Energy-conserving theory of strongly nonlinear plasma wakefields

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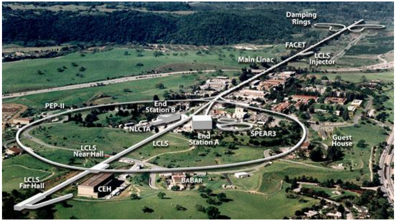
Paper: A. Golovanov et al. *PRL* **130**, 105001 (2023)

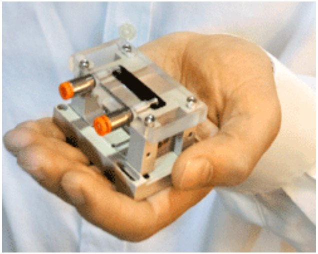
Stanford linear acceleration center (SLAC)



50 GeV at 3 km

Acceleration gradient ~100 MeV/m

Plasma accelerators

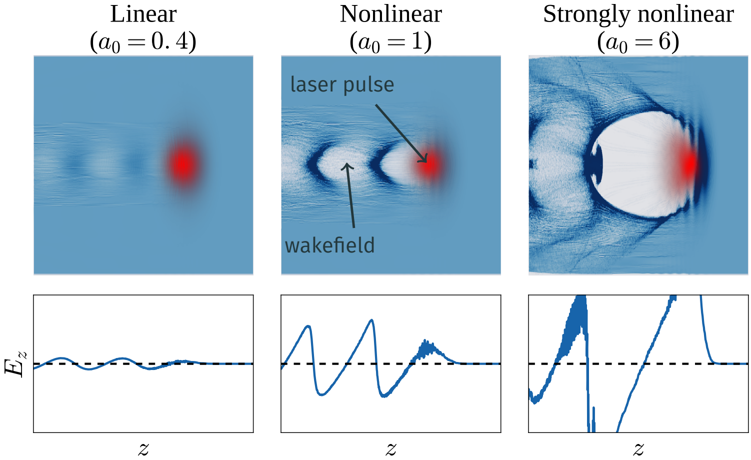


7.8 GeV at 20 cm.

Acceleration gradient ~50 GeV/m

Gonsalves *et al.* PRL **122**, 084801 (2019).

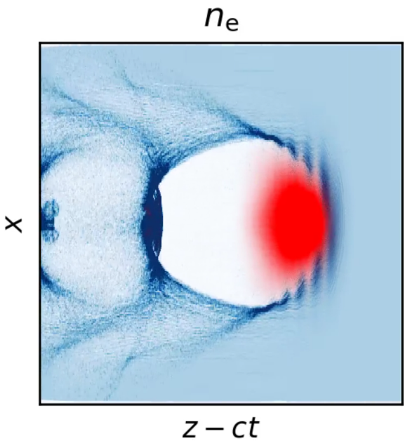
Wakefield in plasma



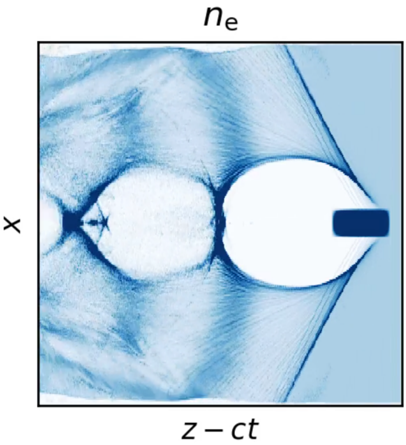
$a_0 = \dfrac{eE_L}{mc\omega_L}$ is the unitless laser amplitude

Strongly nonlinear (“bubble” or “blowout”) regime of wakefield

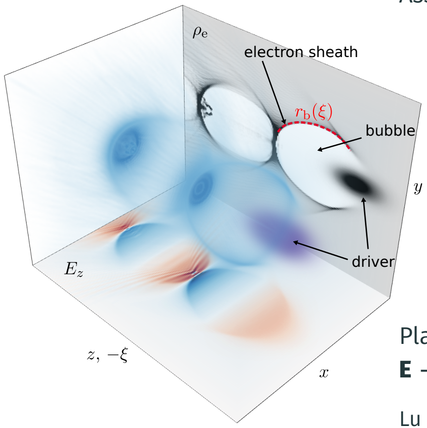
Laser driver ($a_0 \gg 1$)



Electron driver ($n_B \gg n_p$)



Phenomenological models of plasma wakefield



Assumptions

- Axial symmetry, coordinates (r, z)
- Quasi-static approximation: $f(t, z, r) = f(\xi, r), \quad \xi = t - z.$
- The plasma bubble has a border $r_b(\xi)$. For $r < r_b$, there are no plasma electrons; $r \geq r_b$ lies the electron sheath
- Ions are immobile

Plasma units: $t \to \omega_p t$, $\mathbf{r} \to k_p \mathbf{r}$, $\mathbf{E} \to e\mathbf{E}/mc\omega_p$, etc.

