On maximizing laser wake field acceleration (LWFA) by tailoring the plasma density. EAAC23, ID357

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Abstract: We sketch a preliminary analytical procedure [1,2] in 4 steps to tailor the initial density (upramp+downramp+plateau) of a cold diluted plasma to the laser pulse so as to control wave breakings (WBs) of the plasma wave (PW) and maximize the acceleration of the first electrons (e^-s) self-injected in the PW by the first WB at the down-ramp; the corresponding plateau density is uniquely determined. We use as long as possible the improved fully relativistic plane hydrodynamic model (HM) of Ref. [3,4,5], modeling the pulse as a plane wave travelling in the z direction. Our (1+1)-dim results may help also in realistic (3+1)-dim problems.

I. Introduction and set-up

Nowadays the equations (Maxwell + kinetic theory for electrons and ions) ruling plasma dynamics in LWFA can be solved via more and more powerful particle-in-cell (PIC) codes, but running them has huge costs for each choice of the input data. Hence it is crucial to do after a preliminary data selection based on simpler models. Below we sketch one maximizing the above LWFA.

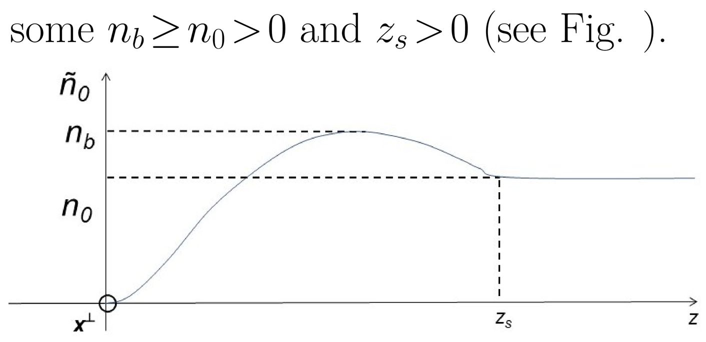
We regard the plasma as long as possible as a static background of ions and fully relativistic collisionless fluid of e^-s. Initial conditions for their Eulerian density n_e, velocity $\mathbf{v}_e$:

$$\mathbf{v}_e(0,\mathbf{x})=\mathbf{0}, \qquad n_e(0,\mathbf{x})=\widetilde{n_0}(z); \tag{1}$$

the initial e^- (and proton) density $\widetilde{n_0}(z)$ satisfies

$$\widetilde{n_0}(z) \le n_b, \qquad \widetilde{n_0}(z)=\begin{cases}0 & \text{if } z\le 0,\\ n_0 & \text{if } z\ge z_s\end{cases} \tag{2}$$

for some $n_b \ge n_0 > 0$ and $z_s > 0$ (see Fig.).



We model the electric and magnetic fields $\mathbf{E}, \mathbf{B}$ as a plane wave propagating in the z-direction,

$$\mathbf{E}(t,\mathbf{x}) = \boldsymbol{\epsilon}^{\perp}(ct-z), \quad \mathbf{B}=\mathbf{k}\times\mathbf{E} \tag{3}$$

($\mathbf{x}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$, c is the speed of light), where the support of $\boldsymbol{\epsilon}^{\perp}(\xi) \perp \mathbf{k}$ is an interval $0\le\xi\le l$ fulfilling $l \lesssim \sqrt{\pi mc^2/n_b e^2}$ (neglect depletion). $\{\widetilde{n_0}(z), \boldsymbol{\epsilon}^{\perp}(\xi)\} \equiv$ **input data** of our problem.

