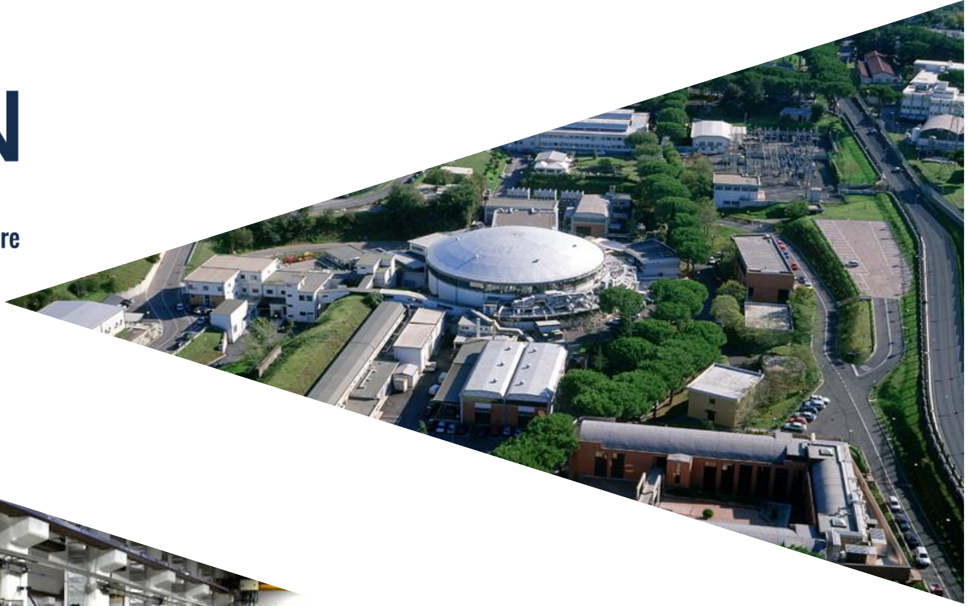
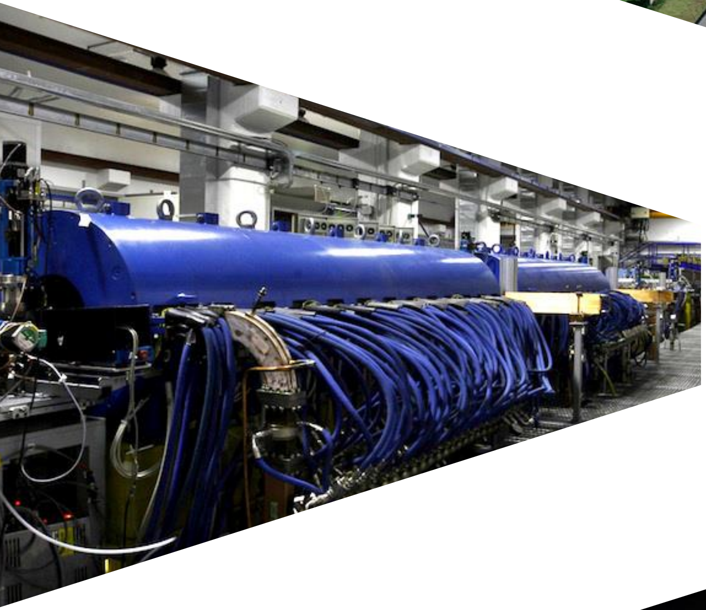




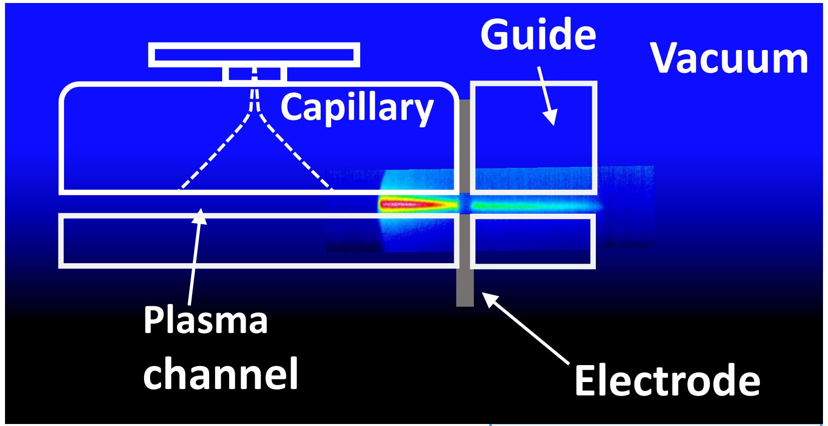
Evaluation of the transfer matrix of a plasma ramp with squared cosine shape via an approximate solution of Mathieu differential equation

Stefano Romeo *stefano.romeo@lnf.infn.it*

Island of Elba 2023-09-18

6th European Advanced Accelerator Concepts Workshop

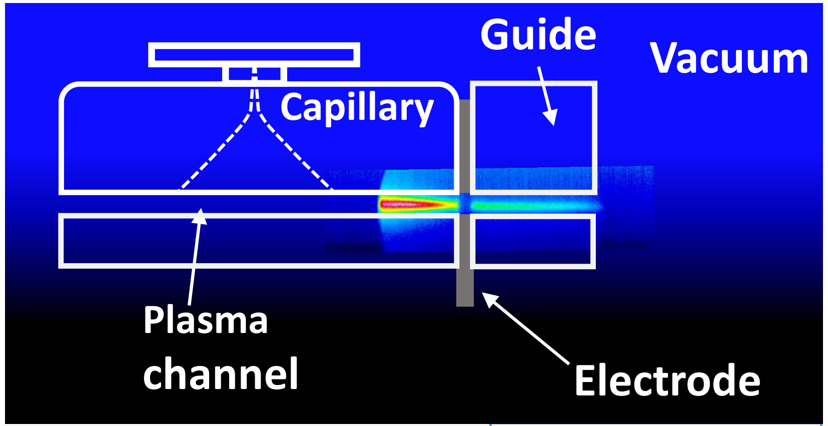
Outline: Plasma transport line optimization



Courtesy of A. Biagioni

- At the edge of discharge capillaries, regions with a gradually decreasing plasma density form, connecting the plasma plateau to the vacuum

Outline: Plasma transport line optimization



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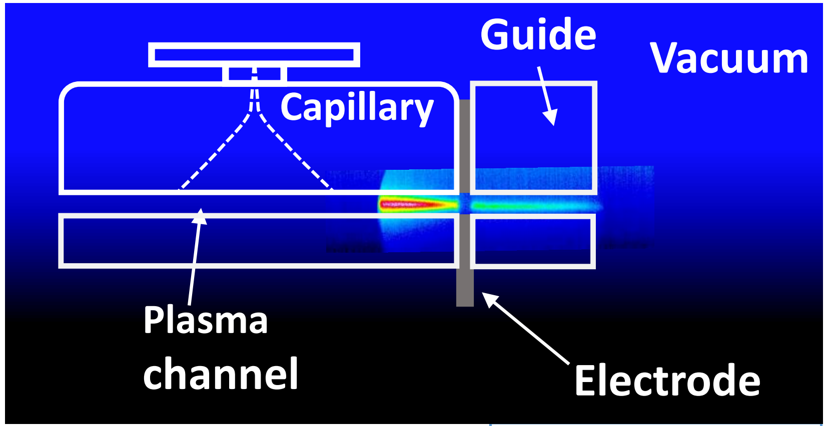

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High sensitivity to transverse instability!

$$\sigma_0 = \sqrt[4]{\frac{2}{\gamma}}\sqrt{\frac{\varepsilon_n}{k_p}}$$

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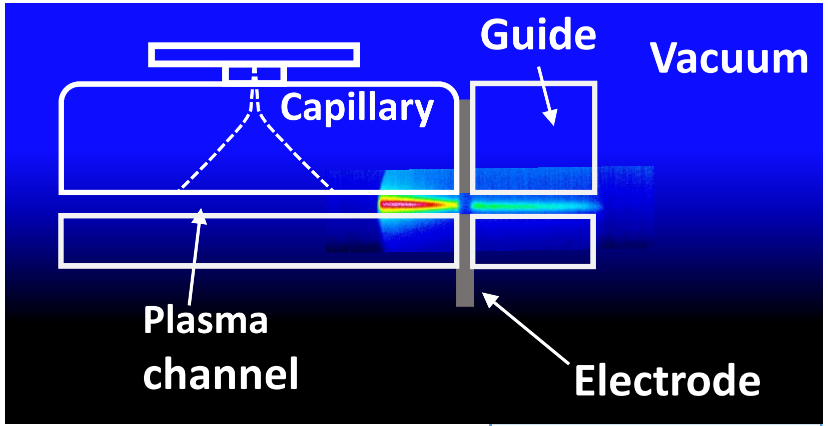
Outline: Plasma transport line optimization