Lu et al. *Phys. Plasmas* **13**, 056709 (2006)

Plasma boundary equation

The boundary of the bubble is described by the equation

$$A(r_\mathrm{b})\frac{\mathrm{d}^2 r_\mathrm{b}}{\mathrm{d}\xi^2} + B(r_\mathrm{b})\left(\frac{\mathrm{d}r_\mathrm{b}}{\mathrm{d}\xi}\right)^2 + C(r_\mathrm{b}) = \lambda(\xi, r_\mathrm{b}),$$

where

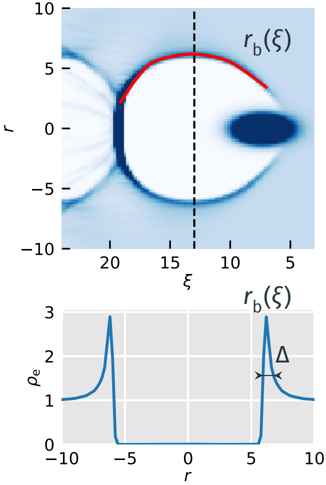
$$A(r_\mathrm{b}) = r_\mathrm{b}\left(1 + \frac{r_\mathrm{b}^2}{4} + \frac{3r_\mathrm{b}\Delta}{4}\right)$$

$$B(r_\mathrm{b}) = \frac{r_\mathrm{b}^2}{2}\left(1 + \frac{\Delta}{r_\mathrm{b}}\right)$$

$$C(r_\mathrm{b}) = \frac{r_\mathrm{b}^2}{4}\,\frac{1 + (1 + r_\mathrm{b}\Delta/2)^2}{(1 + r_\mathrm{b}\Delta/2)^2}$$

$$\lambda(\xi, r_\mathrm{b}) = -\int_0^{r_\mathrm{b}} \rho_\mathrm{e}(\xi, r')r'\,\mathrm{d}r'$$

$$E_z(\xi) = \frac{r_\mathrm{b} + \Delta}{2}\,\frac{\mathrm{d}r_\mathrm{b}}{\mathrm{d}\xi}$$



Lu et al. *Phys. Plasmas* **13**, 056709 (2006); Golovanov et al. *Quantum electron.* **46**, 295 (2016).

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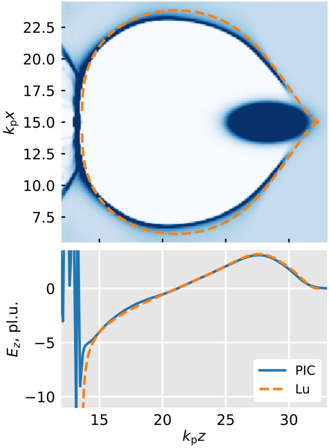

Lu et al. *Phys. Plasmas* **13**, 056709 (2006); Golovanov et al. *Quantum electron.* **46**, 295 (2016).

EM field (wakefield)

$$\frac{\partial W_{\mathrm{EM}}}{\partial t} + \nabla \cdot \mathbf{S}_{\mathrm{EM}} = -\mathbf{j} \cdot \mathbf{E}$$

$W_{\mathrm{EM}} = \dfrac{\mathbf{E}^2 + \mathbf{B}^2}{2}$ EM energy density

$\mathbf{S}_{\mathrm{EM}} = \mathbf{E} \times \mathbf{B}$ Poynting vector

Energy conservation in a plasma wakefield

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Plasma electrons

$$\frac{\partial W_{\mathrm{e}}}{\partial t} + \nabla \cdot \mathbf{S}_{\mathrm{e}} = \mathbf{j}_{\mathrm{e}} \cdot \mathbf{E} + \frac{1}{2} \frac{\partial \langle \mathbf{a}^2 \rangle}{\partial t} n_{\mathrm{e}} \overline{\gamma^{-1}}$$

