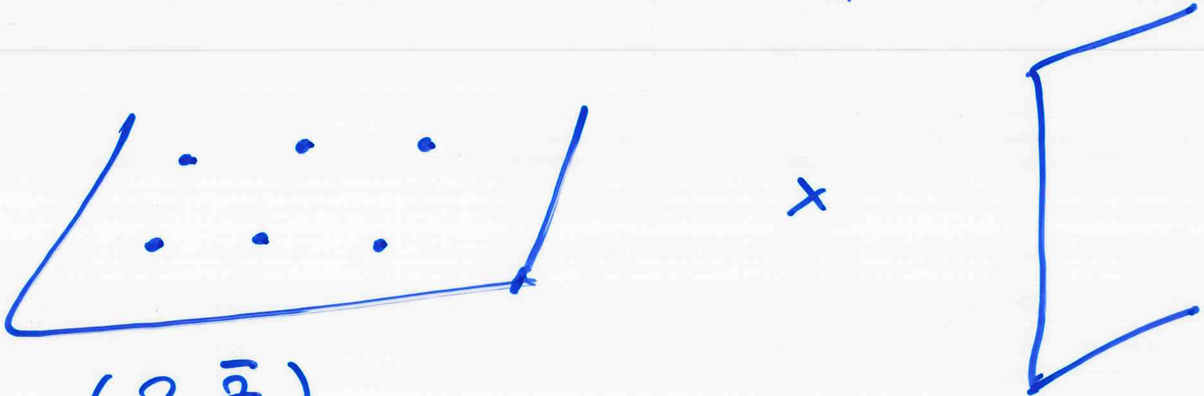
$$Z = \int_{C_{0+}} \delta u\, \delta \bar u\, \delta \mu_{0+}\; e^{-\frac{N 8\pi^2}{g^2} Q - \frac{N 4}{g^2}\int \mathrm{Tr}_f (\mu_{0+} \bar\mu_{0+})\, d^4x}$$

$$\times\ \mathrm{Tr}_f\, P e^{i\int B'_1}\ \delta(-i F_{B'_1} - \theta^{-1} - \mu_{0+})\, \delta(u)\, \delta(\partial_A \bar D')\, \delta A\, \delta\bar A\, \delta D'\, \delta \bar D'$$

Holomorphic ambiguity

$$\delta \mu_{0+} = \frac{\delta \mu_{0+}}{\delta \mu'_{0+}}\, \delta \mu'_{0+}$$

Lattice of surface operators



$(z, \bar z)$ $\times$ $(u, \bar u)$

$$F^{-}_{\alpha\beta}(A) = \sum_p \mu^{-}_{\alpha\beta}(p)\, \delta^{(2)}\left(z - z_p(u, \bar u)\right)$$

Kronheimer Mrowka Topology 32 (1993)

M. B. JHEP 9901 (1999) hep-th/9810015

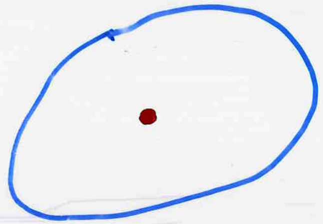
Gukov Witten hep-th/0612073

For vortices $\mu_p = \tilde{\mu}_p = 0$ $\qquad e^{i\lambda_p} = e^{i\frac{2\pi K}{N}}$

't Hooft e/m duality

electric charges characters of Z_N

magnetic charges $\pi_2(SU(N)/Z_N) = \pi_1(Z_N) = Z_N$



Z_N vortex in $d=2$

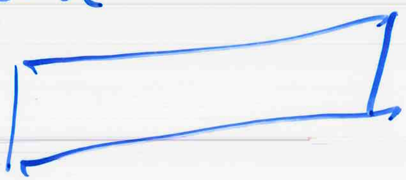
A has a pole $\qquad P e^{i\int A} = e^{\frac{2\pi i k}{N}}$

$F_A \sim \delta^{(2)}$

$d=3$

vortex line

$d=4$



vortex sheet

Holomorphic loop equation in the Nicolai variables

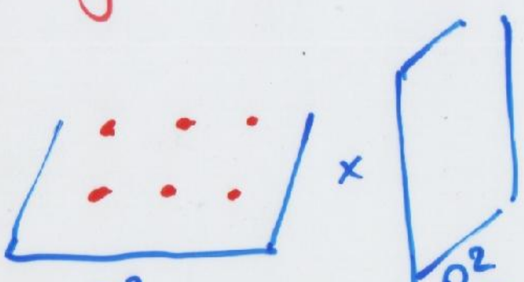
$$Z=\int e^{-\frac{N}{2g^2}\int \mathrm{Tr}\, F_{\alpha\beta}^2}\, DA = \int e^{-\frac{8\pi^2 QN}{g^2}-\frac{N}{4g^2}\int \mathrm{Tr}\, F^{-2}_{\alpha\beta}}\, DA$$

$$1=\int \delta(F^-_{\alpha\beta}-\mu^-_{\alpha\beta})\, D\mu^-_{\alpha\beta} \quad \text{(Nicolai map)}$$

