Quantum-inspired techniques for learning the geometry of data

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QuantHEP 2023, Istituto Nazionale di Fisica Nucleare

September 26, 2023

- **Data analysis tasks are *often* geometric analysis of data**

- **Geometry of HEP data analysis**

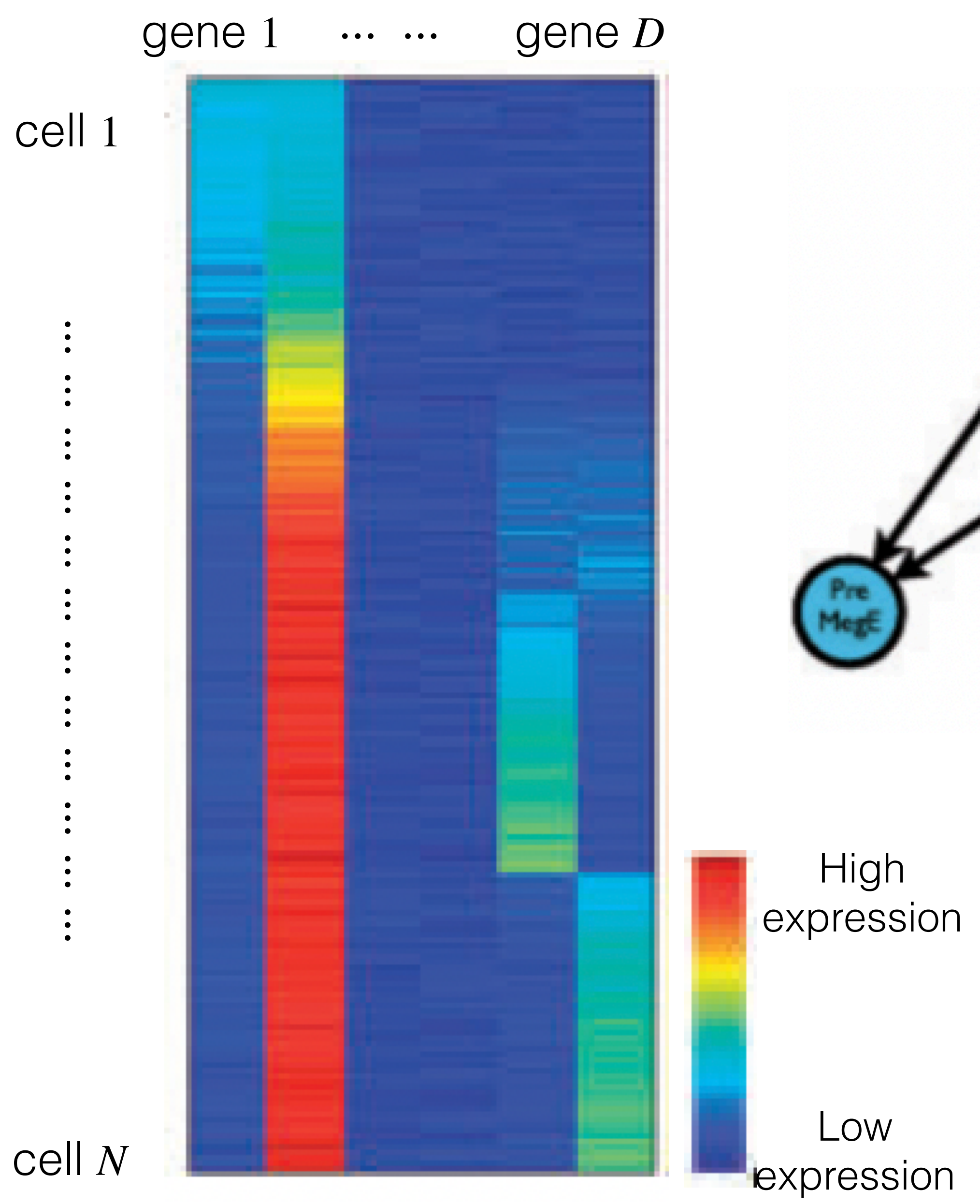
- **Quantum dynamics underlying the geometry of data**
  - **Example applications**

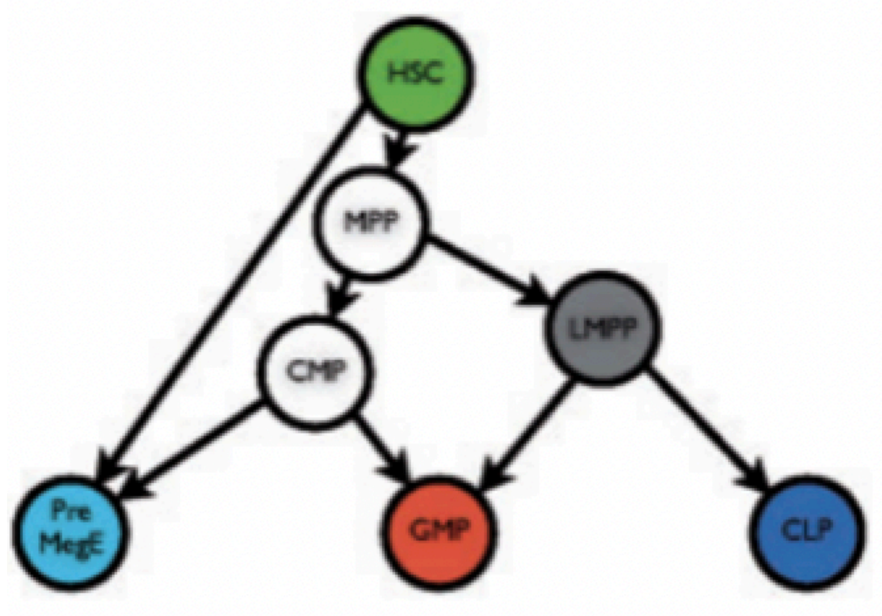
- **Future directions for HEP applications**

Data is often an *organized* point cloud

What is “data”?

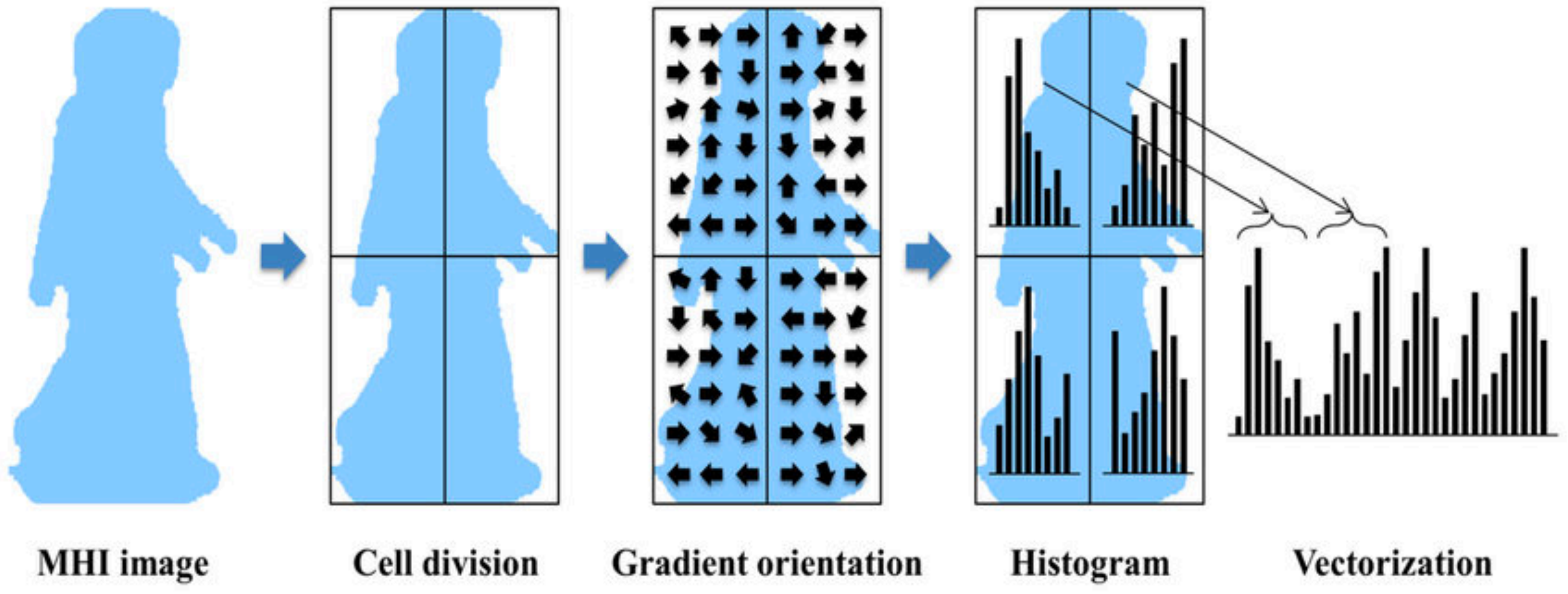
N vectors in $\mathbb{R}^D$





Hierarchy of cell types

Haghverdi et al., *Bioinformatics* (2015)



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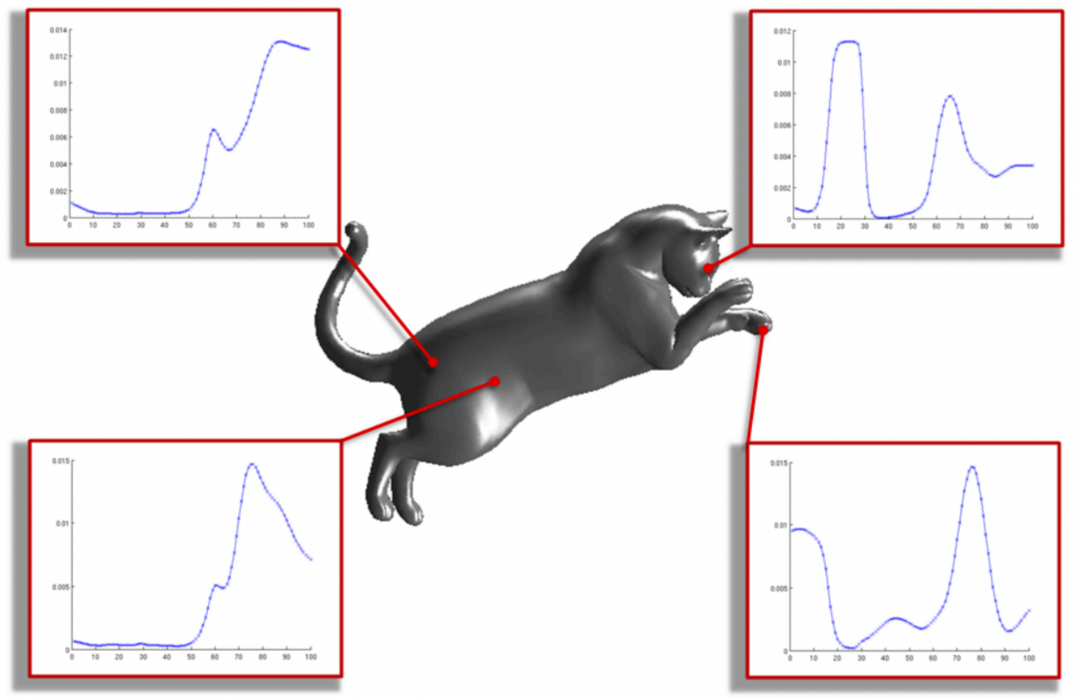

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Measurements, including statistics of natural processes, constitute data

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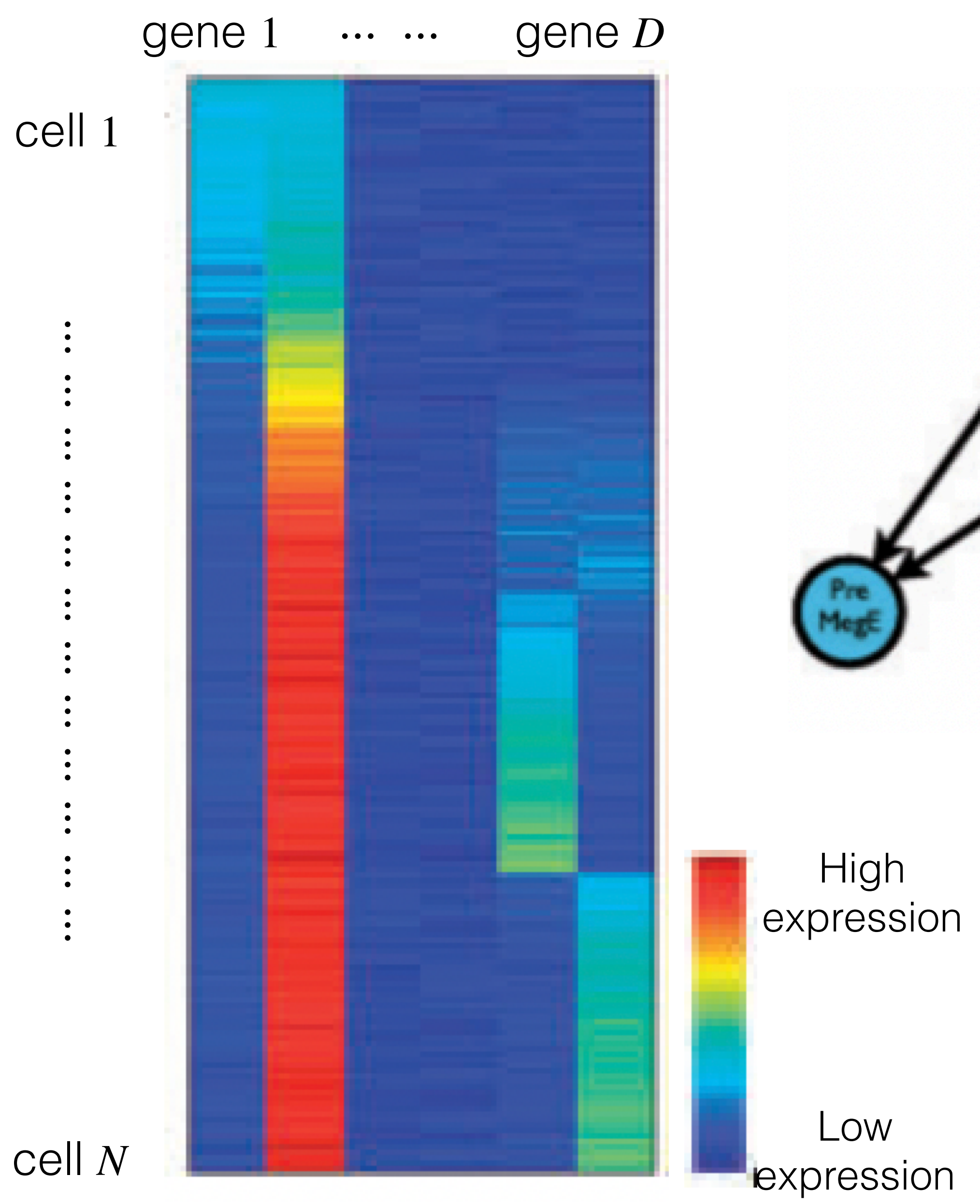

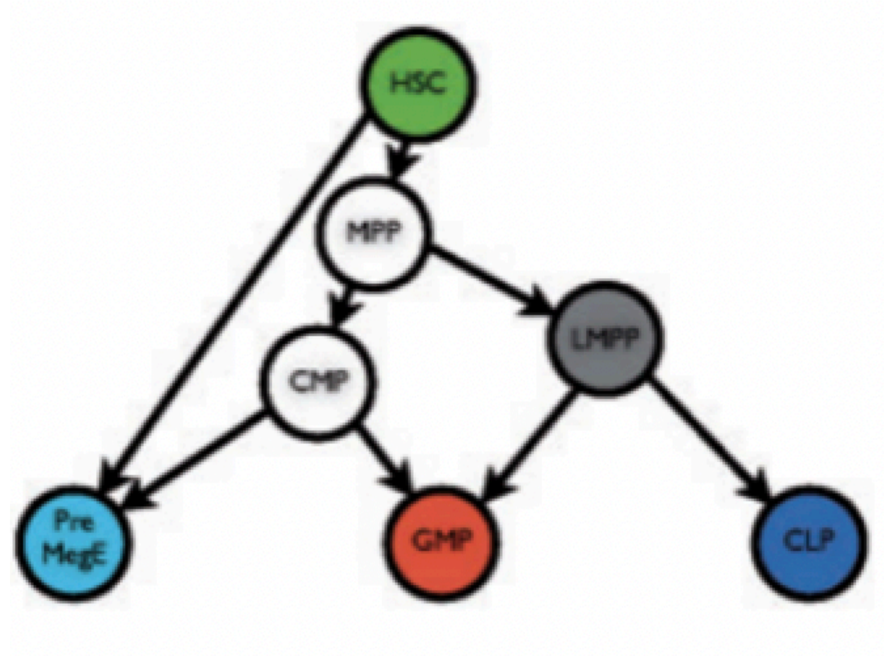

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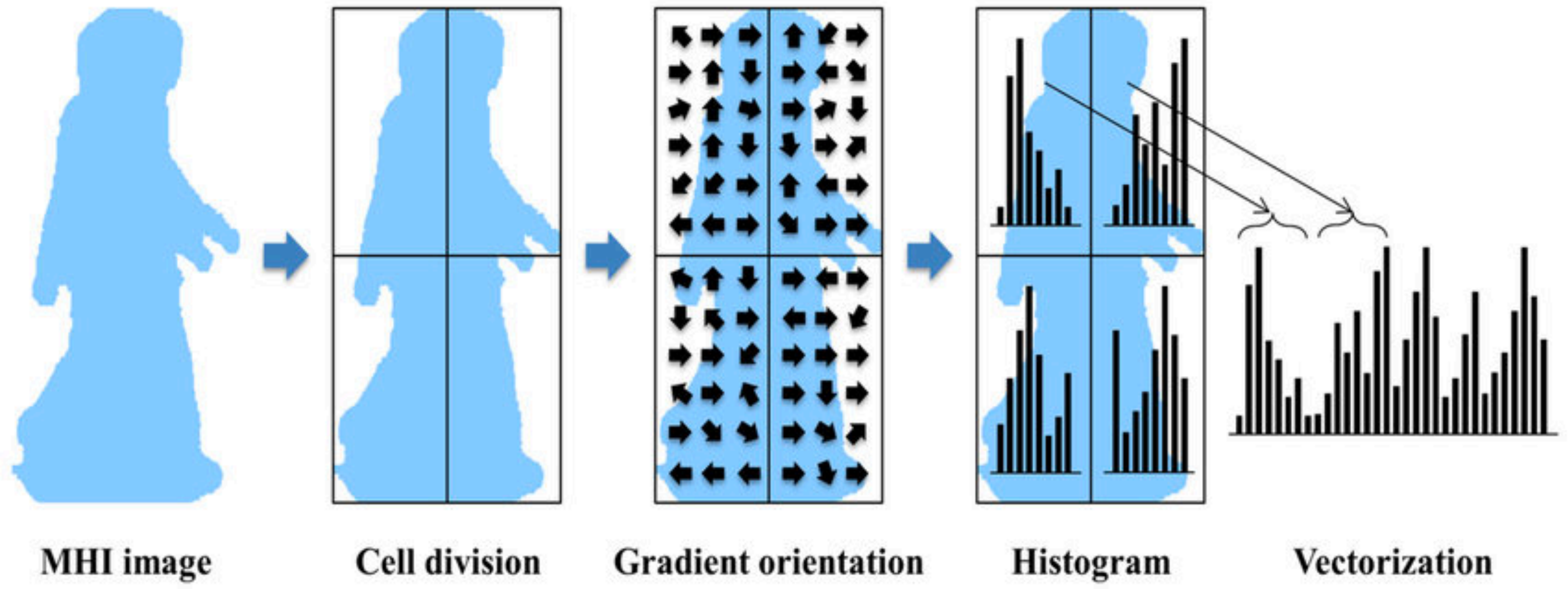

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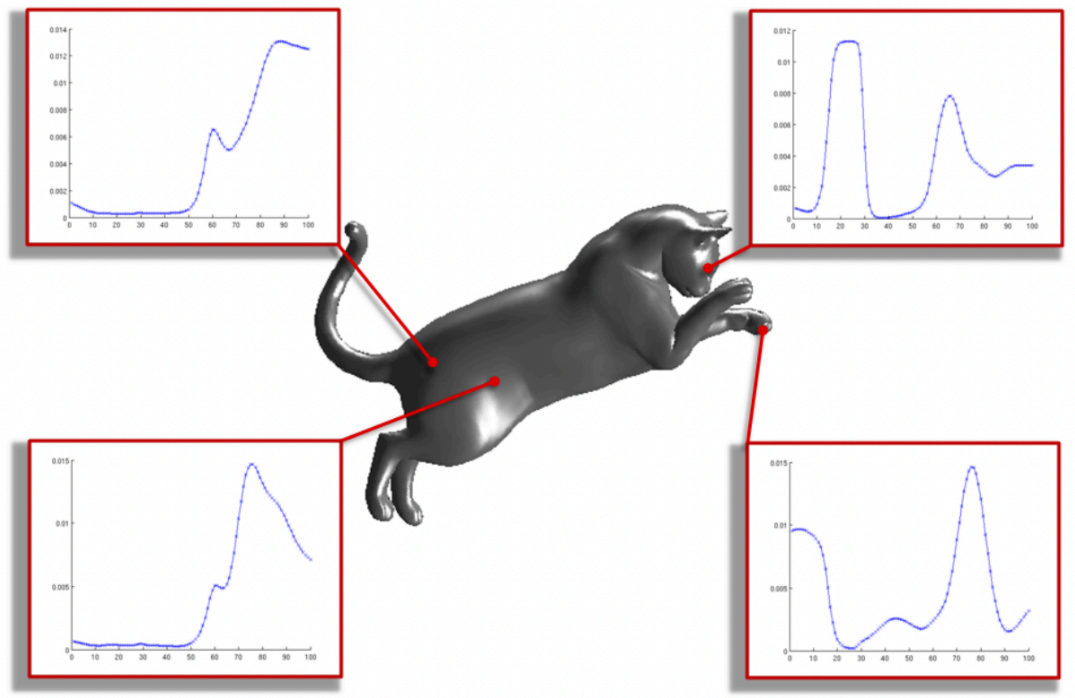

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Measurements, including statistics of natural processes, constitute data

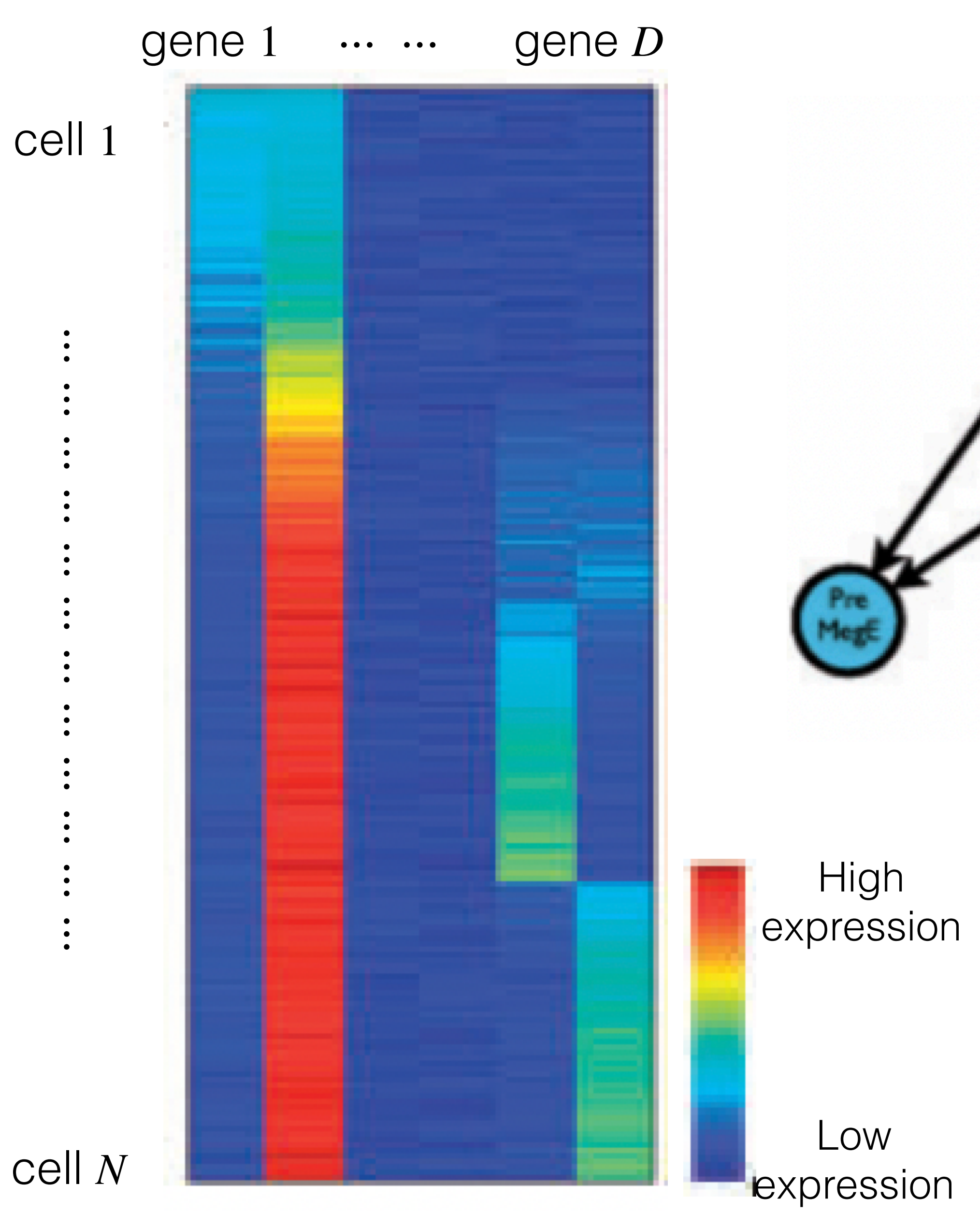
- measuring/sampling D variables in an experiment
- recording the output of D sensors

$$v := (\text{meas.}_1, \ldots, \text{meas.}_D) \in \mathbb{R}^D$$

"feature vector"

What is “data”?