The position $\mathbf{x}(t)$ and momentum $\mathbf{p}(t) = mc\,\mathbf{u}(t)$ of an e^- fulfill the Lorentz eqs. Dimensionless variables: $\boldsymbol{\beta}\equiv\mathbf{v}/c=\dot{\mathbf{x}}/c$, $\gamma\equiv 1/\sqrt{1-\boldsymbol{\beta}^2}=\sqrt{1+\mathbf{u}^2}$, the 4-velocity $u=(u^0,\mathbf{u})\equiv(\gamma,\gamma\boldsymbol{\beta})$. As v $<$ c, we can make the change $t\mapsto\xi=ct-z$ of independent parameter along the worldline (WL) of e^- (see Fig. 1), so that the term $\boldsymbol{\epsilon}^{\perp}[ct-z(t)]$, where the *unknown* $z(t)$ is in the argument of the highly nonlinear and rapidly varying $\boldsymbol{\epsilon}^{\perp}$, becomes the *known* forcing term $\boldsymbol{\epsilon}^{\perp}(\xi)$. We denote as $\hat{\mathbf{x}}(\xi)$ the position of e^- as a function of ξ; it is determined by $\hat{\mathbf{x}}(\xi)=\mathbf{x}(t)$. More generally given any $f(t,\mathbf{x})$ we denote $\hat{f}(\xi,\hat{\mathbf{x}}) \equiv f\left[\hat{t}(\xi),\hat{\mathbf{x}}\right]$ (where $c\hat{t}(\xi)=\xi+\hat{z}(\xi)$), abbreviate $\dot{f}\equiv df/dt$, $\hat{f}'\equiv d\hat{f}/d\xi$ (total derivatives). Convenient change of dependent variable $u^z\mapsto s\equiv$ the lightlike component of u [5]:

$$s\equiv\gamma-u^z=u^-=\gamma(1-\beta^z)=\frac{\gamma\,d\tilde{\xi}}{c\,dt}>0; \tag{4}$$

$\gamma, \mathbf{u}, \boldsymbol{\beta}$ are the *rational* function of $\mathbf{u}^{\perp}, s$

$$\gamma=\frac{1+\mathbf{u}^{\perp 2}+s^2}{2s}, \quad u^z=\frac{1+\mathbf{u}^{\perp 2}-s^2}{2s}, \quad \boldsymbol{\beta}=\frac{\mathbf{u}}{\gamma}; \tag{5}$$

(5) hold also with $\hat{}$. If $\hat{s}(\xi)\to 0$ as $\xi\uparrow\xi_f<\infty$, then $\hat{\gamma}, \hat{u}^z, \hat{t}\to\infty$. Replacing $\gamma\,d/dt\mapsto cs\,d/d\xi$ and putting $\hat{}$ on all variables makes Lorentz eqs *rational* in the unknowns $\hat{\mathbf{u}}^{\perp}, \hat{s}$. Moreover, $\hat{s}$ is practically insensitive to fast oscillations of $\boldsymbol{\epsilon}^{\perp}(\xi)$ (see Fig. 2.b). Let $\mathbf{x}_e(t,\mathbf{X}) \equiv$ position at time t of the e^- fluid element d^3X initially located at $\mathbf{X}\equiv(X,Y,Z)$, $\hat{\mathbf{x}}_e(\xi,\mathbf{X})\equiv$ the same position as a function of ξ. We dub as 'Z electrons' the e^-s in the layer $[Z, Z+dZ]$ for $t\le 0$. In the hydrodynamic regime (HR) the maps $\mathbf{x}_e(t,\cdot):\mathbf{X}\mapsto\mathbf{x}$, $\hat{\mathbf{x}}_e(\xi,\cdot):\mathbf{X}\mapsto\mathbf{x}$ are one-to-one for all t, resp. ξ. The inverses $\mathbf{X}_e(t,\cdot)$, $\hat{\mathbf{X}}_e(\xi,\cdot)$ fulfill