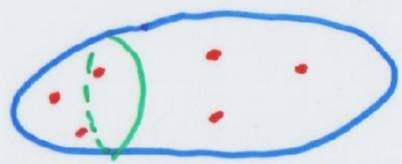
$$Z=\int e^{-\frac{8\pi^2 QN}{g^2}-\frac{N}{4g^2}\int \mathrm{Tr}(\mu^-_{\alpha\beta})^2}\, \mathrm{Det}^{-\frac{1}{2}}(-\Delta_A\delta_{\alpha\beta}+D_\alpha D_\beta+i\,\mathrm{ad}F^-_{\alpha\beta})\, D\mu^-_{\alpha\beta}=\int e^{-\Gamma} D\mu$$

Nicolai map on a lattice of surface operators

$$Z=\int e^{-\frac{8\pi^2 QN}{g^2}-\frac{N}{4g^2}\int \mathrm{Tr}\, F^{-2}_{\alpha\beta}}\, \delta\Big(F^-_{\alpha\beta}-\sum_p \mu^-_{\alpha\beta}(p)\,\delta^{(2)}(z-z_p(u,\bar u))\Big)\prod_p \mu^-_{\alpha\beta}(p)$$



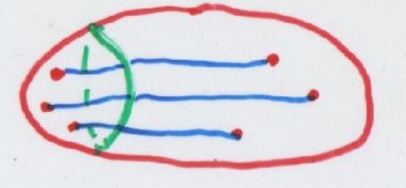
$R^2 \times R^2_\theta$



$$\Big\langle \frac{\delta\Gamma}{\delta\mu_p}\Psi_{z_p z_p}\Big\rangle=\frac{1}{2\pi}\int \frac{dz'}{z_p-z'}\langle\Psi_{z_p z'}\rangle\langle\Psi_{z' z_p}\rangle$$

$$\Psi_{zw}(B)=e^{i\int_{C_{zw}}(A_z+D_u)dz+(A_{\bar z}+D_{\bar u})d\bar z} \; ; \; D_u=\partial_u+iA_u$$

$$\Big\langle \frac{\delta\Gamma}{\delta\mu_p}\Psi_{z_p z_p}(C\cup b)\Big\rangle=0$$



Holomorphic loop equation

$$\Psi(L_{zz}) = P e^{i\int_{L_{zz}} (A_z + D_u)dz + (A_{\bar z} + D_{\bar u})d\bar z}$$

$$B_z = A_z + D_u \; ; \quad A_{\bar z} + D_{\bar u} = B_{\bar z}$$

$$-i F_{z\bar z}(B) = \mu + \theta^{-1} 1$$

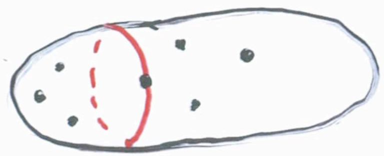
$$F_{z\bar z} = \partial_z B_{\bar z} - \partial_{\bar z} B_z + i [B_z, B_{\bar z}]$$

There is a holomorphic gauge $B_{\bar z} = 0$; $\Psi' = P e^{i\int_L B'_z dz}$

$$i \bar\partial B' = \mu' + \theta^{-1} 1$$

Loop equation

$$\int Tr\left(\frac{\delta}{\delta \mu'} \Psi e^{-\Gamma}\right) D\mu' = 0 \qquad \left\langle Tr \frac{\delta \Gamma}{\delta \mu'(z)} \Psi'(L_{zz}) \right\rangle = \frac{1}{2\pi} \int_{L_{zz}} \frac{dw}{z-w} \langle Tr \Psi'(L_{zw}) \rangle$$



lattice version

$$\left\langle Tr \frac{\delta \Gamma}{\delta \mu'(z_p)} \Psi'(L_{z_p z_p}) \right\rangle = \frac{1}{2\pi} \int_{L_{z_p z_p}} \frac{dw}{z_p - w} \langle Tr \Psi'(L_{z_p w}) \rangle$$

By the Stokes theorem cohomology is dual to homology

$$\int_M d\omega = \int_{\partial M} \omega$$

Can we get localization of functional integrals by homological rather than cohomological deformations?

If Yes we could get localization without SUSY

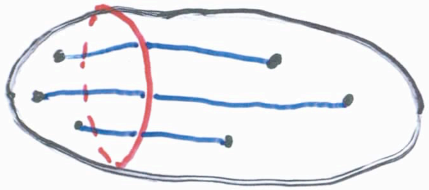
The answer is No in local field theory but it is Yes in loop equations

Localization of the holomorphic loop equation by homology

$$\left\langle \mathrm{Tr}\left(\frac{\delta\Gamma}{\delta\mu'(z_p)}\Psi'(L_{z_pz_p})\right)\right\rangle = \frac{1}{2\pi}\int_{L_{z_pz_p}}\frac{dw}{z_p-w}\left\langle \mathrm{Tr}\,\Psi'(L_{z_pw})\right\rangle$$

cf2 $\int\omega = \int\omega + d\alpha$

$$\frac{dw}{z_p-w}\to\frac{d\varphi_+}{x_+(p)-\varphi_+ + i\epsilon} = \pi\,\delta(x_+-\varphi_+) + P\frac{1}{x_+-\varphi_+}$$



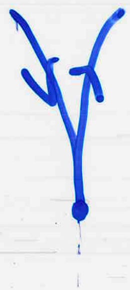
$$\left\langle \mathrm{Tr}\left(\frac{\delta\Gamma}{\delta\mu'(z_p)}\Psi'(L_{z_pz_p}\cup vb)\right)\right\rangle = 0$$