N vectors on low-dim. *submanifold*



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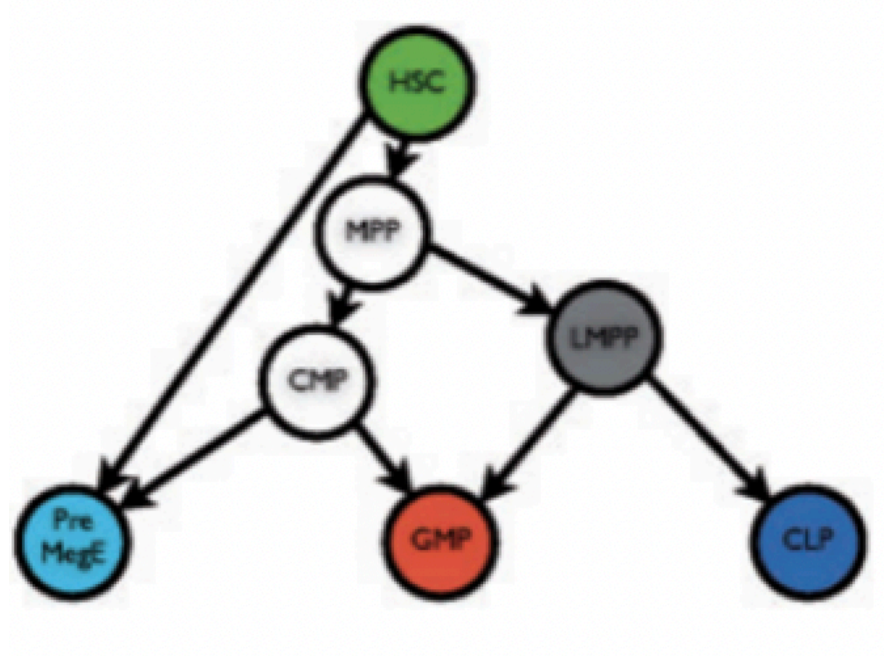

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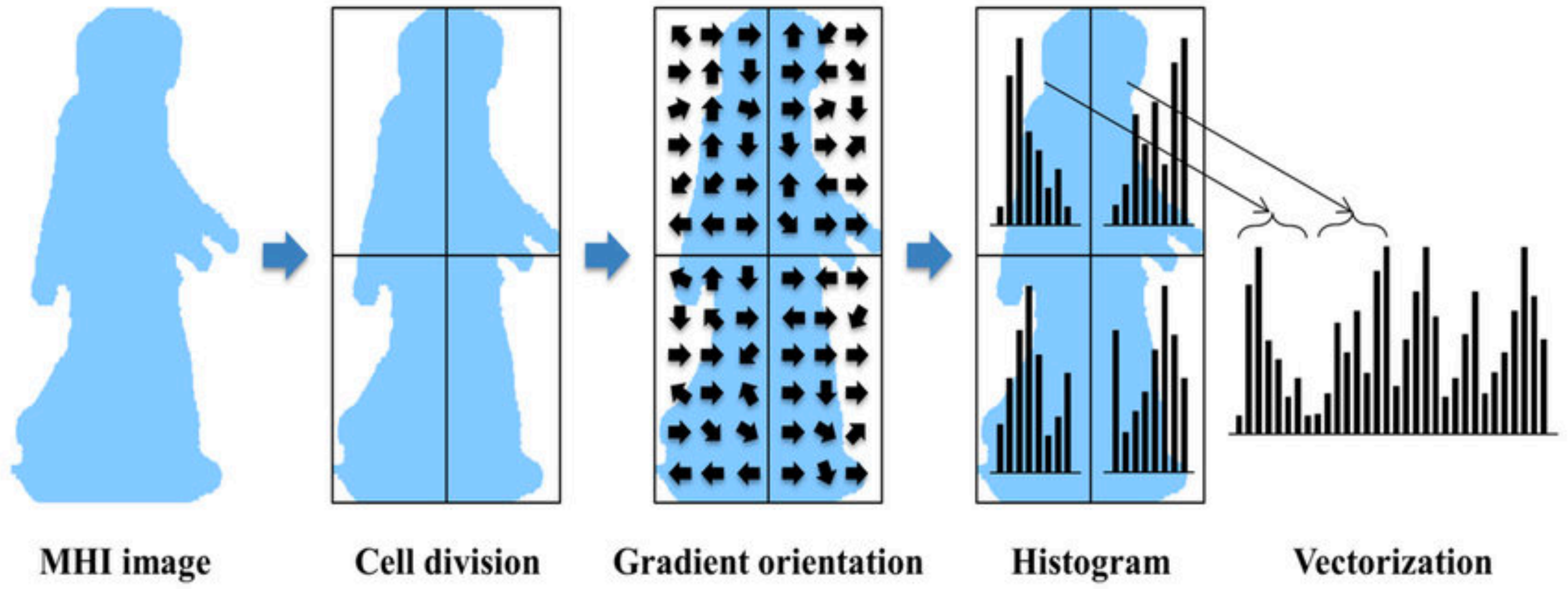

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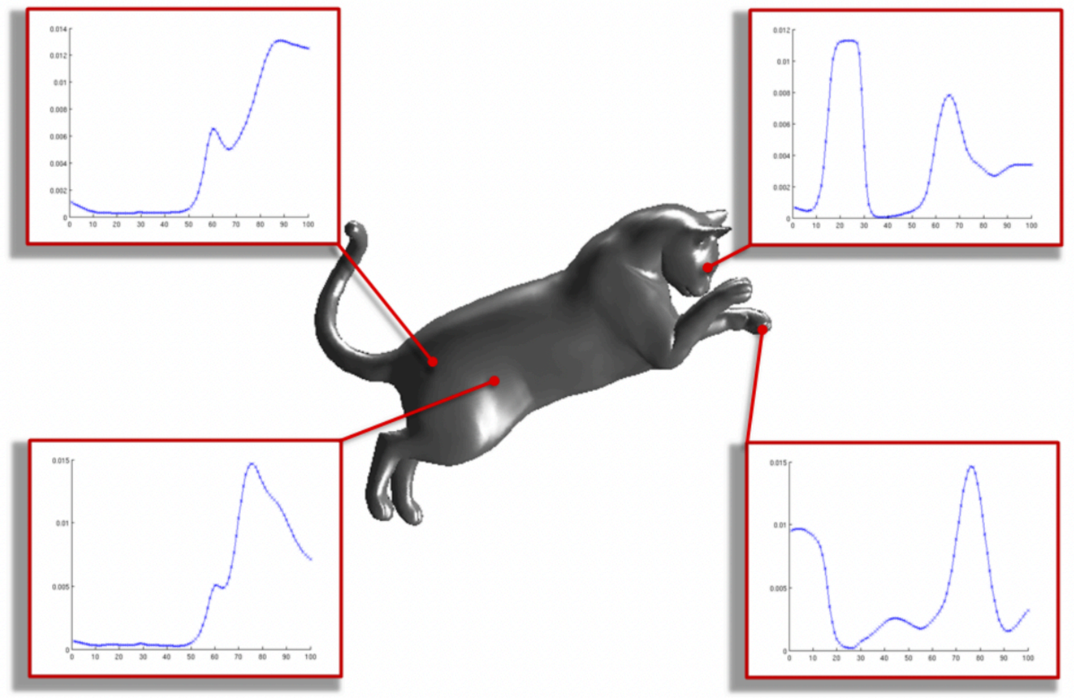

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The manifold hypothesis:
“high dimensional data tend to lie in the vicinity of a low dimensional manifold”.

Fefferman, Mitter, Narayanan. J. Am. Math. Soc., 29, 983 (2016)

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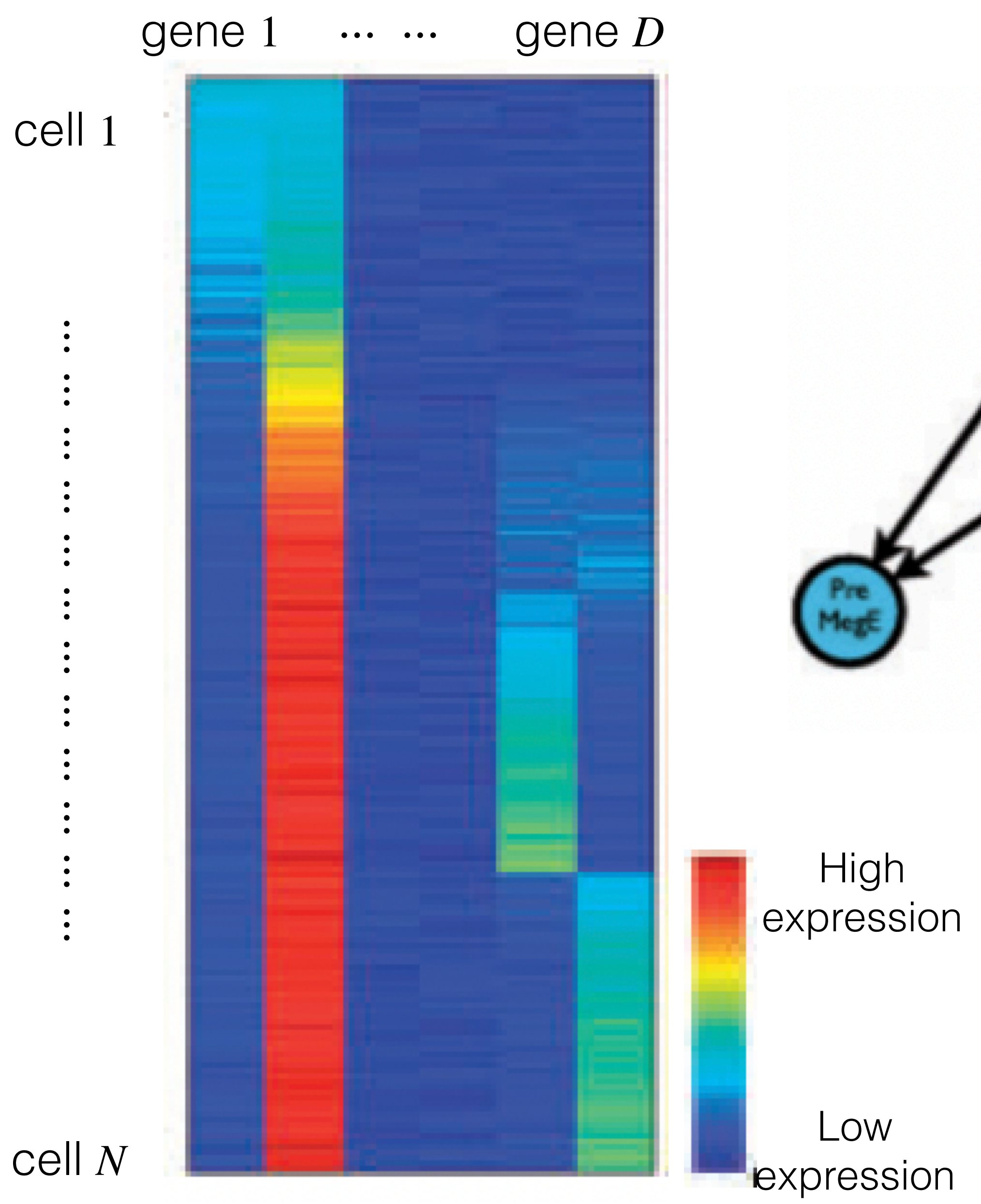

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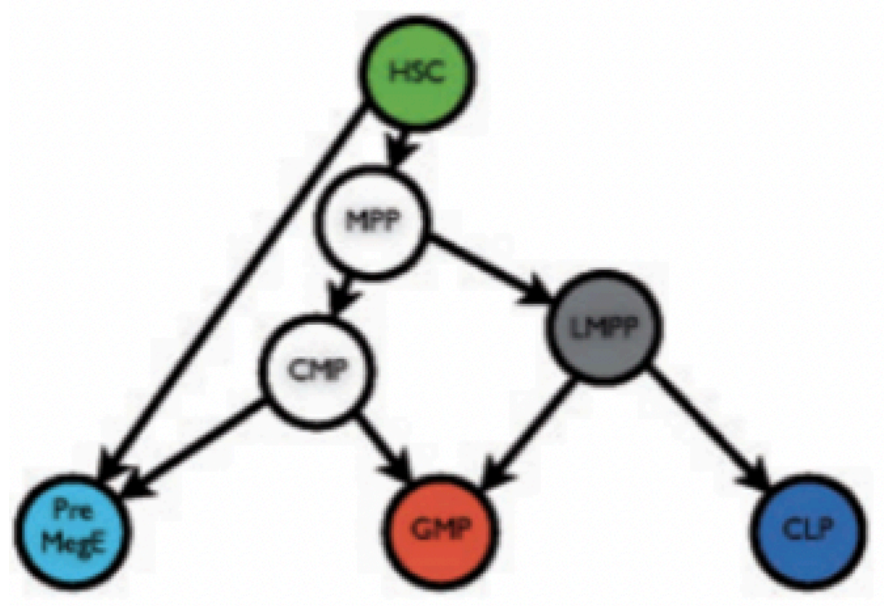

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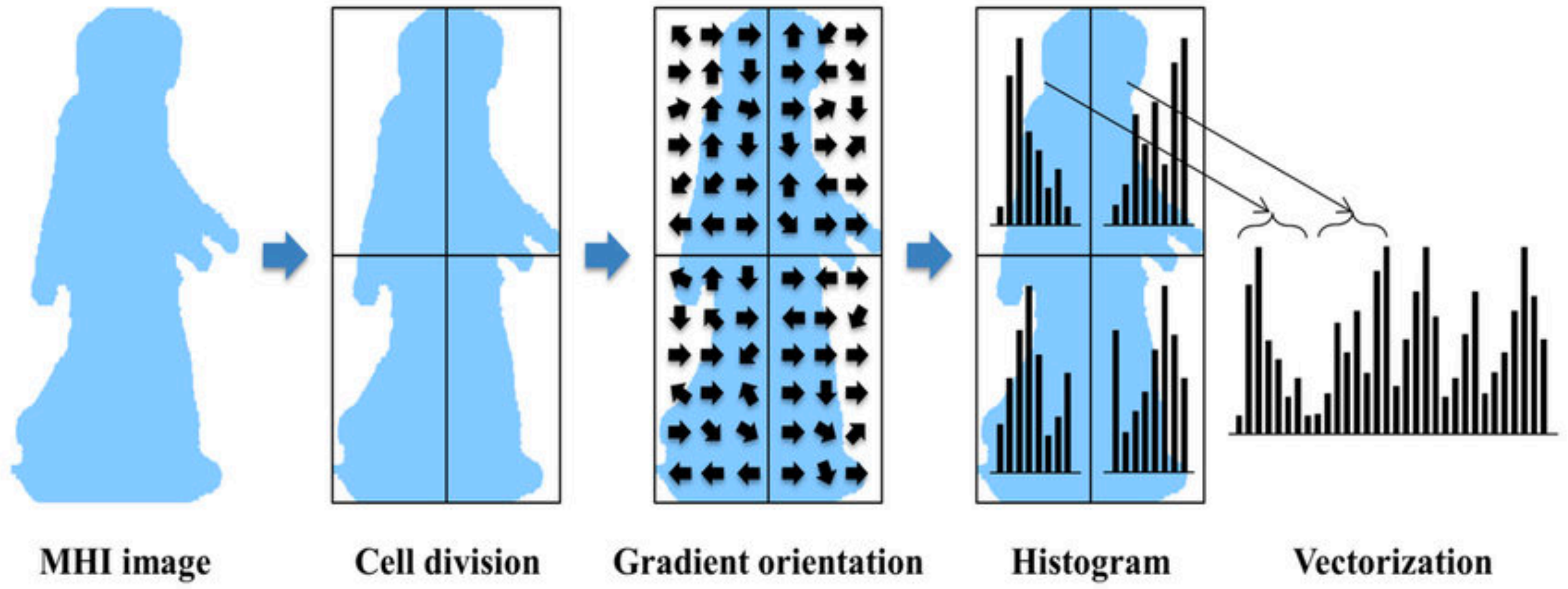

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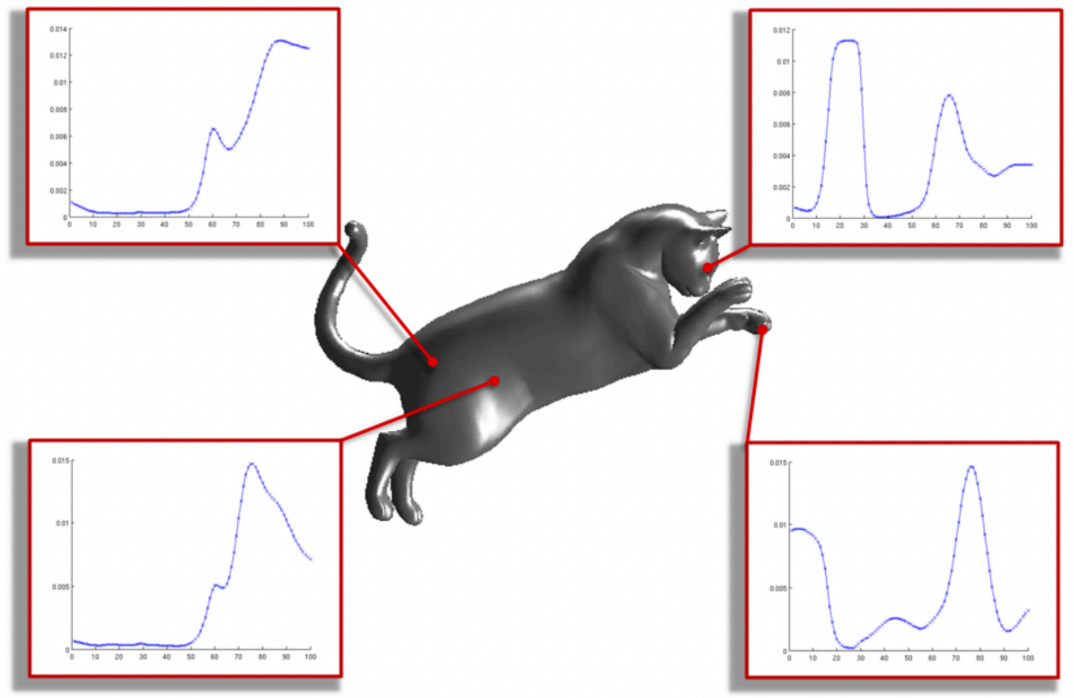

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Real-world data has low-dimensional structure

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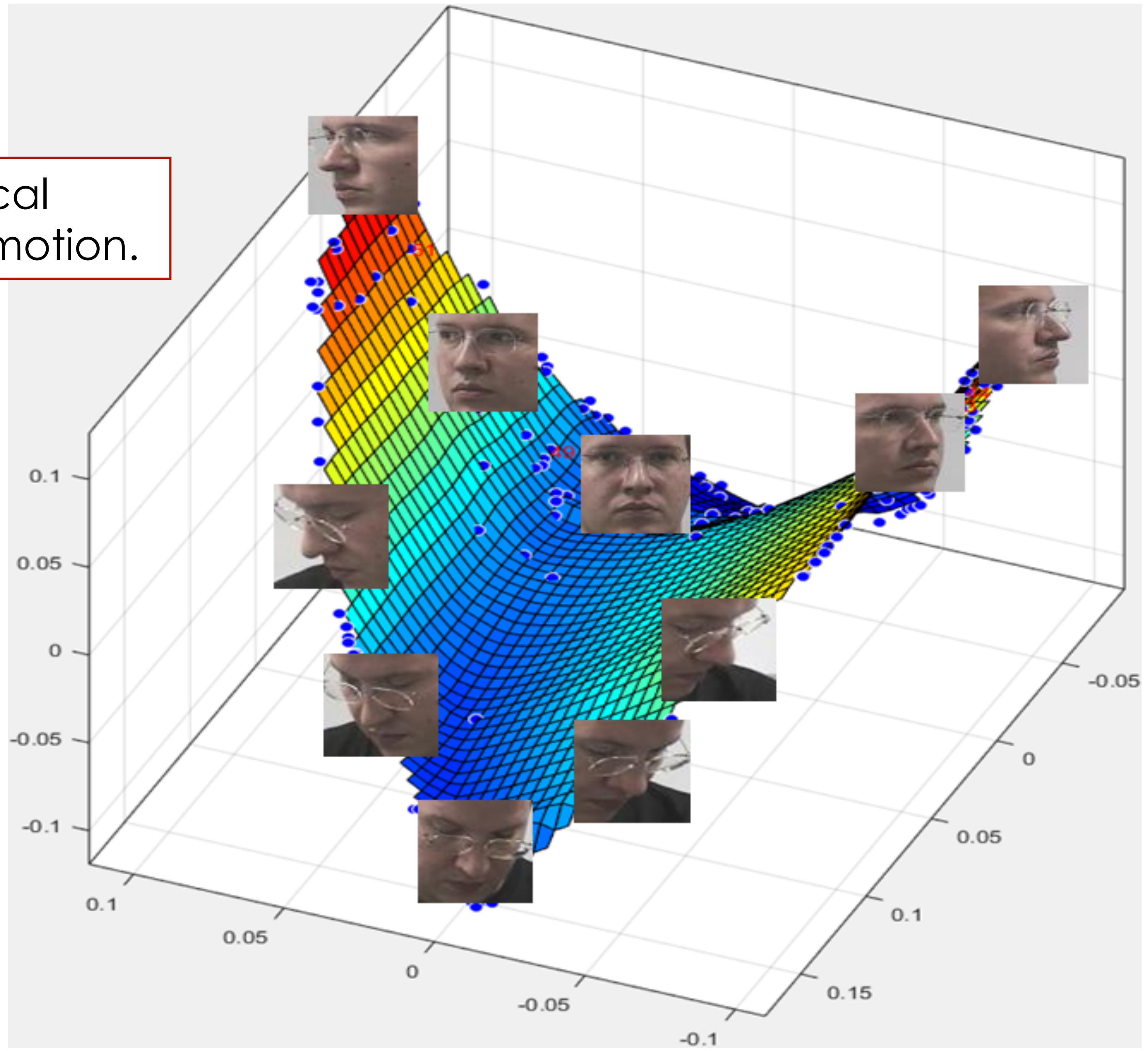
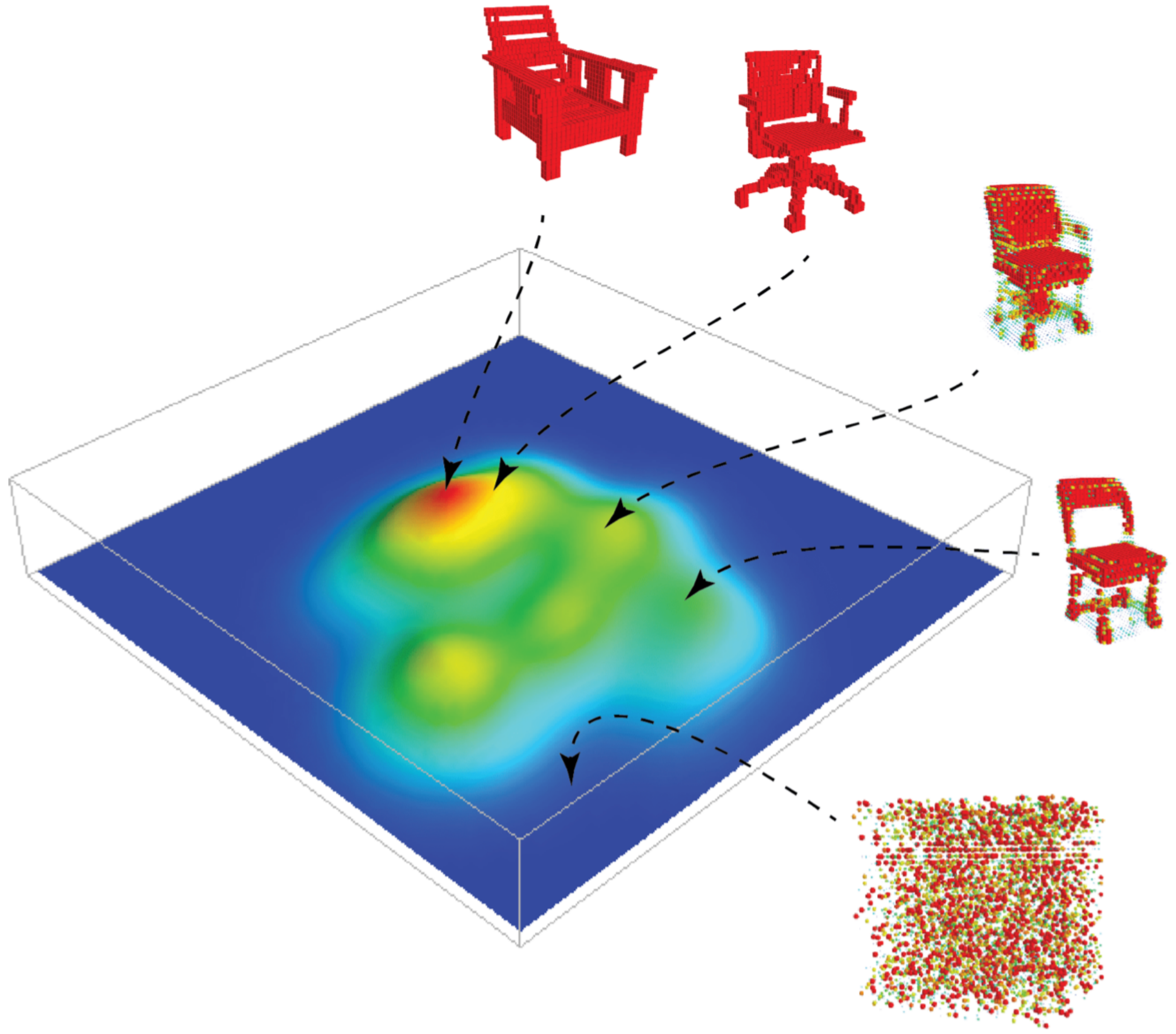
N vectors on low-dim. *submanifold*

Data generated by dynamical system follows equations of motion.



https://www.intechopen.com/books/manifolds-current-research-areas/head-pose-estimation-via-manifold-learning



Randomly generated image of NxN pixels ≠ real world scene.

Gwak et al. *International Conference on 3D Vision (3DV)*, IEEE (2017).

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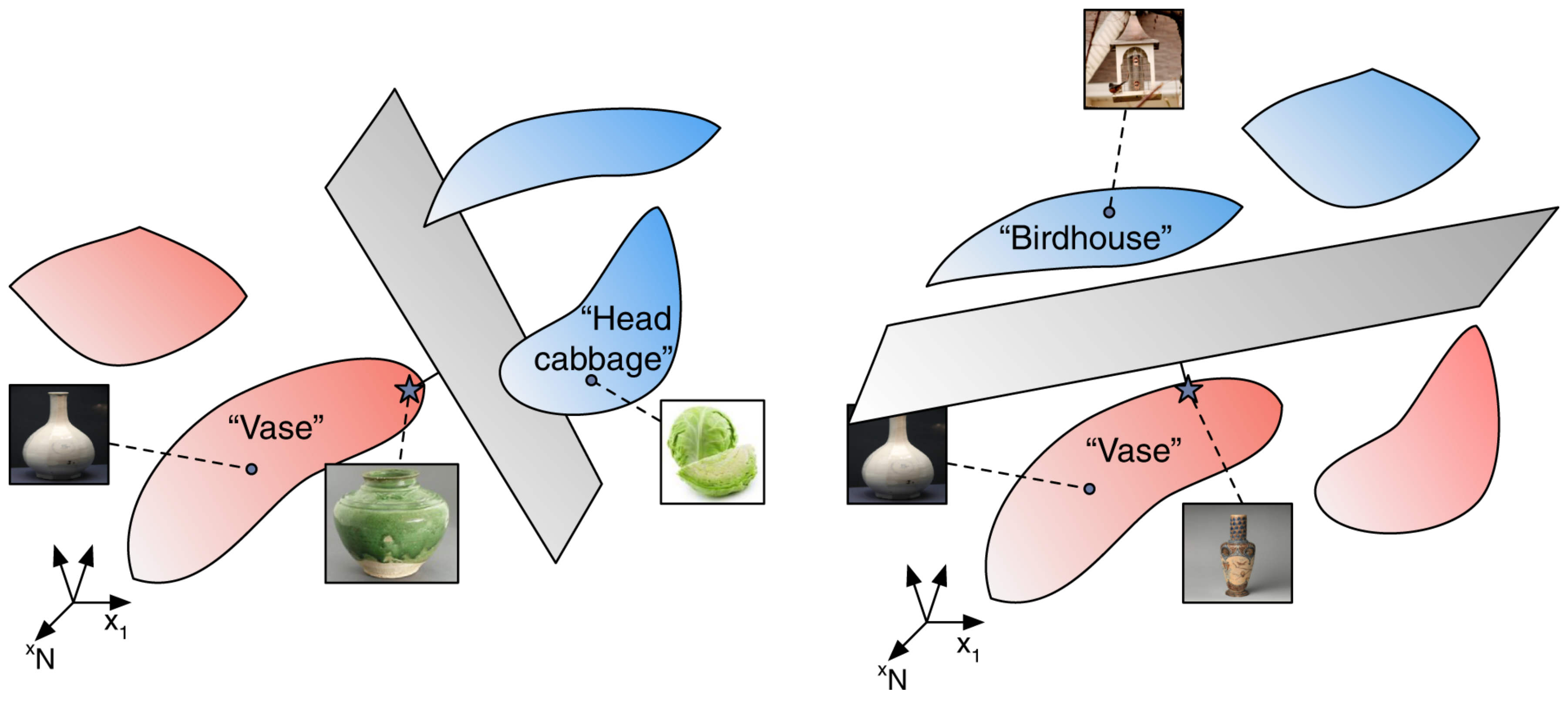
Data analysis $\approx$ Recovering geometry of data

Classification within neural network's *concept space*:



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- Classification

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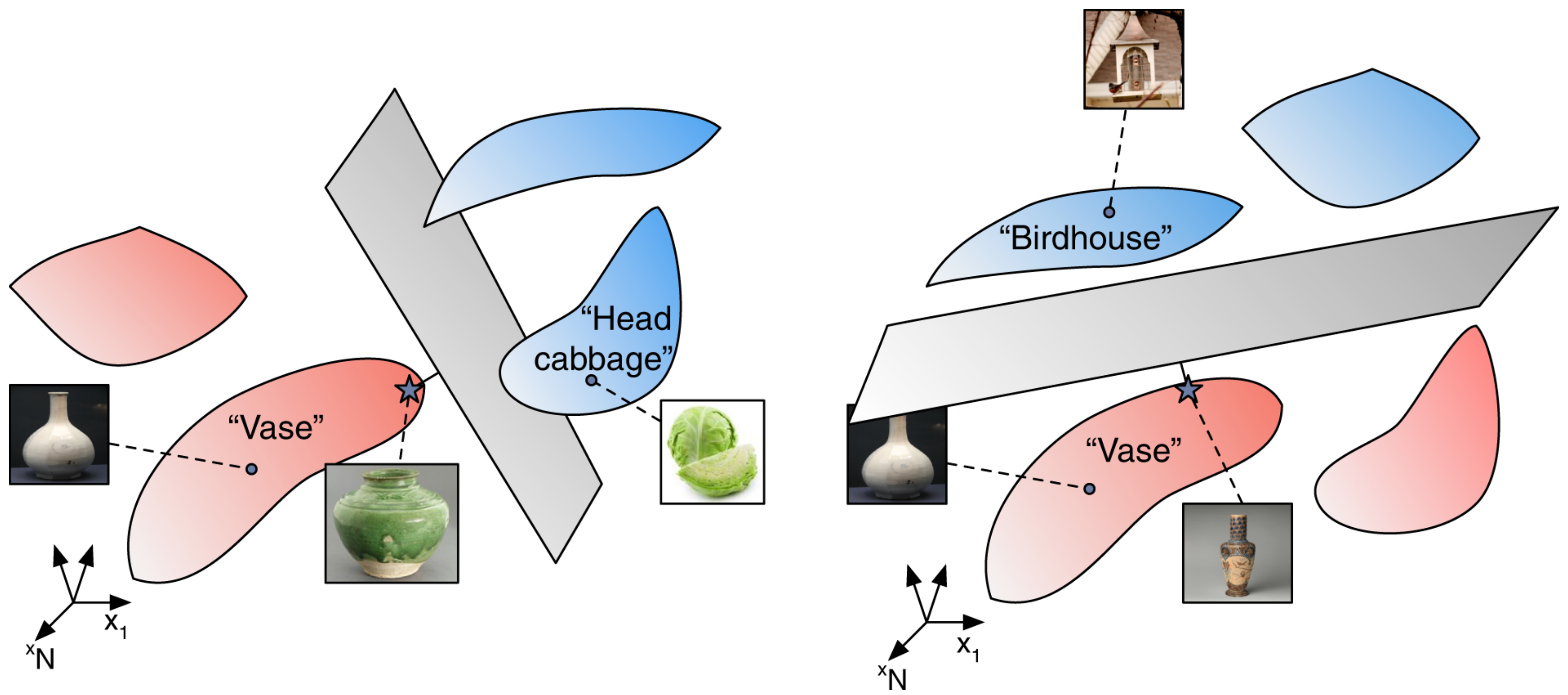

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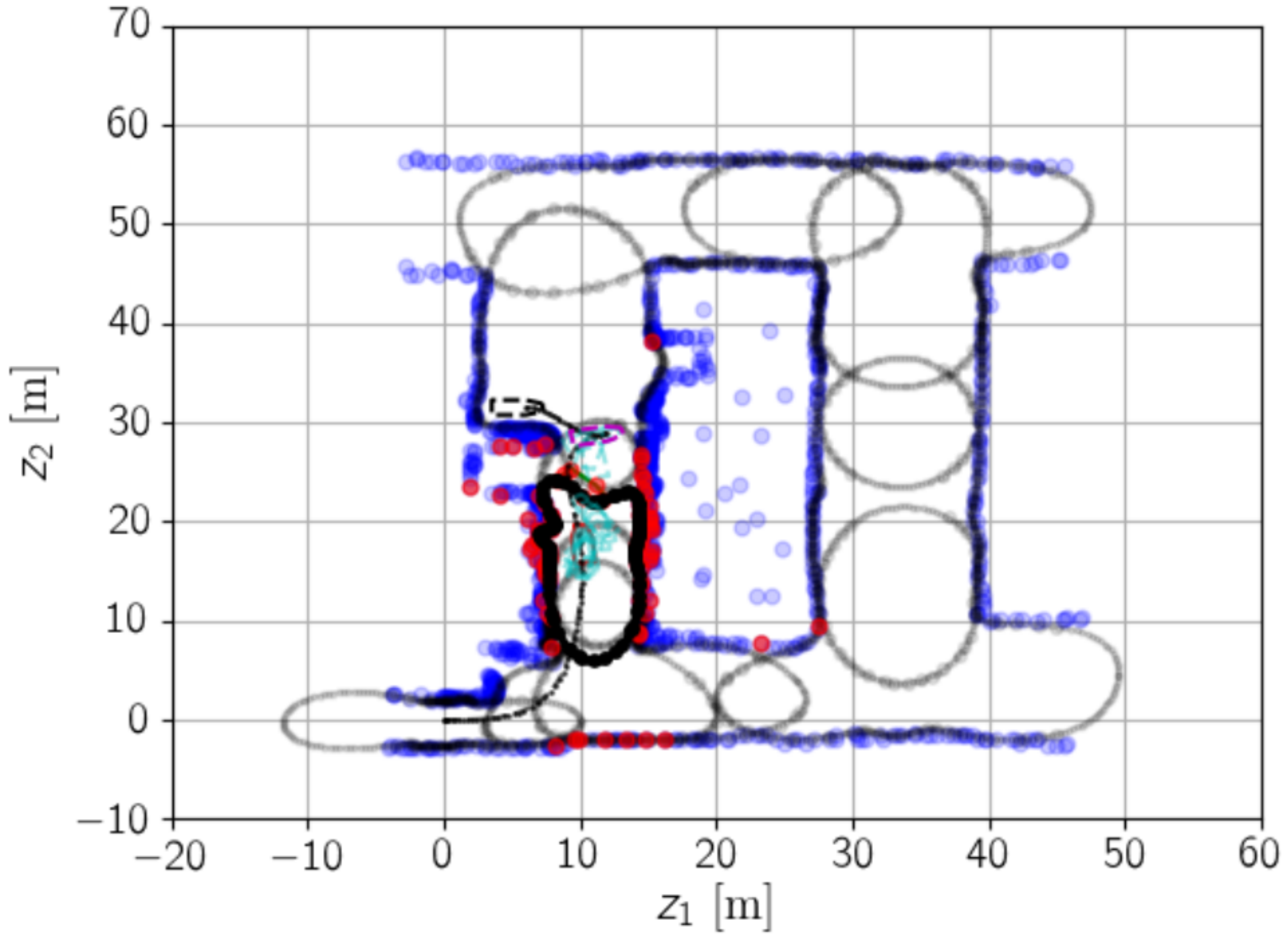
Obstacle avoidance in autonomous driving:



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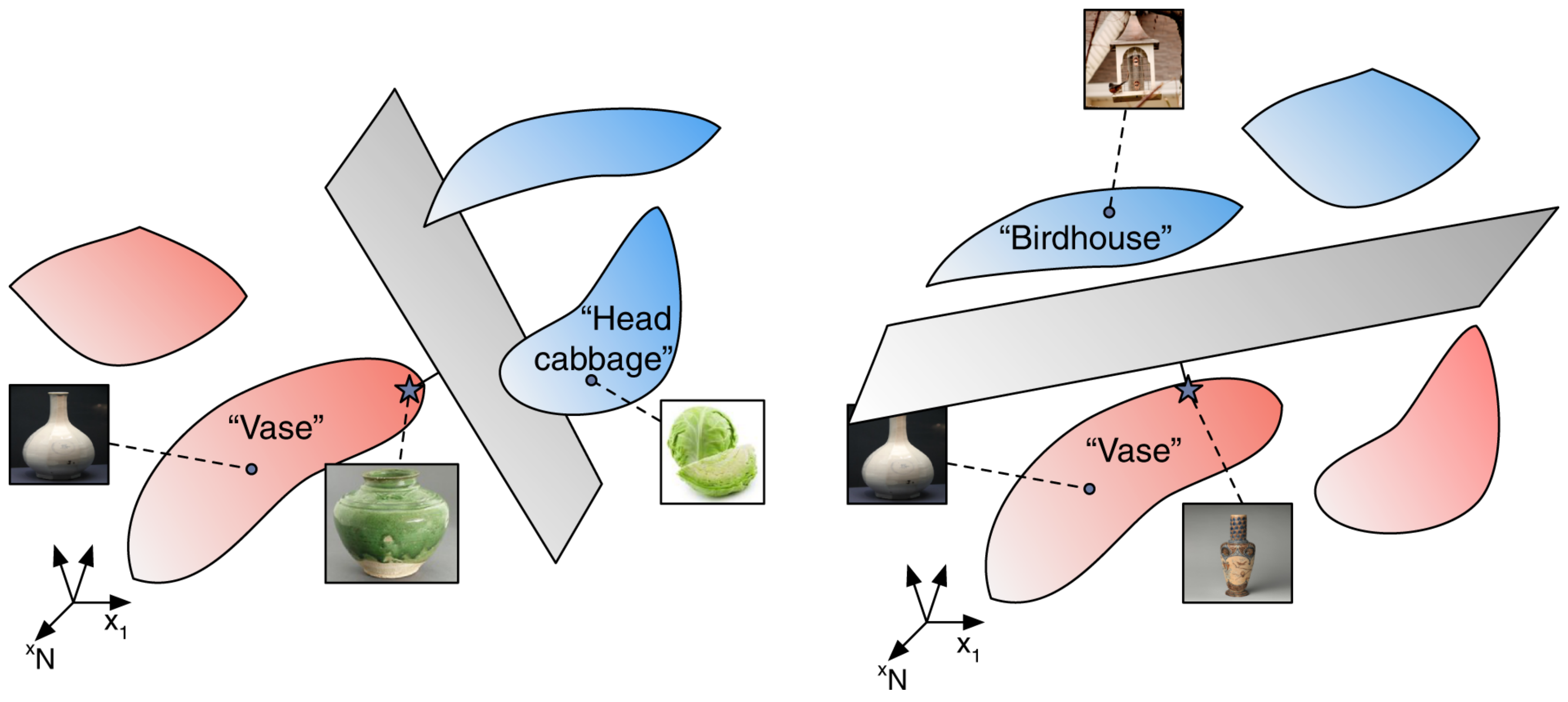

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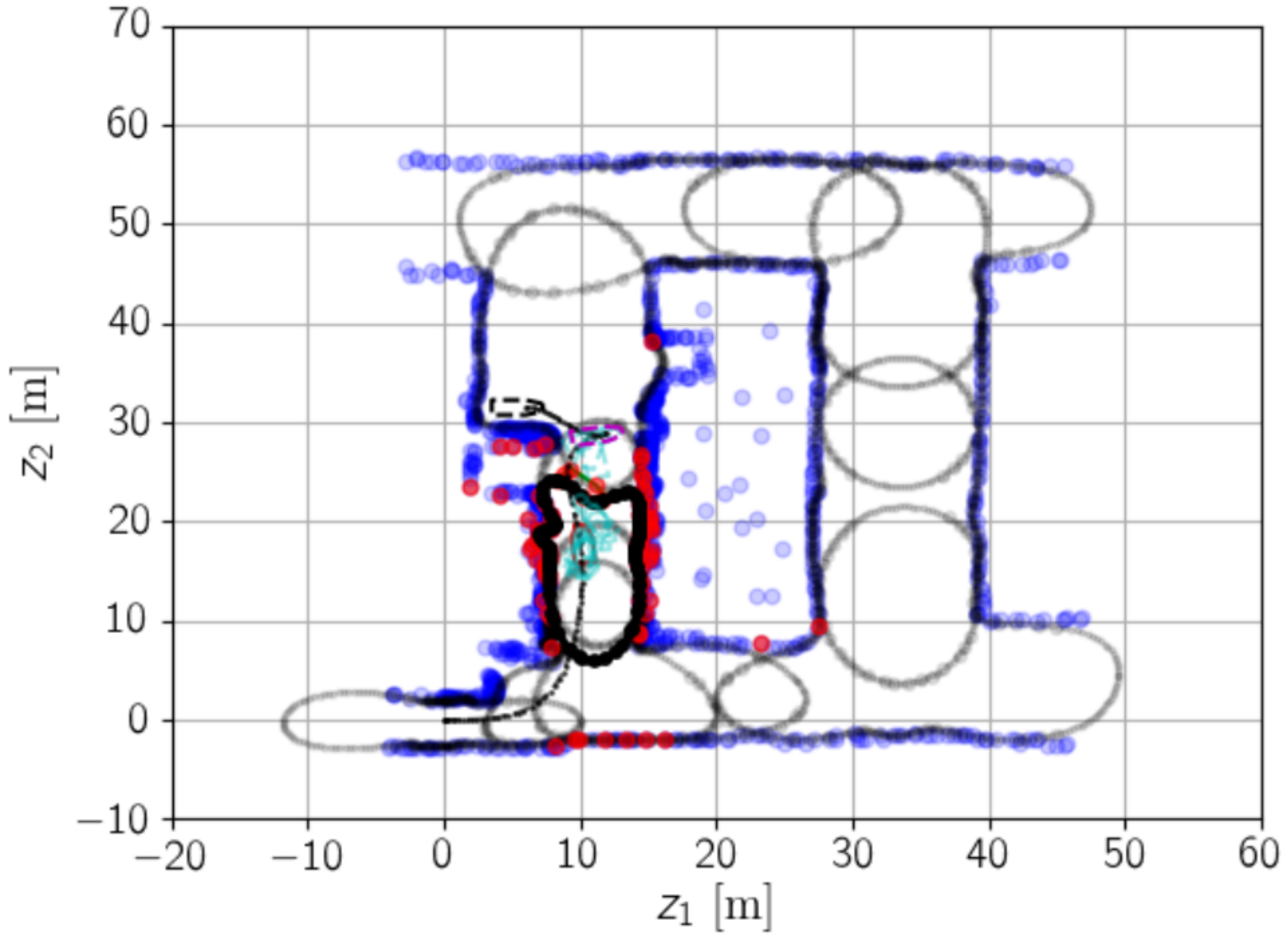

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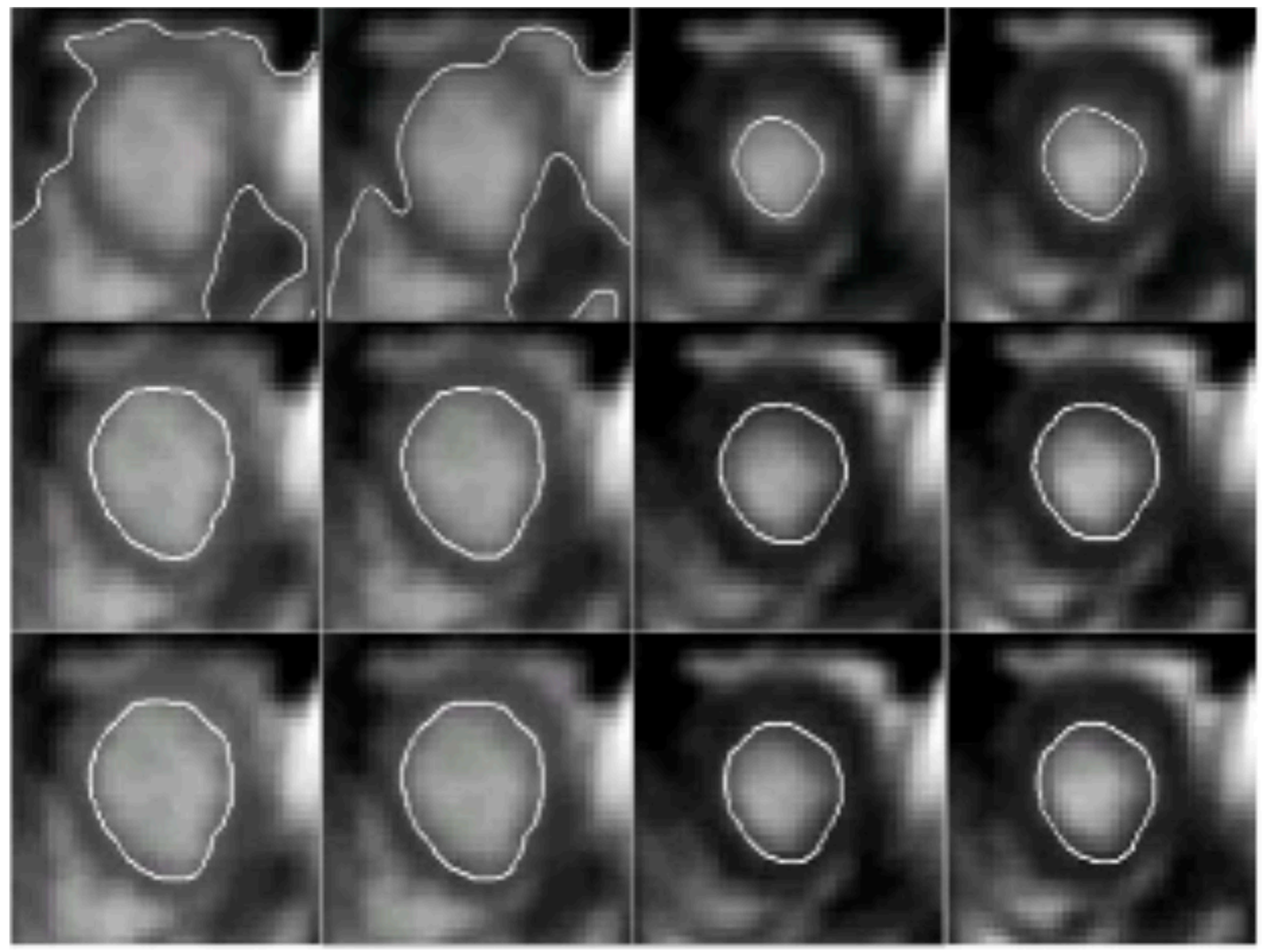
Image segmentation for medical imaging:



Zhang et al., IEEE CVPR, 1092 (2006).

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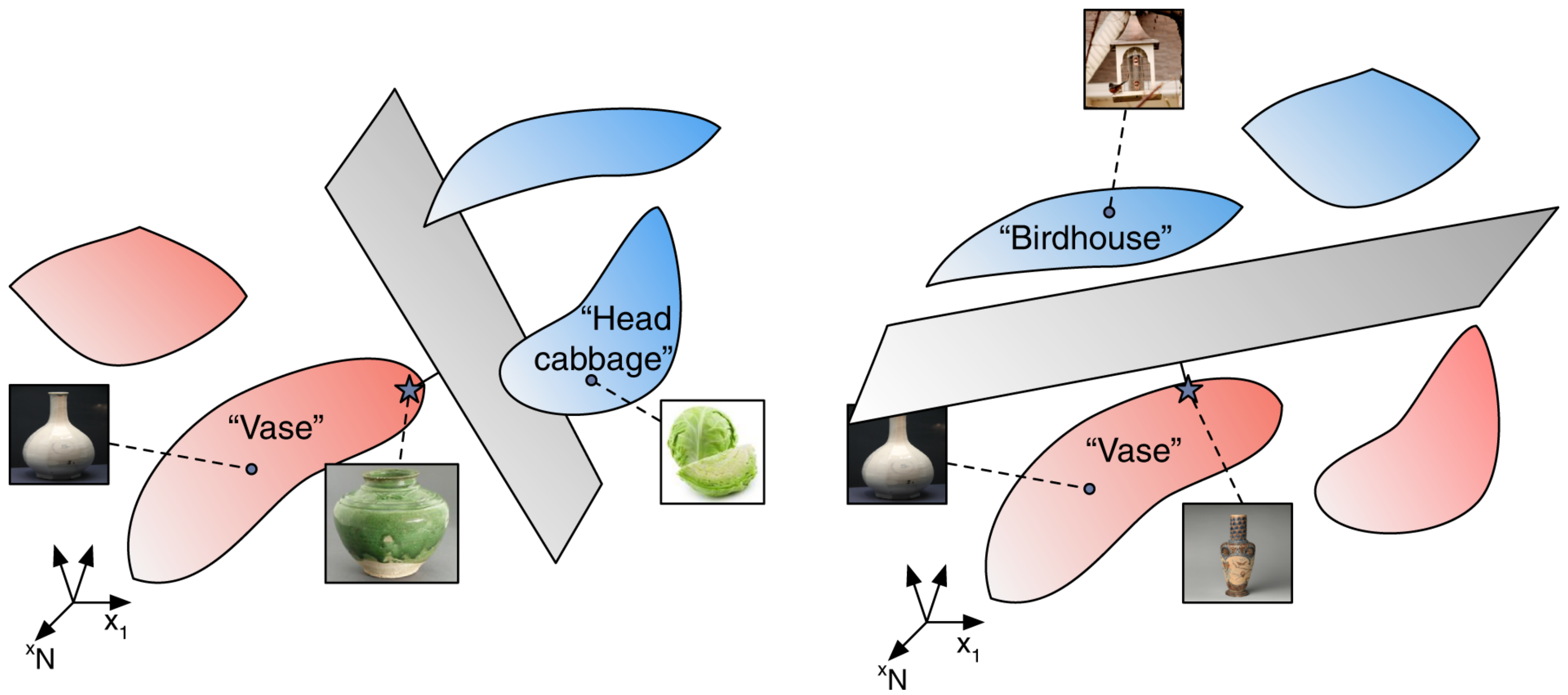

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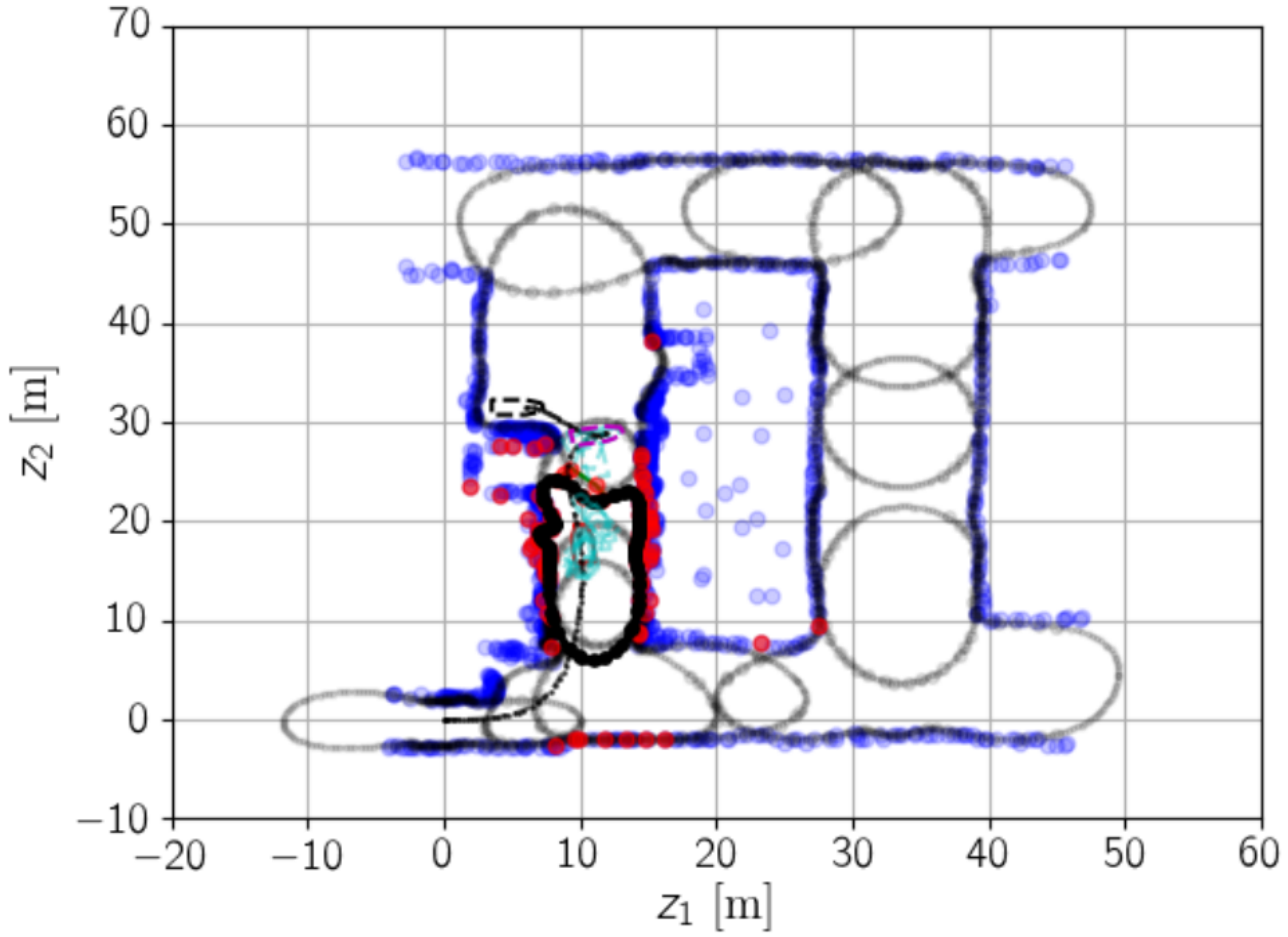

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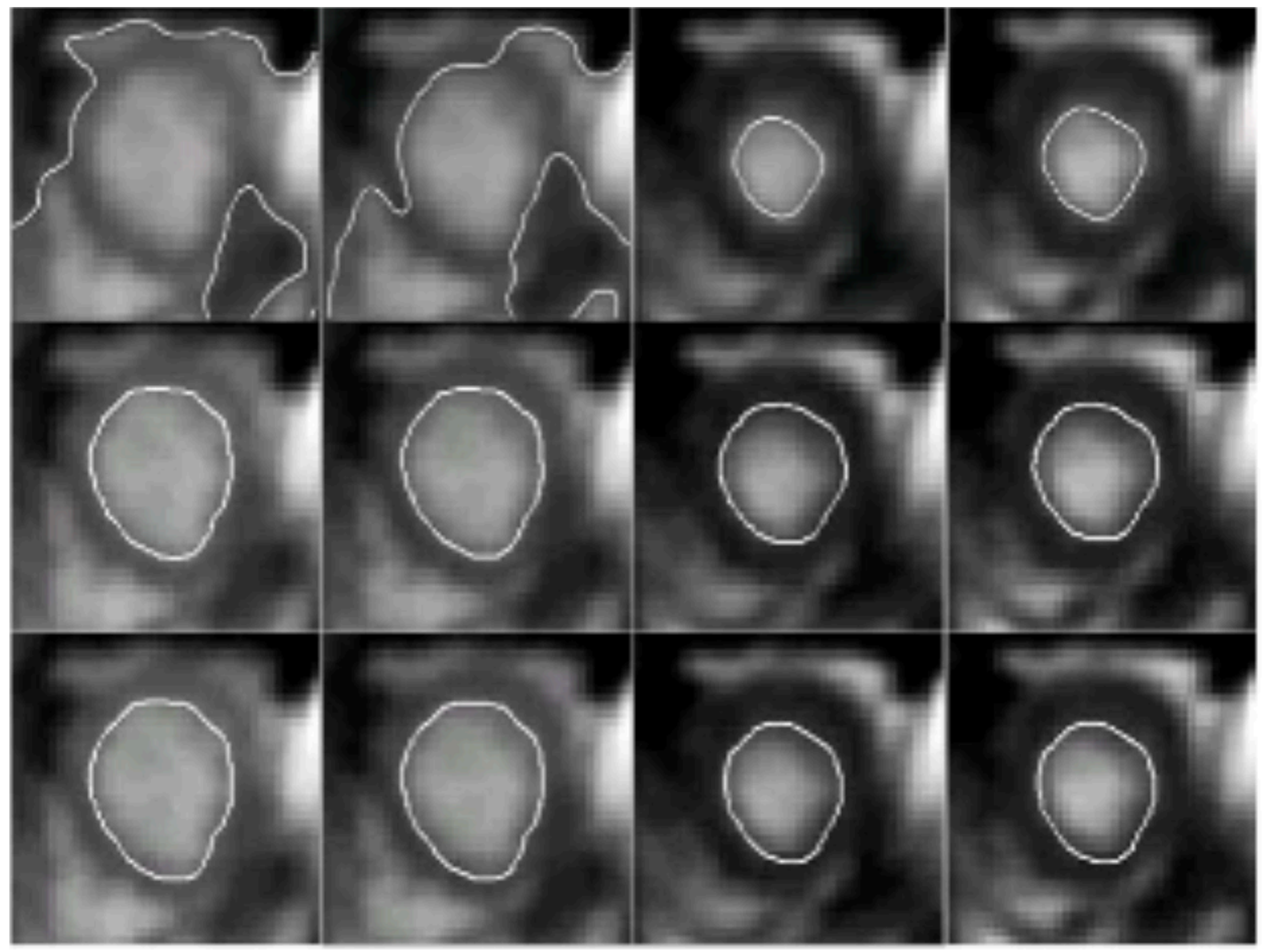

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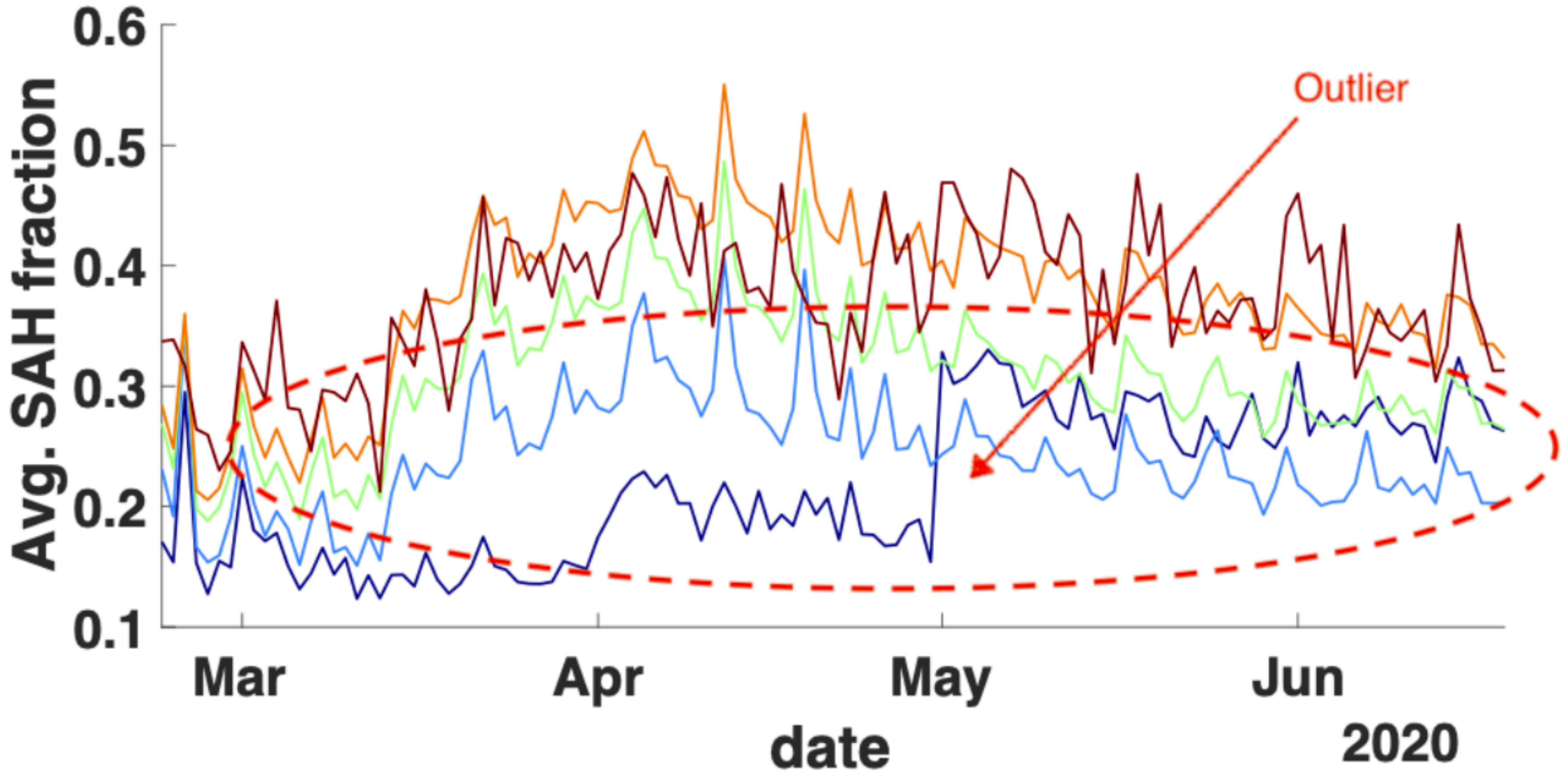
Anomaly detection in statistical dataset:



Kumar & Sarovar, arXiv (2022).