$$\mathbf{X}_e(t,\mathbf{x})=\hat{\mathbf{X}}_e(ct-z,\mathbf{x}). \tag{6}$$

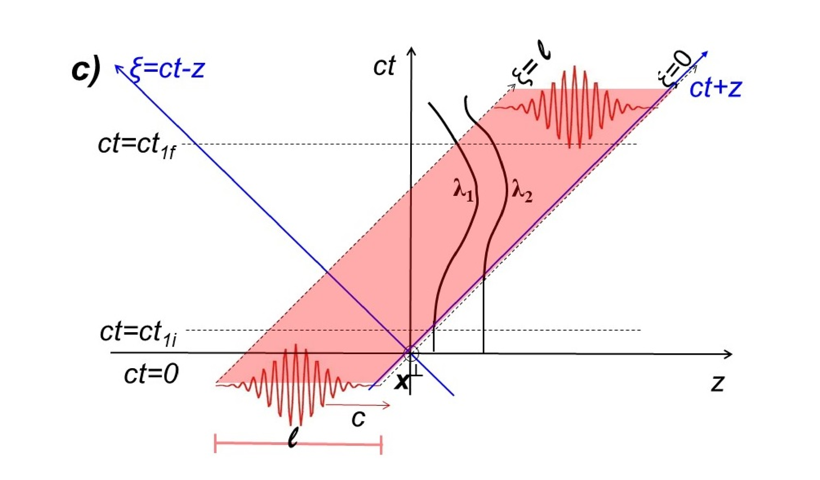


Figure 1: Two particle worldlines (WLs) λ_1, λ_2 in Minkowski space; they intersect the support (pink) of a plane electromagnetic (EM) wave of total length l in the positive z direction. Since each WL intersects once every hyperplane $\xi=$ const (beside every hyperplane $t=$ const), we can use ξ rather than t as a parameter along it. The front, end of the EM wave intersect different WLs at different t-instants ($t_{1i}\neq t_{2i}$, $t_{1f}\neq t_{2f}$), but at the same ξ-instants $\xi_i=0$, $\xi_f=l$.

