Transverse envelope dynamics of beam slices in a uniform charged ellipsoidal model of the plasma bubble regime

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Overview

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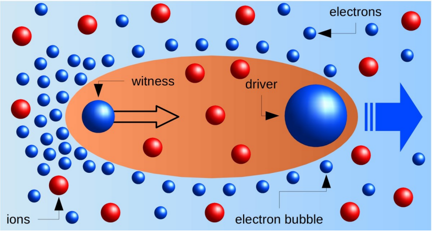
Conventional accelerator or Beam driven plasma wakefield accelerator?

- Conventional accelerators are large (100 metres) and expensive $10 - 100$M
- Conventional accelerators cannot achieve better than a few 10 MV/m or you get breakdown
- Plasma waves are a possible alternative - providing a route to university scale accelerators and radiation sources
- Plasma based accelerators are a possible compact alternative
- in particular we are now quite good at accelerating electrons to ~ 1 GeV with ~ 100 TW lasers



$$E_0 = \frac{m_e c\,\omega_p}{e} \simeq 96\sqrt{n_0(cm^{-3})} \rightarrow E_0 \approx 10\frac{GV}{m} @\, n_0 = 10^{16}\,cm^{-3}$$

The driver:

- Particle bunch (PWFA)
- Laser pulse (LWFA)

The witness can be:

- Self injected
- Externally injected

From conventional Cavity $\sim$ meter to capillary $\sim$ cm

The rms envelope equation describes the rms beam envelope dynamics [1, 2, 3]. Therefore, the complete rms envelope equation is:

$$\sigma_x'' + \frac{p'}{p}\sigma_x' - \frac{1}{\sigma_x}\frac{\langle xF_x\rangle}{\beta cp} = \frac{\varepsilon_{rms}^2}{\sigma_x^3}\,. \tag{1}$$

We introduce normalized emittance $\varepsilon_{n,rms} = \gamma\varepsilon_{rms}$

$$\sigma_x'' + \frac{p'}{p}\sigma_x' - \frac{1}{\sigma_x}\frac{\langle xF_x\rangle}{\beta cp} = \frac{\varepsilon_{n,rms}^2}{\gamma^2\sigma_x^3} \tag{2}$$

In the nonlinear regime, the plasma electrons behind the driver are completely expelled and an ellipsoidal cavity filled by ions is formed. A very simplified model for the plasma behind the driving pulse is illustrated in Fig. 1. We consider an ellipsoidal uniform ion distribution, indicated by the dashed ellipsoid, with particle density n_p. We focus here on the beam driven case, i.e. a plasma wave excited by a driving electron bunch with particle density n_b, but the model can be easily extended to describe the laser driven case.

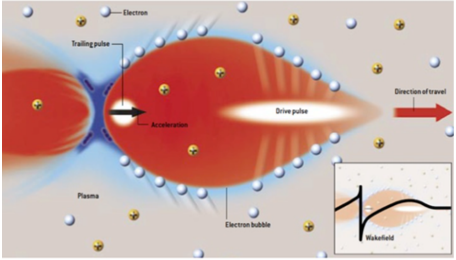


Figure 1: Schematic representation of the longitudinal wake field (black line) and ion distribution (red area) behind a driving laser or particle beam [4].

Ellipsoidal plasma bubble model

This model is justified by the fact that in this regime the fields are linear in both longitudinal and transverse directions, at least in the region of interest for particle acceleration, as the one produced by a uniform ion distribution within an ellipsoid of semi-axes [5]:

$$\left.\begin{aligned} X &= 2\sqrt{\alpha}\sigma_{xd} \\ Y &= 2\sqrt{\alpha}\sigma_{yd} \\ Z &= \frac{\lambda_p}{2} \end{aligned}\right\} \tag{3}$$

where $\alpha = n_b/n_p \geq 1$ in the nonlinear regime, σ_{xd} and σ_{yd} the driving beam spot sizes and $\lambda_p = 2\pi/k_p$ is the plasma wavelength.