The loop equation gets localized by adding to the loop vanishing boundaries (vb) in homology ending with cusps

$$\int dy(s)\, \delta^{(1)}\big(x(s_{cusp}) - y(s)\big) =$$

$$= \frac{1}{2}\left(\frac{\dot{x}(s^{+}_{cusp})}{|\dot{x}(s^{+}_{cusp})|} + \frac{\dot{x}(s^{-}_{cusp})}{|\dot{x}(s^{-}_{cusp})|} \right)$$

$= 0$ if the cusp backtracks



Cohomology

$[\omega] = \int \omega = \int \omega + d\alpha$

$Q S_{SUSY} = 0 \sim d\omega = 0$

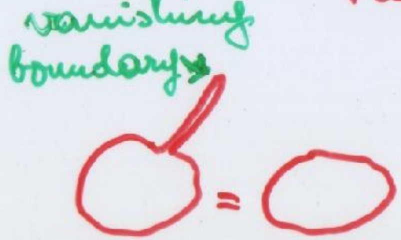
the co-boundary Q is the generator of the (super) symmetry

$Z(t) = \int e^{-S_{SUSY} + tQ\alpha}$

$t \to \infty$

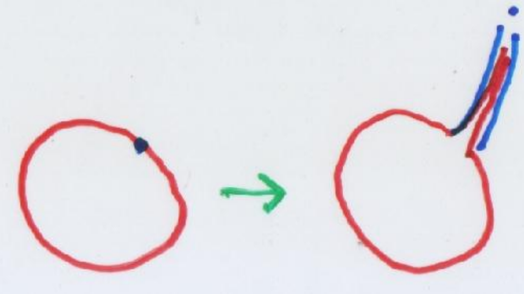
localization on critical points of the co-boundary $Q\alpha$

Homology

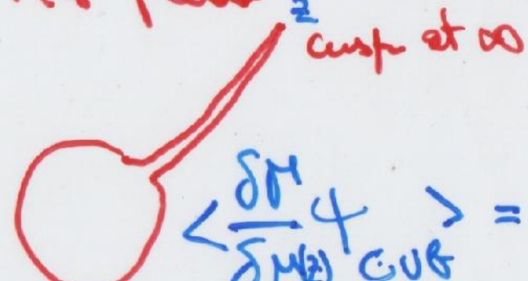


$\langle W \rangle \sim e^{P\Lambda + \gamma \log \Lambda}$

$\langle \varphi_C \rangle = \langle \varphi_{CUG} \rangle$



the vanishing boundary is generated by a conformal transformation = symmetry of the RG flow



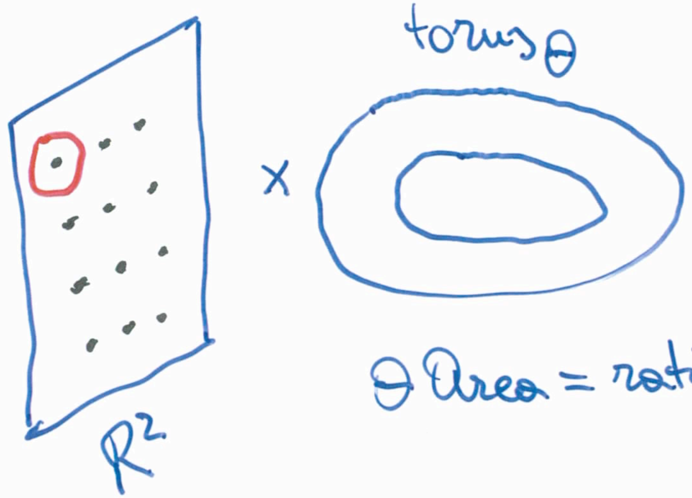
$\langle \frac{\delta M}{\delta \mu(z)} \varphi_{CUG} \rangle = \frac{1}{2\pi} \int_{CUG} \frac{dw}{z-w} \langle \varphi_{CUG} \rangle = 0$

$$\lim_{\lambda\to 0} \langle \mathrm{Tr}_{Adj}\Psi_\lambda(L_{ww})\rangle \overset{\theta\to\infty}{=} \sum_{k} e^{-\Gamma(k)}$$

Twistor Wilson loops are localized on the fixed points of the semigroup that rescales λ

Γ = effective action

The fixed points are a lattice of surface operators with no moduli and Z_N holonomy