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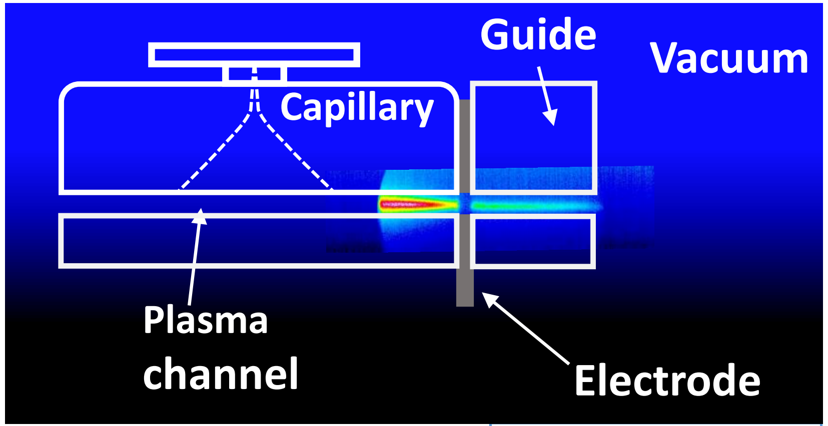

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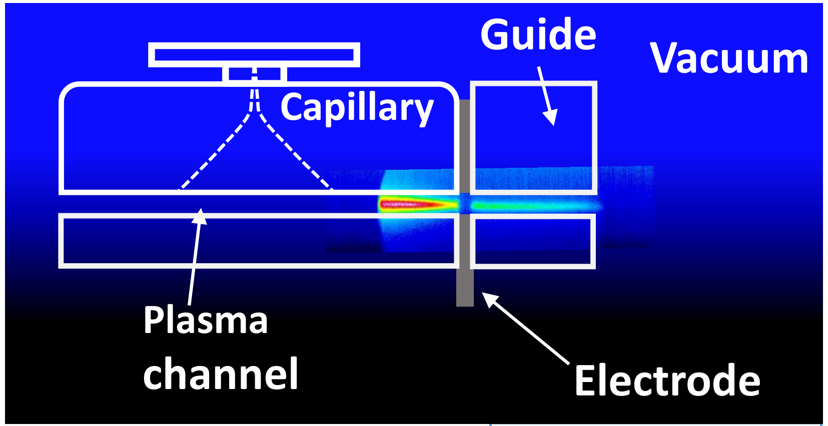

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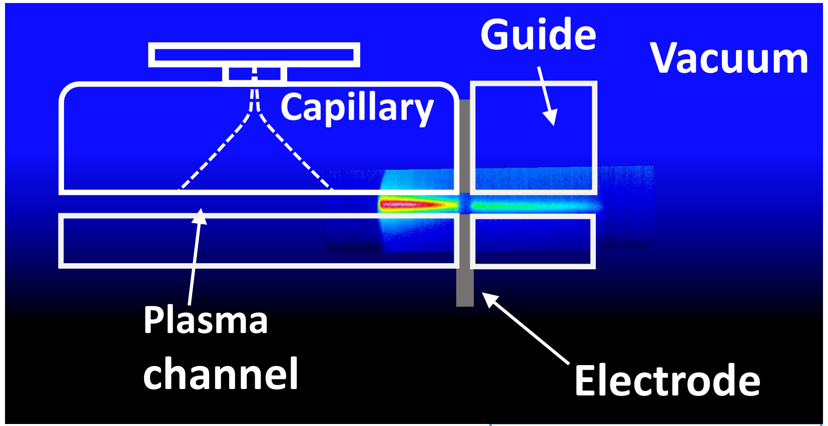
Which plasma ramp?

Numerical solution:
For any working point perform several simulation scans

Outline: Plasma transport line optimization



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Analytical solution:
Find a general rule for a suitable shape

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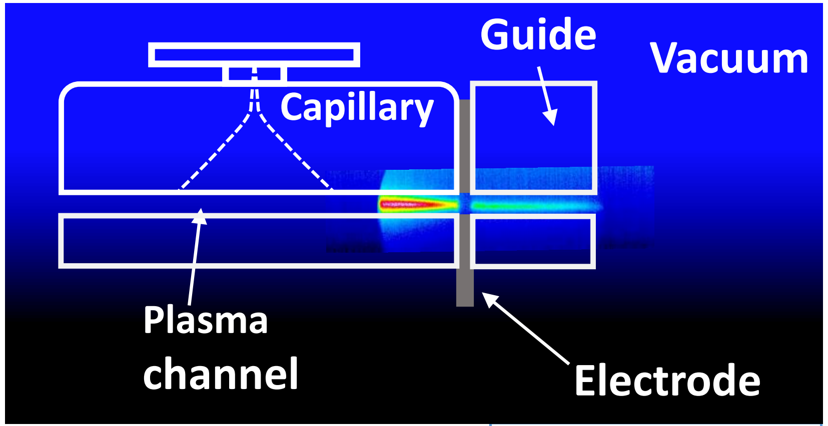
For any working point perform several simulation scans

Outline: **Plasma transport line optimization**

EuPRAXIA



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Which plasma ramp?

Evaluation of ramp transfer matrix

Analytical solution:
Find a general rule for a suitable shape

Numerical solution:
For any working point perform several simulation scans

Theory: Plasma focusing strength

$$\beta''(z) + 2k_{ext}^2(z)\beta(z) = \frac{2}{\beta(z)} + \frac{[\beta'(z)]^2}{2\beta(z)}$$

Envelope equation

Plasma focusing strength?

$$k_{ext}^2(z) = \frac{k_p^2(z)}{2\gamma}$$

Ion column model
(only the particles inside the bubble)

- ❑ $\beta(z) = \frac{\sqrt{2\gamma}}{k_p}$
- ❑ $\alpha(z) = 0$
- ❑ $\gamma(z) = \frac{1}{\beta(z)}$
- ❑ $\beta''(z) = 0$

Matching conditions on the plateau

Theory: Plasma focusing strength

$$\beta''(z) + 2k_{ext}^2(z)\beta(z) = \frac{2}{\beta(z)} + \frac{[\beta'(z)]^2}{2\beta(z)}$$

Envelope equation

Neglecting constants, the focusing force inside a plasma ramp depends only from the plasma density, with the energy taken as a parameter

$$k_{ext}^2(z) = \frac{e^2}{2\epsilon_0 m_e c^2}\frac{n_p(z)}{\gamma} = f(z;\gamma)$$

Plasma focusing strength?

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Matching conditions on the plateau

Method: **Hill's Equation**

$$y''(z) + f(z)y(z) = 0$$

Hill's Equation

$$k_{ext}^2(z) = f(z) \propto n_p(z)$$

Negligible deceleration

$f(z)$ is a continuous and derivable function

$$\begin{bmatrix} \beta(z) \\ \alpha(z) \\ \gamma(z) \end{bmatrix} = \begin{bmatrix} C^2 & -2CS & S^2 \\ -CC' & CS' + C'S & -SS' \\ C'^2 & -2C'S' & S'^2 \end{bmatrix} \begin{bmatrix} \beta_0 \\ 0 \\ \frac{1}{\beta_0} \end{bmatrix}$$

The computation is performed over the inverse matrix in order to have simplified matching conditions

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Hill's Equation

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Negligible deceleration

$f(z)$ is a continuous and derivable function

- $C^2(0)\beta_0 + \frac{S^2(0)}{\beta_0} = \beta_0$
- $C'^2(0)\beta_0 + \frac{S'^2(0)}{\beta_0} = \frac{1}{\beta_0}$
- $-C(0)C'(0)\beta_0 - \frac{S(0)S'(0)}{\beta_0} = 0$
- $\beta''(0) = 0$
- Liouville's Theorem

$$\begin{bmatrix} \beta(z) \\ \alpha(z) \\ \gamma(z) \end{bmatrix} = \begin{bmatrix} C^2 & -2CS & S^2 \\ -CC' & CS' + C'S & -SS' \\ C'^2 & -2C'S' & S'^2 \end{bmatrix} \begin{bmatrix} \beta_0 \\ 0 \\ \frac{1}{\beta_0} \end{bmatrix}$$