Ellipsoidal plasma bubble model

The electrostatic fields are represented with the field distribution produced by a 3D ellipsoidal filled by a charge of $Q_B = 4\pi XYZen_p/3$ [6]:

$$\left.\begin{aligned} E_x &= \frac{3Q_B(1-f)}{4\pi\varepsilon_0(X+Y)Z}\frac{x}{X} = \frac{en_p}{\varepsilon_0}\frac{Y}{X+Y}(1-f)x \\ E_y &= \frac{3Q_B(1-f)}{4\pi\varepsilon_0(X+Y)Z}\frac{y}{Y} = \frac{en_p}{\varepsilon_0}\frac{X}{X+Y}(1-f)y \\ E_z &= \frac{3Q_Bf}{4\pi\varepsilon_0XY}\frac{z}{Z} = \frac{en_p}{\varepsilon_0}fz \end{aligned}\right\} \tag{4}$$

The quantity f is the ellipsoid form factor that can be defined as:

$$f = \frac{1}{3}\frac{\sqrt{XY}}{Z} = \frac{2^{5/4}}{3\pi^{5/4}}\sqrt{\frac{Q_d r_e}{e\sigma_{zd}}}. \tag{5}$$

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Driver Fields

Using the value of f, we obtain

$$E_x = \left(\frac{3\pi^{5/4} - 2^{5/4}\sqrt{\frac{Q_d r_e}{e\sigma_{zd}}}}{12\varepsilon_0 \pi^{9/4} r_e} \right) \frac{\sigma_{yd}}{\sigma_{xd} + \sigma_{yd}} e k_p^2 x \tag{6}$$

$$E_y = \left(\frac{3\pi^{5/4} - 2^{5/4}\sqrt{\frac{Q_d r_e}{e\sigma_{zd}}}}{12\varepsilon_0 \pi^{9/4} r_e} \right) \frac{\sigma_{xd}}{\sigma_{xd} + \sigma_{yd}} e k_p^2 y \tag{7}$$

$$E_z = \frac{k_p^2}{3 \times 2^{3/4} \varepsilon_0 \pi^{9/4}} \sqrt{\frac{eQ_d}{r_e \sigma_{zd}}}\, z \tag{8}$$

$$B_\vartheta = \frac{e n_p}{2\varepsilon_0 c} f x \tag{9}$$

Lorentz Force

Lorentz force is given by

$$F_x = e(E_x - \beta c B_\vartheta) = e\left(\frac{en_p}{\varepsilon_0}\frac{Y}{X+Y}(1-f) - \beta c\frac{en_p}{2\varepsilon_0 c}f\right)x$$

$$F_x = \frac{3\pi^{5/4}\sigma_{yd} - 2^{1/4}\sqrt{\frac{Q_d r_e}{e\sigma_{zd}}}(\sigma_{xd}+3\sigma_{yd})}{12\varepsilon_0\pi^{9/4}r_e}\frac{e^2}{\sigma_{xd}+\sigma_{yd}}k_p^2 x \qquad (10)$$

$$F_y = \frac{1}{12\pi^{9/4}\varepsilon_0 r_e}\left(3\pi^{5/4}\sigma_{xd} - (\sigma_{yd}+3\sigma_{xd})2^{1/4}\sqrt{\frac{Q_d r_e}{e\sigma_{zd}}}\right)\frac{e^2 k_p^2}{\sigma_{xd}+\sigma_{yd}}y \qquad (11)$$