- Image segmentation
- Outlier detection

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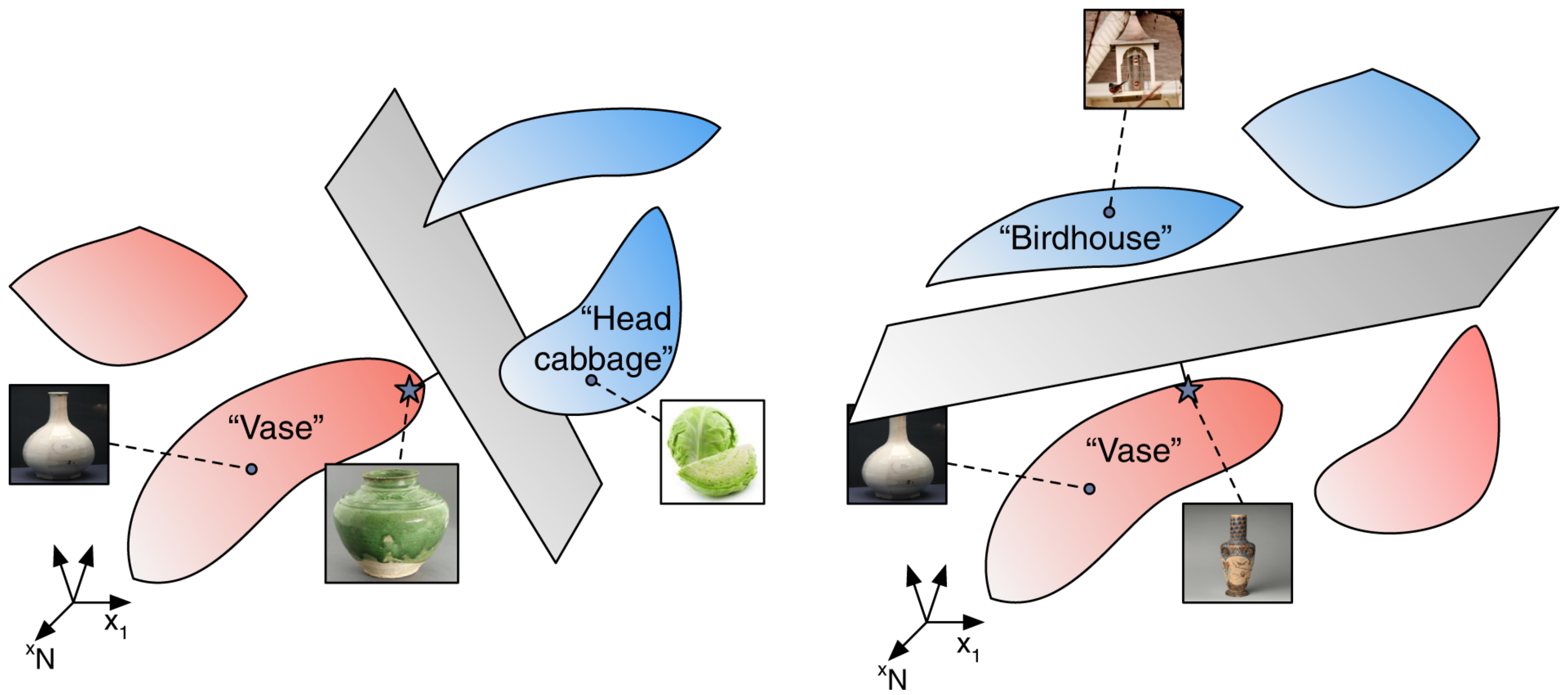

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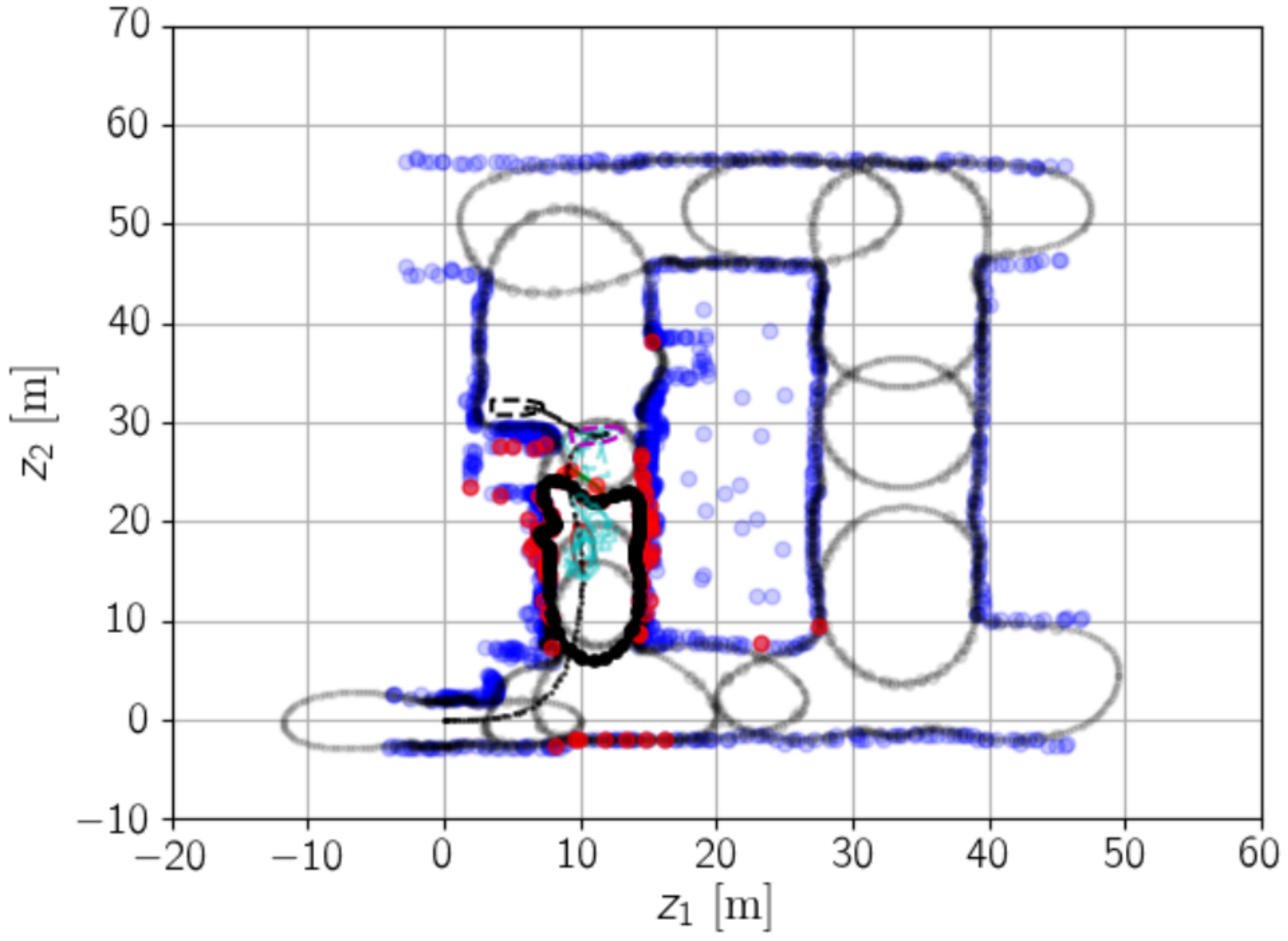

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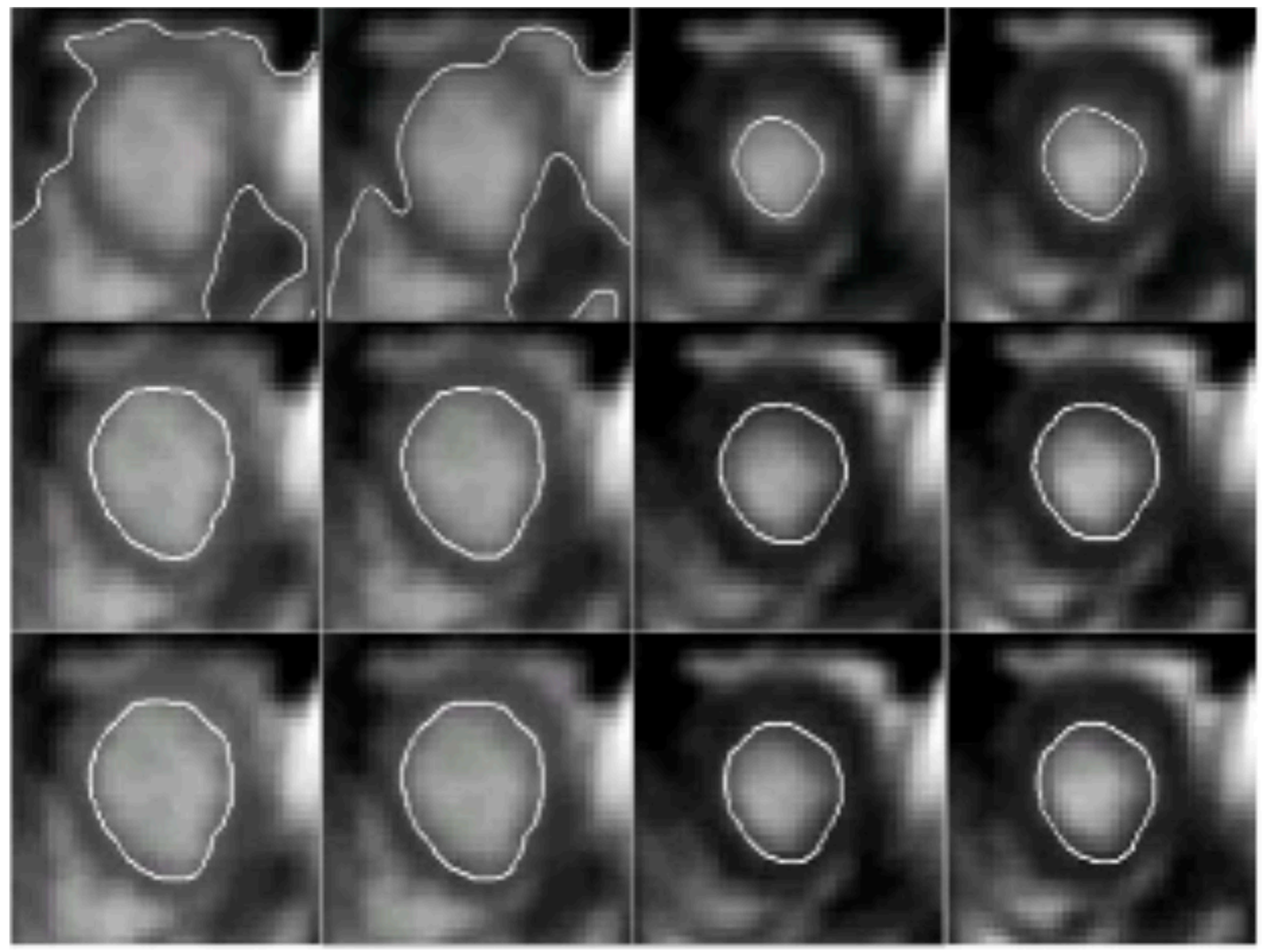

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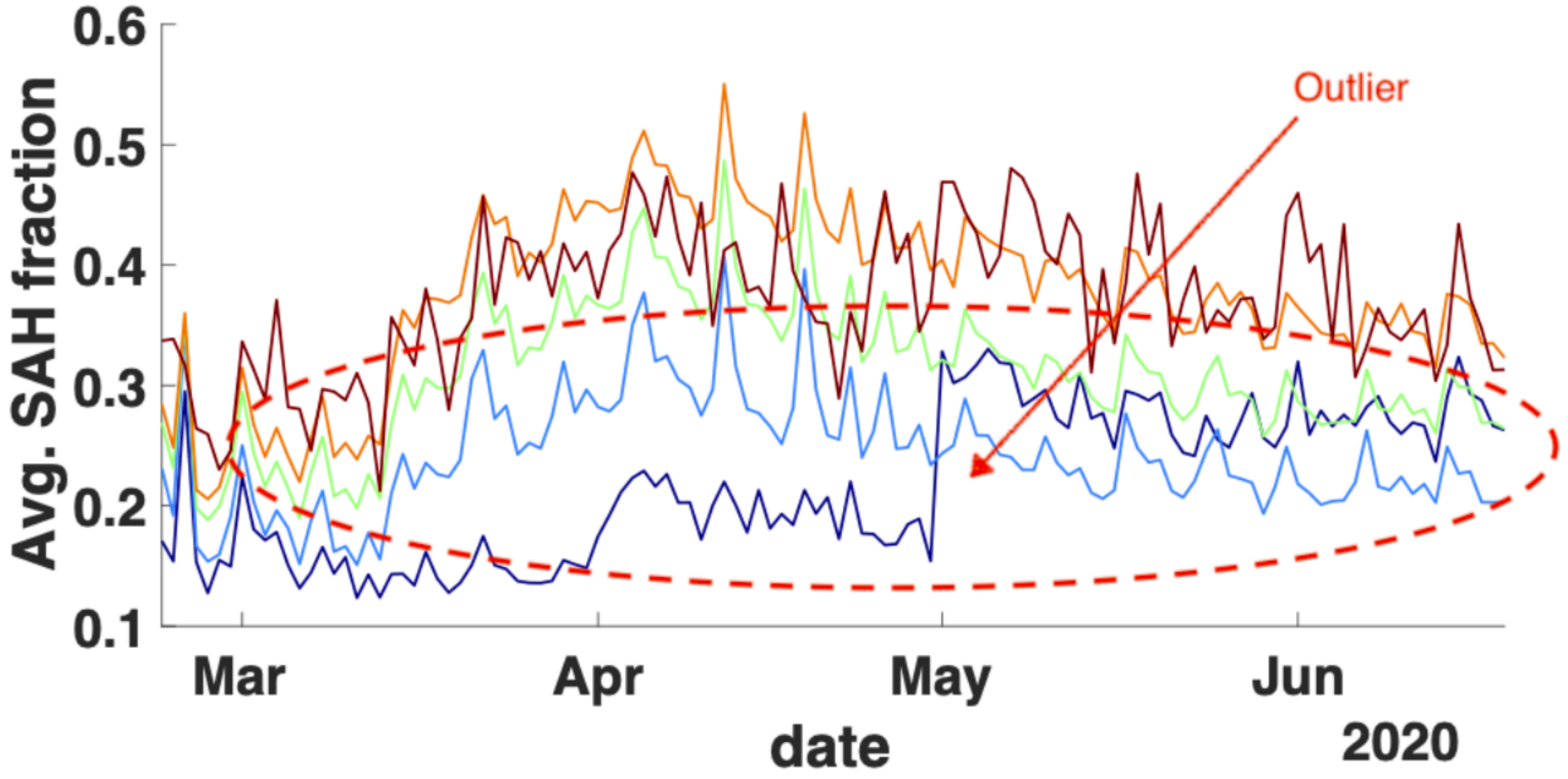

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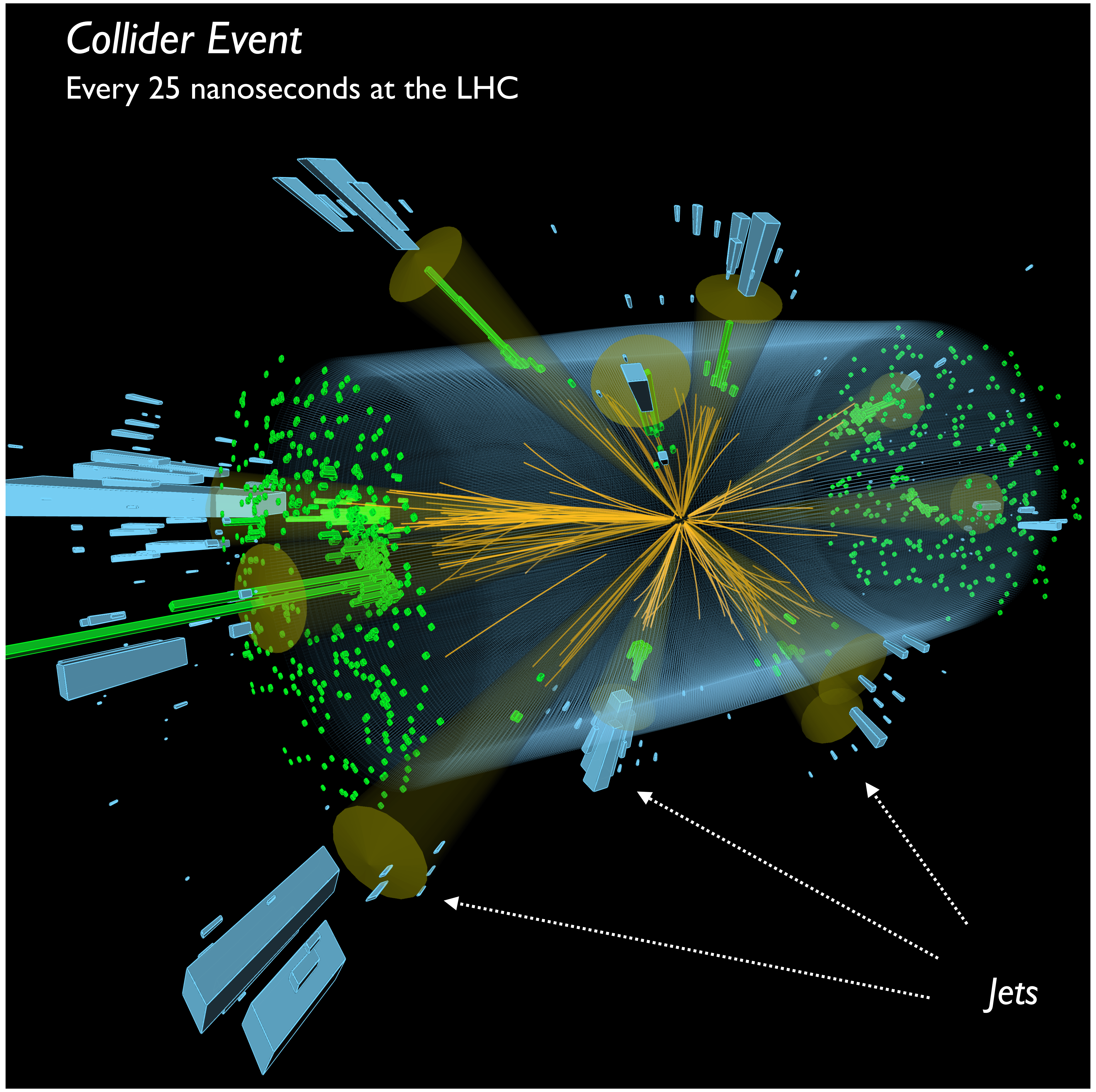
- Classification
- Route planning
- Dimensionality reduction, clustering, augmented reality, regularization of learning methods, visualization, …
- Image segmentation
- Outlier detection



Collider data has geometric structure

Komiske, Metodiev, Thaler, **The hidden geometry of particle collisions**, J. HEP, 6 (2020)

Figures from talk by Jesse Thaler, University of Chicago and Caltech AI+Science: https://www.youtube.com/watch?v=BMBSAWUxBn4

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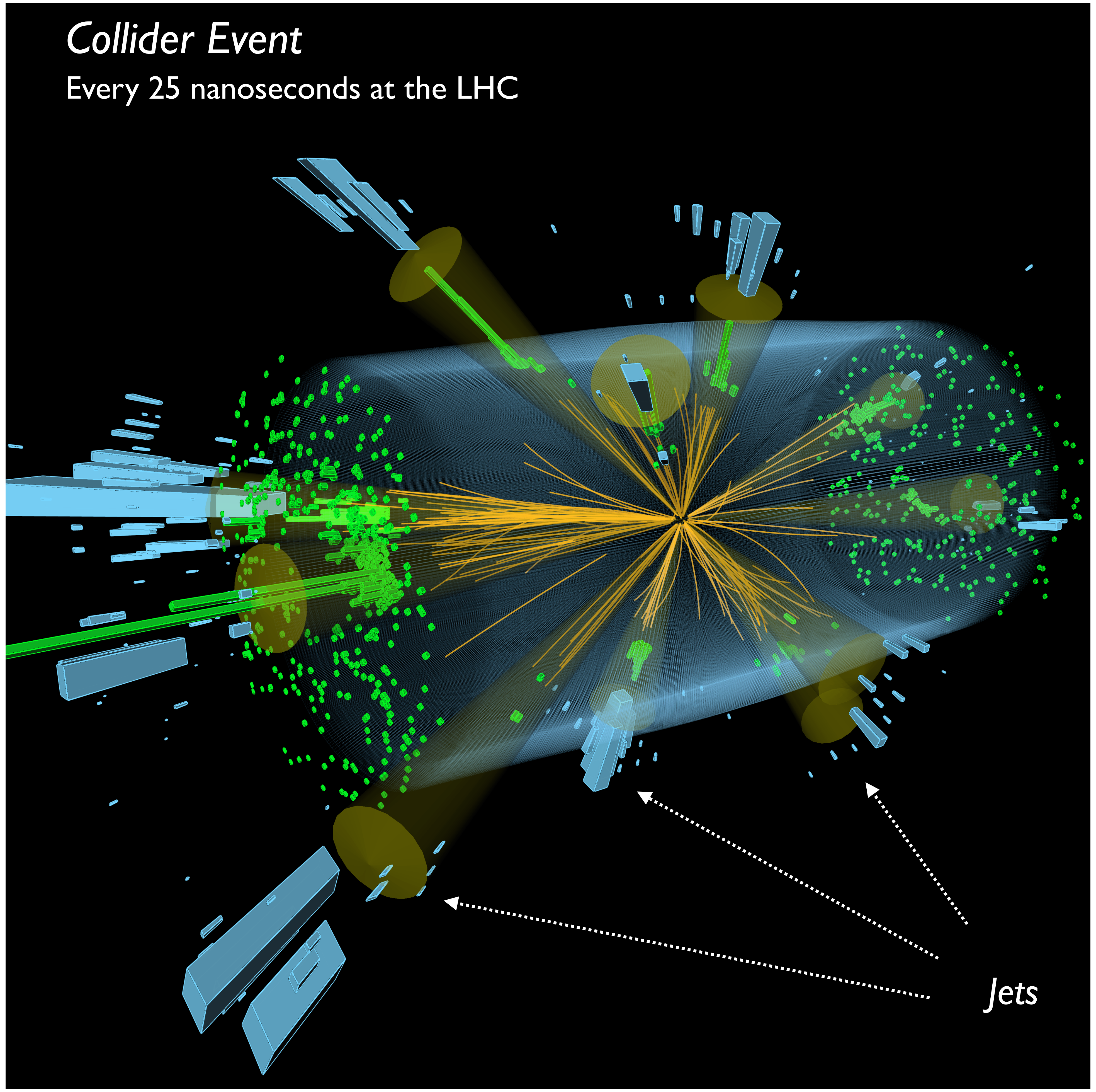

Energy flow (or Event):

$$\mathcal{E}(\hat{n}) = \sum_{j=1}^{N} E_j \delta(\hat{n} - \hat{n}_j)$$

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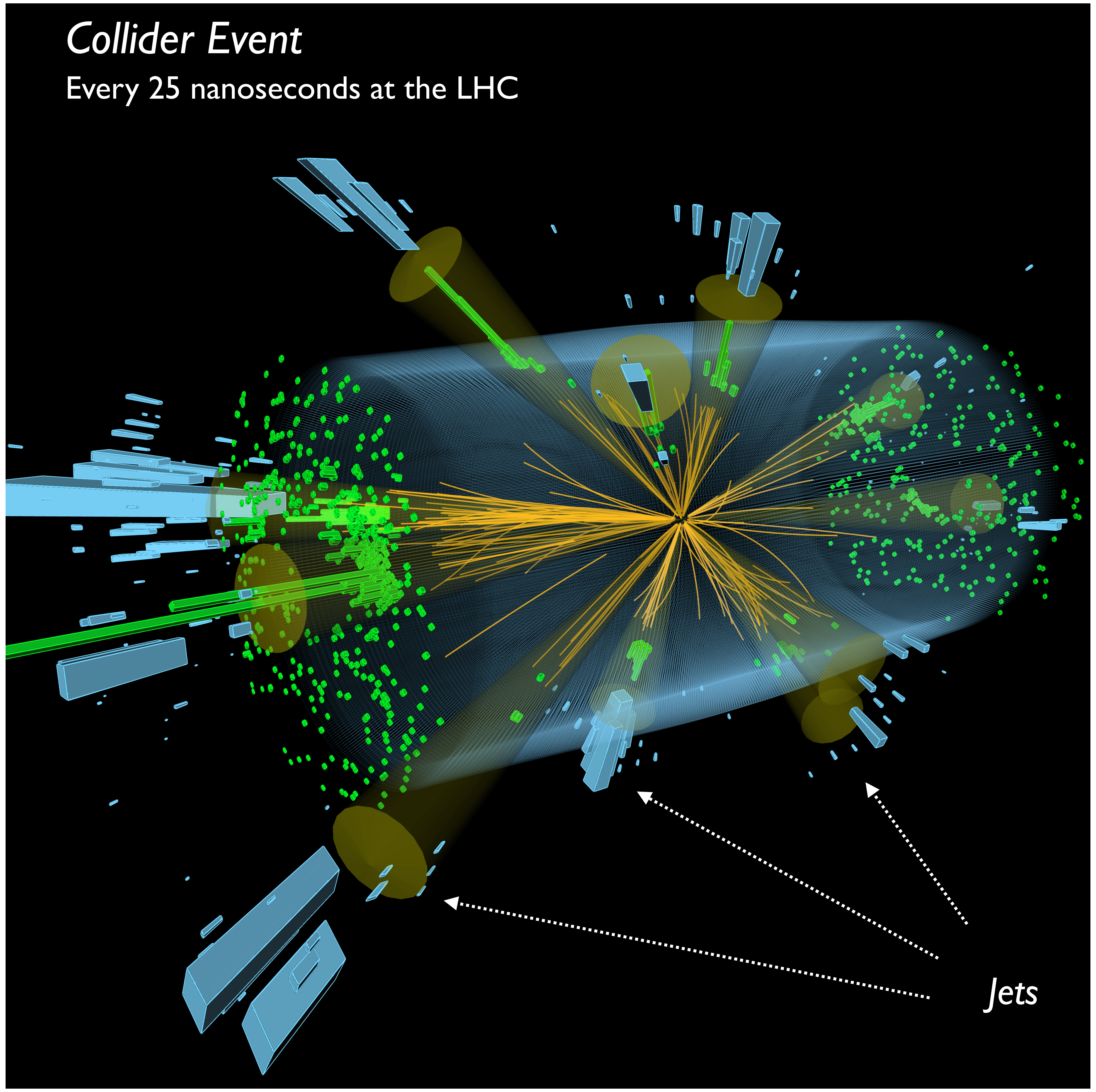