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Ramps with an even functional form guarantee the possibility to respect matching conditions

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Framework: Squared Cosine plasma ramp

Solutions are continuous and derivable

One solution is even, the other is odd

If $f(z)$ is even Hill's equation becomes a symmetric differential equation

Ramps with an even functional form guarantee the possibility to respect matching conditions

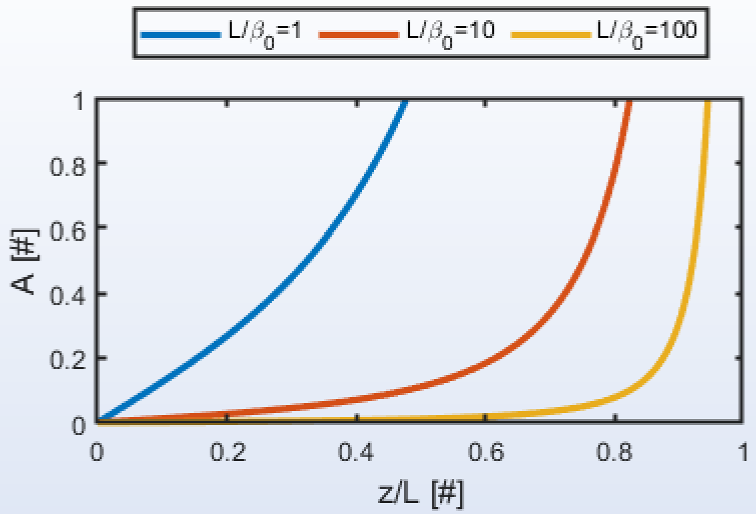
We can normalize the solutions such as $C(0) = 1$ and $S'(0) = 1$

THE CHOICE

$$f(z) = \frac{1}{\beta_0^2}\cos^2\left(\frac{\pi z}{2L}\right)$$

- ❑ Realistic ramp
- ❑ No discontinuity at any derivative order
- ❑ Partially concave, partially convex
- ❑ Never adiabatic

$$\mathcal{A} = \frac{1}{4f^{3/2}(z)}\left|\frac{df(z)}{dz}\right|$$



Expansion: Mathieu Equation

Hill's Equation

$$y''(z) + \frac{1}{\beta_0^2}\cos^2\left(\frac{\pi z}{2L}\right) y(z) = 0$$

Mathieu Equation

$$y''(\xi) + \frac{2L^2}{\pi^2\beta_0^2}[1 + \cos(2\,\xi)]y(\xi) = 0$$

Expansion: Mathieu Equation

Hill's Equation

$$y''(z) + \frac{1}{\beta_0^2}\cos^2\left(\frac{\pi z}{2L}\right)y(z) = 0$$

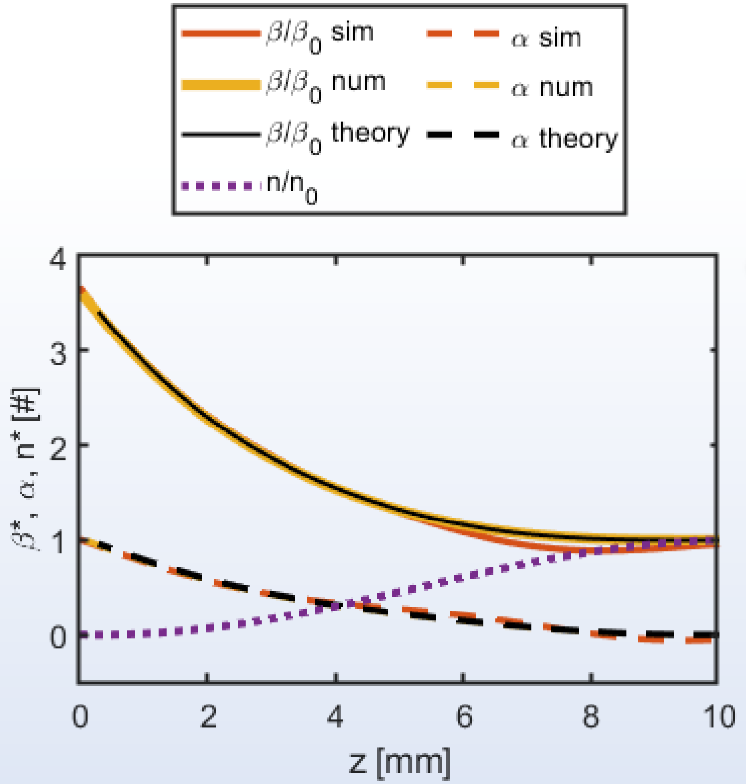
Mathieu Equation

$$y''(\xi) + \frac{2L^2}{\pi^2\beta_0^2}[1+\cos(2\,\xi)]y(\xi) = 0$$

Even and odd normalized Mathieu Functions

$$C(z) = \frac{Ce_n\left(\frac{2L^2}{\pi^2\beta_0^2}, -\frac{L^2}{\pi^2\beta_0^2}, \frac{\pi z}{2L}\right)}{Ce_n\left(\frac{2L^2}{\pi^2\beta_0^2}, -\frac{L^2}{\pi^2\beta_0^2}, 0\right)}$$

$$S(z) = \frac{2L}{\pi}\frac{Se_n\left(\frac{2L^2}{\pi^2\beta_0^2}, -\frac{L^2}{\pi^2\beta_0^2}, \frac{\pi z}{2L}\right)}{Se_n'\left(\frac{2L^2}{\pi^2\beta_0^2}, -\frac{L^2}{\pi^2\beta_0^2}, 0\right)}$$



Simplify: Normalized transfer matrix

Normalization

$$R = \frac{L}{\pi\beta_0}$$

Geometric factor

$$c_n = C\left(\frac{\pi}{2}\right) = \frac{Ce_n\left(2R^2, -R^2, \frac{\pi}{2}\right)}{Ce_n(2R^2, -R^2, 0)}$$

$$s_n = \frac{S\left(\frac{\pi}{2}\right)}{\beta_0} = 2R\frac{Se_n\left(2R^2, -R^2, \frac{\pi}{2}\right)}{Se_n'(2R^2, -R^2, 0)}$$

$$c'_n = C'\left(\frac{\pi}{2}\right)\beta_0 = \frac{1}{2R}\frac{Ce_n'\left(2R^2, -R^2, \frac{\pi}{2}\right)}{Ce_n(2R^2, -R^2, 0)}$$

$$s'_n = S'\left(\frac{\pi}{2}\right) = \frac{Se_n'\left(2R^2, -R^2, \frac{\pi}{2}\right)}{Se_n'(2R^2, -R^2, 0)}$$

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$$s'_n = S'\left(\frac{\pi}{2}\right) = \frac{Se_n'\left(2R^2, -R^2, \frac{\pi}{2}\right)}{Se_n'(2R^2, -R^2, 0)}$$