A relativistic witness electron bunch is propagating in a ionized gas background of opposite charge, a simplified configuration similar to the one produced in a capillary discharge where a plasma oscillation (charge separation) has been excited by a driving pulse. The transverse electric and magnetic fields experienced by a particle can be evaluated by considering a uniform cylindrical beam space charge model [8] that gives:

$$\begin{cases} E_r = \dfrac{I(1-f_e)}{2\pi\varepsilon_0 R^2 v} r\,; & \text{for} \quad r \leq R \\ E_r = \dfrac{I(1-f_e)}{2\pi\varepsilon_0 v}\dfrac{1}{r}\,; & \text{for} \quad r > R \end{cases} \tag{12}$$

$$\begin{cases} B_\vartheta = \dfrac{\mu_0 I(1-f_m)}{2\pi R^2} r\,; & \text{for} \quad r \leq R \\ B_\vartheta = \dfrac{\mu_0 I(1-f_m)}{2\pi}\dfrac{1}{r}\,; & \text{for} \quad r > R \end{cases} \tag{13}$$

Using the fields (12) and (13) the Lorentz force acting on the electrons can be written as:

$$F_r = e(E_r - \beta c B_\vartheta) = \frac{eE_r}{\gamma^2}\left(\frac{1-\gamma^2 f_e + \beta^2\gamma^2 f_m}{1-f_e}\right) \tag{14}$$

$$\frac{\langle rF_r\rangle}{\beta cp} = \frac{2I(1-\gamma^2 f_e + \beta^2\gamma^2 f_m)}{I_A\gamma^3} = \frac{k_{sc}^0}{\gamma^3}\,.$$

Here, the alfvén current, $I_A = 4\pi\varepsilon_0 m_0 c^3/e$ and beam current, $I = en_e\pi\sigma^2 c$. The factor f_e accounts for a stationary charge distribution of opposite sign that results in a partial neutralization of the space charge of the primary beam particles. The f_m factor accounts for a partial magnetic neutralization produced by the plasma return current.

It is worth mentioning that the new "space charge" term on the right hand side of (14) can contribute to focus or defocus the beam depending on the sign of the term between brackets:

$$(1 - \gamma^2 f_e + \beta^2\gamma^2 f_m) \begin{cases} < 0 & \Rightarrow \text{ focusing effect} \\ = 0 & \Rightarrow \text{ screening effect} \\ > 0 & \Rightarrow \text{ defocusing effect} \end{cases} \tag{15}$$

Let us consider the charge screening effect of the plasma background with particle density n_p , by defining $f_e = n_p/n_e$, where n_e is the bunch particle density and $f_m = 0$.

$$\sigma_{xd}'' + \frac{\gamma'}{\gamma}\sigma_{xd}' + \frac{1}{\gamma}\left(\frac{2^{\frac{1}{4}}}{3\pi^{\frac{5}{4}}}\sqrt{\frac{Q_d r_e}{e\sigma_{zd}}}(\sigma_{xd}+3\sigma_{yd}) - \sigma_{yd}\right)\frac{k_p^2\sigma_{xd}}{(\sigma_{xd}+\sigma_{yd})} = \frac{\varepsilon_n^2}{\gamma^2\sigma_{xd}^3} + \frac{k_{sc}^0}{\gamma^3\sigma_{xd}} \tag{16}$$

$$\sigma_{yd}'' + \frac{\gamma'}{\gamma}\sigma_{yd}' + \frac{1}{\gamma}\left(\frac{2^{\frac{1}{4}}}{3\pi^{\frac{5}{4}}}\sqrt{\frac{Q_d r_e}{e\sigma_{zd}}}(\sigma_{yd}+3\sigma_{xd}) - \sigma_{xd}\right)\frac{k_p^2\sigma_{yd}}{(\sigma_{xd}+\sigma_{yd})} = \frac{\varepsilon_n^2}{\gamma^2\sigma_{yd}^3} + \frac{k_{sc}^0}{\gamma^3\sigma_{yd}} \tag{17}$$