θ Area = rational

$$\begin{cases} F^-_{01} = \sum_p \lambda_p^{(k)} \delta^{(2)}(z - z_p(u,\bar u)) + 1\,\theta^{-1} \\ F^-_{02} = 0 \\ F^-_{03} = 0 \end{cases}$$

$$z_p(u,\bar u) = z_p$$

$$F^-_{\alpha\beta} = F_{\alpha\beta} - \frac{1}{2}\varepsilon_{\alpha\beta\gamma\delta}F_{\gamma\delta}$$

$$e^{i\lambda_p^{(k)}} = e^{\frac{2\pi i k}{N}}\,1$$

$$\langle \mathrm{Tr}_{Adj}\Psi_\lambda\rangle \overset{N\to\infty}{=} \langle \mathrm{Tr}_{f}\Psi_\lambda\rangle\,\langle \mathrm{Tr}_{f}\Psi_\lambda\rangle$$

$$\Gamma_{gf} = \Big[\frac{N_4}{g^2} \int \mathrm{Tr}_f(\mu_{0^+} \bar{\mu}_{0^+})\, d^4x - \log\Big(\mathrm{Det}^{\cancel{1}-\frac{1}{2}}\big(-\Delta_A \delta_{\alpha\beta} + i\, \mathrm{ad}_{\mu_{\bar{\alpha\beta}}}\big)\, \Delta_{FP}\Big)$$

$$- \mu_f \log\Big(\frac{\Lambda}{2\mu}\Big) - \log\Big(\mathrm{Det}^{\frac{1}{2}}_{\omega} \frac{\delta \mu_{0^+}}{\delta \mu'_{0^+}}\Big) + c.c. \Big]_{\mu = \bar{\mu} = 0}$$

We must understand regularization and renormalization of Γ_{gf} on surface operators!

$$\int d^4x\, \mathrm{Tr}(F_{\alpha\beta}^2) \sim \int d^4x\, \delta^{(2)}(z-z_p)\,\delta^{(2)}(z-z_p) \sim \delta^{(2)}(0)\int d^4x\,\delta^{(2)}(z-z_p) \sim \left(\frac{\Lambda}{2\pi}\right)^2 \text{Area} = N_2$$

Large N non-commutative Eguchi-Kawai reduction

$$Z_{EK} = \int e^{-\frac{N}{2g^2}\left(\frac{2\pi}{\Lambda}\right)^d \sum_{\alpha\neq\beta} \mathrm{Tr}\left(-i[C_\alpha, C_\beta] - (\Theta^{-1})_{\alpha\beta} I\right)^2} \delta C_\alpha$$

$$C_\alpha \to D_\alpha = \partial_\alpha + i A_\alpha$$

$$N_d = \left(\frac{\Lambda}{2\pi}\right)^d V_d$$

$$\left(\frac{2\pi}{\Lambda}\right)^d = \frac{V_d}{N_d} = \frac{1}{N_d}\int d^dx \quad ; \quad S_{YM} \to \frac{1}{N_2} S_{YM} = S_{EK}$$

Regularization of logarithmic divergences

$$\frac{1}{(4\pi^2)^2}\sum_{p\neq p'}\int |d^2u|\,|d^2v|\, \frac{N\,\mathrm{Tr}(\mu_p \bar{\mu}_{p'})}{\left(|z_p - z_{p'}|^2 + |u-v|^2\right)^2}$$

$$\int \mathrm{Tr}(F_{\alpha\beta}^2) = \frac{1}{2}\int \mathrm{Tr}(F^-_{\alpha\beta})^2 + \int F_{\alpha\beta}\tilde{F}_{\alpha\beta} = 2\int \mathrm{Tr}\left(\frac{F^-_{\alpha\beta}}{2}\right)^2 + \int F_{\alpha\beta}\tilde{F}_{\alpha\beta}$$

topological term

Background field $F_{\alpha\beta} = \mu_{\alpha\beta}$ (*), Feynman gauge (**)

$$\int \mathrm{Tr}(F_{\alpha\beta}^2) \rightarrow -\Delta_A \mathbb{1}_4 + 2i\,\mathrm{ad}\,\mu_{\alpha\beta} \rightarrow \beta_0 = \frac{1}{(4\pi)^2}\Big[\underset{(*)}{-\frac{1}{3}} + \underset{(**)}{4}\Big]$$

$$\int \mathrm{Tr}(F^-_{\alpha\beta})^2 \rightarrow -\Delta_A \mathbb{1}_4 + 2i\,\mathrm{ad}\frac{\mu^-_{\alpha\beta}}{2} + 2i\,\mathrm{ad}\left(\frac{\mu^+_{\alpha\beta}}{2}\right)$$

$$\delta(F^-_{\alpha\beta} - \mu^-_{\alpha\beta}) \rightarrow \int \mathrm{Tr}(F^-_{\alpha\beta} - \mu^-_{\alpha\beta})^2 \rightarrow -\Delta_A \mathbb{1}_4 + 2i\,\mathrm{ad}\frac{\mu^+_{\alpha\beta}}{2}$$

$$(1)\rightarrow \mathrm{Det}^{-\frac{1}{2}}\left(-\Delta_A \mathbb{1}_4 + 2i\,\mathrm{ad}\frac{\mu^+_{\alpha\beta}}{2}\right) \rightarrow (2)\rightarrow \mathrm{Det}^{-1/2}\left(-\Delta_A \mathbb{1}_4 + 2i\,\mathrm{ad}\frac{\mu^-_{\alpha\beta}}{2}\right)\mathrm{Det}(\text{zero modes})$$

$$\not{D} = \sigma_\mu D_\mu \;;\quad \bar{\not{D}} = \bar{\sigma}_\mu D_\mu \;;\quad -\not{D}\bar{\not{D}}\,\mathbb{1}_2 = (-\Delta_A - \sigma_{\mu\nu}F_{\mu\nu})\mathbb{1}_2 \quad (1)$$

$$\delta F^-_{\rho\sigma} \sim \bar{\not{D}}^{\dot\alpha\alpha}\,\delta A_{\alpha\dot\beta} \;;\quad -\bar{\not{D}}\not{D}\,\mathbb{1}_2 = (-\Delta_A - \bar{\sigma}_{\mu\nu}F_{\mu\nu})\mathbb{1}_2 \quad (2)$$

$$\sigma_{\mu\nu}F_{\mu\nu} = \sigma_{\mu\nu}\frac{F^+_{\mu\nu}}{2} \;;\quad \bar{\sigma}_{\mu\nu}F_{\mu\nu} = \bar{\sigma}_{\mu\nu}\frac{F^-_{\mu\nu}}{2}$$