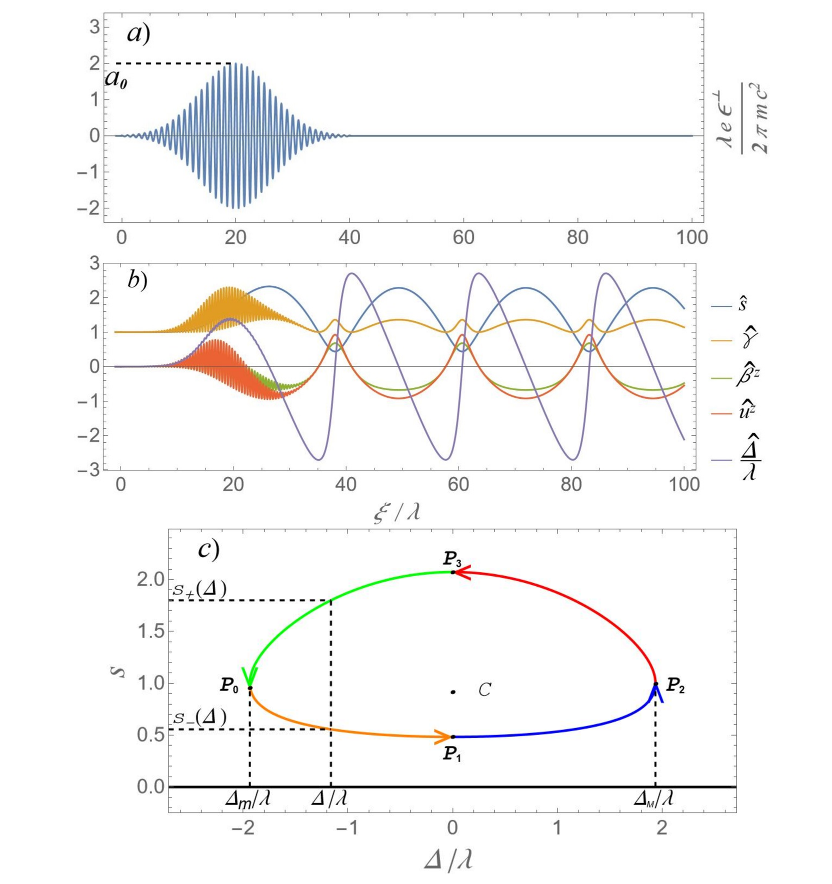


Figure 2: (a) Normalized gaussian pulse of FWHM $l'=10.5\lambda$, linear polarization, peak amplitude $a_0\equiv\lambda e E_{\scriptscriptstyle M}^{\perp}/2\pi mc^2=2$. (b) Corresponding solution of (12) if $\widetilde{n_0}(z)=n_0^j\equiv n_{cr}/267$ ($n_{cr}=\pi mc^2/e^2\lambda^2$ is the critical density); as a result, $E/mc^2\equiv h=1.28$. Adopting $n_0=n_0^j$ as the plateau density maximizes the maximal $|E^z|$ in the PW, hence also the LWFA of test electrons with the 'right' phase. If $\lambda=0.8\mu$m, then peak intensity is $I=1.7{\times}10^{19}$W/cm^2 and $n_0^j=6.5{\times}10^{18}$cm^{-3}. (c) The corresponding electron phase portrait (at $\xi>l$).

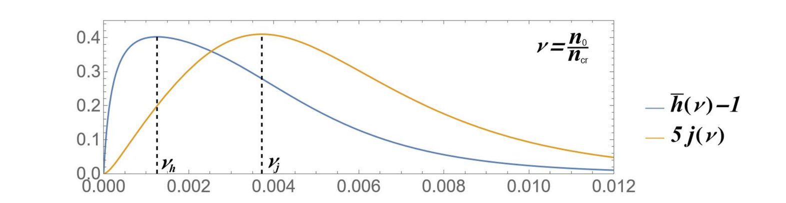


Figure 3: $\bar{h}$–1 (energy gain per e^-), j vs. the density n_0.

$\Delta\mathbf{x}_e \equiv \mathbf{x}_e(t,\mathbf{X})-\mathbf{X}$ actually depends only on t, Z [and $\Delta\hat{\mathbf{x}}_e\equiv\hat{\mathbf{x}}_e(\xi,\mathbf{X})-\mathbf{X}$ only on ξ, Z] and by causality vanishes if $ct\le Z$. We adopt the x,y-independent physical observable

$$\mathbf{A}^{\perp}(t,z)\equiv -c\int_{-\infty}^{t} dt'\,\mathbf{E}^{\perp}(t',z); \tag{7}$$

as the transverse component of the EM potential: $c\mathbf{E}^{\perp}=-\partial_t\mathbf{A}^{\perp}$, $\mathbf{B}=\mathbf{k}\wedge\partial_z\mathbf{A}^{\perp}$. As usual, (Lorentz eq.)$^{\perp}$ and $\mathbf{p}_e^{\perp}(0,\mathbf{x})=\mathbf{0}=\boldsymbol{\alpha}^{\perp}(-z)$ if $z\ge 0$ imply

$$\mathbf{p}_e^{\perp}=\frac{e}{c}\mathbf{A}^{\perp} \qquad \text{i.e.} \qquad \mathbf{u}_e^{\perp}=\frac{e}{mc^2}\mathbf{A}^{\perp}, \tag{8}$$

for the Eulerian e^- momentum $\mathbf{p}_e$, allowing to trade $\mathbf{u}_e^{\perp}$ for $\mathbf{A}^{\perp}$ as an unknown. By (3), for $t\le 0$

$$\mathbf{A}^{\perp}(t,z)=\boldsymbol{\alpha}^{\perp}(ct-z), \quad \boldsymbol{\alpha}^{\perp}(\xi)\equiv\int_{-\infty}^{\xi}-d\eta\,\boldsymbol{\epsilon}^{\perp}(\eta); \tag{9}$$

(9) approximately holds also for small $t>0$. The conservation $n_e\,dz=\widetilde{n_0}\,dZ$ of number of e^- gives:

$$n_e(t,z)=\widetilde{n_0}[Z_e(t,z)]\;\partial_z Z_e(t,z). \tag{10}$$

Maxwell eqs $\nabla\cdot\mathbf{E}-4\pi j^0=\partial_z E^z-4\pi e(n_p-n_e)=0$, $\partial_t E^z/c+4\pi j^z=(\nabla\wedge\mathbf{B})^z=0$ & in. cond. imply [5]

$$E^z(t,z)=4\pi e\left\{\widetilde{N}(z)-\widetilde{N}[Z_e(t,z)]\right\}, \tag{11}$$