$$\sigma_{xw}'' + \frac{\gamma'}{\gamma}\sigma_{xw}' + \frac{1}{\gamma}\left(\frac{2^{\frac{1}{4}}}{3\pi^{\frac{5}{4}}}\sqrt{\frac{Q_d r_e}{e\sigma_{zd}}}(\sigma_{xd}+3\sigma_{yd}) - \sigma_{yd}\right)\frac{k_p^2\sigma_{xw}}{(\sigma_{xd}+\sigma_{yd})} + \frac{1}{\gamma}\left(\frac{2^{\frac{1}{4}}}{3\pi^{\frac{5}{4}}}\sqrt{\frac{Q_w r_e}{e\sigma_{zw}}}(\sigma_{xw}+3\sigma_{yw}) - \sigma_{yw}\right)\frac{k_p^2\sigma_{xw}}{(\sigma_{xw}+\sigma_{yw})} = \frac{\varepsilon_n^2}{\gamma^2\sigma_{xw}^3} + \frac{k_{sc}^0}{\gamma^3\sigma_{xw}} \tag{18}$$

$$\sigma_{yw}'' + \frac{\gamma'}{\gamma}\sigma_{yw}' + \frac{1}{\gamma}\left(\frac{2^{\frac{1}{4}}}{3\pi^{\frac{5}{4}}}\sqrt{\frac{Q_d r_e}{e\sigma_{zd}}}(\sigma_{yd}+3\sigma_{xd}) - \sigma_{xd}\right)\frac{k_p^2\sigma_{yw}}{(\sigma_{xd}+\sigma_{yd})} + \frac{1}{\gamma}\left(\frac{2^{\frac{1}{4}}}{3\pi^{\frac{5}{4}}}\sqrt{\frac{Q_w r_e}{e\sigma_{zw}}}(\sigma_{yw}+3\sigma_{xw}) - \sigma_{xw}\right)\frac{k_p^2\sigma_{yw}}{(\sigma_{xw}+\sigma_{yw})} = \frac{\varepsilon_n^2}{\gamma^2\sigma_{yw}^3} + \frac{k_{sc}^0}{\gamma^3\sigma_{yw}} \tag{19}$$

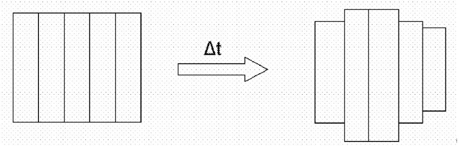

Figure 2: Multi-slice approximation: when beam is travelling on axis. [9].

Total rms emittance

In order to evaluate the degradation of the rms emittance produced by longitudinal correlation in space charge effect and transverse external forces, we use the following expression for the correlated emittance [9]

$$\varepsilon_{n,x}^{\text{cor}} = \sqrt{\langle \sigma_x^2 \rangle \langle (\gamma \sigma_x')^2 \rangle - \langle \sigma_x \gamma \sigma_x' \rangle^2}\,, \tag{20}$$

where the average $\langle\ \rangle = \frac{1}{N} \sum_{s=1}^{N}$ is performed over the N slices and an analogous equation holds for the σ_y envelope. The total rms emittance will be given by

$$\varepsilon_n = \sqrt{(\varepsilon_n^{\text{th}})^2 + (\varepsilon_n^{\text{cor}})^2} \tag{21}$$

The energy spread is defined as [9]:

$$\frac{\Delta\gamma}{\gamma} = \frac{\sqrt{\langle (\gamma - \langle \gamma \rangle)^2 \rangle}}{\langle \gamma \rangle} \tag{22}$$

Plasma	Driver
Density $= 1.0 \times 10^{16}$ cm^{-3}	Charge $= 140.$ pC
Wavelength $= 333.894$ μm 1.11375 ps	N electrons $= 8.73811 \times 10^{8}$
Wave number $k_p = 18817.9$	$\sigma_z = 183.46$ fs 55 μm
Ramp Length $= 0.1$ cm	$\sigma_x = 10.$ μm
Capillary Length $= 60.$ cm	$\sigma_y = 10.$ μm
Capillary Radius $= 500.$ μm	Current $= 305.243$ A
Capillary Discharge Current $= 0.$ A	rms norm. emittance $= 2.0$ μm
Magnetic Gradient $= 0.$ T/m	Energy $= 600$ MeV
Drift Length $= 0.$ cm	Energy Spread $= 0.$ MeV 0. %
	Q tilde $= 0.582278$
	alpha $= 1.00875$
	Bubble axis $= 20.0873$ 20.0873 μm