Vortices

$$[D_z\, D_{\bar z}] - [D_u\, D_{\bar u}] = i\Big(\sum_p g_p \lambda_p g_p^{-1} \delta_p^{(2)} - \theta^{-1} \mathbb{1}\Big)$$

$$[D_z\, D_{\bar u}] = 0$$

$$[D_{\bar z}\, D_u] = 0$$

$$\lambda_p \qquad [N-k] \quad \frac{2\pi k}{N} \qquad\qquad [k] \; -\frac{2\pi(N-k)}{N}$$

$$\mathrm{Tr}\,\lambda^2 = (N-k)\Big(\frac{2\pi k}{N}\Big)^2 + k\Big(\frac{2\pi(k-N)}{N}\Big)^2 = (2\pi)^2\, \frac{k(N-k)}{N}$$

number of vortex zero modes (complex) $= \frac{1}{2}\left(N^2 - \sum m_i^2\right)$

$$= \frac{1}{2}\left(N^2 - k^2 - (N-k)^2\right) = k(N-k)$$

Vortex action $\quad \dfrac{8\pi^2}{2g^2} k(N-k) + \dfrac{8\pi^2}{2g^2} k(N-k) = \dfrac{16\pi^2}{2g^2} k(N-k)$

$$\Gamma(k) = \frac{N}{2g_W^2} S_{YM} - \log \mathrm{Det}' - \log \Lambda^{n_B(\text{zero modes})}$$

$$= \frac{k(N-k)}{2g_W^2} 16\pi^2 - k(N-k)\frac{5}{3}\log\Lambda - k(N-k)\,2\log\Lambda + \text{c.c.} + \dots$$

$$\frac{\partial g_W}{\partial \log\Lambda} = -\beta_0 g_W^3 \quad ; \quad \beta_0 = \frac{1}{(4\pi)^2}\frac{11}{3}$$

Problems 1) and 2) of Clay Math. Inst. have a much simpler form at large N in the Wilsonian scheme since:

$$\Lambda_W = \Lambda e^{-\frac{1}{2\beta_0 g_W^2}}$$

Beta function from localization

$\mathcal{N}=1$ SUSY YM

$$Z_{\text{instantons}} = \int e^{-\frac{(4\pi)^2 Q}{2 g_w^2}} \Lambda^{n_B - \frac{n_F}{2}} \frac{\operatorname{Det}\omega_B}{\operatorname{Det}\omega_F} \; ;$$

$$\frac{(4\pi)^2 Q}{2 g_w^2(\mu)} = \frac{(4\pi)^2 Q}{2 g_w^2(\Lambda)} - \left(n_B - \frac{n_F}{2}\right)\log\left(\frac{\Lambda}{\mu}\right) \; ;$$

$$A \to g A_c \qquad \frac{\operatorname{Det}\omega_B}{\operatorname{Det}\omega_F} \to g^{n_B - n_F} \; ;$$

$$-\frac{1}{2g^2} = -\frac{1}{2g_w^2} + \frac{n_B - n_F}{(4\pi)^2 Q}\log g \; ;$$

$SU(N)$ YM $N=\infty$

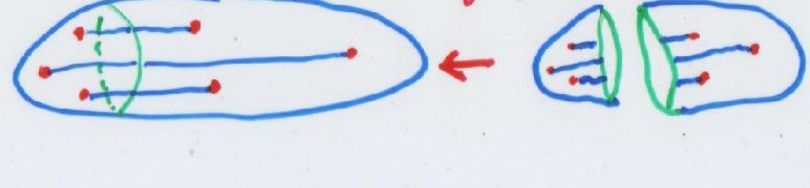
$$Z_{\text{vertices}} = \int e^{-\frac{N}{2g_w^2} S_{YM}(\text{vertices})} \Delta_{FP} \operatorname{Det}^{-\frac{1}{2}}(\text{non-zero modes}) \times \Lambda^{n_B} \operatorname{Det}\omega_B$$

$$\frac{S_{YM}}{2 g_w^2(\mu)} = \frac{Z^{-1} S_{YM}}{2 g_w^2(\Lambda)} - n_B \log\left(\frac{\Lambda}{\mu}\right)$$

$$A \to g Z^{\frac{1}{2}} A_c$$

$$-\frac{1}{2g^2} = -\frac{1}{2g_w^2} + \frac{n_B}{S_{YM}}\log g + \frac{n_B}{4 S_{YM}}\log Z$$

gluing



Exact beta function

$\mathcal{N}=1$ SUSY YM

$$\frac{\partial g_W}{\partial \log\Lambda} = -\frac{3}{(4\pi)^2}\, g_W^3$$

$$\frac{\partial g}{\partial \log\Lambda} = \frac{-\frac{3}{(4\pi)^2}\, g^3}{1-\frac{2}{(4\pi)^2}\, g^2}$$