$\mathbf{j}=-en_e\boldsymbol{\beta}_e$, $\widetilde{N}(z)\equiv\int_0^z d\eta\,\widetilde{n_0}(\eta)$. Via (10-11) we express n_e, E^z through $\widetilde{n_0}$ and the still unknown $Z_e(t,z)$. (5c) amounts to $\hat{\mathbf{x}}_e^{\perp\prime}=\hat{\mathbf{u}}_e^{\perp}/\hat{s}$, which integrated yields $\hat{\mathbf{x}}_e^{\perp}$ in terms of $\hat{s}\equiv\hat{s}_e$. The remaining unknowns $\hat{\Delta}(\xi,Z)\equiv\hat{z}_e(\xi,Z)-Z$, $\hat{s}$ satisfy

$$\hat{\Delta}'=\frac{1+v}{2\hat{s}^2}-\frac{1}{2}, \quad \hat{s}'=K\left\{\widetilde{N}\left[Z+\hat{\Delta}\right]-\widetilde{N}(Z)\right\}, \quad \hat{\Delta}(0,Z)=0, \quad \hat{s}(0,Z)=1, \tag{12}$$

$$v\equiv\hat{\mathbf{u}}_e^{\perp 2}=\left[\frac{e\hat{\mathbf{A}}^{\perp}}{mc^2}\right]^2=\left[\frac{e\boldsymbol{\alpha}^{\perp}}{mc^2}\right]^2, \qquad K\equiv\frac{4\pi e^2}{mc^2}$$

Eqs (12a) are a Z-family of *decoupled ODEs*, Hamilton eqs $\hat{\Delta}'=-\partial\hat{H}/\partial\hat{s}$, $\hat{s}'=\partial\hat{H}/\partial\hat{\Delta}$ of a 1-dim system: $\xi, \hat{\Delta}, -\hat{s}$ play the role of t, q, p,

$$\hat{H}(\hat{\Delta},\hat{s},\xi;Z)\equiv\gamma(\hat{s};\xi)+\mathcal{U}(\hat{\Delta};Z), \tag{13}$$

$$\gamma(s;\xi)\equiv\frac{s^2+1+v(\xi)}{2s}, \quad \frac{\mathcal{U}(\Delta;z)}{K}\equiv\int_z^{z+\Delta}d\zeta\,\widetilde{N}(\zeta)-\widetilde{N}(z)\Delta;$$

γ–1, $\mathcal{U}$ act as kinetic,potential energy (mc^2 units). We can easily solve (12) in the unknown $\hat{P}\equiv(\hat{\Delta},\hat{s})$ numerically, or by quadrature for $\xi\ge l$.

Hydrodynamic regime up to WB

The HR holds as long as, for all Z,

$$\hat{J}\equiv\left|\frac{\partial\hat{\mathbf{x}}_e}{\partial\mathbf{X}}\right|=\frac{\partial\hat{z}_e}{\partial Z}>0. \tag{14}$$

The identity $\hat{z}_e[\xi+i\xi_{\scriptscriptstyle H}(Z),Z]=\hat{z}_e(\xi,Z)$ holds for $i\in\mathbb{N}$, $\xi>l$; differentiating w.r.t. Z one finds [3]

$$\hat{J}(\xi+i\xi_{\scriptscriptstyle H},Z)=\hat{J}(\xi,Z)-i\,\Phi(Z)\,\Delta'(\xi,Z); \tag{15}$$