#particles

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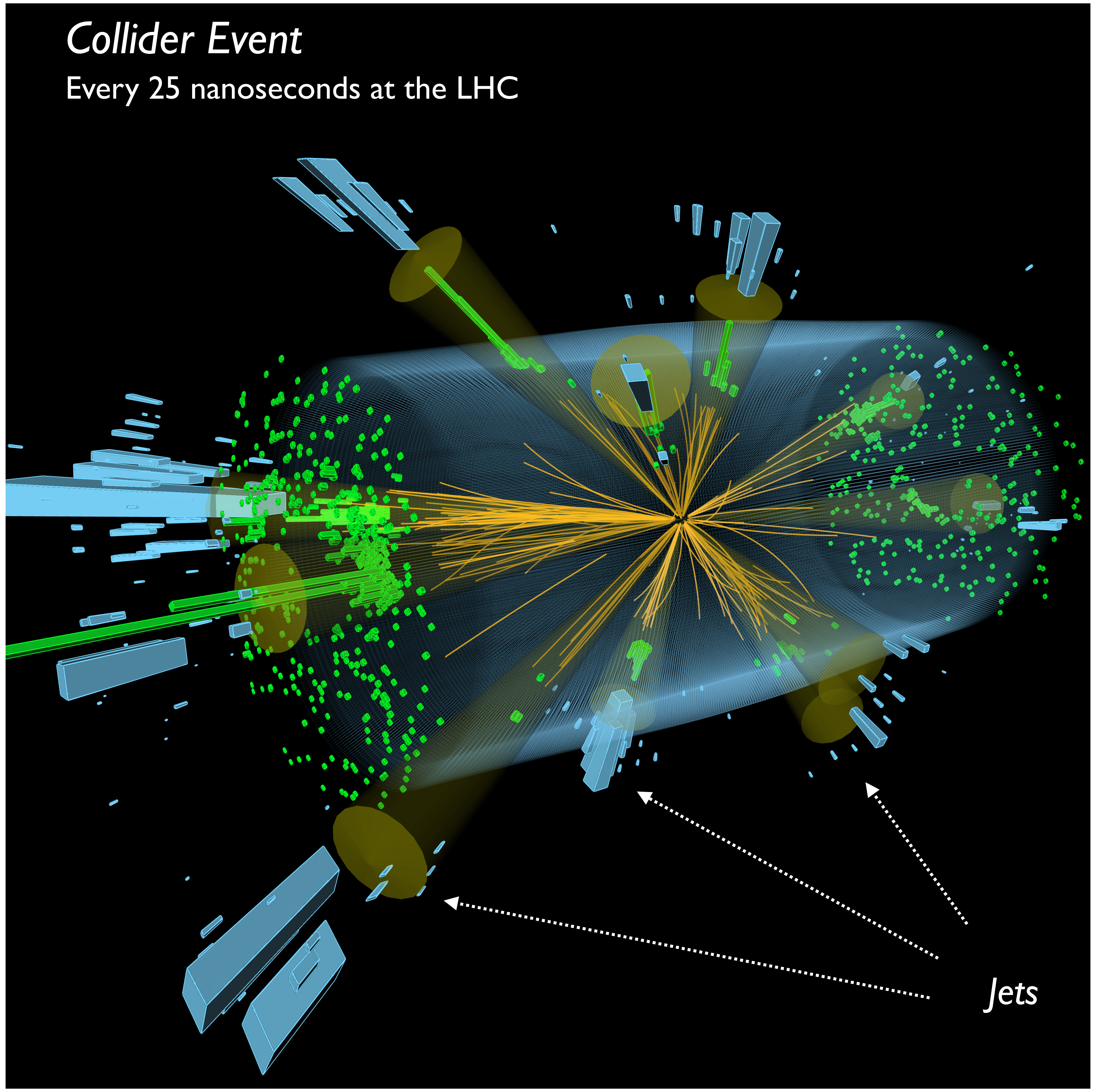

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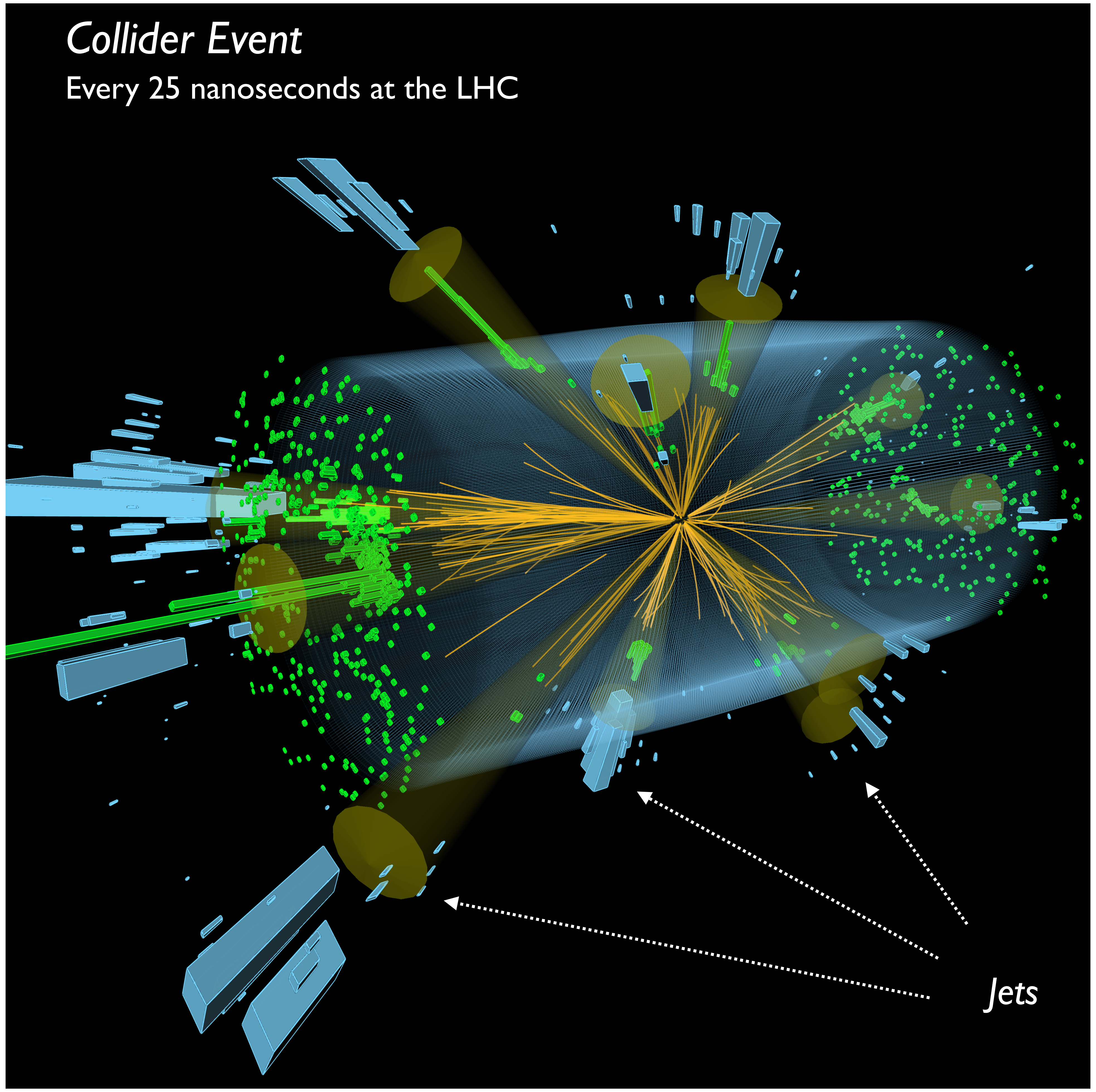

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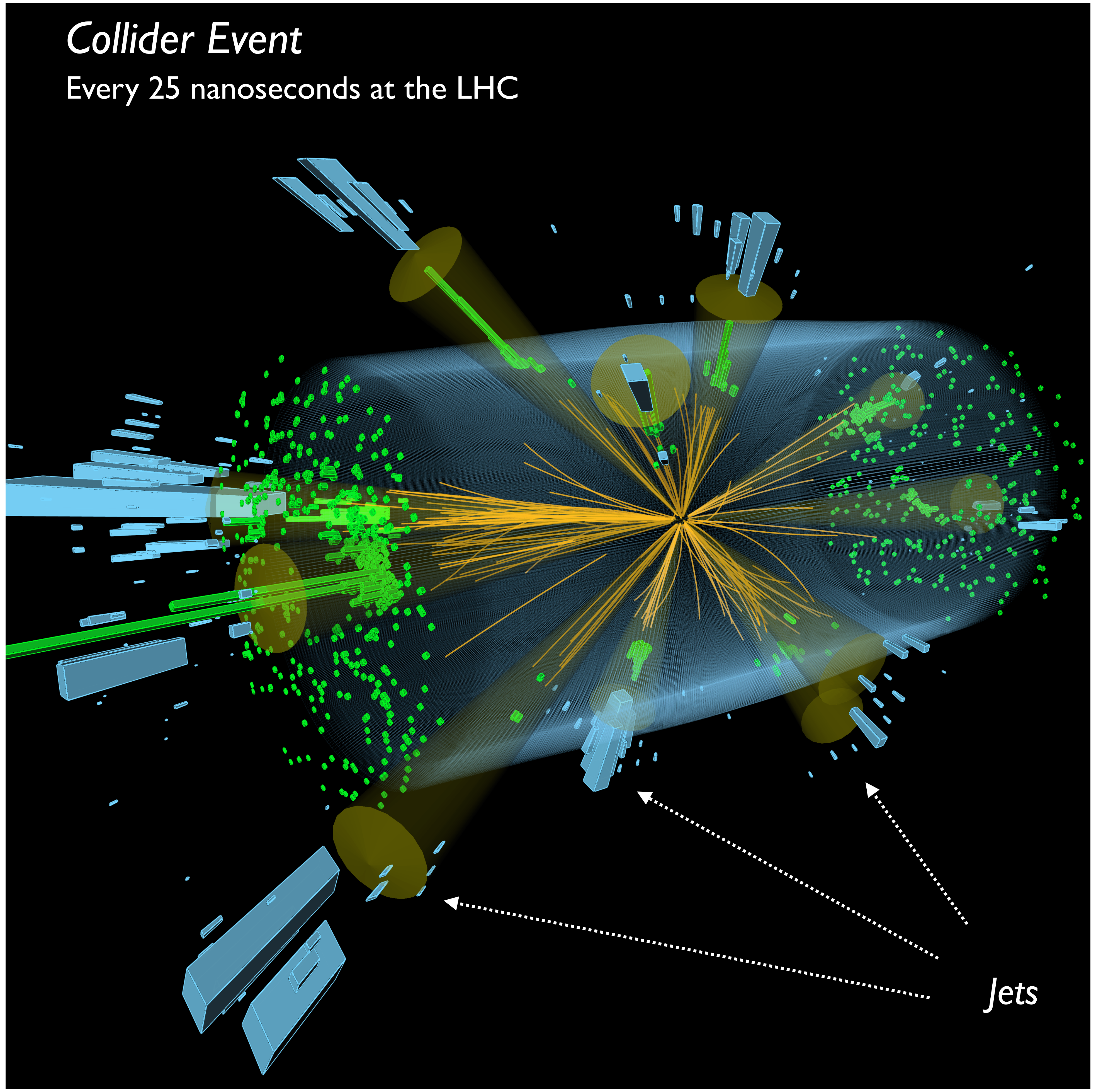

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∠directions

$$\wr\!\wr$$

Energy-weighted point cloud:

$$(E_j, \hat{n}_{x,j}, \hat{n}_{y,j}, \hat{n}_{z,j})_{j=1}^{N}$$

$$\cap$$

$$\mathbb{R}_+ \times \mathbb{S}^2$$

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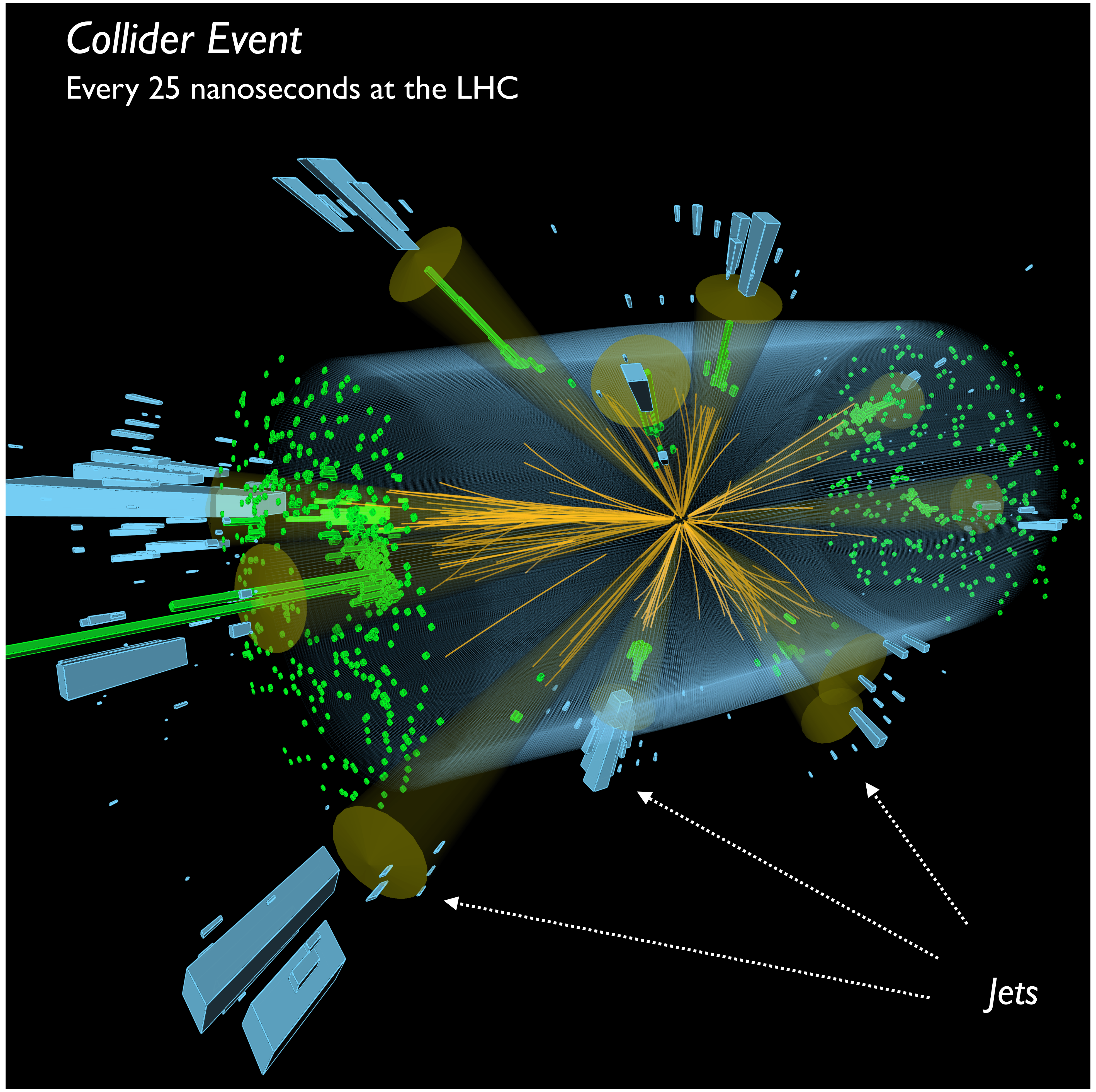

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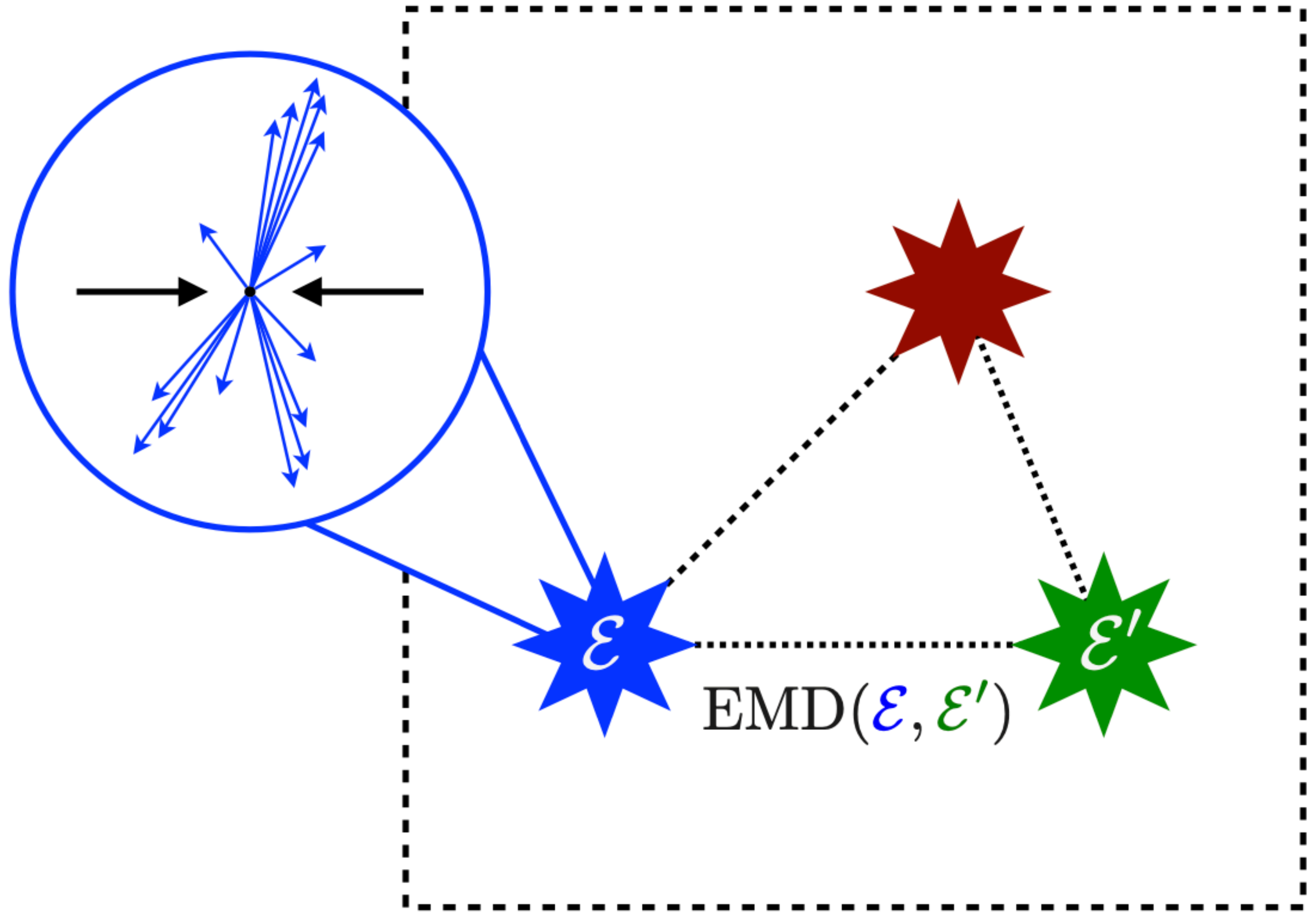
Metric* space of possible events



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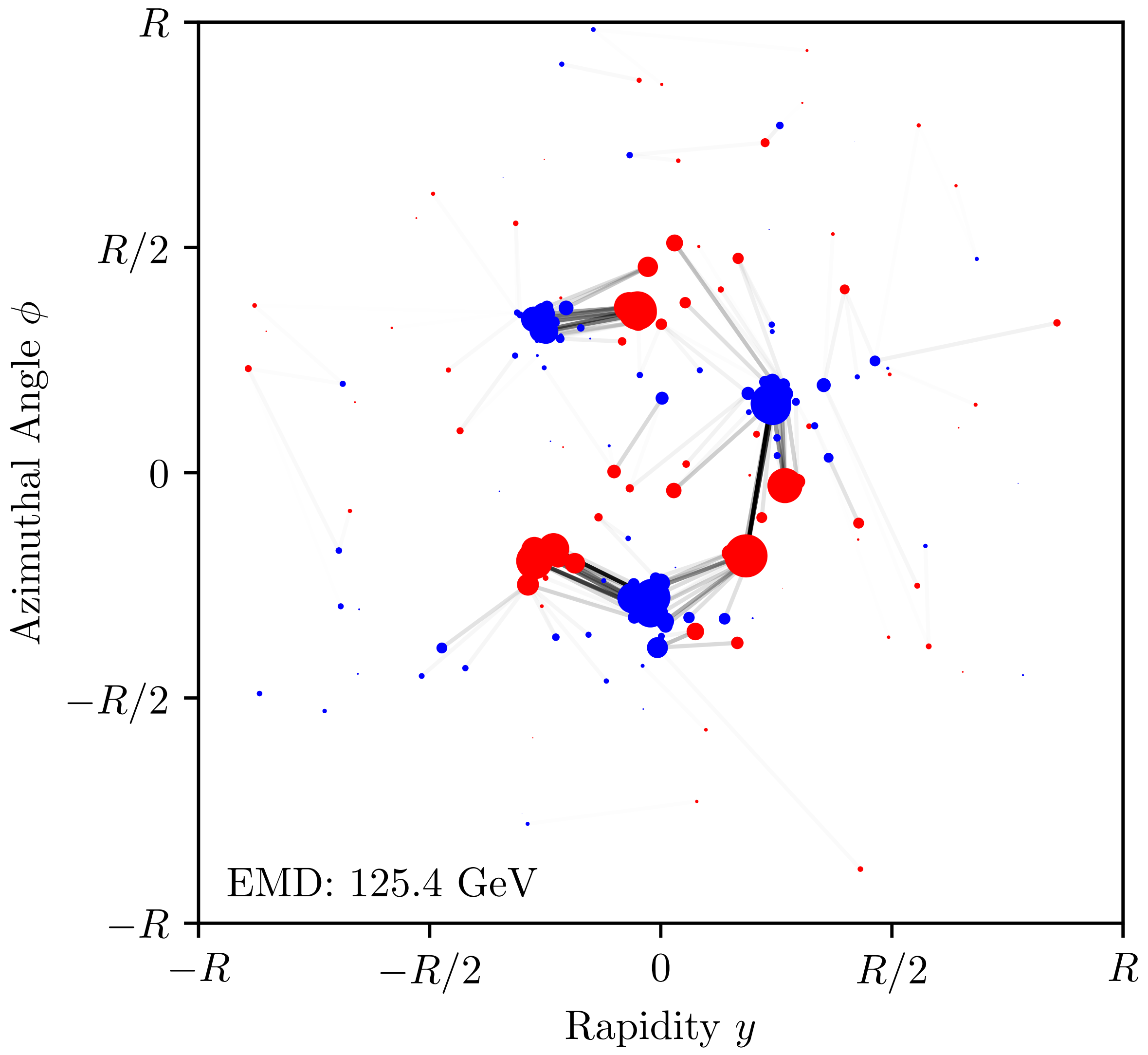

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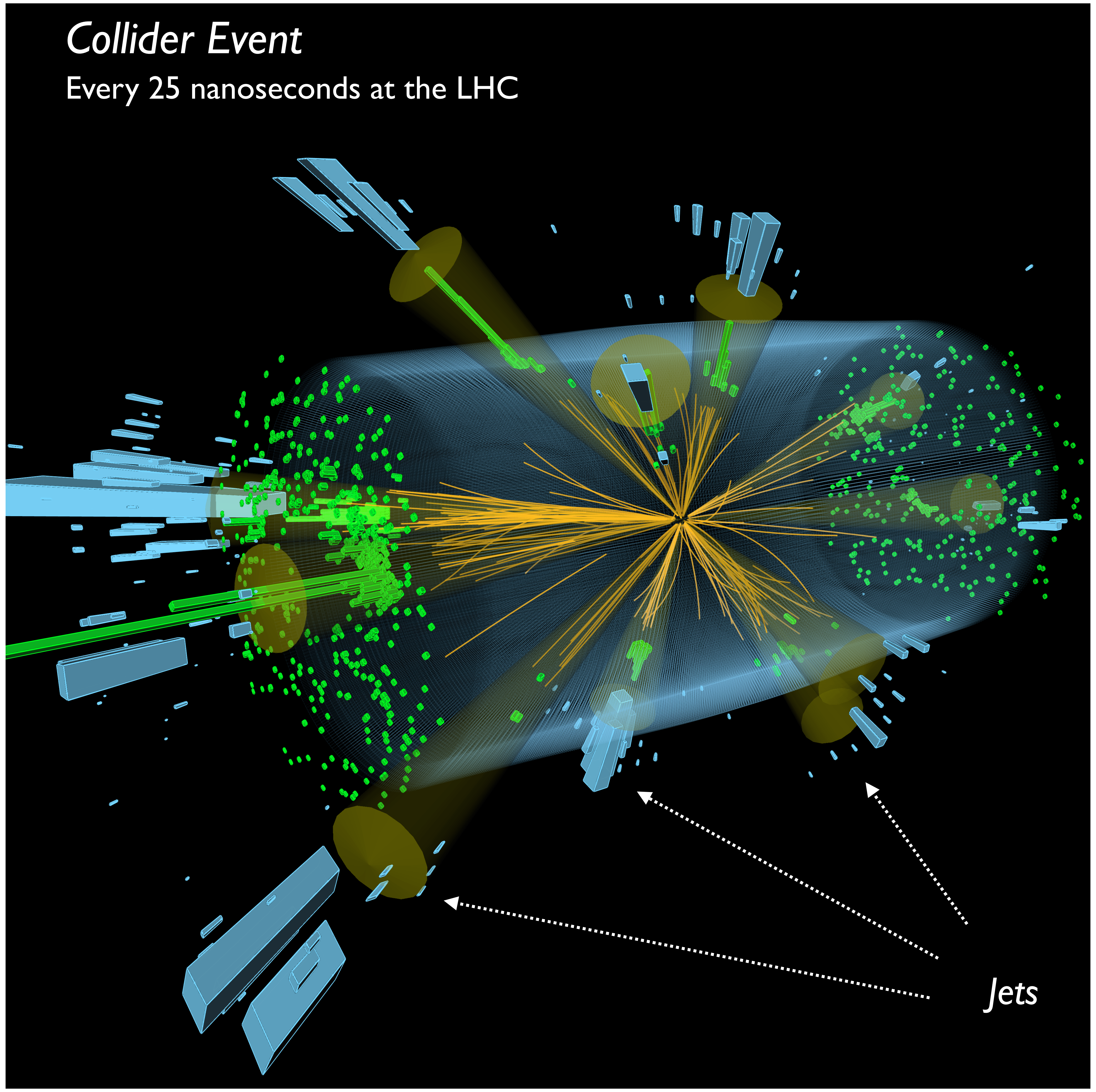

energy (GeV)