$W_{\mathrm{e}} = n_{\mathrm{e}} \overline{(\gamma - 1)}$ electron energy density

$\mathbf{S}_{\mathrm{e}} = n_{\mathrm{e}} \overline{\mathbf{v}(\gamma - 1)}$ energy density current

$\langle \mathbf{a}^2 \rangle$ ponderomotive potential

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$\langle \mathbf{a}^2 \rangle$ ponderomotive potential

Total wake energy

$$\frac{\partial W}{\partial t} + \nabla \cdot \mathbf{S} = -\mathbf{j}_{\mathrm{B}} \cdot \mathbf{E} + \frac{1}{2} \frac{\partial \langle \mathbf{a}^2 \rangle}{\partial t} n_{\mathrm{e}} \overline{\gamma^{-1}}$$

$$W = W_{\mathrm{EM}} + W_{\mathrm{e}} \quad \mathbf{S} = \mathbf{S}_{\mathrm{EM}} + \mathbf{S}_{\mathrm{e}}$$

Currents $\mathbf{j} = \mathbf{j}_{\mathrm{e}} + \mathbf{j}_{\mathrm{B}}$ belong to plasma electrons ($\mathbf{j}_{\mathrm{e}}$) or to external bunches ($\mathbf{j}_{\mathrm{B}}$).

Quasistatic approximation

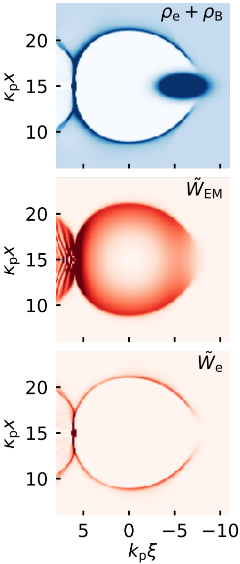
Total wake energy

$$\frac{\partial W}{\partial t} + \nabla \cdot \mathbf{S} = -\mathbf{j}_B \cdot \mathbf{E} + \frac{1}{2}\frac{\partial \langle \mathbf{a}^2 \rangle}{\partial t} n_e \overline{\gamma^{-1}}$$

Quasistatic approximation ($\xi = t - z$)

$$\frac{\partial \tilde{W}}{\partial \xi} + \nabla_\perp \cdot \mathbf{S}_\perp = -\rho_B E_z + \frac{1}{2}\frac{\partial \langle \mathbf{a}^2 \rangle}{\partial \xi} n_e \overline{\gamma^{-1}}$$

$\tilde{W} = W - S_z$ is the quasi-energy density.
Ultrarelativistic bunches: $\mathbf{j}_B \approx \rho_B \mathbf{z}_0$



Total wake energy

$$\frac{\partial W}{\partial t} + \nabla \cdot \mathbf{S} = -\mathbf{j}_\mathrm{B} \cdot \mathbf{E} + \frac{1}{2}\frac{\partial \langle \mathbf{a}^2 \rangle}{\partial t} n_\mathrm{e}\overline{\gamma^{-1}}$$

Quasistatic approximation ($\xi = t - z$)

$$\frac{\partial \tilde{W}}{\partial \xi} + \nabla_\perp \cdot \mathbf{S}_\perp = -\rho_\mathrm{B} E_z + \frac{1}{2}\frac{\partial \langle \mathbf{a}^2 \rangle}{\partial \xi} n_\mathrm{e}\overline{\gamma^{-1}}$$

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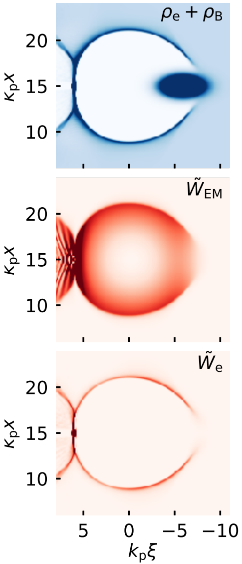
Ultrarelativistic bunches: $\mathbf{j}_\mathrm{B} \approx \rho_\mathrm{B}\mathbf{z}_0$

$$\tilde{W}_\mathrm{EM} = \frac{1}{2}\left[(\nabla\psi_\mathrm{w})^2 + B_z^2\right] \quad \tilde{W}_\mathrm{e} = n_\mathrm{e}\overline{(\gamma - 1)(1 - v_z)}$$

$\psi_\mathrm{w} = \varphi - A_z$ is the wakefield potential.