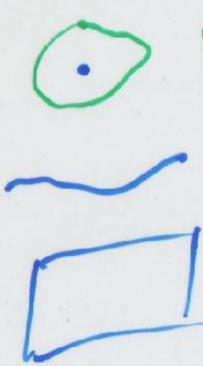
gluino condensate $= \mathrm{Tr}\,\lambda^2$

localized on instantons

$$F^-_{\alpha\beta}=0 \ ; \ F^-_{\alpha\beta} = F_{\alpha\beta} - \tilde{F}_{\alpha\beta}$$

Localization = one-loop is exact

Surface operators → Z_N vortices



$$P e^{i\int A} = e^{\frac{2\pi i k}{N}}$$

d=2

d=3

d=4

SU(N) pure YM $N=\infty$

$$\frac{\partial g_W}{\partial \log\Lambda} = -\beta_0\, g_W^3 \ ; \ \beta_0 = \frac{11}{3}\frac{1}{(4\pi)^2}$$

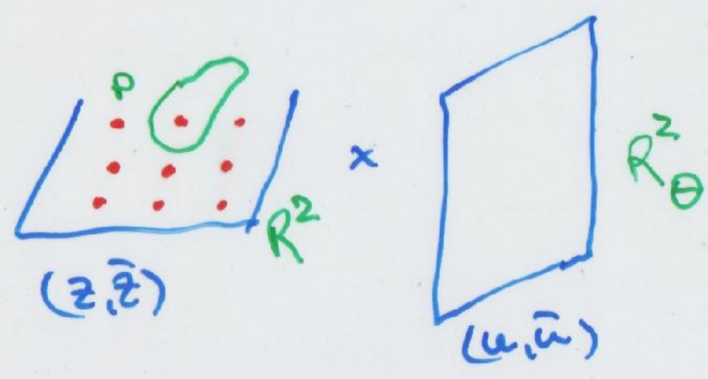
$$\frac{\partial g}{\partial \log\Lambda} = \frac{-\beta_0 g^3 + \frac{g^3}{(4\pi)^2}\frac{\partial \log Z}{\partial \log\Lambda}}{1-\frac{4}{(4\pi)^2}\, g^2}$$

$$\frac{\partial \log Z}{\partial \log\Lambda} = \frac{\gamma\, g_W^2}{1+c\, g_W^2} \ ; \ \gamma = \frac{10}{3}\frac{1}{(4\pi)^2}$$

special Wilson loop $= P\exp i\int (A_z + D_u)\,dz + c.c.$

localized on surface operators

$$F^-_{\alpha\beta} = \sum_p \mu^-_{\alpha\beta}(p)\, \delta^{(2)}\big(z - z_p(u,\bar{u})\big) + \theta^{-1}_{\alpha\beta}\mathbb{1}$$



Large N effective action Γ

$$\Gamma = \frac{16\pi^2}{2g^2} Q N + \frac{N}{4g^2} \sum_{\alpha\neq\beta} \int \mathrm{Tr}(\mu^-_{\alpha\beta})^2 d^4x \quad (\mathrm{I})$$

(I) classical YM action

$$+ \log\left[\mathrm{Det}^{1-\frac{1}{2}}(-\Delta_A \delta_{\alpha\beta} + i\,\mathrm{ad}\,\mu^-_{\alpha\beta})\Delta_{FP}\right] \quad (\mathrm{II})$$

Jacobian of Nicolai map + gauge fixing

$$- \log \Lambda^{n_B(\text{zero-modes})} \quad (\mathrm{III})$$

(III) divergent contribution of zero modes via Pauli-Villars regularization

$$+ \mathrm{Tr} \log |\Delta(\mu)|^2 \quad (\mathrm{IV})$$

$$+ \mathrm{Tr}\log\left|\frac{\mathrm{Det}\left(\frac{\delta \omega_{MC}}{\delta\mu}\right)}{\mathrm{Det}^{\frac{1}{2}}\,\omega_{\text{zero-modes}}}\right|^2 \quad (\mathrm{V})$$

(IV) + (V) Jacobian to the holomorphic gauge + finite contribution of zero modes

It is convenient to introduce the density of vortices

$$\rho = N_v^{-1} \sum_p \delta^{(2)}(z - z_p)$$

ρ is determined in terms of Λ_w^2 by the critical equation

$$\frac{\delta \Gamma}{\delta \rho} = 0$$

Then we can compute the glueball propagator $\langle \mathrm{Tr}\,\mu^2 \; \mathrm{Tr}\,\bar{\mu}^2 \rangle$
in the $\mu/\bar{\mu}$ sector by small fluctuations
around the vortex condensate

$$e^{-\Gamma(\rho) - \frac{\delta^2 \Gamma(\rho)}{\delta\mu\,\delta\bar{\mu}} \delta\mu\,\delta\bar{\mu}}$$

at the critical point of the effective action

Γ is computed on a background of surface operators

$$F_{\alpha\bar{\beta}}(A) = \sum_p \mu^-_{\alpha\beta}(p)\, \delta^{(2)}(z - z_p(u,\bar{u})) + 1(\Theta^{-1})_{\alpha\beta}$$