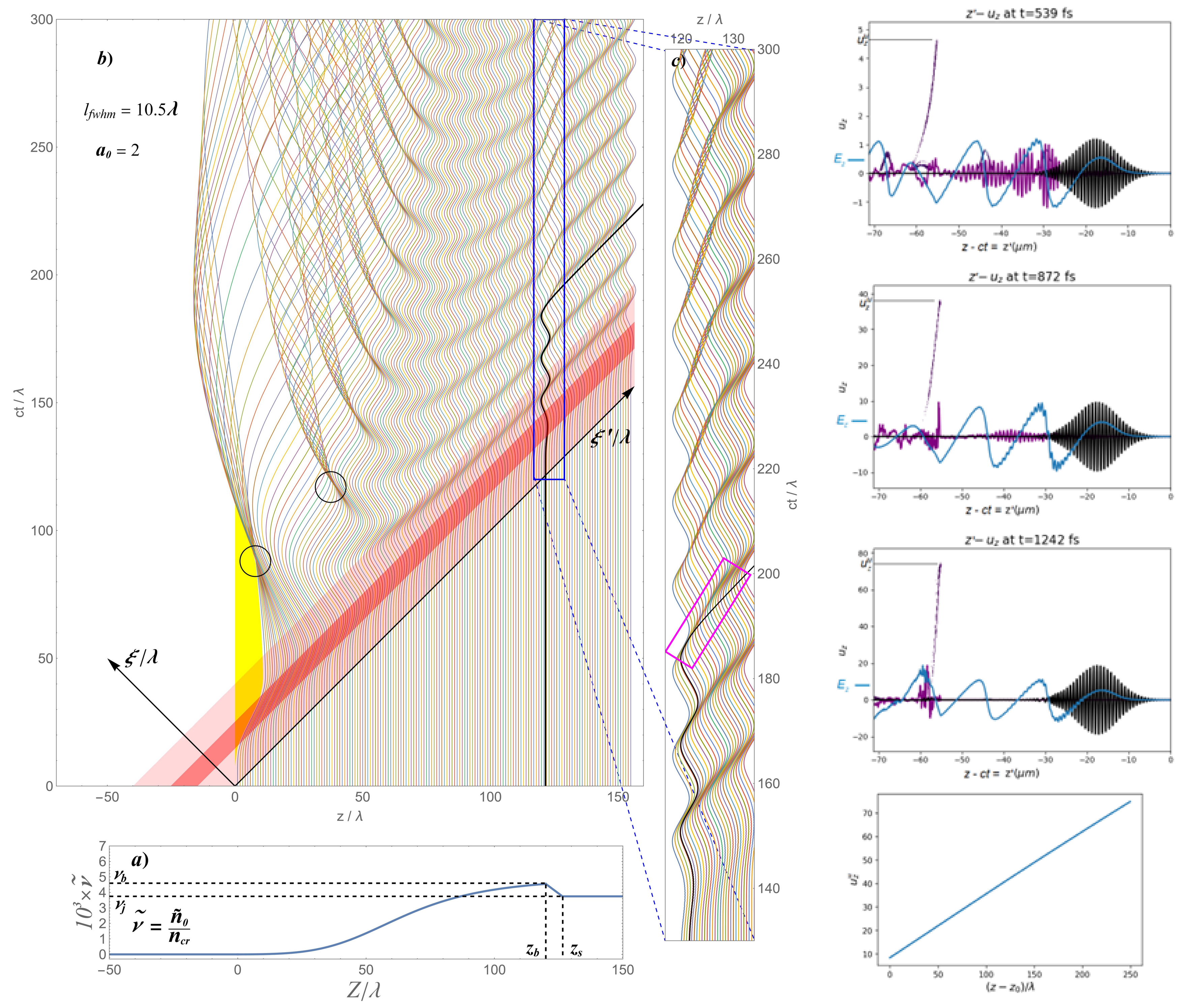


Figure 4: **Left:** a) Optimal initial plasma density $\widetilde{n_0}(Z)$ for the pulse of Fig. 2.a: $n=n_0^j=n_{cr}/267$, $n_b=1.21\times n_0^j$, $z_b=120\lambda$, $z_s-z_b=6.6\lambda$. b) Projections onto the z, ct plane of the corresponding WLs (in Minkowski space) of the Z electrons for $Z=0,\lambda,...,156\lambda$. We have studied the down-ramp Z electrons more in detail, determining their WLs for $Z=120\lambda, 120.1\lambda,...,140\lambda$: in c) we zoom the blue box of a). Here: $\xi'\equiv ct+z$; in the dark yellow region only ions are present; we have painted pink, red the support of $\boldsymbol{\epsilon}^{\perp}(ct-z)$ (considering $\boldsymbol{\epsilon}^{\perp}(\xi)=0$ outside $0<\xi<40\lambda$) and the region where the modulating intensity is above half maximum, i.e. $-l'/2<\xi-20\lambda<l'/2$, with $l'=10.5\lambda$. **Right:** Three longitudinal phase-space plots $z-u^z$ (of injected e^-s) obtained via a FB-PIC simulation (courtesy of P. Tomassini); $u_z^{\scriptscriptstyle M}\equiv$ maximal u^z. Bottom: plot $u_z^{\scriptscriptstyle M}\simeq\gamma_i^{\scriptscriptstyle M}$ vs. z_i is linear with growth rate $F\simeq 0.27$; agrees well with our prediction (19-20), where $F=0.286$!

the period $\xi_{\scriptscriptstyle H}(Z)$ of the Z e^- is computed by quadrature, $\Phi\equiv\frac{\partial\xi_H}{\partial Z}$. Via (15) we can extend our knowledge of $\hat{J}$ from $[l, l+\xi_{\scriptscriptstyle H}[$ to all $\xi\ge l$ and determine the first WB [3].