Quasistatic approximation

Total wake energy

$$\frac{\partial W}{\partial t} + \nabla \cdot \mathbf{S} = -\mathbf{j}_\mathrm{B} \cdot \mathbf{E} + \frac{1}{2}\frac{\partial \langle \mathbf{a}^2 \rangle}{\partial t} n_\mathrm{e} \overline{\gamma^{-1}}$$

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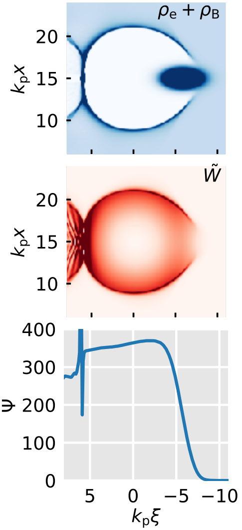
$\tilde{W} = W - S_z$ is the quasi-energy density.

Integrating over the transverse plane

$$\frac{d\Psi}{d\xi} = -\int \rho_\mathrm{B} E_z \, d^2\mathbf{r}_\perp + \frac{1}{2}\int \frac{\partial \langle \mathbf{a}^2 \rangle}{\partial \xi} n_\mathrm{e} \overline{\gamma^{-1}} \, d^2\mathbf{r}_\perp$$

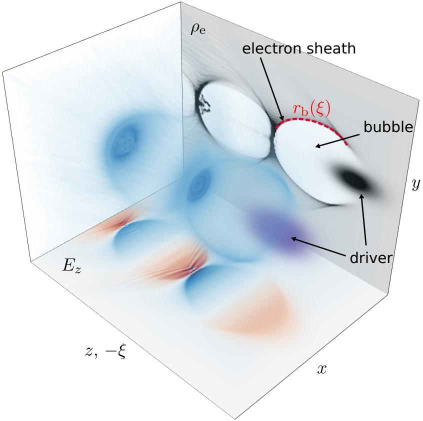
$$\Psi(\xi) = \int \tilde{W} \, d^2\mathbf{r}_\perp \quad \text{(slice quasi-energy)}$$

Behind the driver, no witness bunches: $\Psi(\xi) = \text{const}$.



Lotov, *Phys. Rev. E* **69**, 046405 (2004)

Model of the bubble



- Axial symmetry (r, z), only E_z, E_r, B_ϕ components.
- The bubble has a boundary, $r_b(\xi)$.
- Inside the boundary, no plasma electrons.
- The electron sheath on the boundary is infinitely thin.

$$\Psi = \Psi_{EM} + \Psi_e$$

EM slice quasi-energy

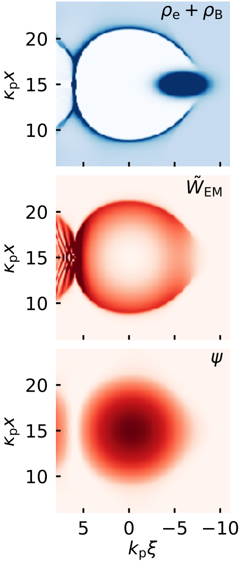
$$\tilde{W}_{\mathrm{EM}} = \frac{(\nabla\psi_{\mathrm{w}})^2}{2} \qquad \Psi_{\mathrm{EM}}(\xi) = \pi\int_0^{r_{\mathrm{b}}(\xi)} (\nabla\psi_{\mathrm{w}})^2 r\,\mathrm{d}r$$

The wakefield potential inside the bubble is

$$\psi_{\mathrm{w}}(\xi, r) = \frac{r_{\mathrm{b}}^2(\xi) - r^2}{4}$$

Therefore

$$\Psi_{\mathrm{EM}}(\xi) = \frac{\pi r_{\mathrm{b}}^4}{16}\left[1 + 2\left(\frac{\mathrm{d}r_{\mathrm{b}}}{\mathrm{d}\xi}\right)^2\right]$$