Table 1: Parameters

Driver

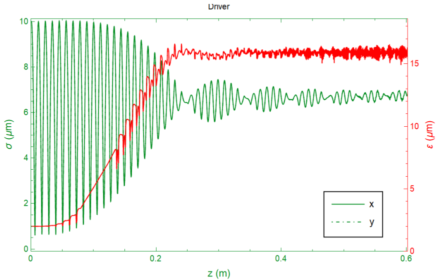

Figure 3: Theoretical

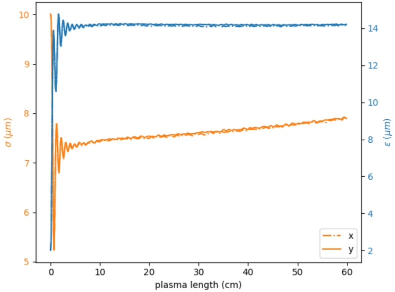

Figure 4: Simulation (Architect code)

Driver

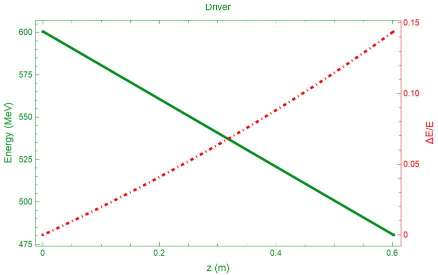

Figure 5: Theoretical

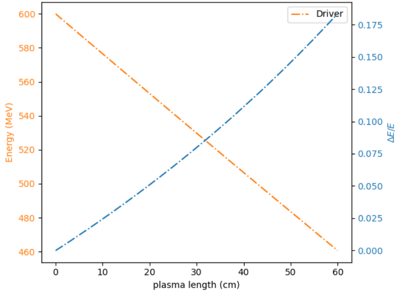

Figure 6: Simulation (Architect code)

Parameters of Driver and witness

Driver	Witness
$\sigma_z = 133.426$ fs 40 μm	$\sigma_z = 50.0346$ fs 15 μm
$\sigma_x = 3.\ \mu$m	$\sigma_x = 3.\ \mu$m
$\sigma_y = 3.\ \mu$m	$\sigma_y = 3.\ \mu$m
rms norm. emittance $= 2.0\ \mu$m	rms norm. emittance $= 1.0\ \mu$m
Energy = 600 MeV	Energy = 600 MeV
Energy Spread = 0. MeV 0. %	Energy Spread = -2.4 MeV -0.4 %
D Current = 419.709 A	Current = 359.751 A
	Location = 67. μm 0.223488 ps

Table 2: Parameters

Witness

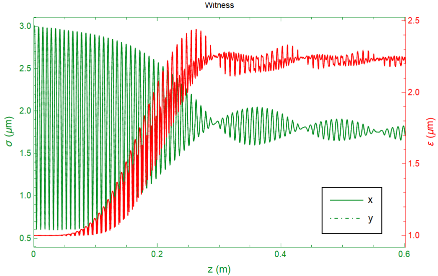

Figure 7: Theoretical

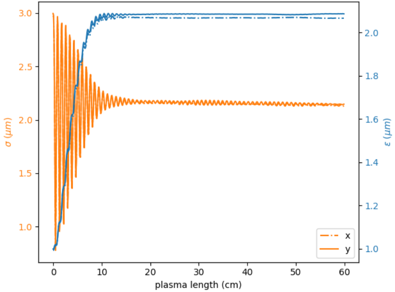

Figure 8: Simulation (Architect code)

- We have assumed in the nonlinear regime, the plasma electrons behind the driver are completely expelled and an ellipsoidal cavity filled with ions is formed.
- It is justified that the fields are linear in both longitudinal and transverse directions, at least in the region of interest for particle acceleration, as the one produced by a uniform ion distribution within a uniformly charged ellipsoidal distribution.
- The fields produced by the ions and experienced by a witness electron beam are purely electrostatic, being the ions at rest in the laboratory frame on the time scale of interest and it can be represented with the field distribution produced by a 3D charged ellipsoidal.