WFA of (self-)injected electrons

If a test e^- is injected with $\left(\hat{z}_i,\hat{s}_i\right)_{\xi=\xi_0}=(z_{i0},s_{i0})$, $\xi_0>l$, $s_{i0}>0$, $\hat{\mathbf{u}}_i^{\perp}(\xi_0)=0$, its $\hat{z}_i, \hat{s}_i$ evolve after

$$\hat{z}_i'=\frac{1-\hat{s}_i^2}{2\hat{s}_i^2}, \quad \hat{s}_i'(\xi)=K\left\{\widetilde{N}[\hat{z}_i(\xi)]-\widetilde{N}\left[\hat{Z}_e(\xi,\hat{z}_i(\xi))\right]\right\}. \tag{16}$$

Along the plateau (16b) is $\hat{s}_i'=M\Delta$. Hence

$$\hat{s}_i(\xi)=\delta s+s(\xi), \quad \hat{z}_i(\xi)=z_{i0}+\int_{\xi_0}^{\xi}\frac{dy}{2}\left[\frac{1}{\hat{s}_i^2(y)}-1\right], \tag{17}$$

if $z_{i0}\ge z_q\equiv z_s+\Delta_{\scriptscriptstyle M}(n_0)$. Here $s=\hat{s}$ when $\widetilde{n_0}(z)=n_0$, and $\delta s\equiv s_{i0}-s(\xi_0)$. If the **trapping condition** $s_i^m\equiv s_m+\delta s<0$ is fulfilled, then $\exists\xi_f>\xi_0$ such that $\hat{s}_i(\xi_f)=0$, $s'(\xi_f)<0$, $\hat{t}(\xi_f)=\infty$; the e^- is **trapped in a trough of the PW and accelerated**: for $\xi\simeq\xi_f$ we have $\hat{s}_i(\xi)\simeq|s'(\xi_f)|\,(\xi_f-\xi)=M\,|\Delta(\xi_f)|\,(\xi_f-\xi)$,

$$\hat{z}_i(\xi)\simeq\frac{1}{2\left[M\Delta(\xi_f)\right]^2(\xi_f-\xi)}\xrightarrow{\xi\to\xi_f}\infty. \tag{18}$$

Solving (18) for $\xi_f-\xi$ we express $\hat{s}_i, \hat{\gamma}_i$ vs. z_i:

$$\gamma_i=\frac{1}{2s_i}+\frac{s_i}{2}\simeq F\,\frac{z_i}{\lambda}\xrightarrow{z_i\to\infty}\infty, \tag{19}$$