at large N the background of surface operators has the structure

$$F(B) = \mu + 1\Theta^{-1} \quad ; \quad \text{In a singular gauge}$$

$$\mu = \sum_p \lambda_p \delta^{(2)}(z - z_p) + \sum_p \delta\mu_p\, \delta^{(2)}(z - z_p(u,\bar{u}))$$

with $\quad z_p(u,\bar{u}) = u \quad , \quad \bar{z}_p(u,\bar{u}) = \bar{u}$

$$= \sum_p \lambda_p \delta^{(2)}(z - z_p) + \sum_p \delta\mu_p(u,\bar{u})\, \delta^{(2)}(z - u)$$

The terms (I) + (II) + (III) contribute to the renormalization of the local part of the effective action; the subtraction scale is $\sqrt{e_{eff}}$

$$e^2_{eff} \equiv e^2\, \mathrm{Tr}(\lambda^2) = e^2 \frac{K(N-K)}{N}$$

The term IV is the log of the square of a holomorphic function of μ and thus does not contribute to $\frac{\delta^2 \mu}{\delta\mu\, \delta\bar{\mu}}$ in the $\mu/\bar{\mu}$ sector

$$IV = \mathrm{Tr}\log|\Delta(\mu)|^2 \propto \sum_{i\neq j} \log|\mu_i - \mu_j|^2$$ is holomorphic

but at coinciding eigenvalues, thus it contributes to $\frac{\delta^2 \mu}{\delta\mu_i\, \delta\bar{\mu}_j}$ by a term proportional to $\delta^{(2)}(\mu_i - \mu_j)$ that is non-trivial on the vortex condensate because of Bose condensation of the vortices eigenvalues

$$\lambda_p \sim \begin{pmatrix} \frac{2\pi K}{N} & \\ & \frac{2\pi(K-N)}{N} \end{pmatrix} \begin{array}{l} N-K \\ K \end{array}$$

The local divergent part of Γ gets renormalized in the following way

$$N_v^2\Big[\frac{N}{2g^2}(4\pi)^2\int e^2\, T_2(\mu\bar{\mu}) - N(4\pi)^2\gamma_0\int e^2\, T_2(\mu\bar{\mu})\log\frac{\Lambda}{\sqrt{e_{eff}}}$$

$$-\frac{2}{(4\pi)^2}(4\pi)^2 k(N-k)\int e^2\log\frac{\Lambda}{\sqrt{e_{eff}}}\Big] = N_v^2\Big[-\beta_0(4\pi)^2\int \underbrace{e^2 k(N-k)}_{\substack{\shortparallel\\ e_{eff}^2 N}}\log\frac{\Lambda_w}{\sqrt{e_{eff}}}\Big]$$

$$\gamma_0 = \frac{1}{(4\pi)^2}\frac{5}{3}\,;\quad T_2(\mu\bar{\mu}) = T_2(\lambda^2) = k\frac{(N-k)}{N} \text{ for vortices}$$

$$\frac{\delta\Gamma}{\delta e} = 0 \rightarrow \log\frac{\Lambda_w}{\sqrt{e_{eff}}} = \frac{1}{4} \rightarrow e_{eff}\sim\Lambda_w^2 \rightarrow e^2\sim\frac{N}{k(N-k)}\Lambda_w^4 \sim \frac{1}{k}\Lambda_w^4$$

Γ contains a term of the form $-\gamma_0 N(4\pi)^2\int\frac{\Lambda_w^2}{k}T_2(\delta\mu\,\delta\bar{\mu})\,d^2x \longrightarrow$

$\longrightarrow$ instability due to AF

However there is a non-perturbative contribution to the local part of the effective action from $\underline{\text{IV}} = \mathrm{Tr}\log|\Delta(\mu)|^2$

$$\rightarrow N_v^2 \int e^2 \log|\Delta(\mu)|^2$$

$$\sum_{ik} \frac{\delta^2}{\delta\mu_i \delta\bar\mu_k} \log\Big|\prod_{\ell>j}(\mu_\ell-\mu_j)\Big|^2 \delta\mu_i \delta\bar\mu_k \propto$$

$$\propto \Big[\sum_{j\neq k} \delta^{(2)}(\mu_j-\mu_k)\,\delta_{ik} - \delta^{(2)}(\mu_i-\mu_k)\Big]\delta\mu_i\delta\bar\mu_k$$

$$\sim \underset{\underset{N}{\text{III}}}{\delta^{(2)}(0)}\, K\, \mathrm{Tr}(\delta\mu\,\delta\bar\mu) + O(1)$$