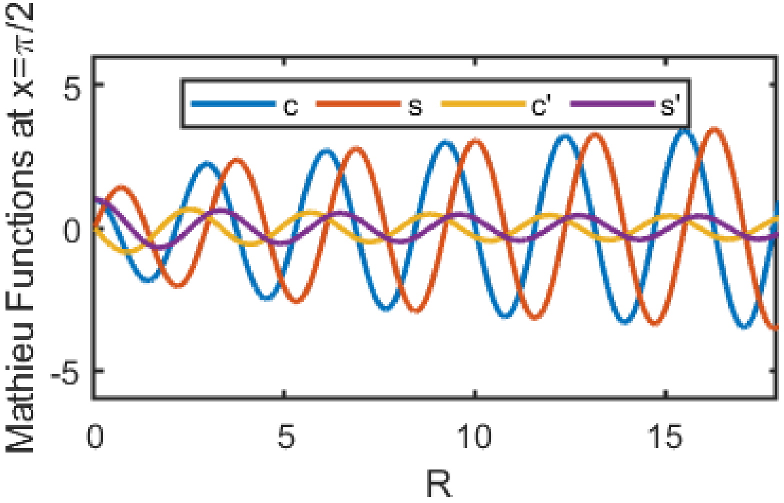
Extraction

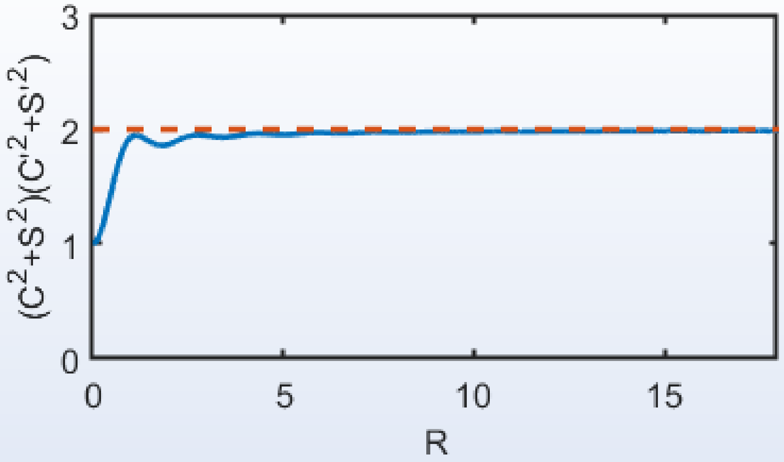
$$\begin{bmatrix} \beta_1/\beta_0 \\ \alpha_1 \\ \gamma_1\beta_0 \end{bmatrix} = \begin{bmatrix} c^2 & -2cs & s^2 \\ -cc' & cs' + c's & -ss' \\ c'^2 & -2c's' & s'^2 \end{bmatrix} \begin{bmatrix} \beta_2/\beta_0 \\ \alpha_2 \\ \gamma_2\beta_0 \end{bmatrix}$$

Normalized transfer matrices

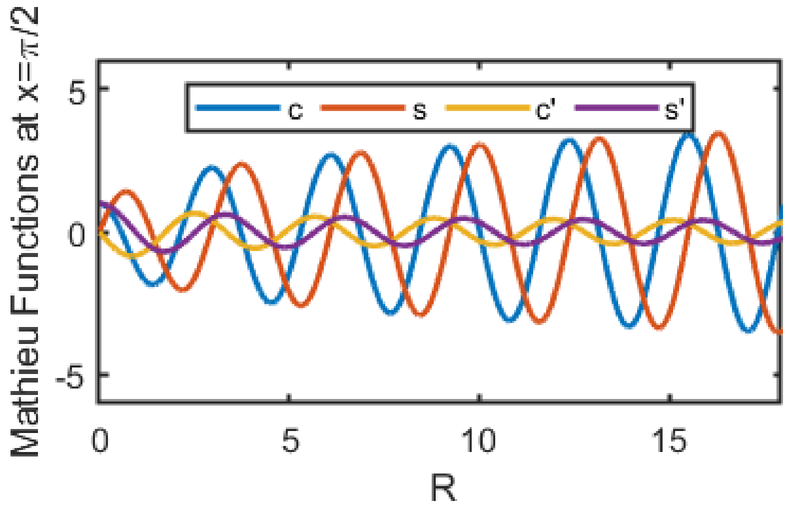
Injection

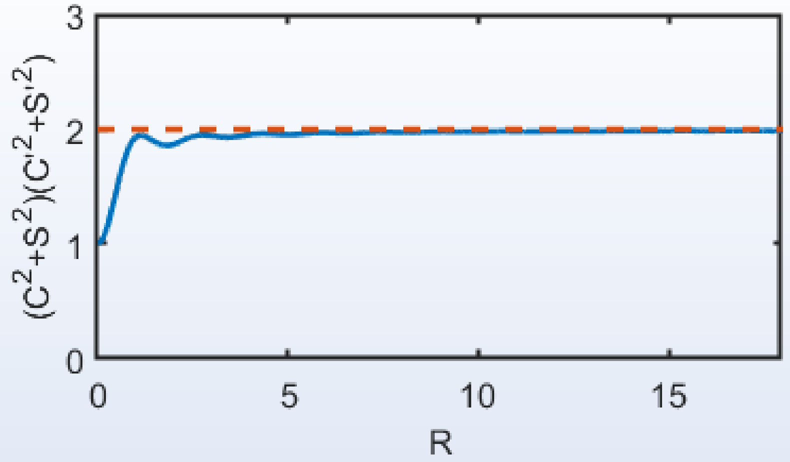
$$\begin{bmatrix} \beta_2/\beta_0 \\ \alpha_2 \\ \gamma_2\beta_0 \end{bmatrix} = \begin{bmatrix} s'^2 & -2ss' & s^2 \\ -c's' & cs' + c's & -cs \\ c'^2 & -2cc' & c^2 \end{bmatrix} \begin{bmatrix} \beta_1/\beta_0 \\ \alpha_1 \\ \gamma_1\beta_0 \end{bmatrix}$$


Mathieu Functions at x=π/2
c
s
c'
s'
5
0
-5
0
5
10
15
R


(C2+S2)(C'2+S'2)
3
2
1
0
0
5
10
15
R

Approximation: Behavior of Mathieu functions





- **Sinusoidal behavior**
- **Phase delay between Mathieu functions and their first derivative**
- **The product $\left(c_n^2 + s_n^2\right)\left({c'}_n^2 + {s'}_n^2\right)$ saturates for $R \gg 1$**
- **Transfer matrix is unimodular (Liouville's theorem)**

$$c_n \approx a\,R^b \cos(2R + \omega)$$

$$s_n \approx a\,R^b \sin(2R + \omega)$$

$$c'_n \approx -\frac{1}{a}\frac{1}{\cos^2\omega - \sin^2\omega} R^{-b} \sin(2R - \omega)$$

$$s'_n \approx \frac{1}{a}\frac{1}{\cos^2\omega - \sin^2\omega} R^{-b} \cos(2R - \omega)$$

Alignment: Evaluation of the constants

$$(c_n^2 + s_n^2)(c'^2_n + s'^2_n) = \left(\frac{1}{\cos^2\omega - \sin^2\omega}\right)^2 \approx \mathbf{2}$$

$$\boldsymbol{\omega = \frac{\pi}{8}}$$

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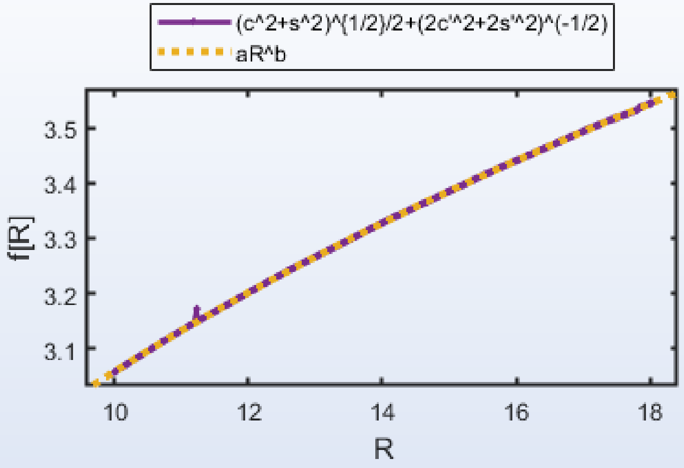
$$\boldsymbol{\omega = \frac{\pi}{8}}$$

$$\frac{1}{2}\sqrt{c_n^2 + s_n^2} + \frac{1}{2}\sqrt{\frac{2}{c'^2_n + s'^2_n}} = aR^b$$



$$\boldsymbol{a = \sqrt{3}}$$
$$\boldsymbol{b = \frac{1}{4}}$$

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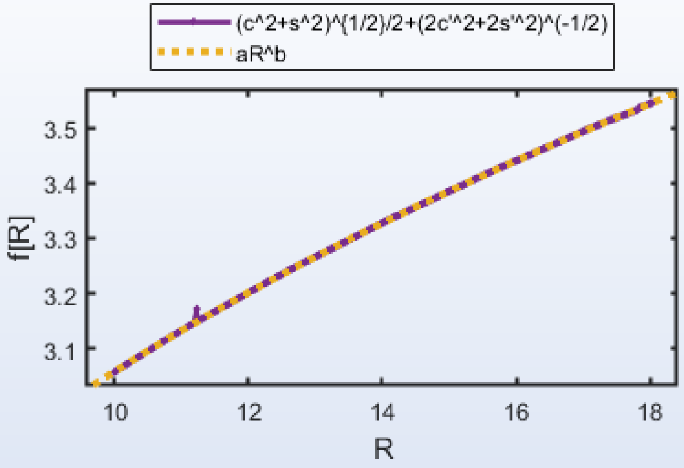
$$\frac{1}{2}\sqrt{c_n^2 + s_n^2} + \frac{1}{2}\sqrt{\frac{2}{c'^2_n + s'^2_n}} = aR^b$$