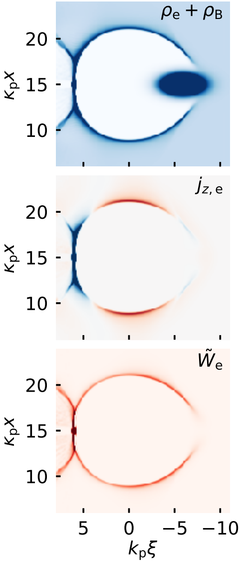
Electron slice quasi-energy

We define the electron sheath as

$$j_{z,\mathrm{e}} = j_0(\xi) r_\mathrm{b} \delta(r - r_\mathrm{b})$$

Then,

$$\Psi_\mathrm{e}(\xi) = 2\pi \int_{r_\mathrm{b}-0}^{r_\mathrm{b}+0} \tilde{W}_\mathrm{e} r \, \mathrm{d}r = \frac{\pi}{2} r_\mathrm{b}^2 \left(\frac{\mathrm{d}r_\mathrm{b}}{\mathrm{d}\xi} \right)^2$$



Equation for the bubble boundary

Finally, the total quasienergy is

$$\Psi = \frac{\pi r_b^2}{16}\left[r_b^2 + (2r_b^2 + 8)\left(\frac{dr_b}{d\xi}\right)^2\right], \qquad \frac{d\Psi}{d\xi} = -\pi r_b \frac{dr_b}{d\xi}\int_0^{r_b} \rho_B r\, dr$$

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And we get the equation

$$\left(\frac{r_b^3}{4} + r_b\right)\frac{d^2 r_b}{d\xi^2} + \left(\frac{r_b^2}{2} + 1\right)\left(\frac{dr_b}{d\xi}\right)^2 + \frac{r_b^2}{2} = \lambda(\xi, r_b), \quad \lambda(\xi, r_b) = \int_0^{r_b} \rho_B r\,dr$$

Red terms — EM energy, blue — plasma electrons energy.

Bubble equation = energy conservation

Comparison of equations

$$A(r_b)\frac{d^2 r_b}{d\xi^2} + B(r_b)\left(\frac{dr_b}{d\xi}\right)^2 + C(r_b) = \lambda(\xi, r_b), \quad \lambda(\xi, r_b) = \int_0^{r_b} \rho_B r\, dr$$

Energy conservation

$$A = \frac{r_b^3}{4} + r_b \qquad B = \frac{r_b^2}{2} + 1$$

$$C = \frac{r_b^2}{4} \qquad E_z = \frac{r_b}{2}\frac{dr_b}{d\xi}$$

Lu's model

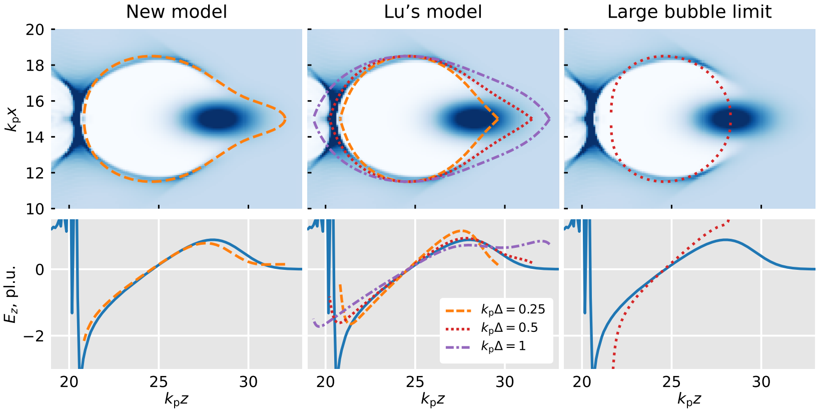
$$A = \frac{r_b^3}{4} + r_b + \frac{3}{4}r_b^2\Delta, \qquad B = \frac{r_b^2}{2} + \frac{r_b\Delta}{2},$$

$$C = \frac{r_b^2}{4}\left[1 + \left(1 + \frac{r_b\Delta}{2}\right)^{-2}\right], \qquad E_z = \frac{r_b + \Delta}{2}\frac{dr_b}{d\xi}.$$

Does not correspond to energy conservation

For large bubbles $r_b \gg 1$, we get the same equation.