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- The energy spread and emittance degradation have been studied by slicing the bunch in an array of cylinders and solving envelope equations for each bunch slice.
- The properties of the transverse envelope and emittance oscillations and energy spread degradation have been analyzed and compared with the simulation results. It is found that there is a good agreement between our results and simulation results.

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Thank you for your kind attention

Derivation of rms envelope equation

We are now interested to follow the evolution of the particle distribution during beam transport and acceleration. One can take profit of the first collective variable defined in eq. (6), the second moment of the distribution termed rms beam envelope, to derive a differential equation suitable to describe the rms beam envelope dynamics [1]. To this end lets compute the first and second derivative of σ_x [2]:

$$\frac{d\sigma_x}{dz} = \frac{d}{dz}\sqrt{\langle x^2 \rangle} = \frac{\sigma_{xx'}}{\sigma_x}$$

$$\frac{d^2\sigma_x}{dz^2} = \frac{\sigma_{x'}^2 + \langle xx'' \rangle}{\sigma_x} - \frac{\sigma_{xx'}^2}{\sigma_x^3}$$

$$\sigma_x'' - \frac{\langle xx'' \rangle}{\sigma_x} = \frac{\sigma_x^2\sigma_{x'}^2 - \sigma_{xx'}^2}{\sigma_x^3} = \frac{\varepsilon_{rms}^2}{\sigma_x^3}$$

$$\boxed{\sigma_x'' - \frac{\langle xx'' \rangle}{\sigma_x} = \frac{\varepsilon_{rms}^2}{\sigma_x^3}} \quad (23)$$

The definition of rms emittance in terms of the second moments of the distribution is

$$\varepsilon_{rms} = \sqrt{\sigma_x^2 \sigma_{x'}^2 - \sigma_{xx'}^2} = \sqrt{(\langle x^2 \rangle \langle x'^2 \rangle - \langle xx' \rangle^2)} \quad (24)$$

For an effective transport of a beam with finite emittance is mandatory to make use of some external force providing beam confinement in the transport or accelerating line. The term $\langle xx'' \rangle$ accounts for external forces when we know x'' given by the single particle equation of motion:

$$\frac{dp_x}{dt} = F_x \quad (25)$$

Under the paraxial approximation $p_x \ll p = \beta\gamma mc$ the transverse momentum p_x can be written as $p_x = px' = \beta\gamma m_0 c x'$, and and the transverse acceleration results to be:

$$x'' = -\frac{p'}{p}x' + \frac{F_x}{\beta c p} \tag{26}$$

It follows that

$$\langle xx'' \rangle = -\frac{p'}{p}\langle xx' \rangle + \frac{\langle xF_x \rangle}{\beta c p} = -\frac{p'}{p}\sigma_{xx'} + \frac{\langle xF_x \rangle}{\beta c p} \tag{27}$$

Inserting (27) into (23) and using $\sigma'_x \sigma_x = \sigma_{xx'}$, the complete rms envelope equation results to be:

$$\sigma''_x + \frac{p'}{p}\sigma'_x - \frac{1}{\sigma_x}\frac{\langle xF_x \rangle}{\beta c p} = \frac{\varepsilon^2_{rms}}{\sigma^3_x} \tag{28}$$

We introduce normalized emittance $\varepsilon_{n,rms} = \gamma\varepsilon_{rms}$

$$\sigma''_x + \frac{p'}{p}\sigma'_x - \frac{1}{\sigma_x}\frac{\langle xF_x \rangle}{\beta c p} = \frac{\varepsilon^2_{n,rms}}{\gamma^2\sigma^3_x} \tag{29}$$