$$c_n \approx \sqrt{3}\,R^{\frac{1}{4}}\cos\left(2R + \frac{\pi}{8}\right)$$

$$s_n \approx \sqrt{3}\,R^{\frac{1}{4}}\sin\left(2R + \frac{\pi}{8}\right)$$

$$c'_n \approx -\sqrt{\frac{2}{3}}\,R^{-\frac{1}{4}}\sin\left(2R - \frac{\pi}{8}\right)$$

$$s'_n \approx \sqrt{\frac{2}{3}}\,R^{-\frac{1}{4}}\cos\left(2R - \frac{\pi}{8}\right)$$



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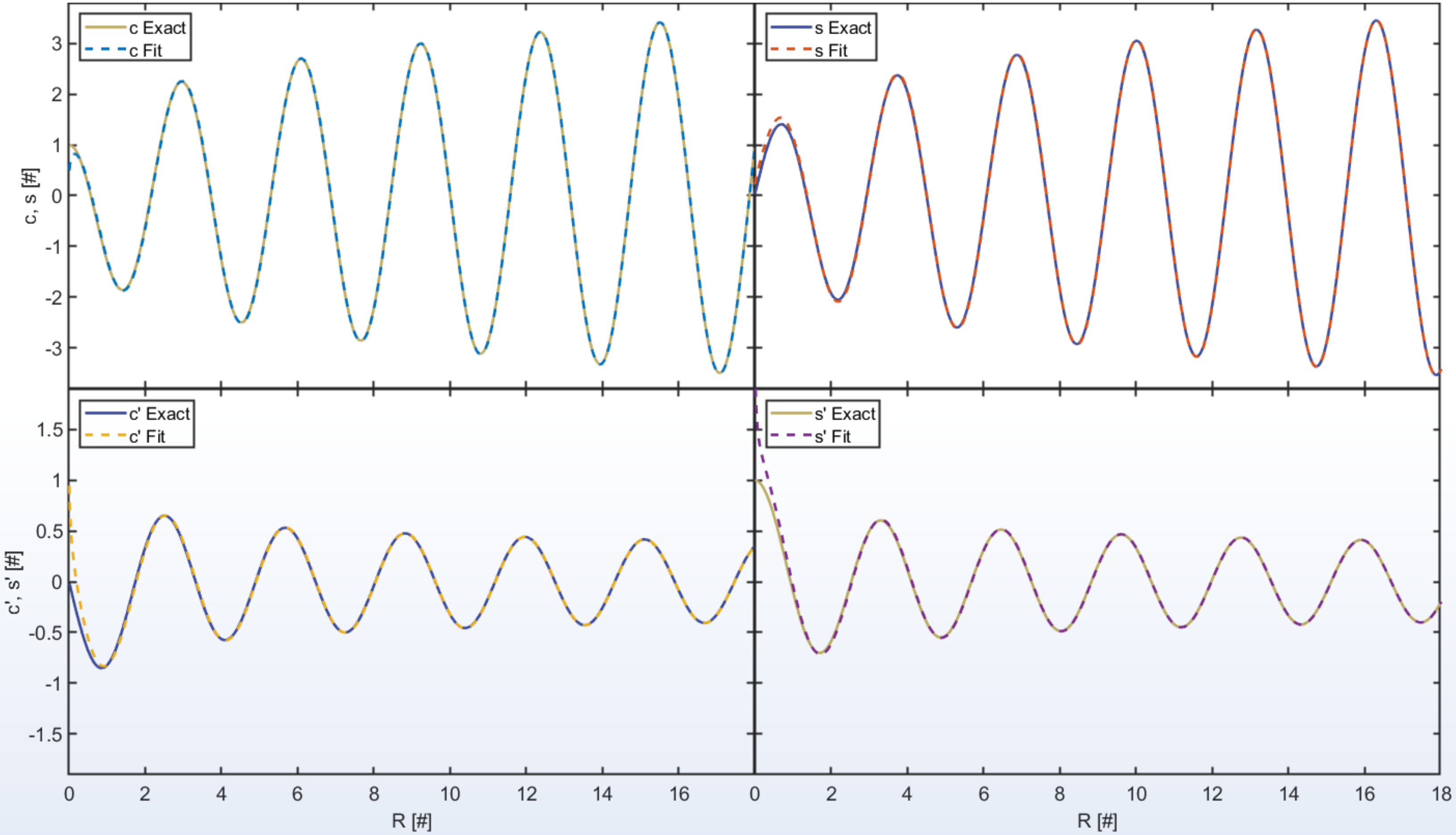
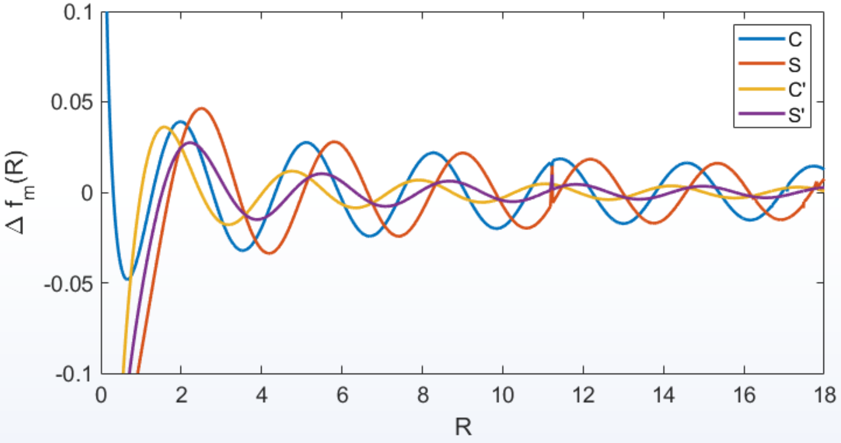

c Exact
c Fit
s Exact
s Fit
c' Exact
c' Fit
s' Exact
s' Fit
c, s [#]
c', s' [#]
R [#]
R [#]
3
2
1
0
-1
-2
-3
1.5
1
0.5
0
-0.5
-1
-1.5
0
2
4
6
8
10
12
14
16
18