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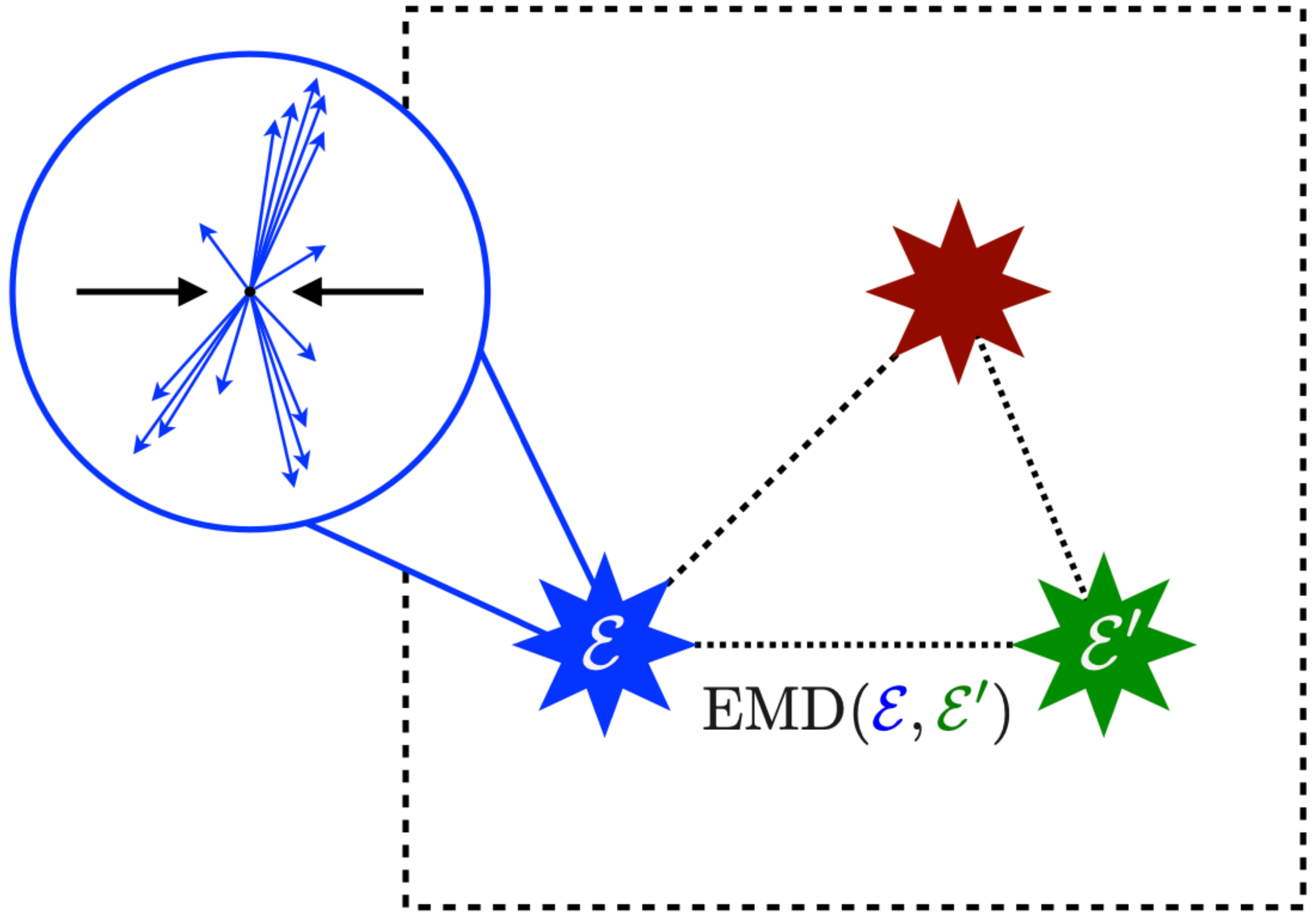

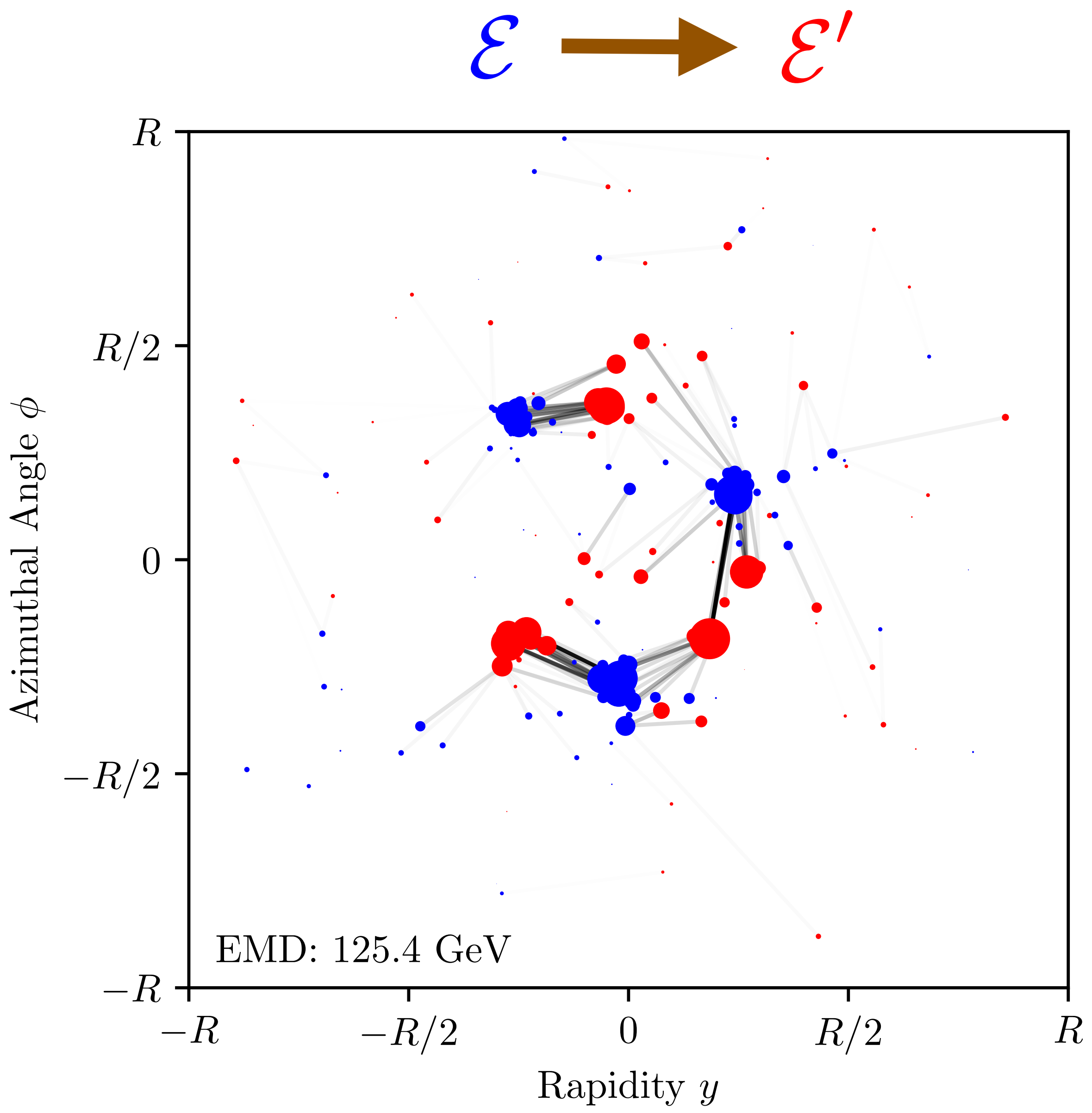

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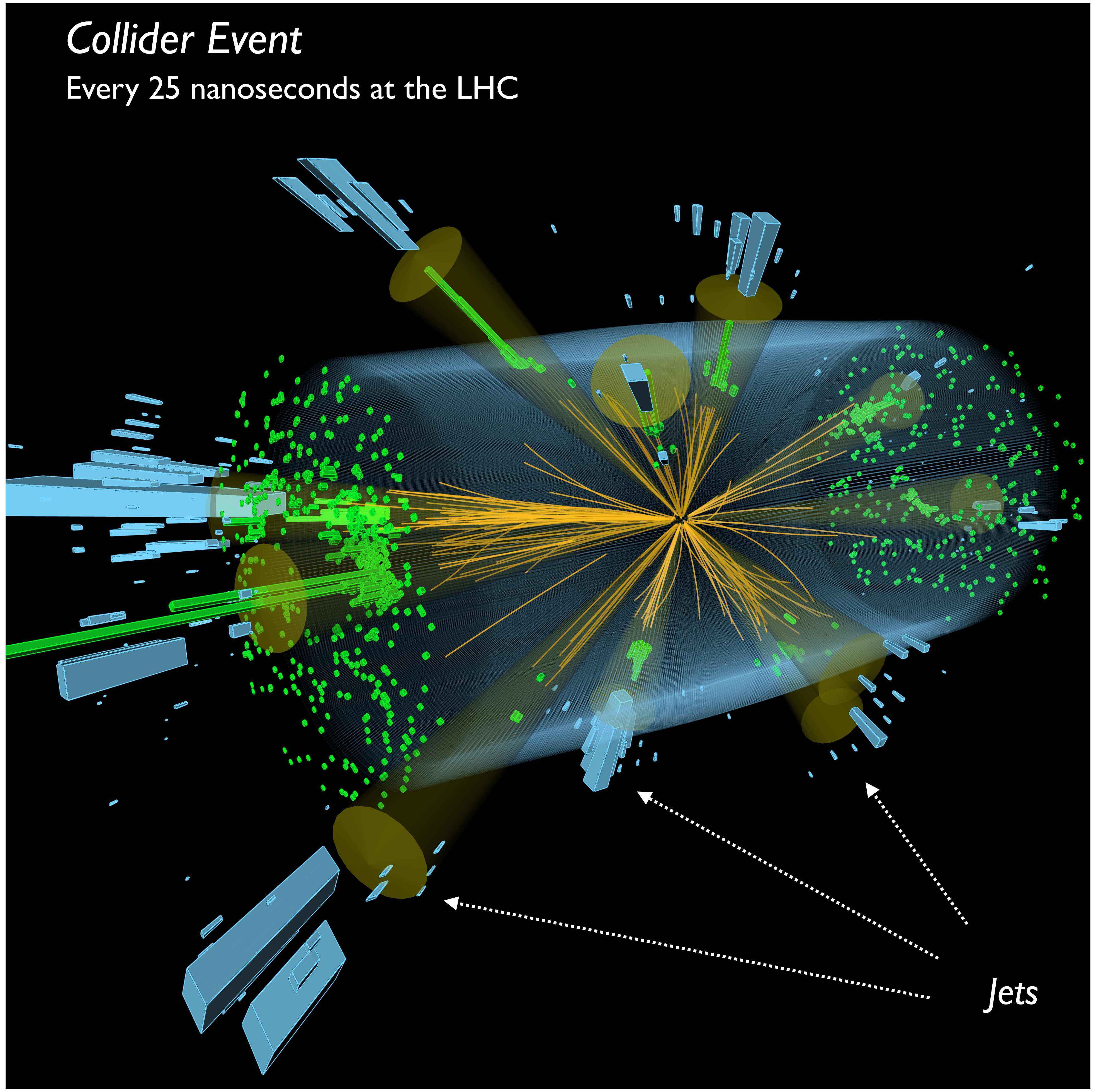

energy (GeV)

cost to move energy

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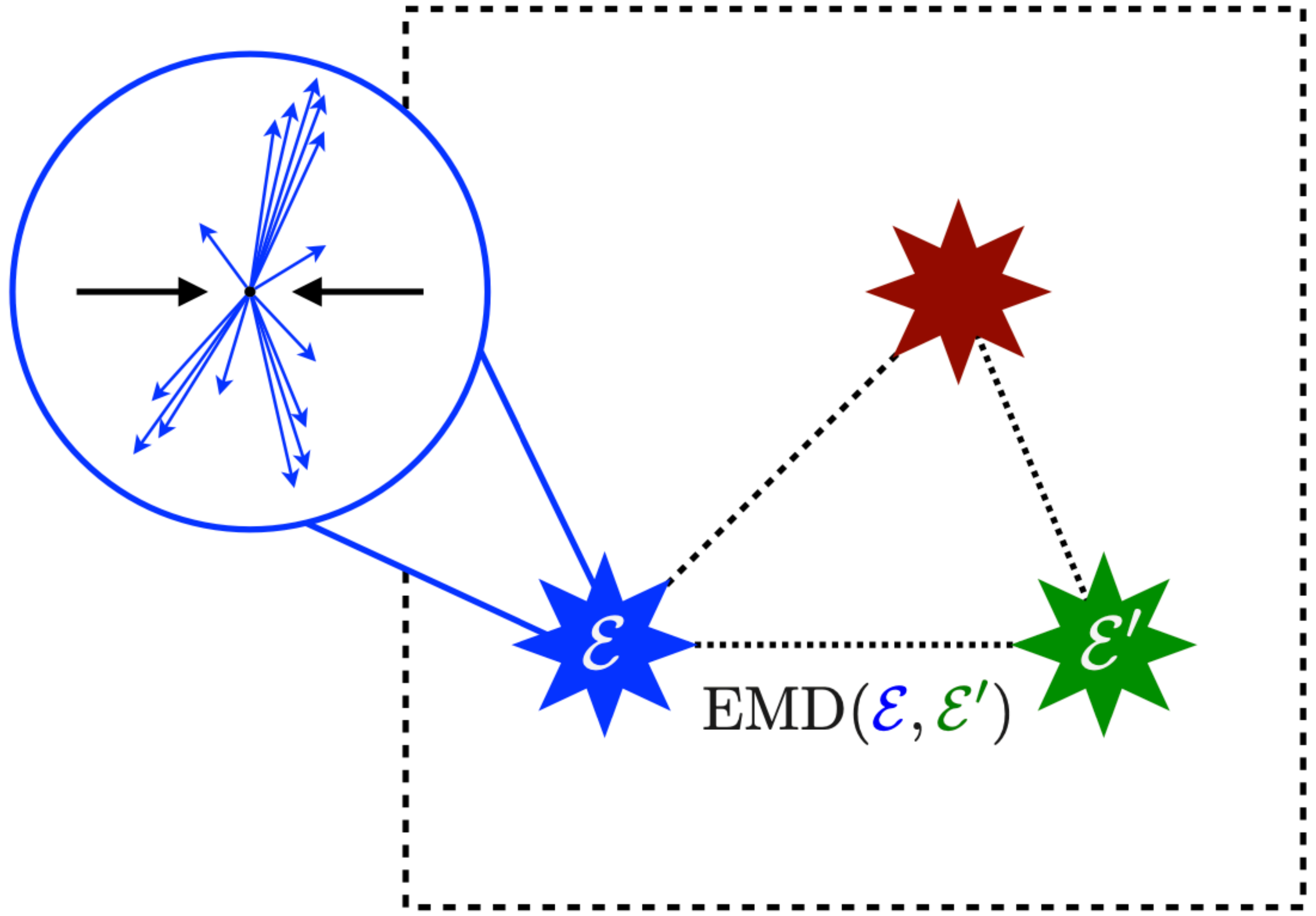

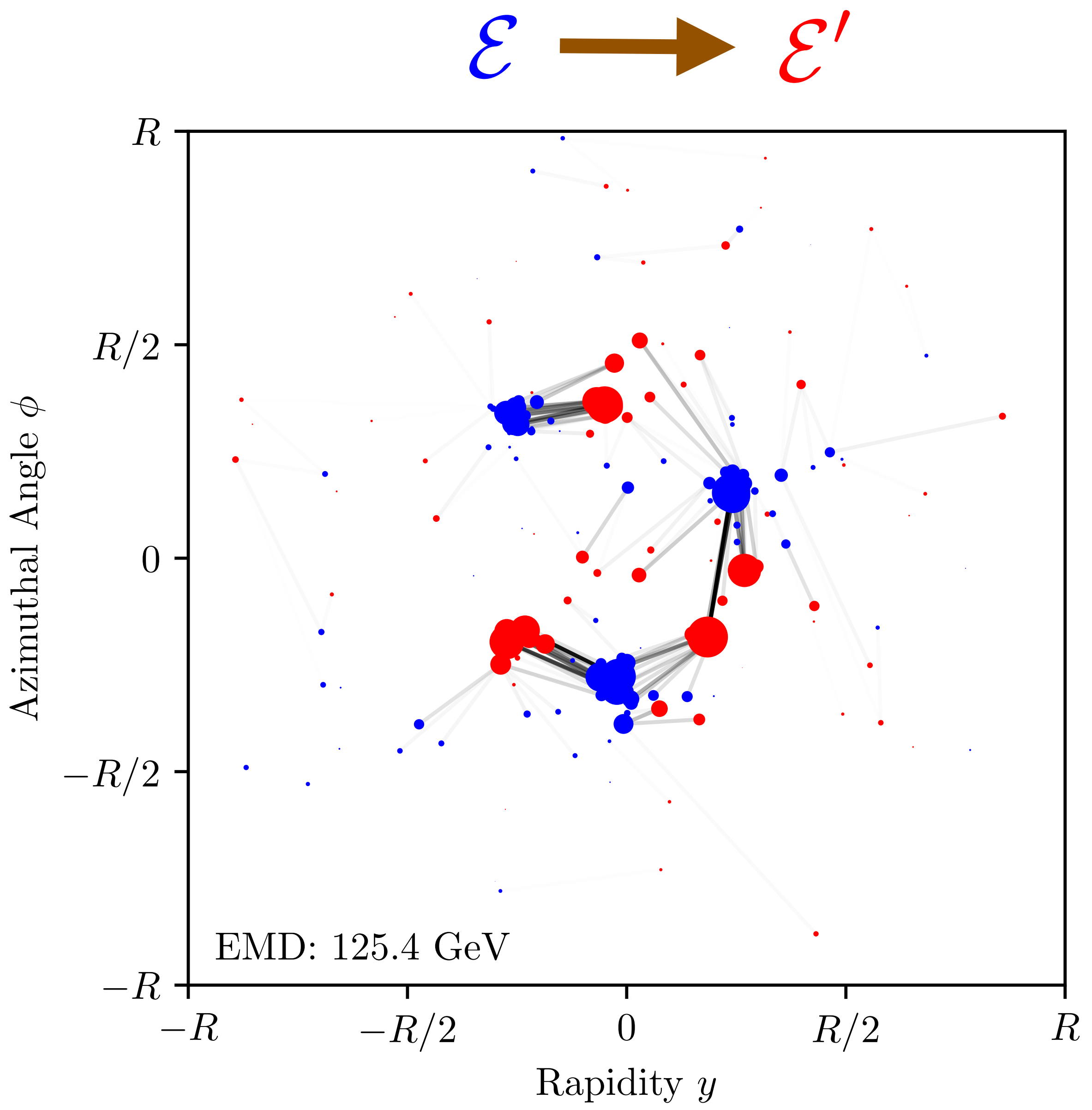

* for R sufficiently large
i.e., $R \geq$ jet radius for conical jets

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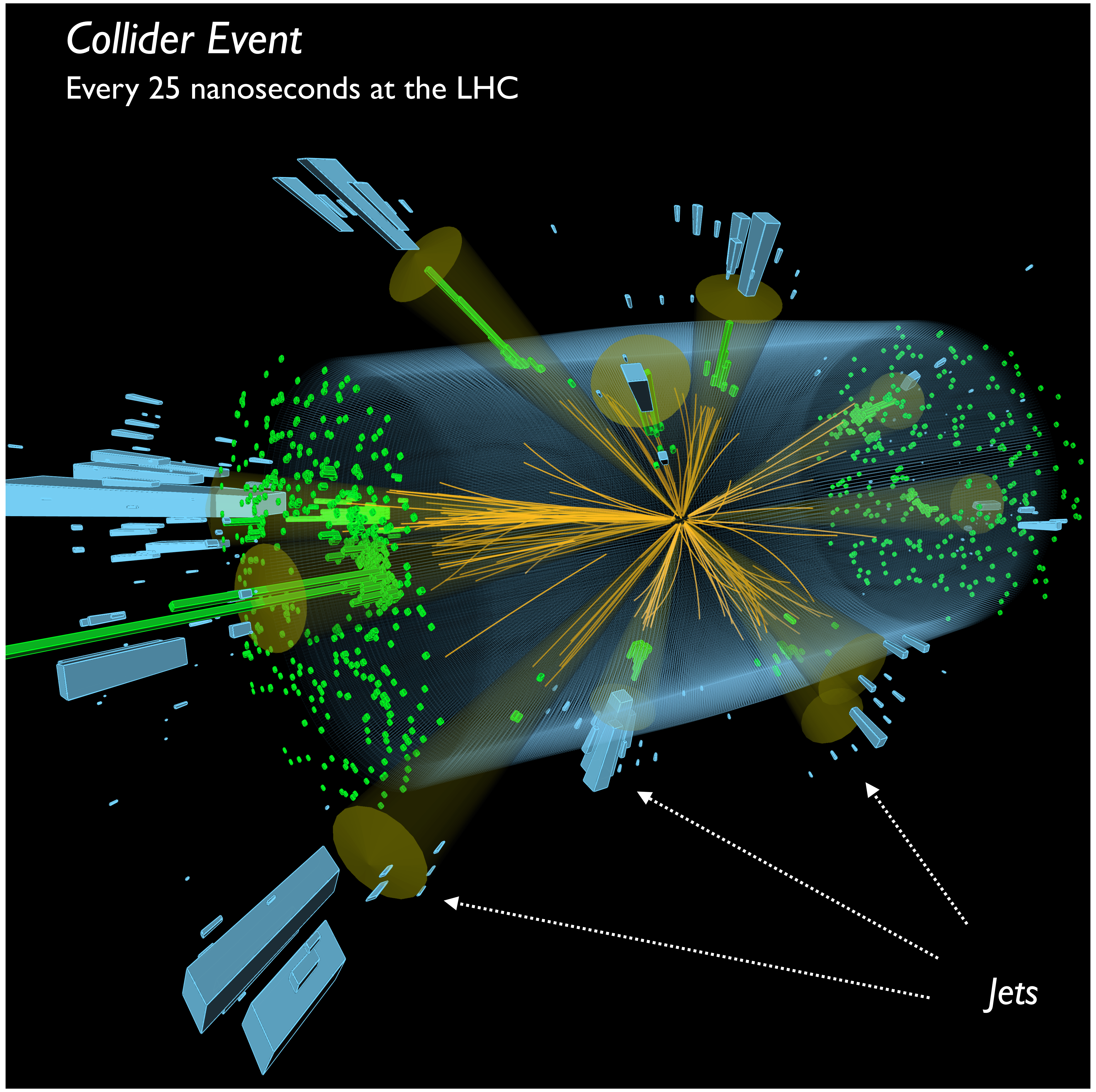

energy (GeV)

cost to move energy

cost to create energy

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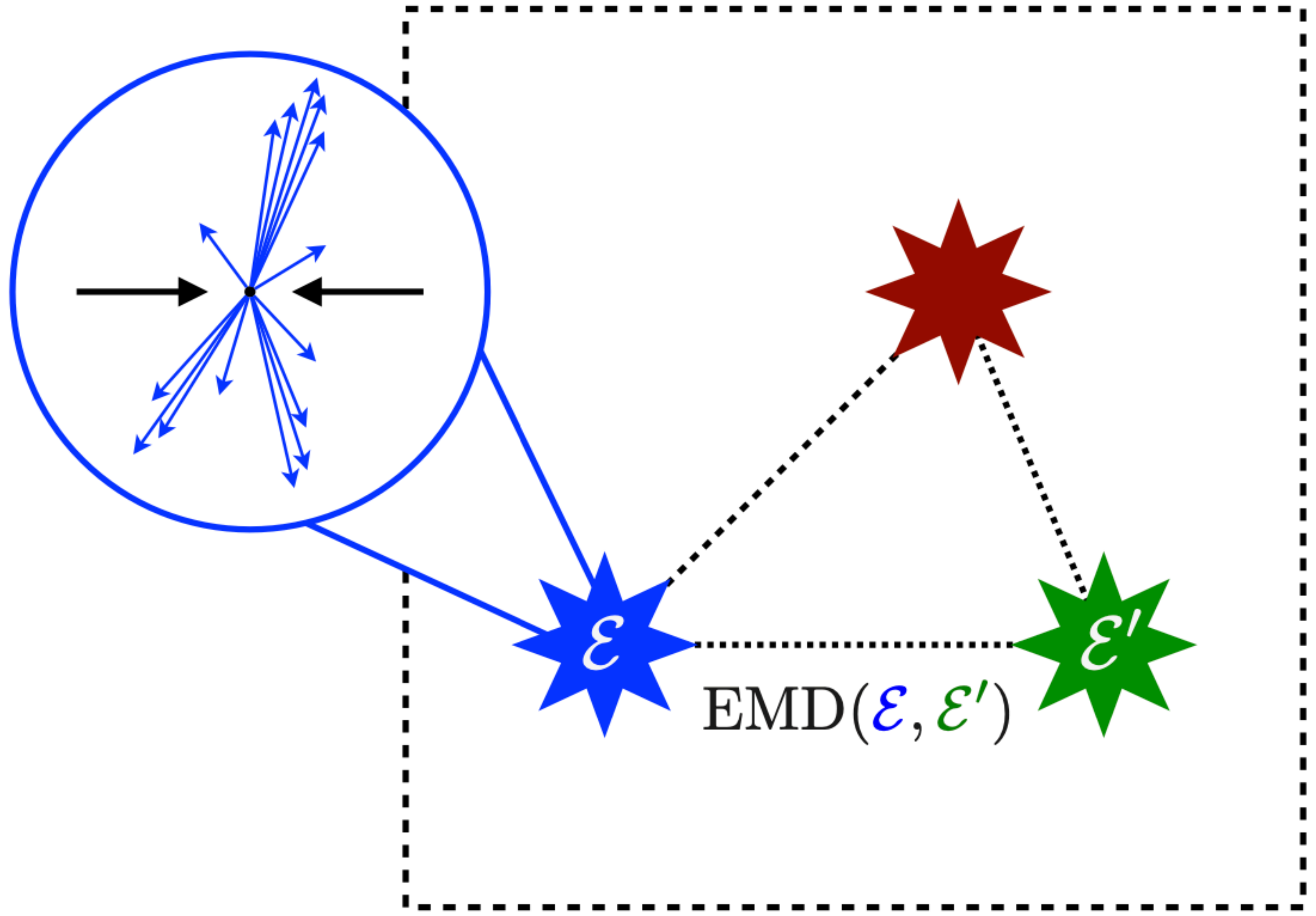

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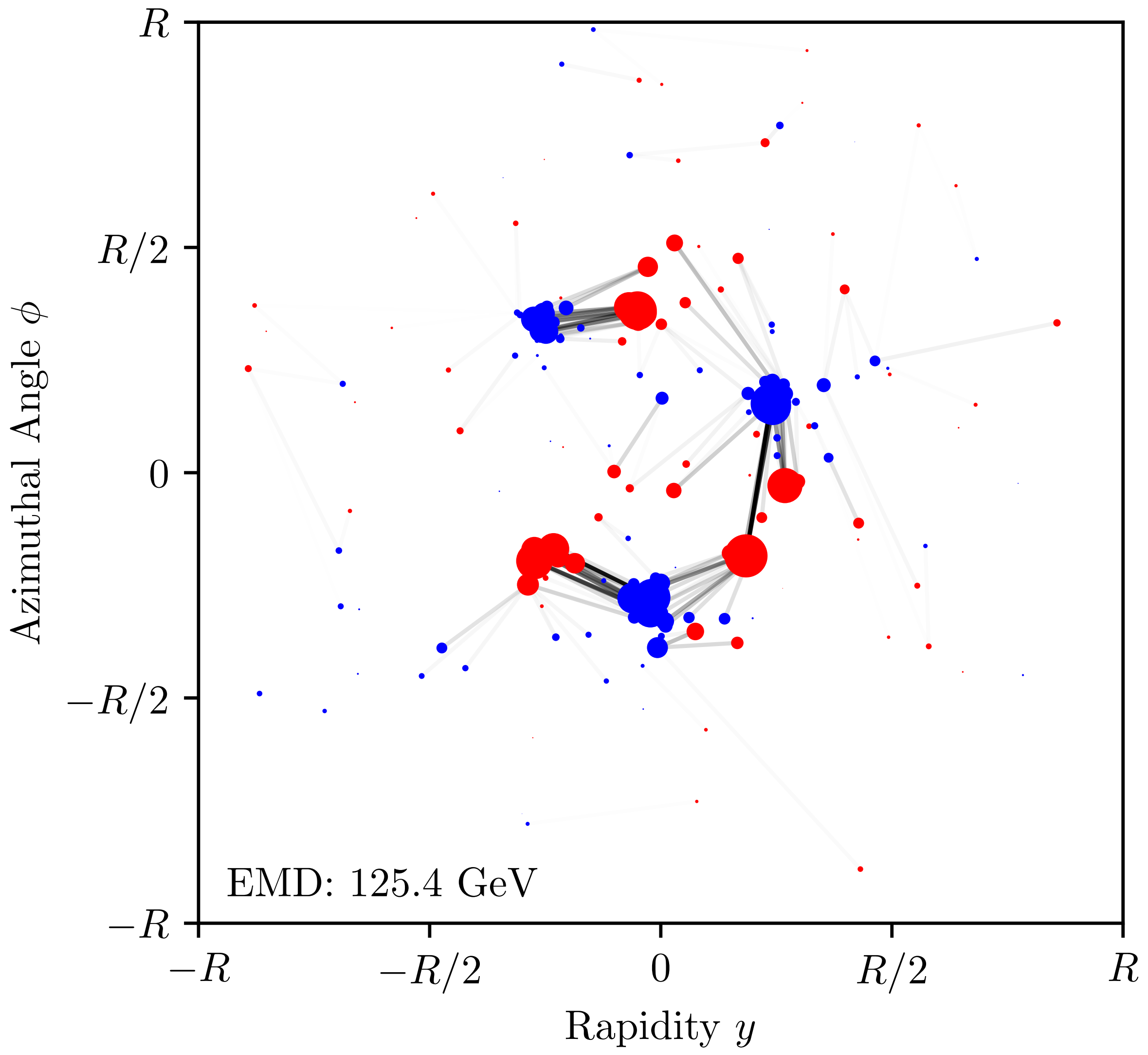