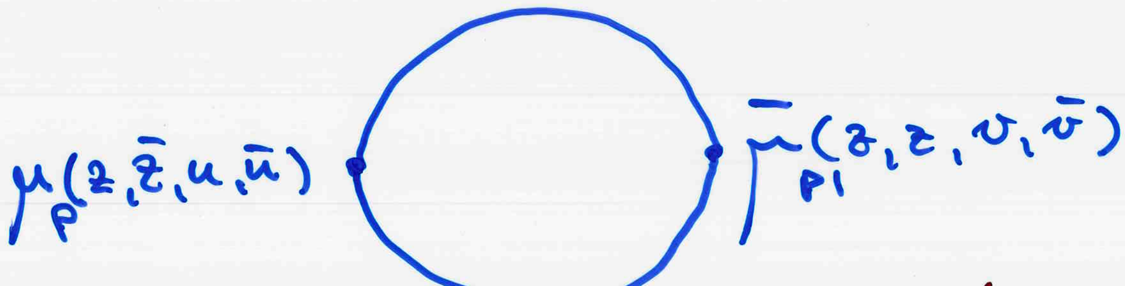
Thus $\dfrac{\delta^2\mu}{\delta\mu\,\delta\bar\mu} \sim \int \left(\delta - \frac{\gamma_0}{K}\right) N\, \mathrm{Tr}(\delta\mu\,\delta\bar\mu)\, \Lambda_\omega^2\, d^2x$

that is $\underline{\text{stable}}$ at least for large K

What about the Kinetic term?

A typical contribution from II to γ' in the background of surface operators is:

$$\frac{1}{(4\pi^2)^2}\sum_{p\neq p'}\int d^2u\, d^2v\, \frac{N\, \mathrm{Tr}(\mu_p \bar{\mu}_{p'})}{\left(|z_p(u,\bar{u}) - z_{p'}(v,\bar{v})|^2 + |u-v|^2\right)^2} \sim$$

$\sim$ 

For example for $z_p(u,\bar{u}) = z_p = \text{const}$; $\mu_p = \lambda = \text{const}$

$$\sim \sum_{p\neq p'}\int d^2u\, d^2v\, \frac{N\, \mathrm{Tr}(\lambda^2)}{\left(|z_p - z_{p'}|^2 + |u-v|^2\right)^2} \sim \cancel{\log\left(\frac{\Lambda}{a^{-1}}\right)} \log\left(\frac{a^{-1}}{\mu}\right)$$

For the fluctuations of diagonally supported surface operators, after introducing the density ρ we get

$$\frac{1}{\kappa}\int d^2u\, d^2v \; \frac{N\, \mathrm{Tr}(\delta\mu(u,\bar u)\, \delta\bar\mu(v,\bar v))}{(|u-v|^2 + |u-v|^2)^2}$$

and by analytic continuation to Minkowski

$$\frac{1}{\kappa}\int du_+\, du_-\, dv_+\, dv_- \; \frac{N\, \mathrm{Tr}(\delta\mu(u_+,u_-)\, \delta\bar\mu(v_+,v_-))}{(u_+-v_+)^2(u_- - v_-)^2}$$

$$\sim \frac{1}{\kappa}\int N\, \mathrm{Tr}\left(\delta\mu\, \partial_+\partial_-\, \delta\bar\mu\right) du_+\, du_-$$

where we are using the Cauchy formula $\partial_+\delta\mu \propto \int \frac{\delta\mu(v_+)\, dv_+}{(u_+ - v_+ + i\varepsilon)^2}$

The final result for the glueball propagator in the $\mu/\bar{\mu}$ sector is thus:

$$\int \left\langle \frac{1}{N} \mathrm{Tr}(\mu^2)(x_+, x_-) \, \frac{1}{N} \mathrm{Tr}(\bar{\mu}^2)(0) \right\rangle_{\mathrm{conn}} e^{i(p_+ x_- + p_- x_+)} \, d^2x$$

$$= \sum_{n=1}^{\infty} \frac{k \, \Lambda_w^6}{\frac{1}{k}\alpha' p_+ p_- + (\delta - \frac{1}{k}\gamma)\Lambda_w^2}$$

asymptotically for large k

Corollary of localization in pure large N YM

$$\int e^{-S_{YM}} \delta(F^-_{\alpha\beta} - \mu^-_{\alpha\beta}) \delta\mu^-_{\alpha\beta} \delta A \rightarrow \int e^{-S_{YM}} \delta(F^-_{\alpha\beta} - \mu^-_{\alpha\beta}) \, \mathrm{Det}^{N_f}(\not{D} + m) \, \delta\mu^-_{\alpha\beta} \delta A$$

Pure YM — QCD with N_F flavours

$$Z^{-1} = \left(1 - \frac{5}{3} g_{\mu}^2 \log\left(\frac{\Lambda}{\mu}\right) \frac{1}{(4\pi)^2}\right) \rightarrow Z^{-1} = \left(1 - \left(\frac{5}{3} - \frac{2}{3}\frac{N_f}{N}\right) g_{\mu}^2 \log\left(\frac{\Lambda}{\mu}\right) \frac{1}{(4\pi)^2}\right)$$

$$m \to 0 \longrightarrow \alpha' \propto \frac{1}{N}\left(\frac{5}{3}N - \frac{2}{3}N_f\right)$$

one loop $\mathcal{N}=1$ SUSY QCD with N_F flavours

$$\int e^{-S_{\mathcal{N}=1}} \, \mathrm{Det}^{N_f}(\not{D} + m) \, \mathrm{Det}^{-N_f}(\Box + m); \quad m \to 0 \to \alpha' \propto \frac{1}{N}\left(\frac{3}{3}N - \left(\frac{2}{3} + \frac{1}{3}\right) N_f\right)$$

$$\frac{N_f}{N} = 1$$