C
S
C'
S'
Δ f_m(R)
R
0.1
0.05
0
-0.05
-0.1
0
2
4
6
8
10
12
14
16
18

Application: Stabilizing effect of the ramp

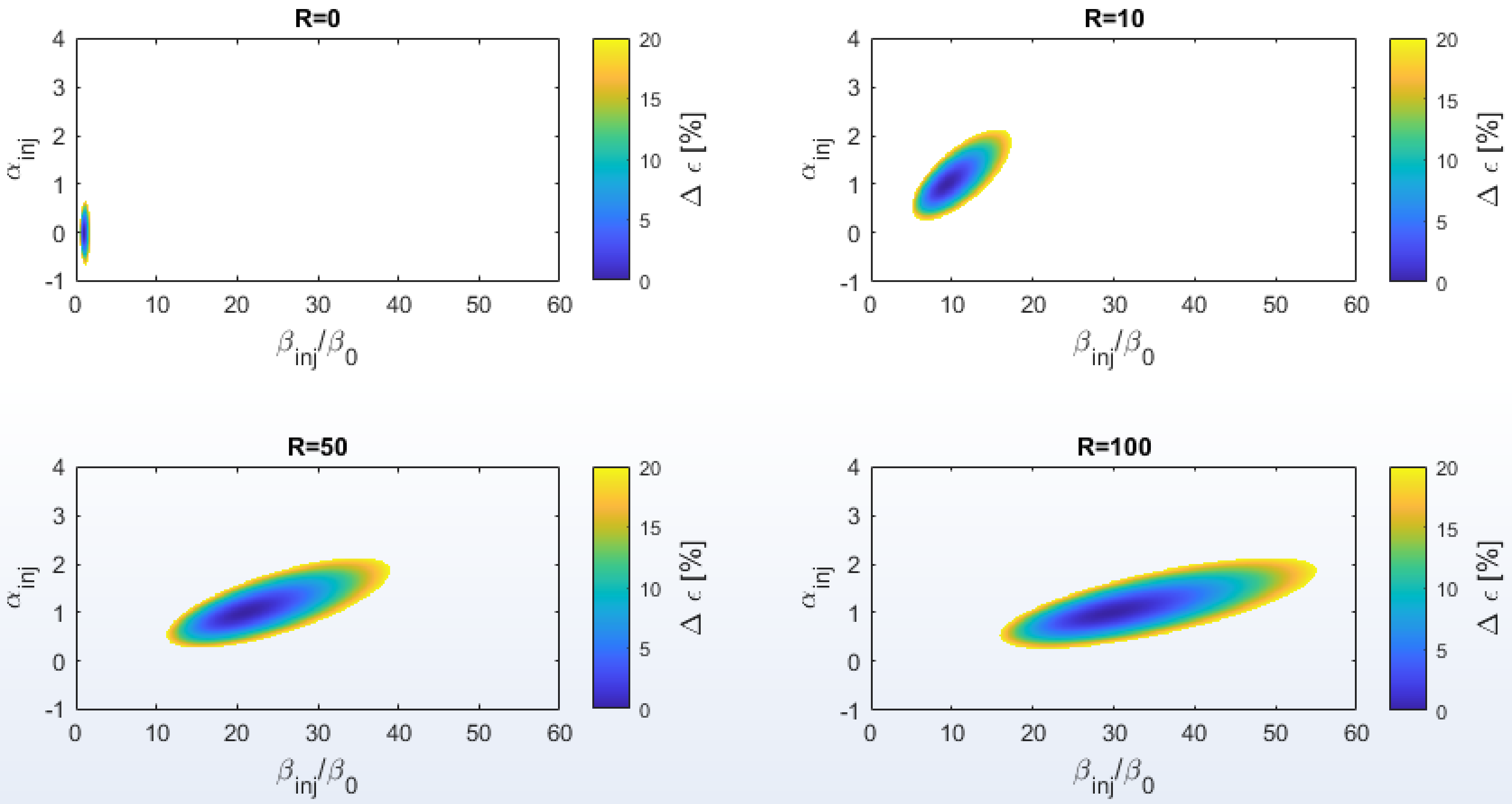
Transfer matrix

$$\begin{bmatrix}\beta^* \\ \alpha \\ \gamma^*\end{bmatrix} = \boldsymbol{M}\begin{bmatrix}\beta_{inj}/\beta_0 \\ \alpha_{inj} \\ \gamma_{inj}\beta_0\end{bmatrix}$$

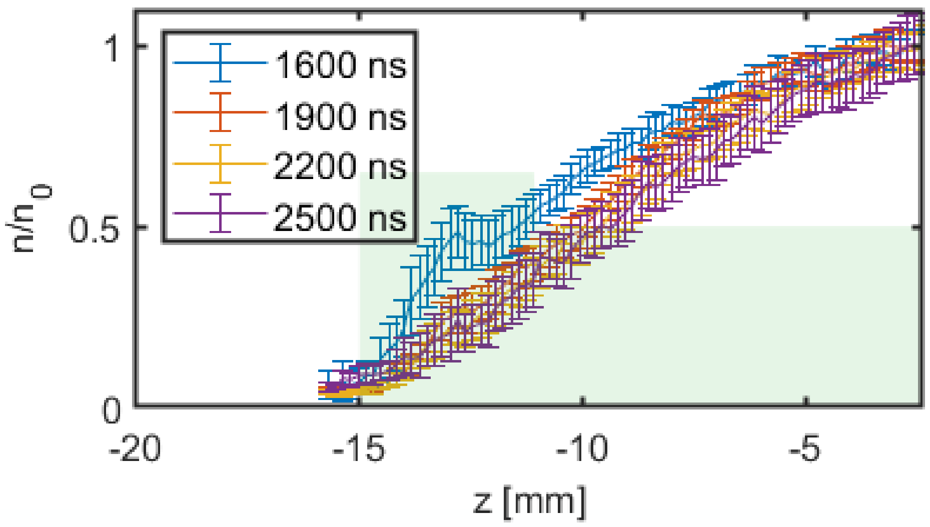
Mehrling-Floettmann equation for emittance growth

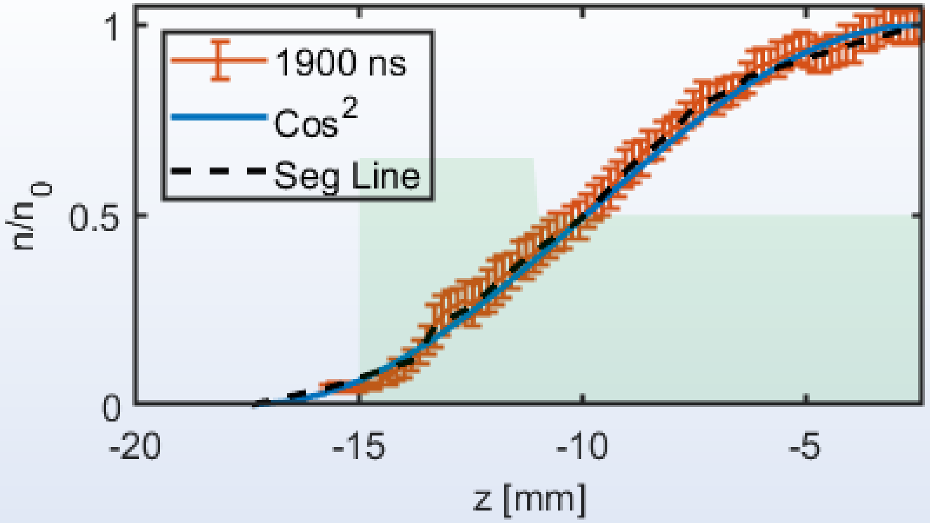
$$\varepsilon_{n,fin} = \frac{\varepsilon_{n,inj}}{2}\left(\frac{1+\alpha^2}{\beta^*} + \beta^*\right)$$

Mehrling, T., et al. "Transverse emittance growth in staged laser-wakefield acceleration." *Physical Review Special Topics-Accelerators and Beams* 15.11 (2012): 111303.