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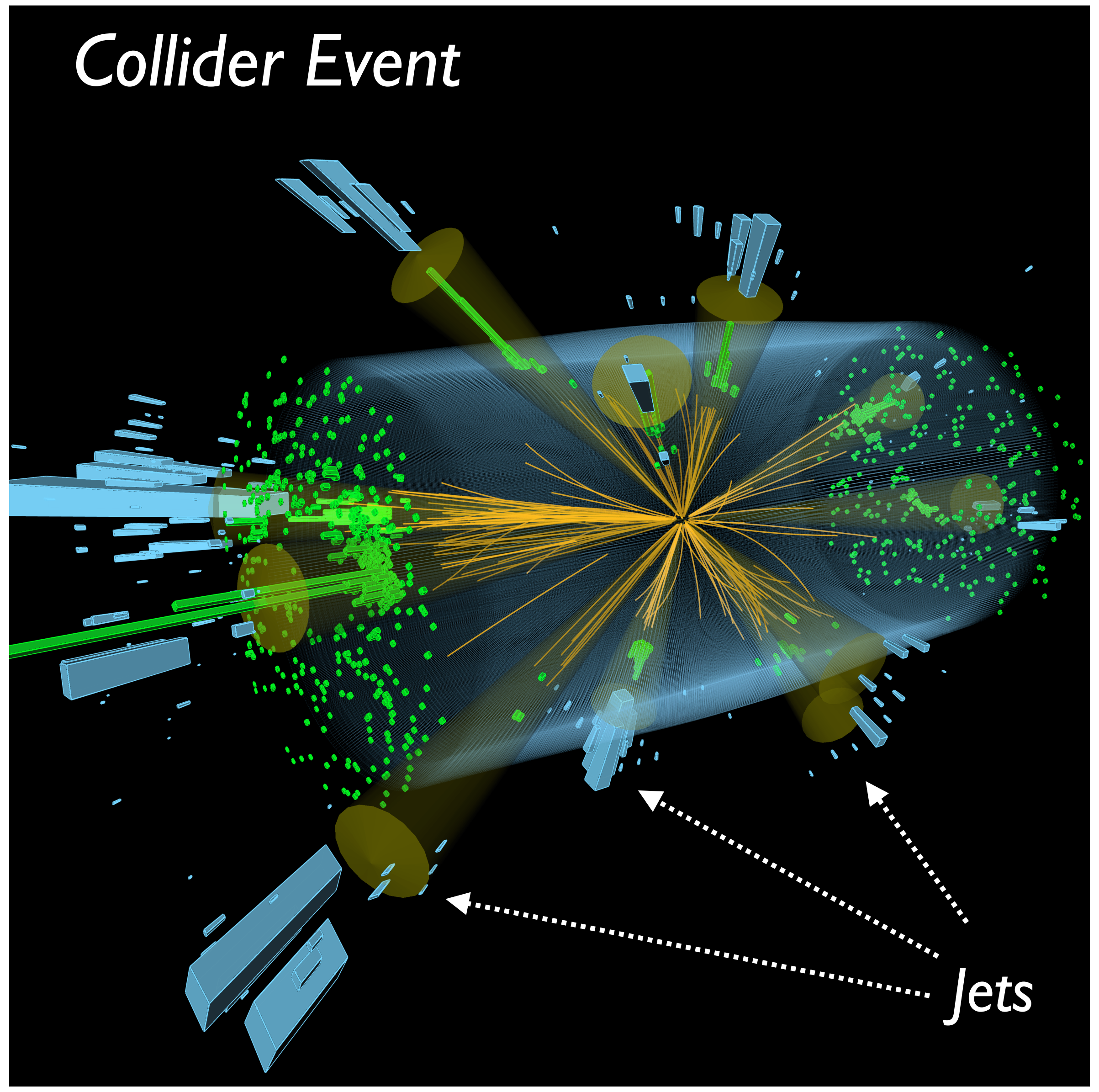
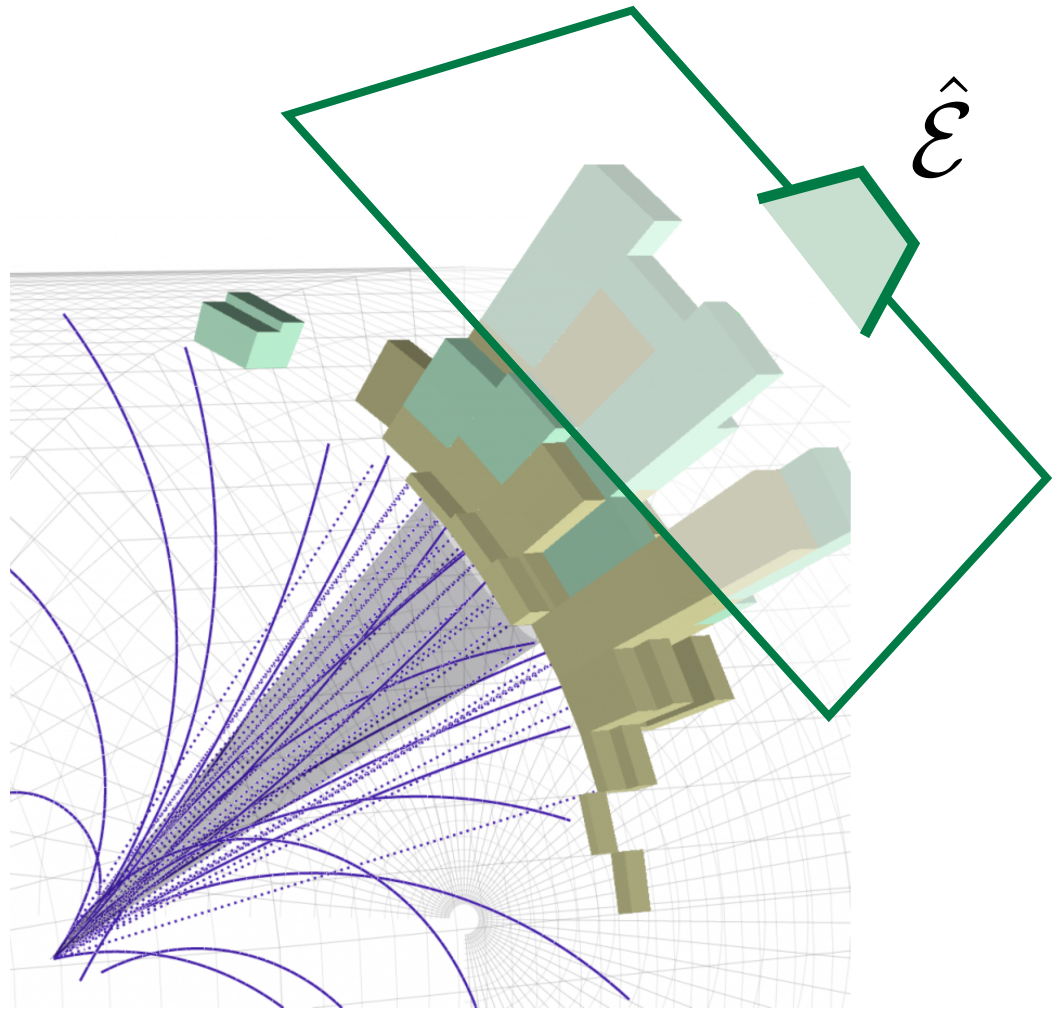
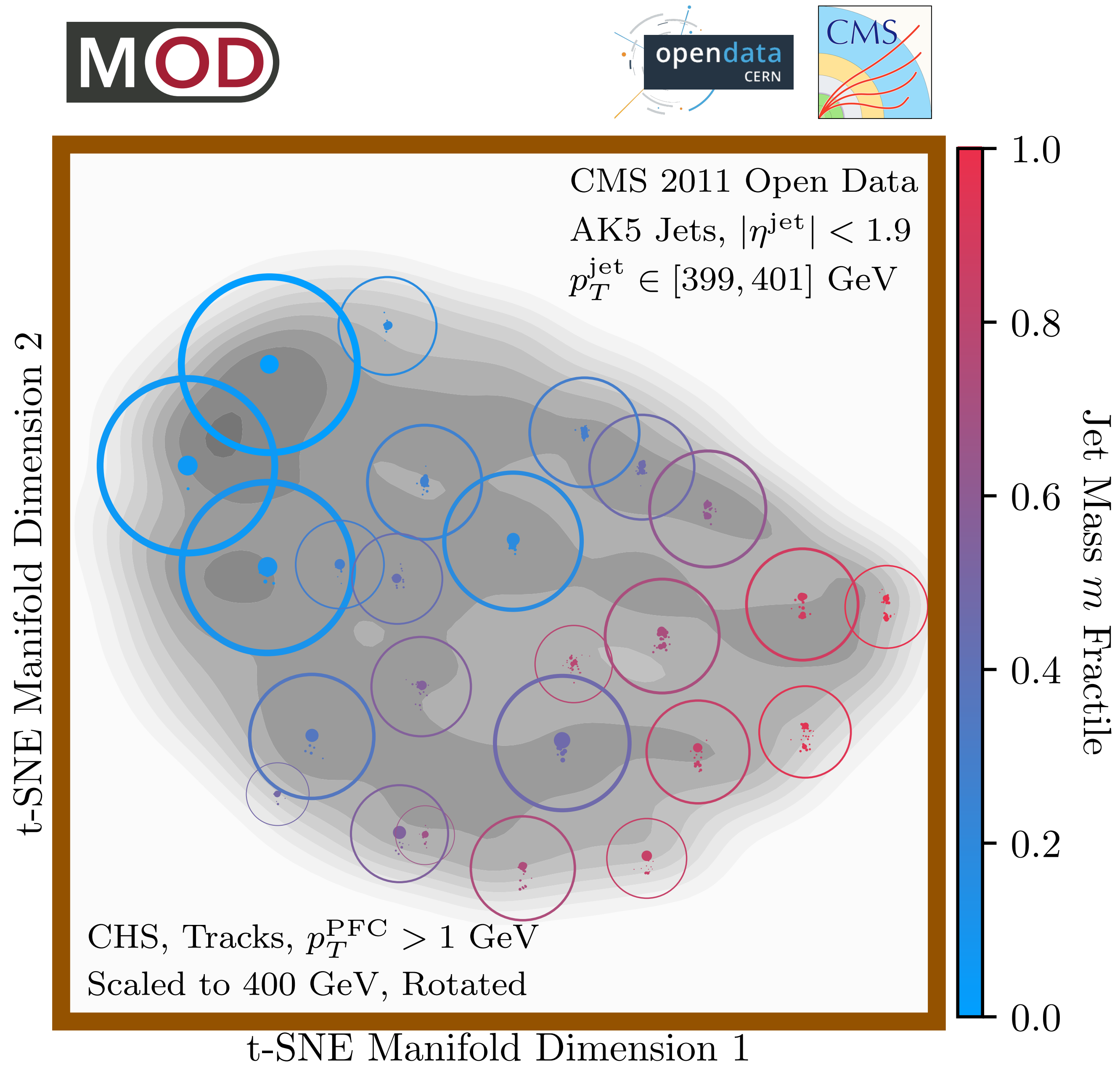


Events summarized by
"energy flow"

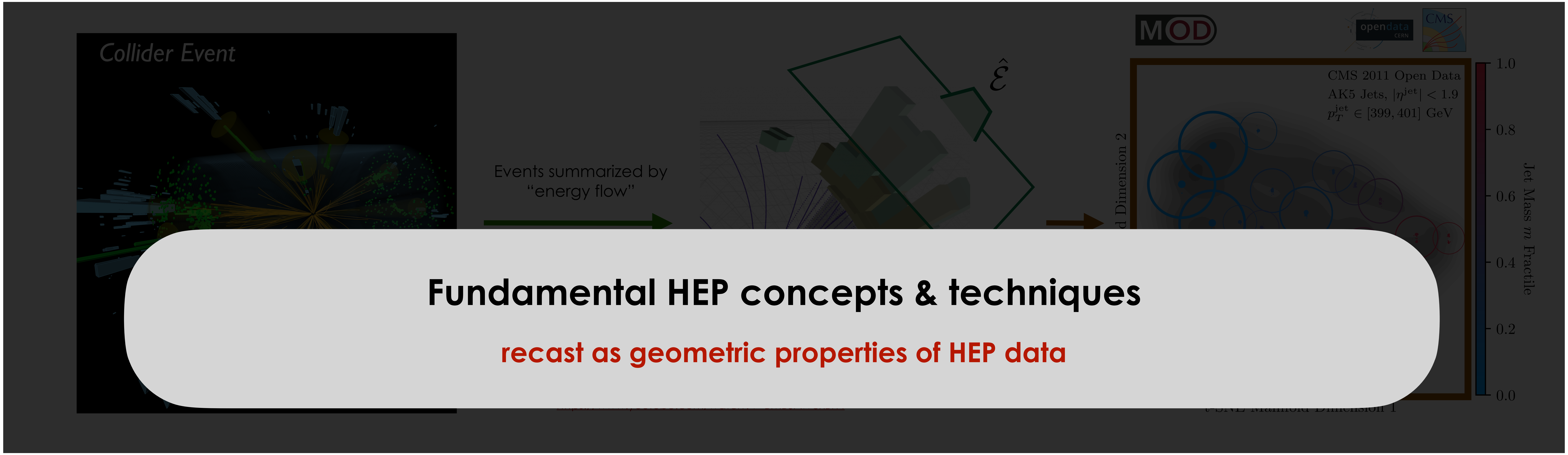




Figures from talk by Jesse Thaler, University of Chicago and Caltech AI+Science:
https://www.youtube.com/watch?v=BMBSAWUxBn4

Komiske, Metodiev, Thaler, The hidden geometry of particle collisions, J. HEP, 6 (2020)

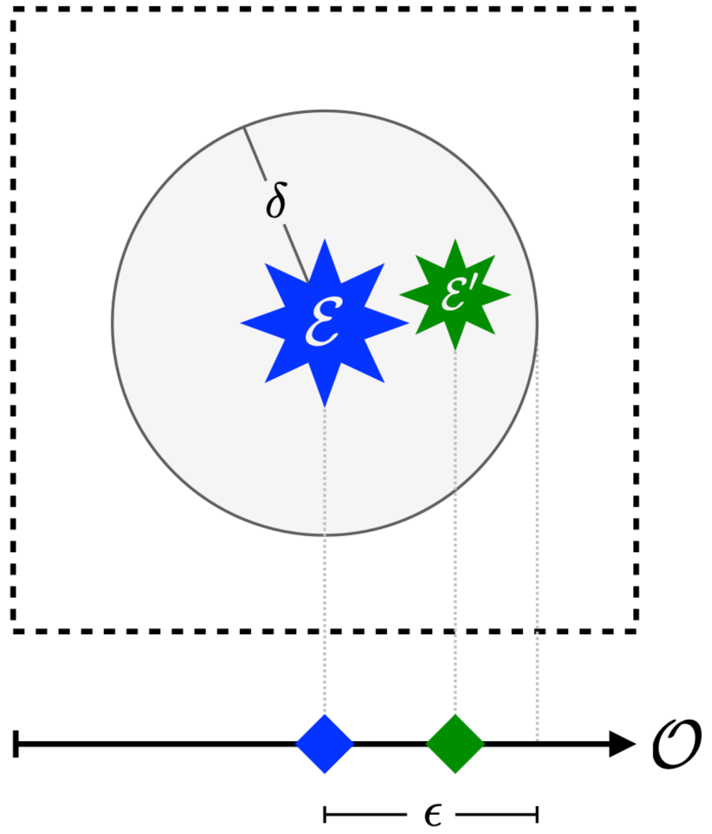
Collider data has geometric structure



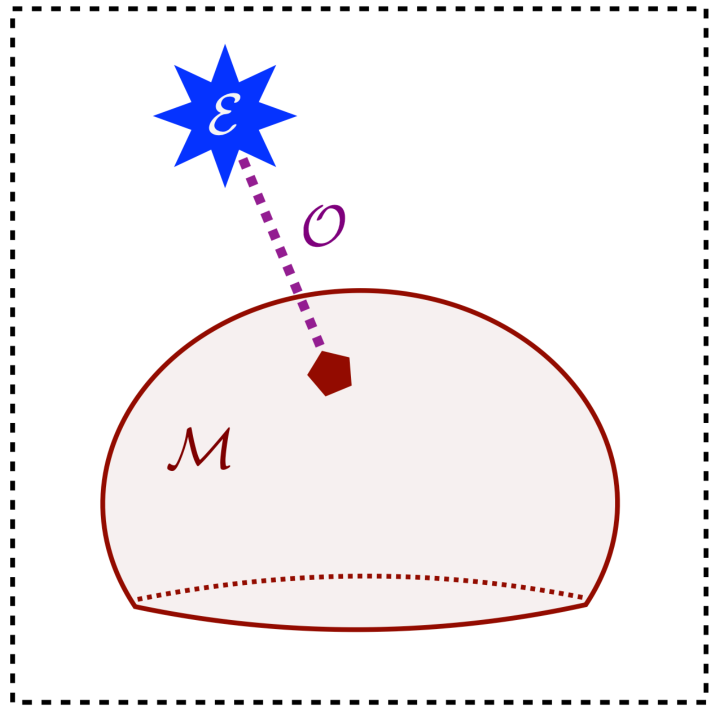
Fundamental HEP concepts & techniques

recast as geometric properties of HEP data

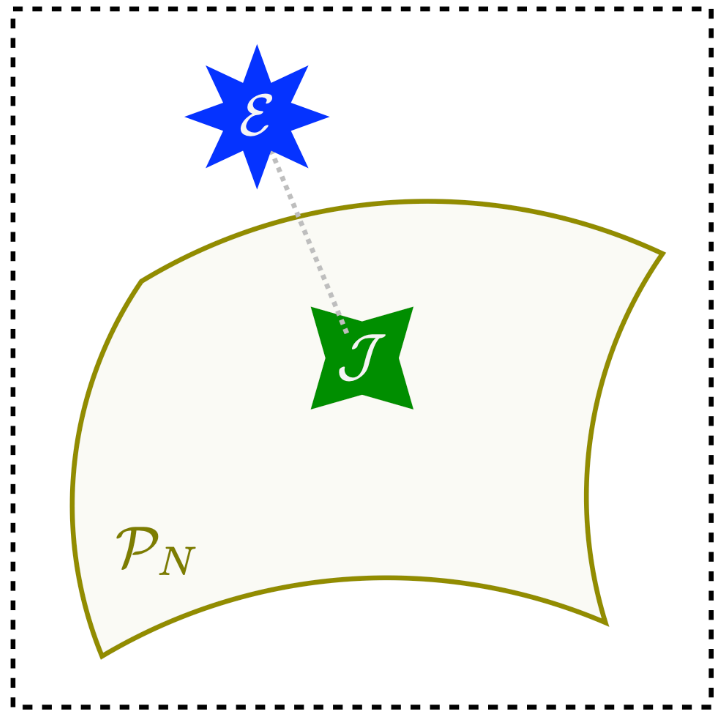
Infrared & Collinear Safety



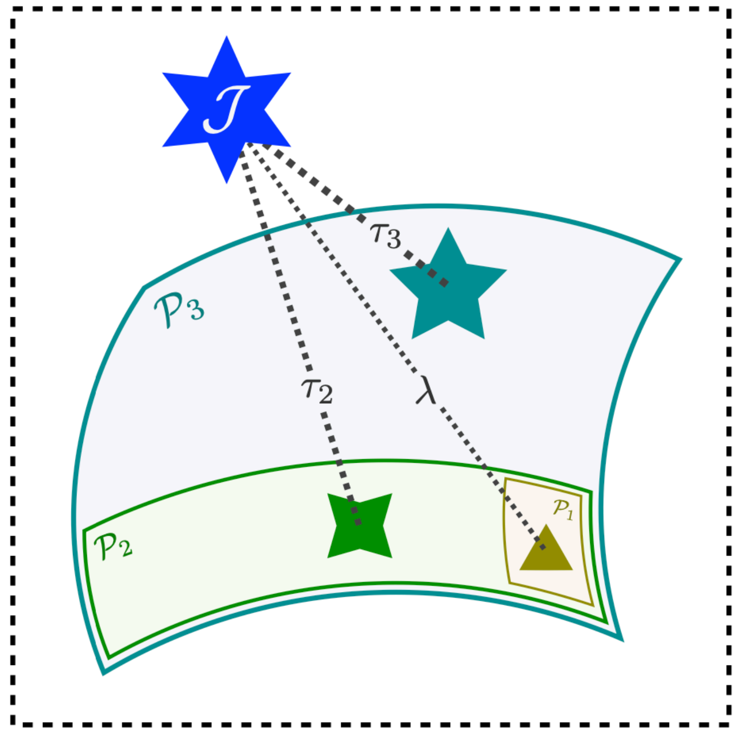
Observables



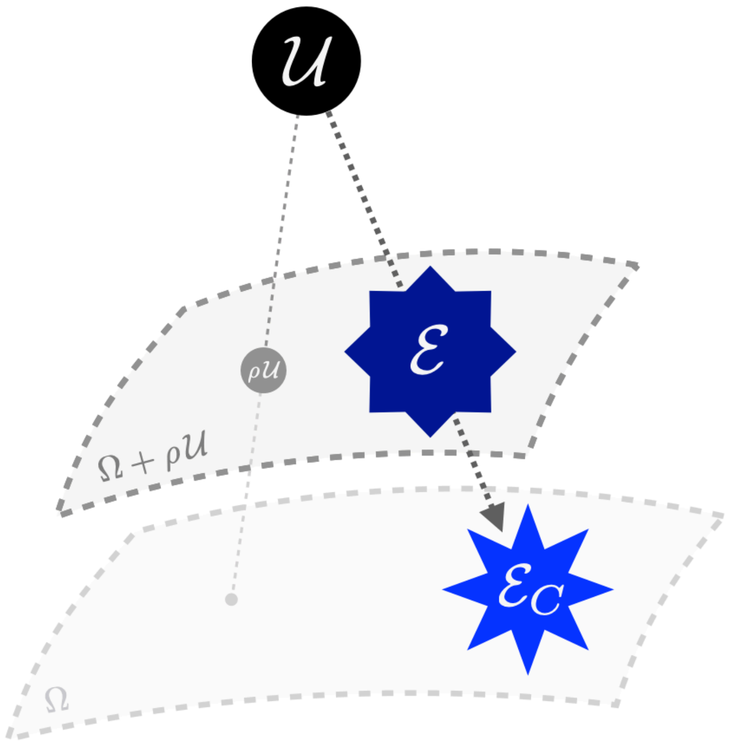
Jets



Jet substructure observables



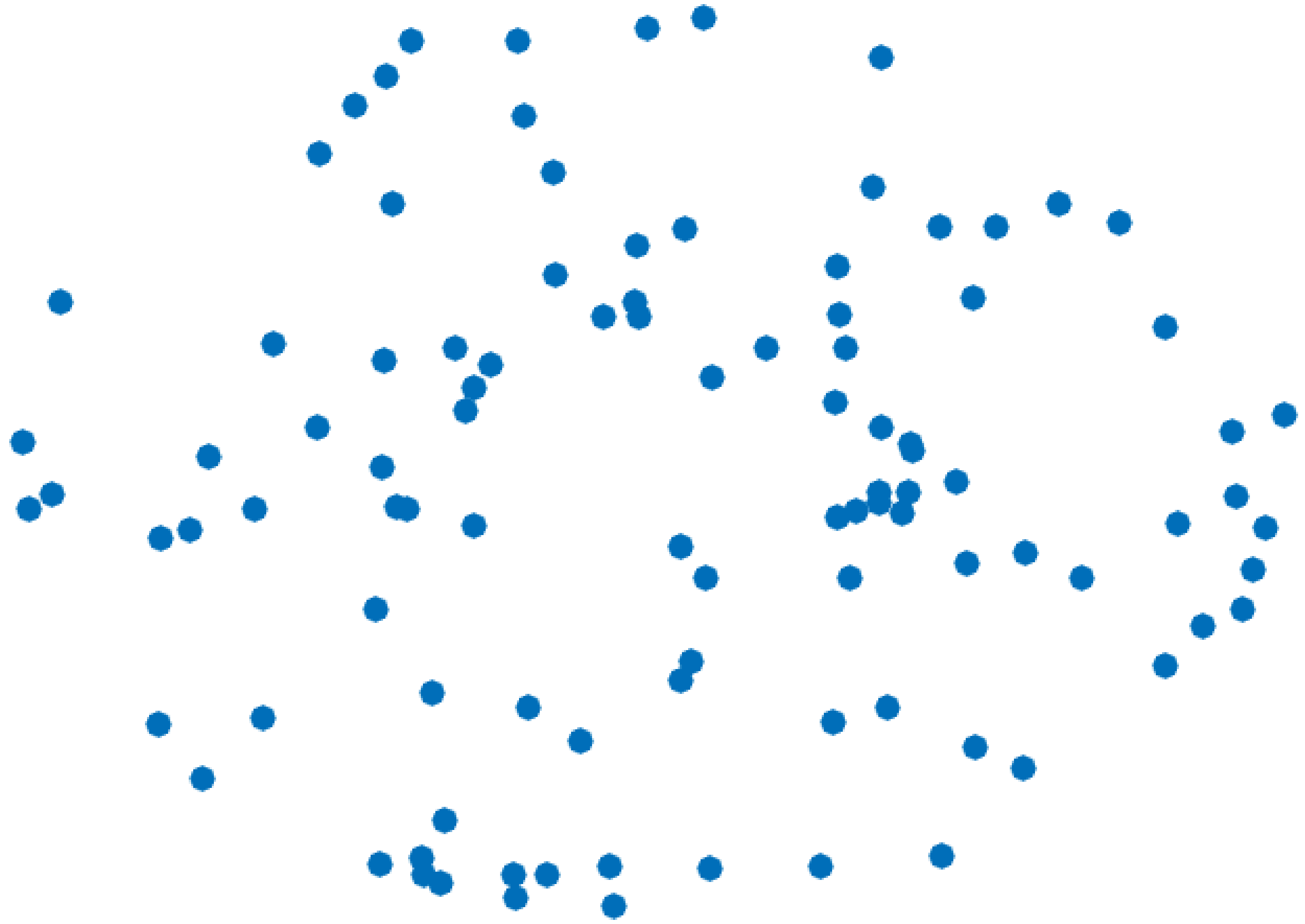
Pileup subtraction



Komiske, Metodiev, Thaler, The hidden geometry of particle collisions, J. HEP, 6 (2020)

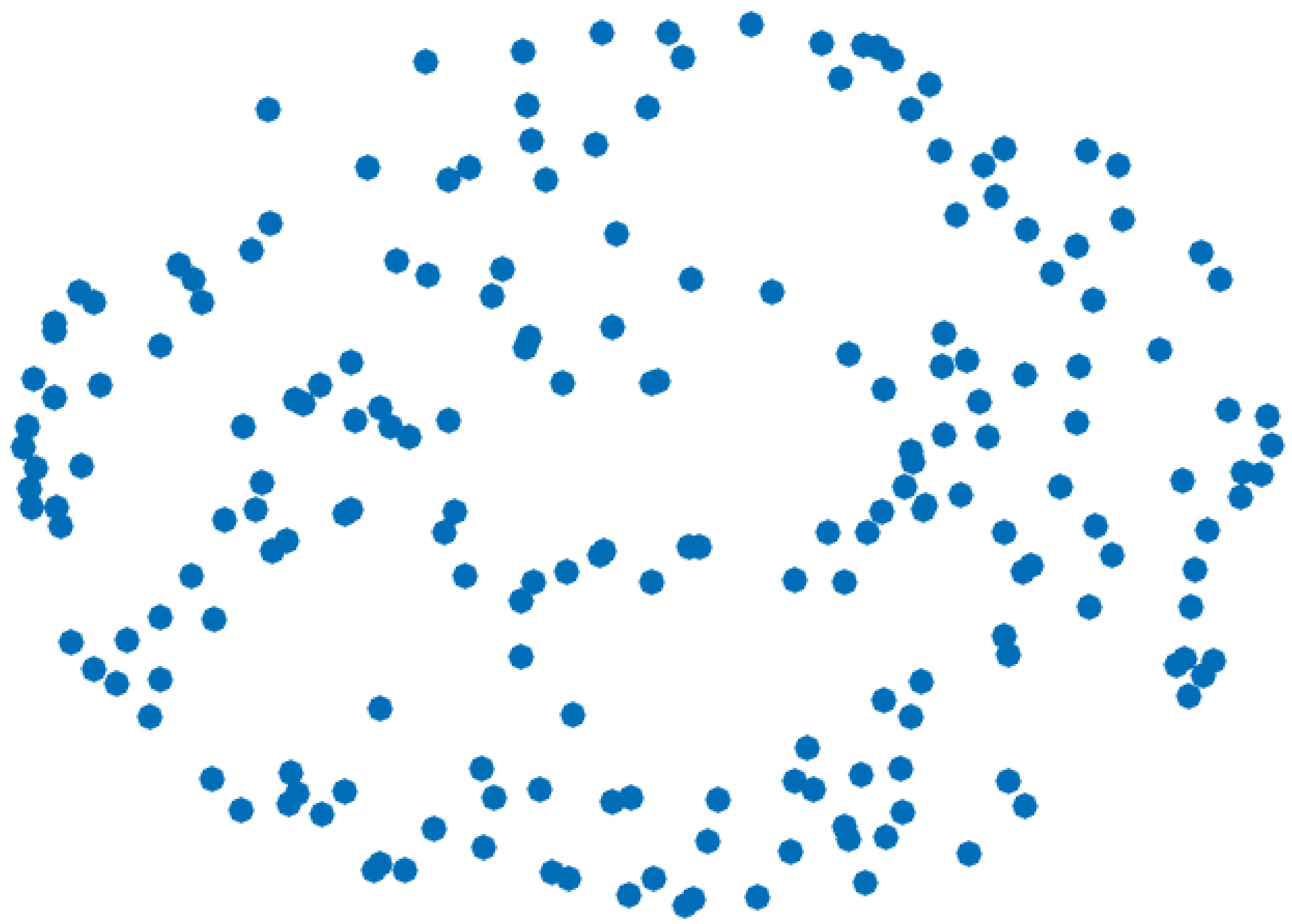
Recovering geometry of data:
problem of connecting data in the *right* way

What do you see?