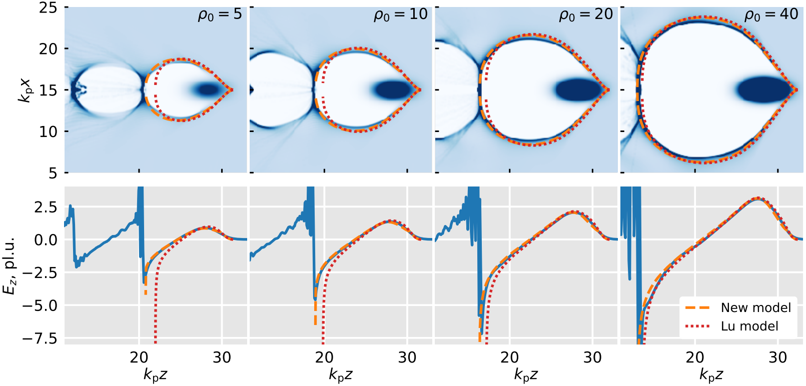
Comparison of models



Bubble excitation

Self-consistent excitation of the bubble by an electron driver.

Golovanov et al. *PPCF* **63**, 085004 (2021).



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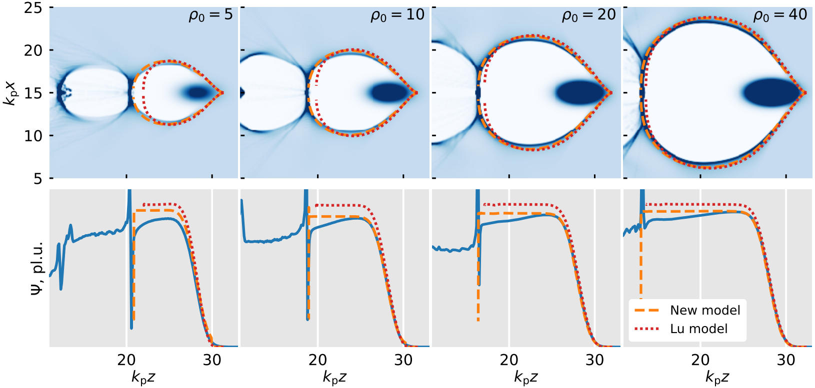
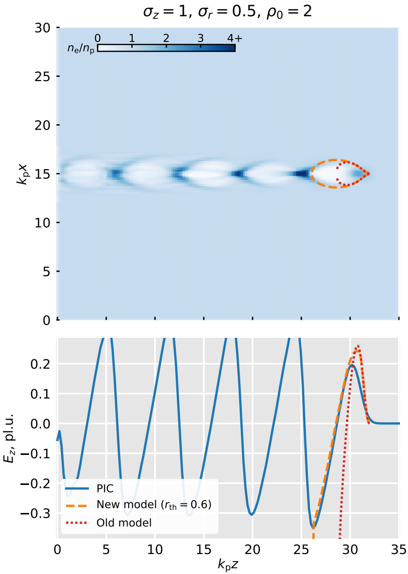
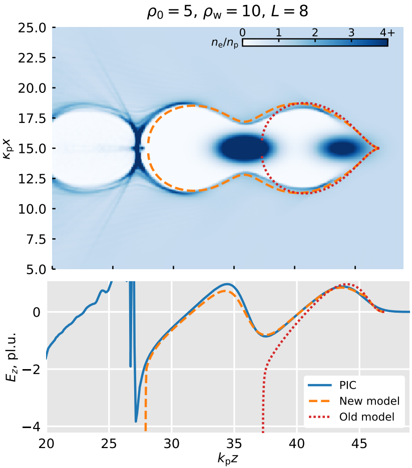
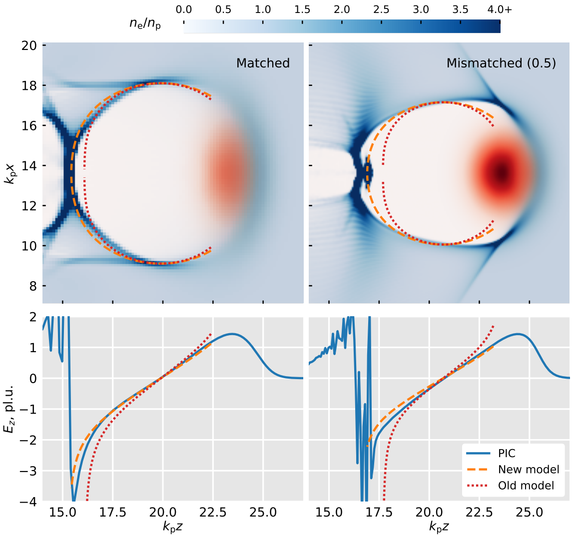


Other examples





Laser driver



Thank you for your attention!

Paper: A. Golovanov et al. *PRL* 130, 105001 (2023)