Perspective: **Experimental ramps**





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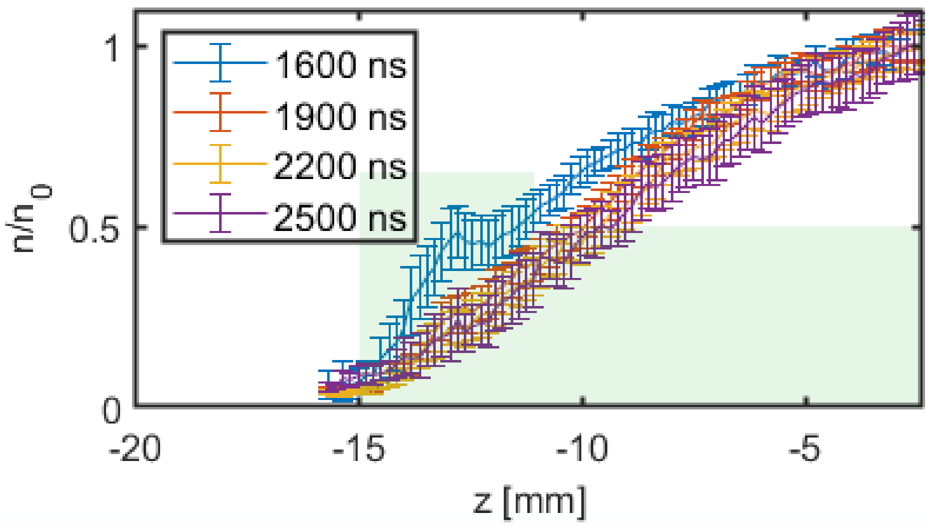
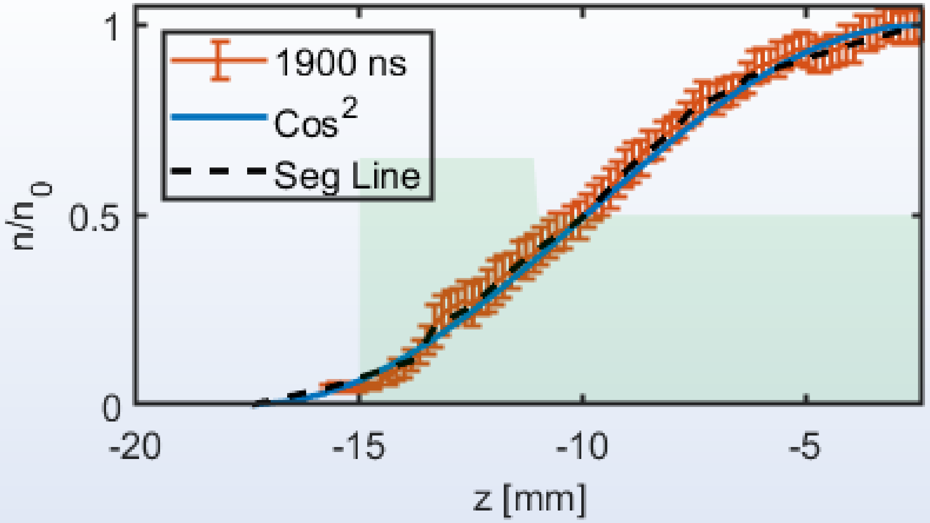
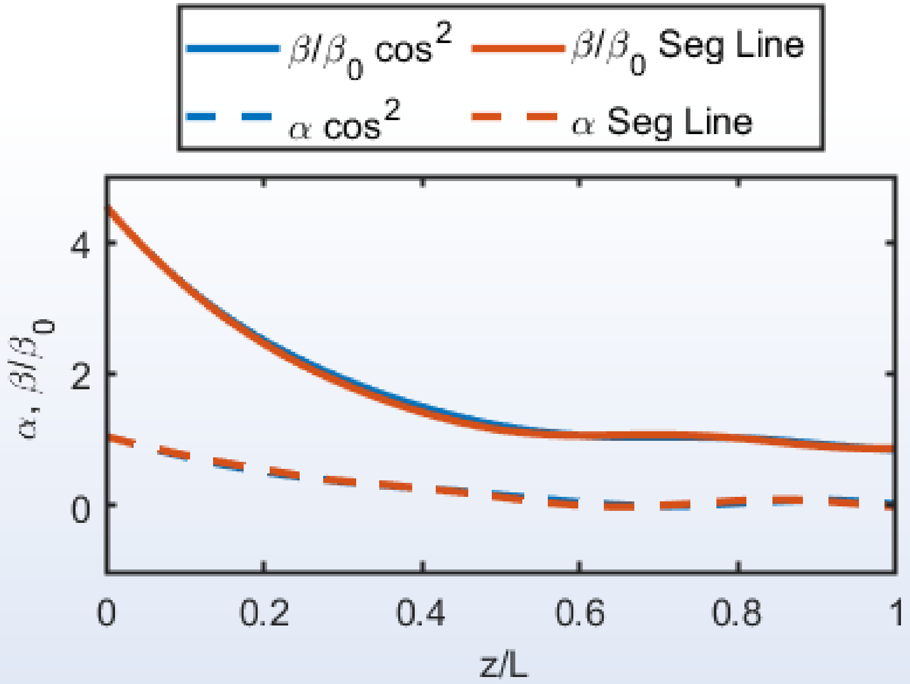


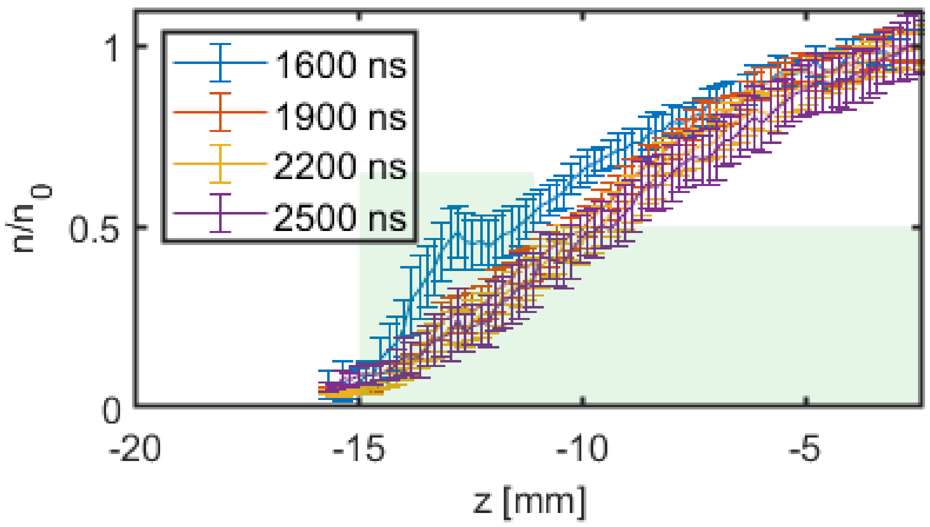


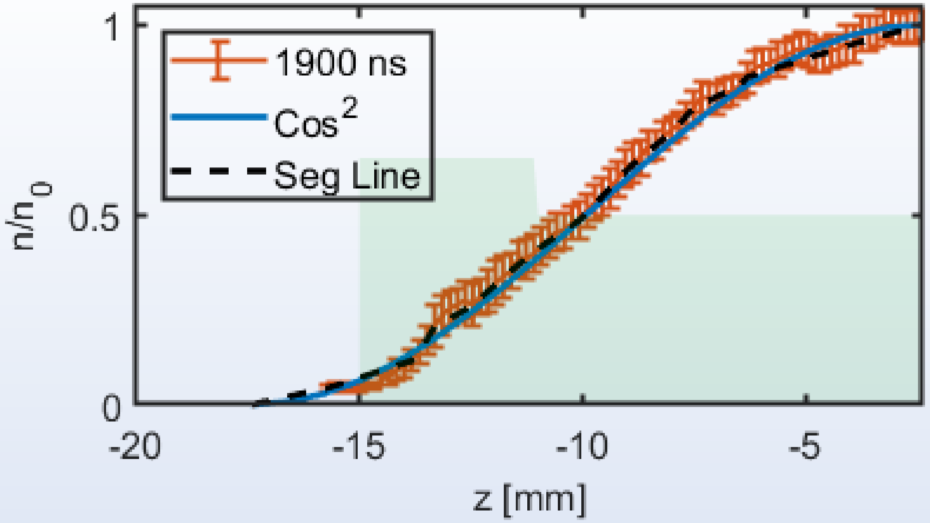
Plasma ramps with squared cosine shape were actually realized at the plasma lab of SPARC_LAB facility by means of tapered capillaries

The measured shapes were replied in numerical simulations by means of segmented lines and tested in comparison with analytical squared cosine shape



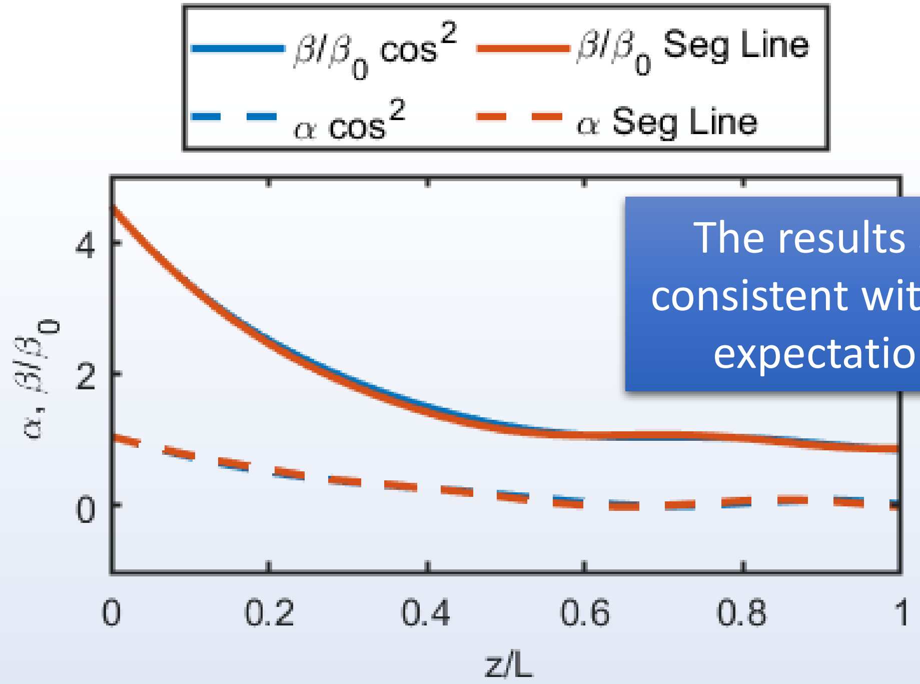
Perspective: **Experimental ramps**





Plasma ramps with squared cosine shape were actually realized at the plasma lab of SPARC_LAB facility by means of tapered capillaries