$F\equiv M\lambda\,|\Delta(\xi_f)|$. In this model the PW phase velocity is c, trapped test e^- *cannot dephase*, **their energy grows $\propto$ travelled distance**. (19) is reliable where pulse depletion is negligible, $0\le z_i\le z_{pd}$. Fixed z_i, n_0, if ξ_0, z_0, s_0 lead to $\delta s=-1$, then $|\Delta(\xi_f)|=|\Delta_m|$, and γ_i is maximized:

$$\gamma_i(z_i,n_0)\simeq\sqrt{j(\nu)}\;z_i/\lambda; \tag{20}$$

here $j(\nu)\equiv 8\pi^2\nu\left[\bar{h}(\nu)-1\right]$, and $\bar{h}(\nu)$ is the final e^- energy transfered by the pulse if $\widetilde{n_0}(z)=n_0$, vs. $\nu\equiv n_0/n_{cr}$. Our 4-steps optimization procedure:

Step A: Computing $\bar{\xi}_{\scriptscriptstyle H}(\nu)$, $\bar{h}(\nu)$, $j(\nu)$ for the given pulse. Done in few seconds using *Mathematica*. In Fig. 3 we plot $\bar{h}(\nu), j(\nu)$ and their maxima ν_h, ν_j for the pulse of Fig. 2.a.

Step B: Optimal choice for the plateau density n_0. If the plasma longitudinal thickness available for WFA is $z_i\le z_{pd}(\nu_j)$, choose $\nu=\nu_j$:

$$\gamma_i^{\scriptscriptstyle M}(z_i)\simeq\sqrt{j(\nu_j)}\,z_i/\lambda. \tag{21}$$

Step C: Optimal linear down-ramp for self-injection, LWFA. For all Z, all Z e^- co-move. We stick to linear downramps

$$\widetilde{n_0}(z)=n_0+\Upsilon(z-z_s), \qquad z_b\le z\le z_s, \tag{22}$$

$\Upsilon=\frac{n_0-n_b}{z_s-z_b}<0$. Let (ξ_{br}, Z_{br}) be the pair (ξ,Z) with $Z\in[z_b,z_s]$ and the smallest ξ such that $\hat{J}(\xi,Z)=0$. For $\xi>\xi_{br}$ a bunch of $Z\sim Z_{br}$ electron layers start breaking the PW locally. $\hat{P}(\xi,Z_{br})$ fulfills (16). The Z_{br} layer earliest crosses other ones; at each $\xi>\xi_{br}$ it overshoots a new layer that up to ξ has evolved via (12a) and contributed to the PW. It does for ever, helped by their mutual repulsive forces; hence the Z_{br} are the fastest electrons injected and trapped in a trough of the PW by the first WB. Fixed $z_{i0}\ge z_q$, let $\xi_0>\xi_{br}$ be the 'instant' when $\hat{z}_e(\xi_0,Z_{br})=z_{i0}$. For $\xi\ge\xi_0$ $(\hat{z}_i,\hat{s}_i)\equiv\left(\hat{z}_e(\cdot,Z_{br}),\hat{s}(\cdot,Z_{br})\right)$ is given by (17) and has $s_i^m<0$. We determine parameters Υ, z_b requiring that: $P(\xi_0)$ is in the upper part of the cycle of Fig. 2.c, i.e. at $\xi=\xi_0$ the Z_{br} layer crosses plateau ones having negative velocity Δ'; δs as close as possible to -1, so that (20) applies.

Step D: choose an up-ramp growing from 0 to n_b in a short interval $0\le z\le z_b$ and preventing WB at $\xi<\xi_{br}$; that $\widetilde{n_0}(z)\simeq O(z^2)$ helps [3,4].

Applying our optimization procedure to the pulse of Fig. 2.a we obtain density and results of Fig. 4. If $\lambda=0.8\mu$m, $F=0.28$ leads to the remarkable energy gain $0.35mc^2\simeq 0.1785$MeV per μm.

If the pulse is cylindrically symmetric around $\vec{z}$ with waist R, by causality our results hold strictly in the causal cone (of axis $\vec{z}$, radius R) trailing the pulse, approximately in a neighbourhood thereof.

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