The measured shapes were replied in numerical simulations by means of segmented lines and tested in comparison with analytical squared cosine shape



The results are consistent with the expectations

Conclusion: Summary

- We opted for a purely analytical approach trying to evaluate the transfer matrix of a plasma ramp
- We concluded that a ramp with a symmetric form could simplify the problem
- We opted for a ramp with a squared cosine shape
- We evaluated that this ramp is never adiabatic and its application lead to Mathieu Equation
- We compared the solution of the Mathieu Equation with actual envelope evolution
- We deduced that the ramp transfer matrix can be normalized to a geometric parameter
- We found an analytical approximation that allows to write the transfer matrix in a simple way

Tons of applications on their way!

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Accepted for publication!
Plasma Phisics and Controlled Fusion

Tons of applications on their way!

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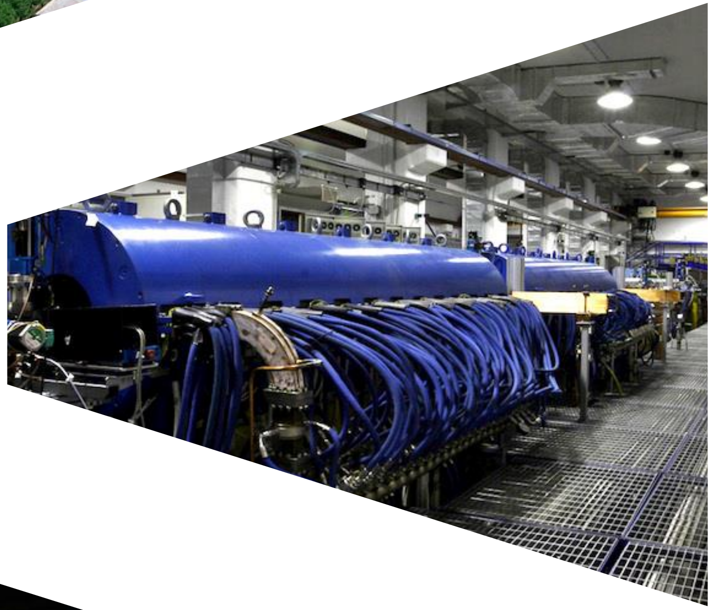
Thank you

for

the attention!



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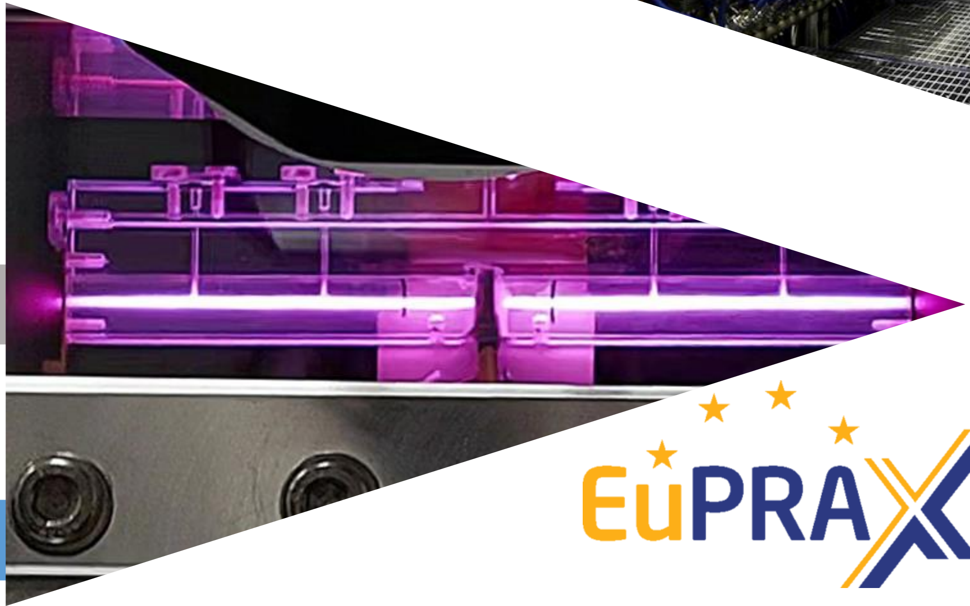

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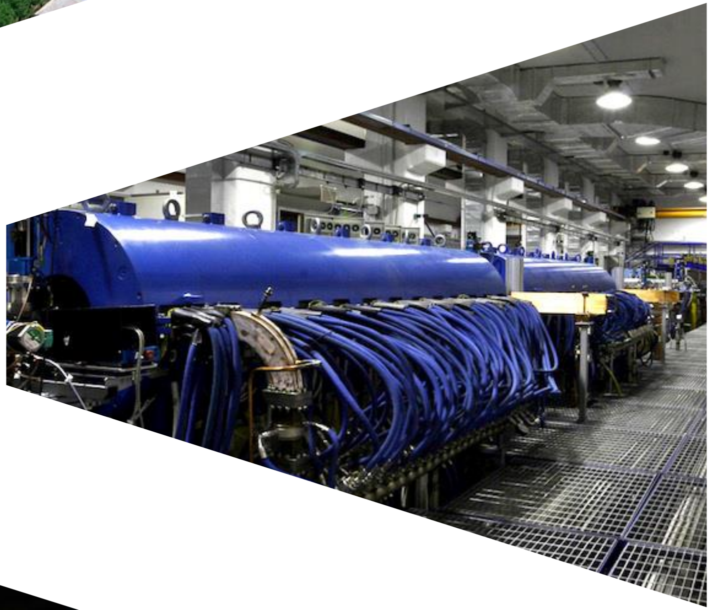

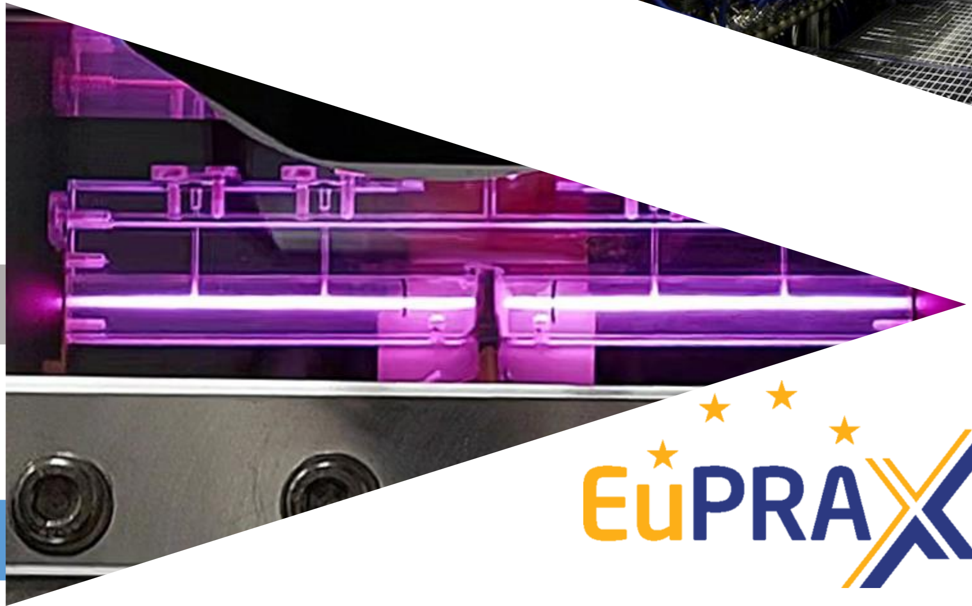

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