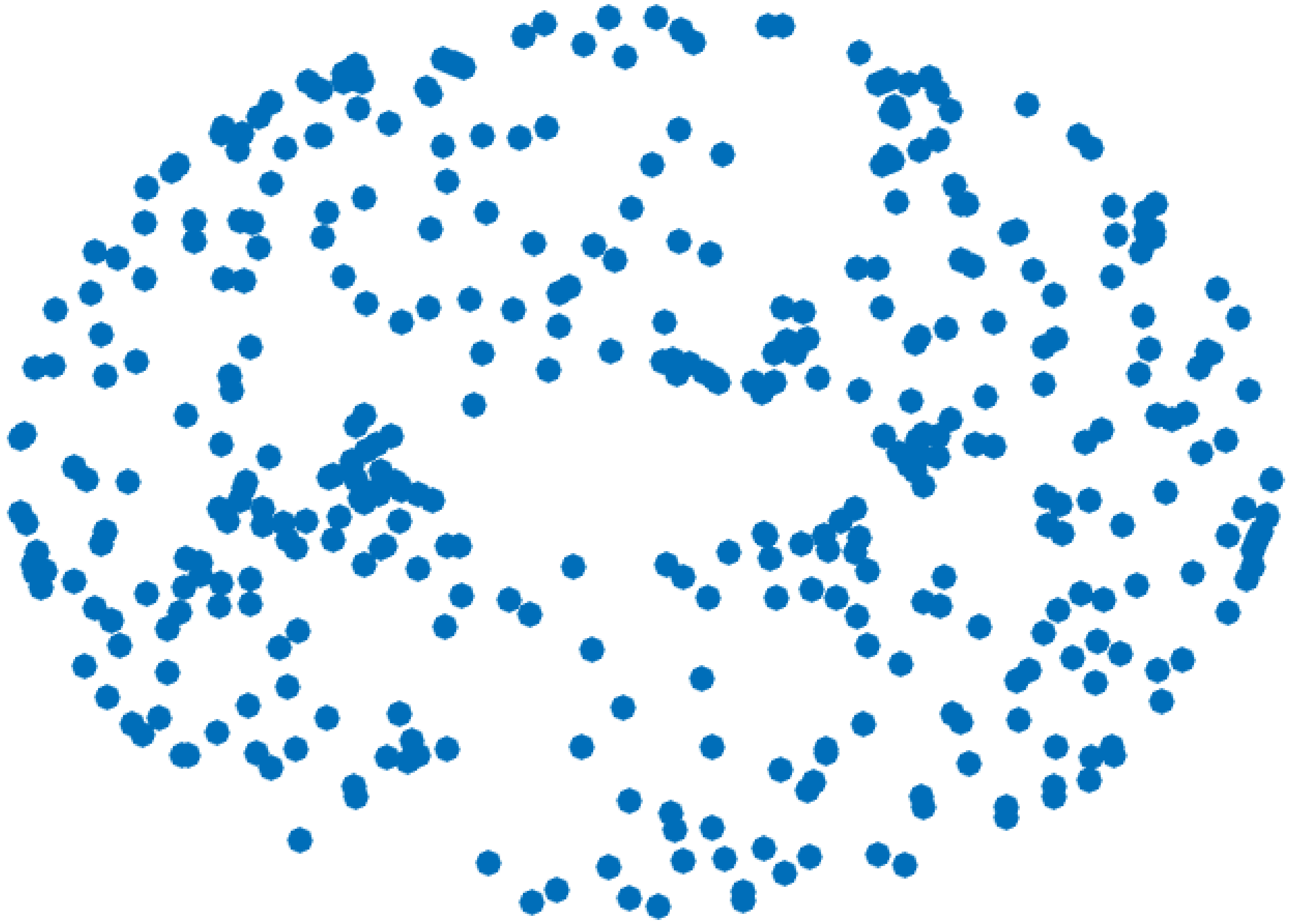
$N = 100$

What do you see?



$N = 200$

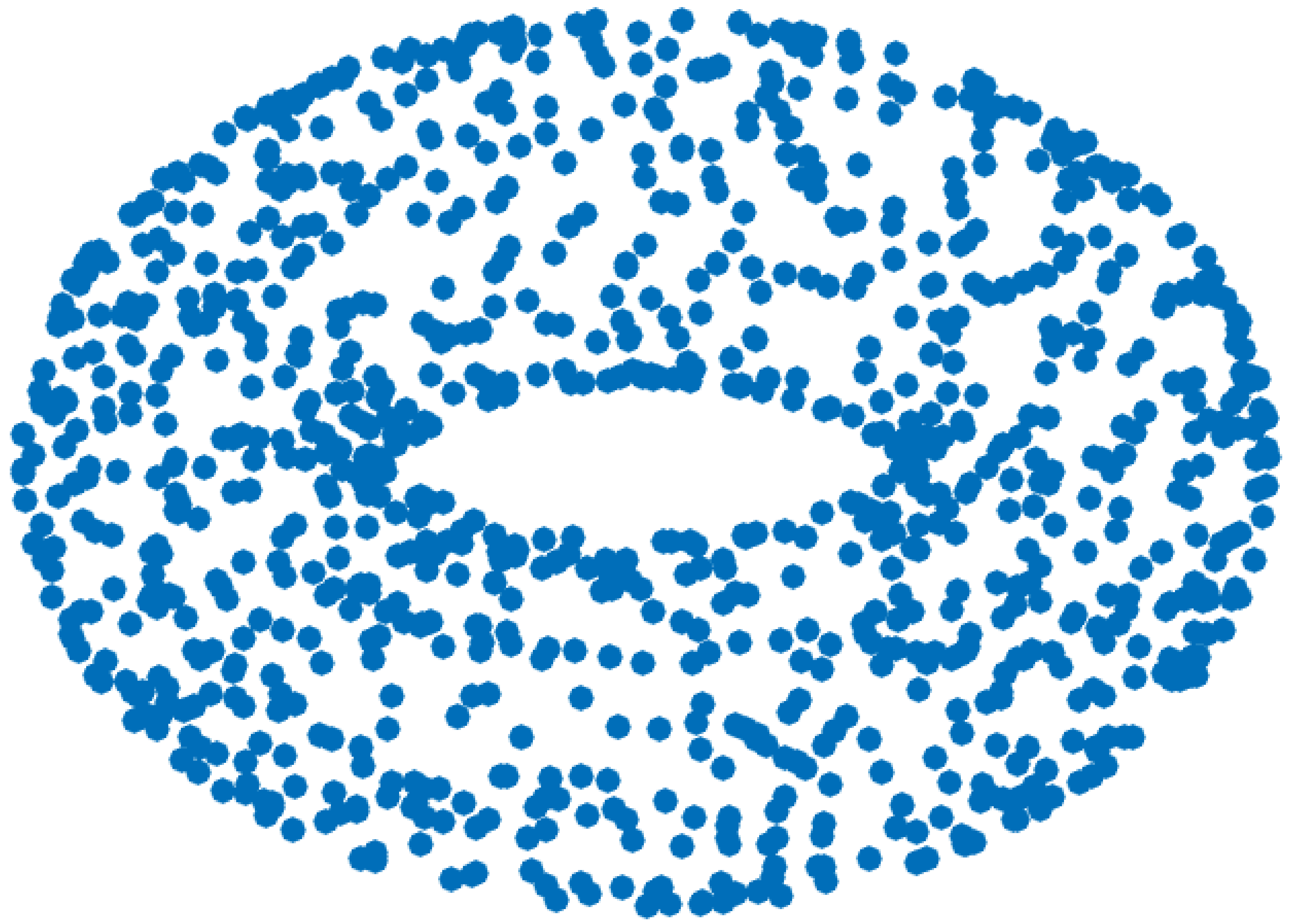
What do you see?



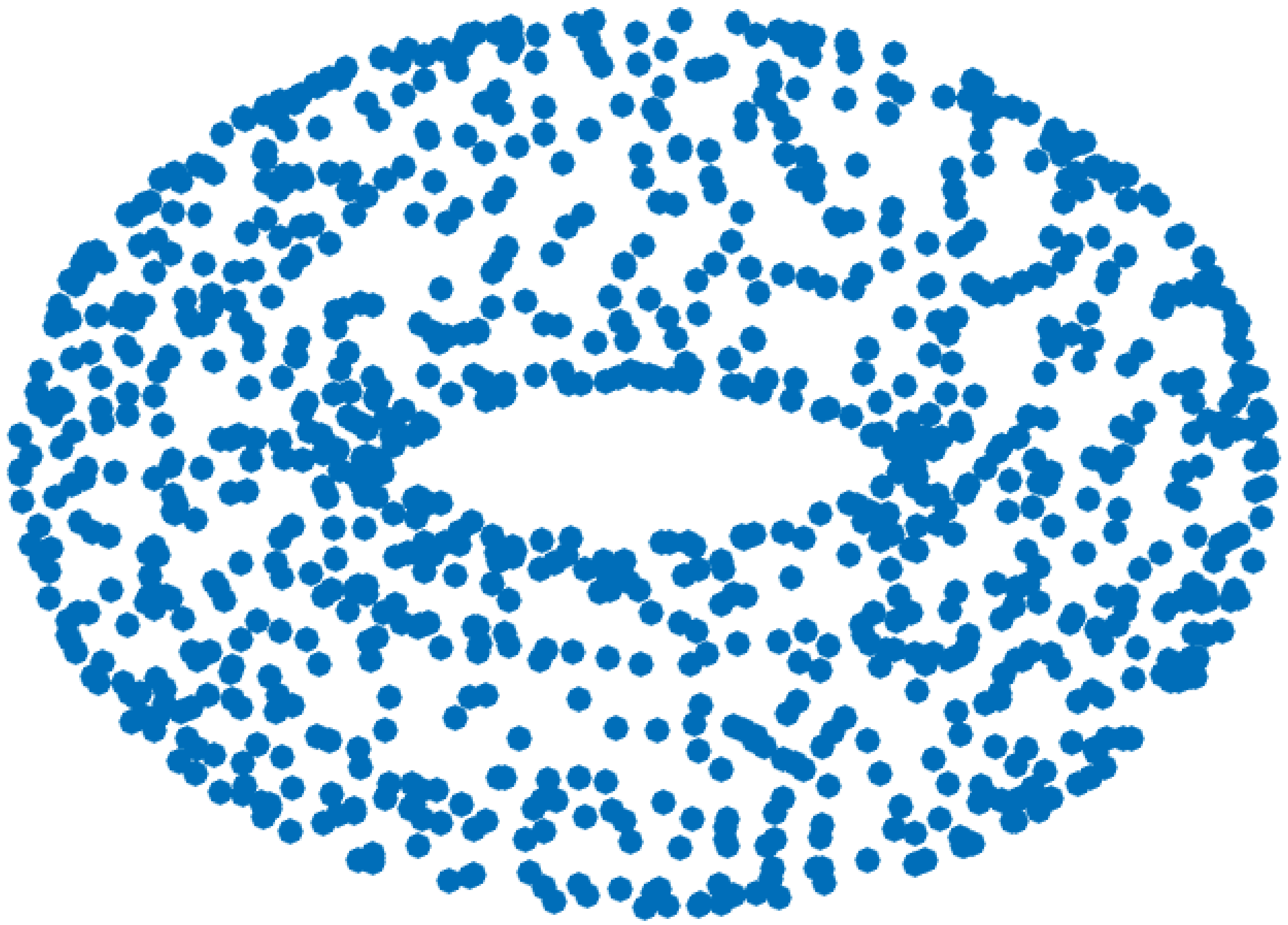
$N = 400$

What do you see?

$N = 1000$



What do you see?

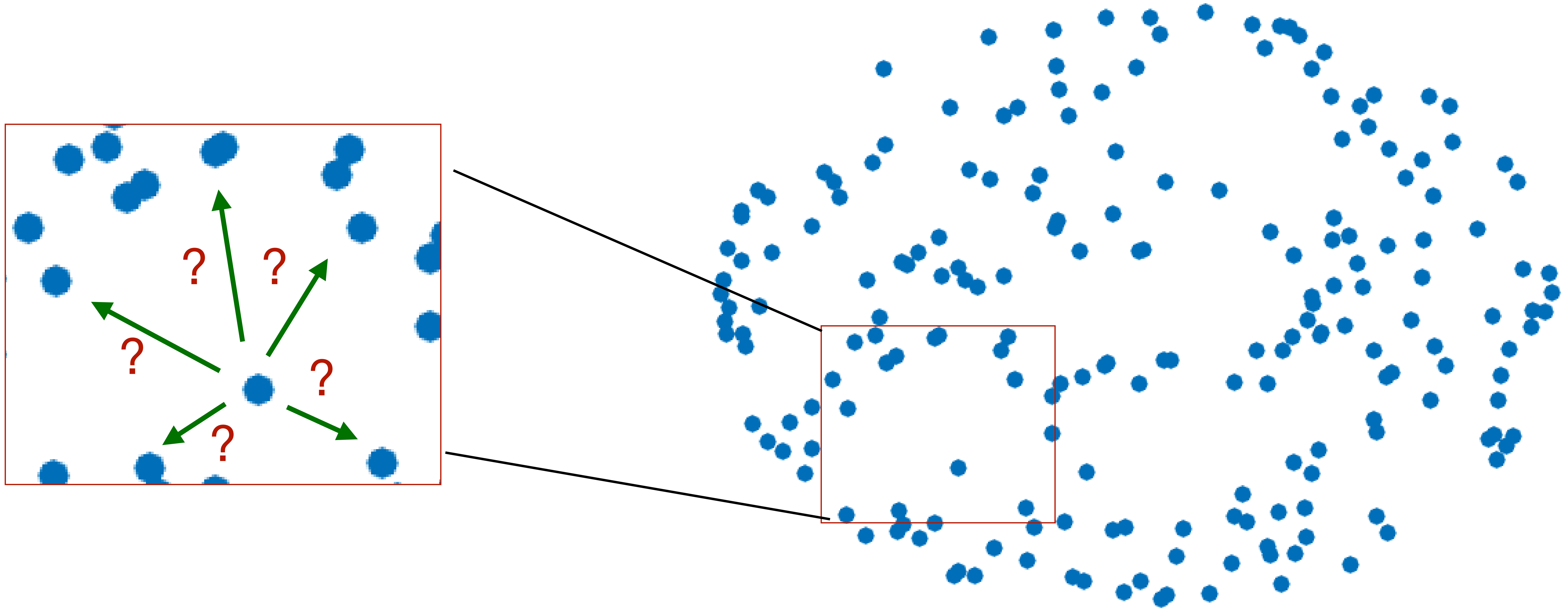


$N = 1000$

N data points in $\mathbb{R}^3$ sampled from a distribution constrained to a *submanifold* $T^2 \subset \mathbb{R}^3$

How is data **interconnected**?

$N = 200$



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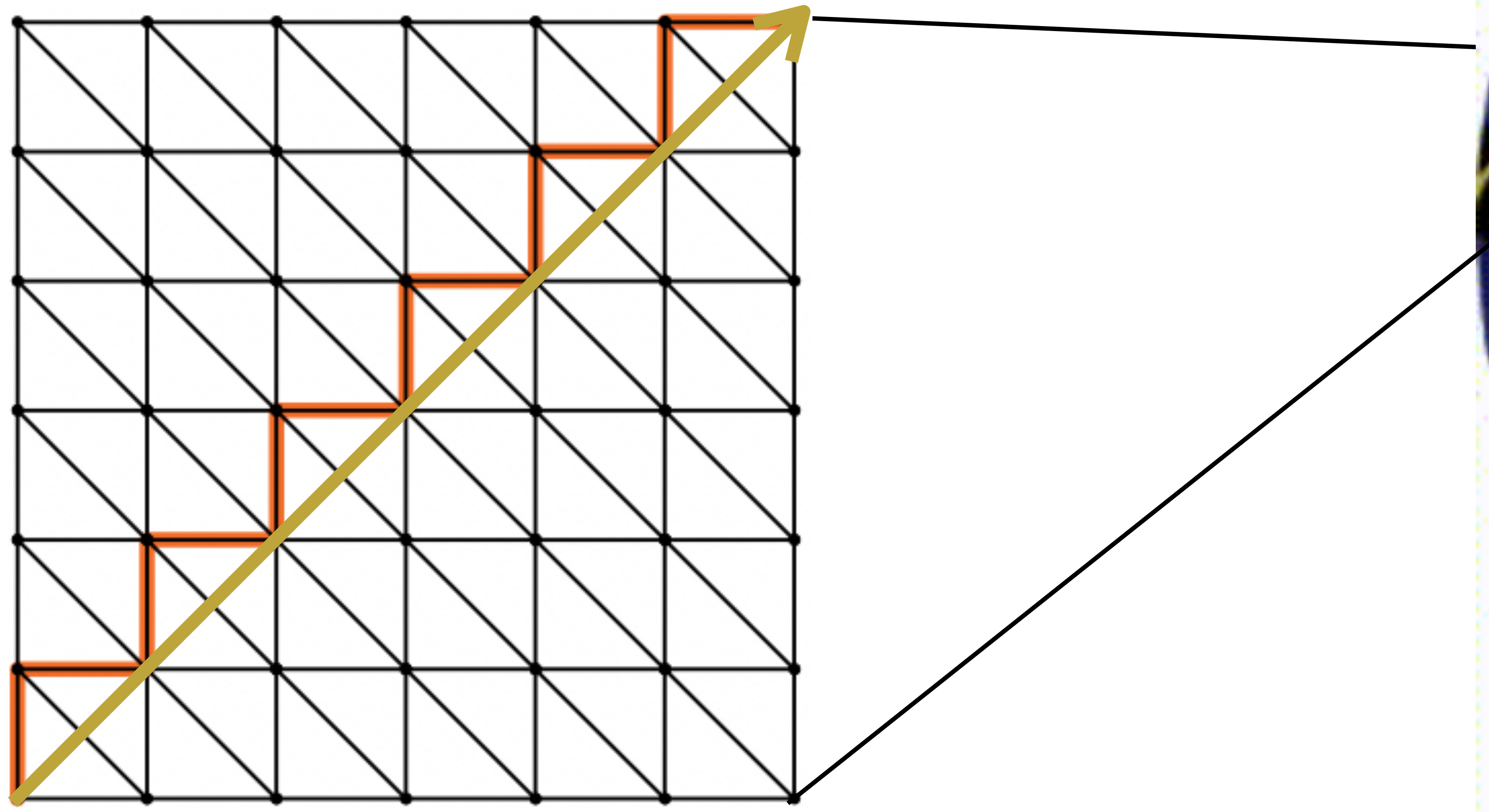
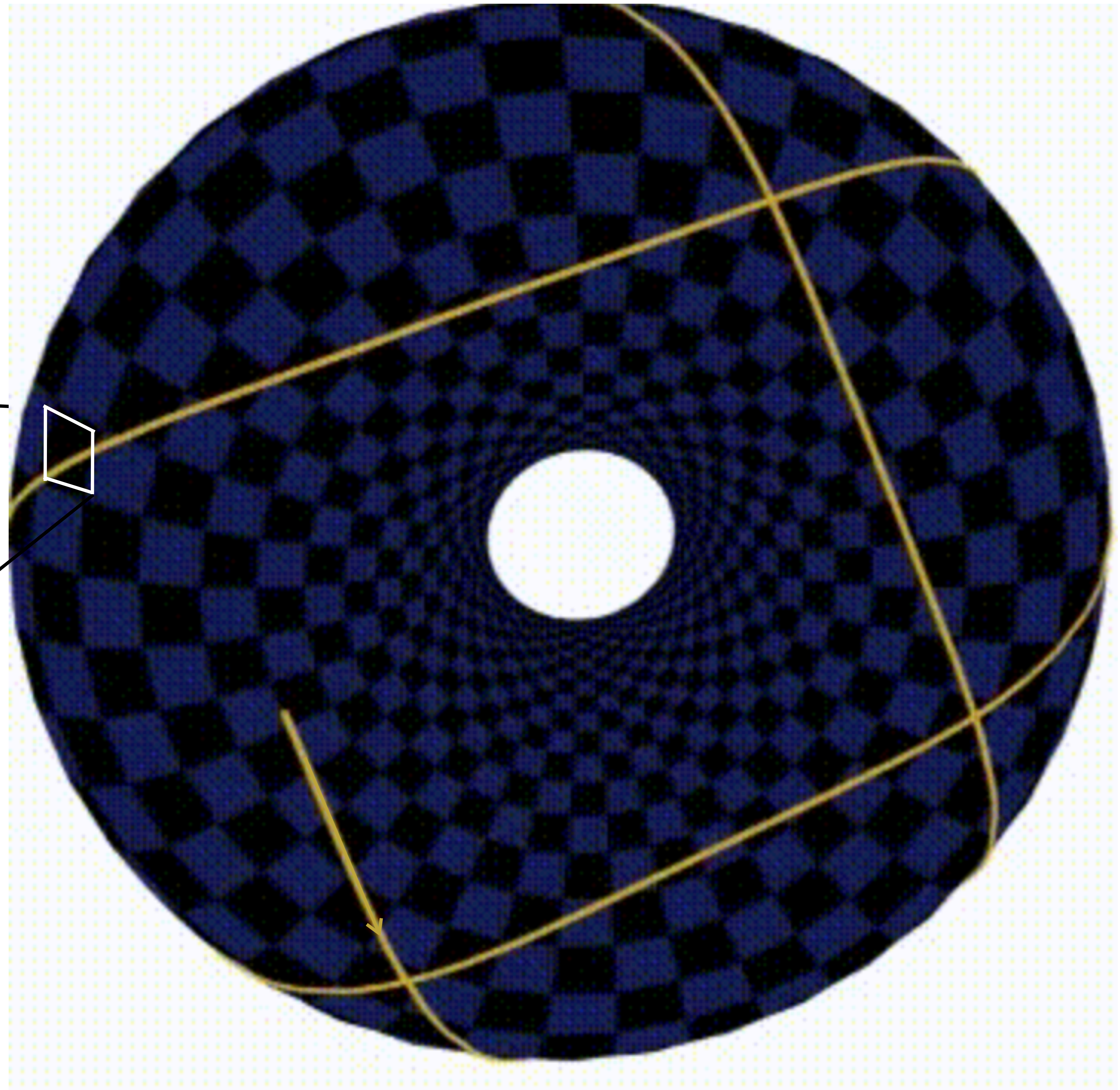


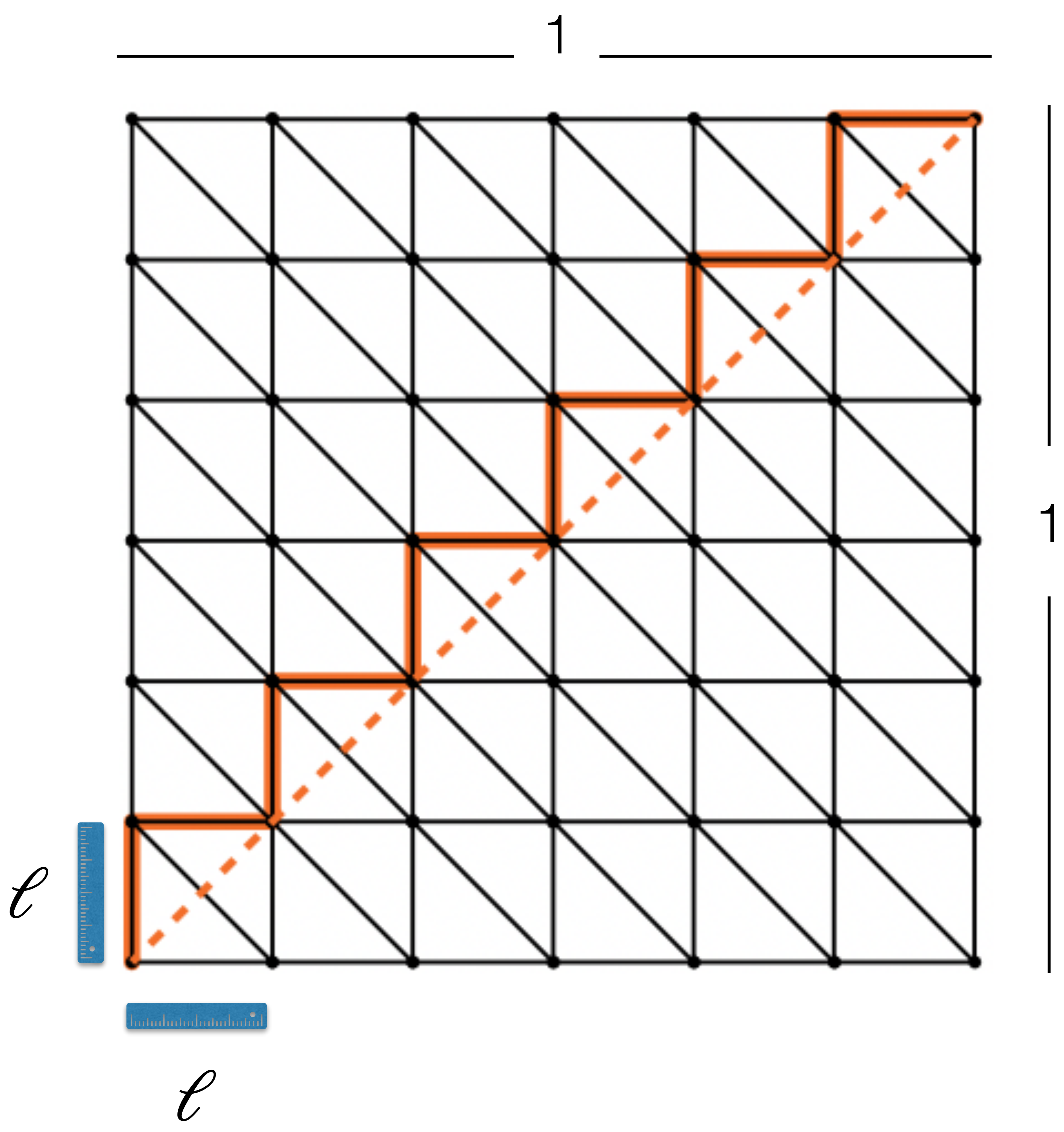
Crane et al., *Comm. ACM* (2017)



Wikipedia: *Geodesics*

intrinsic local and **non-local** relationships between data points

How is data **interconnected**?



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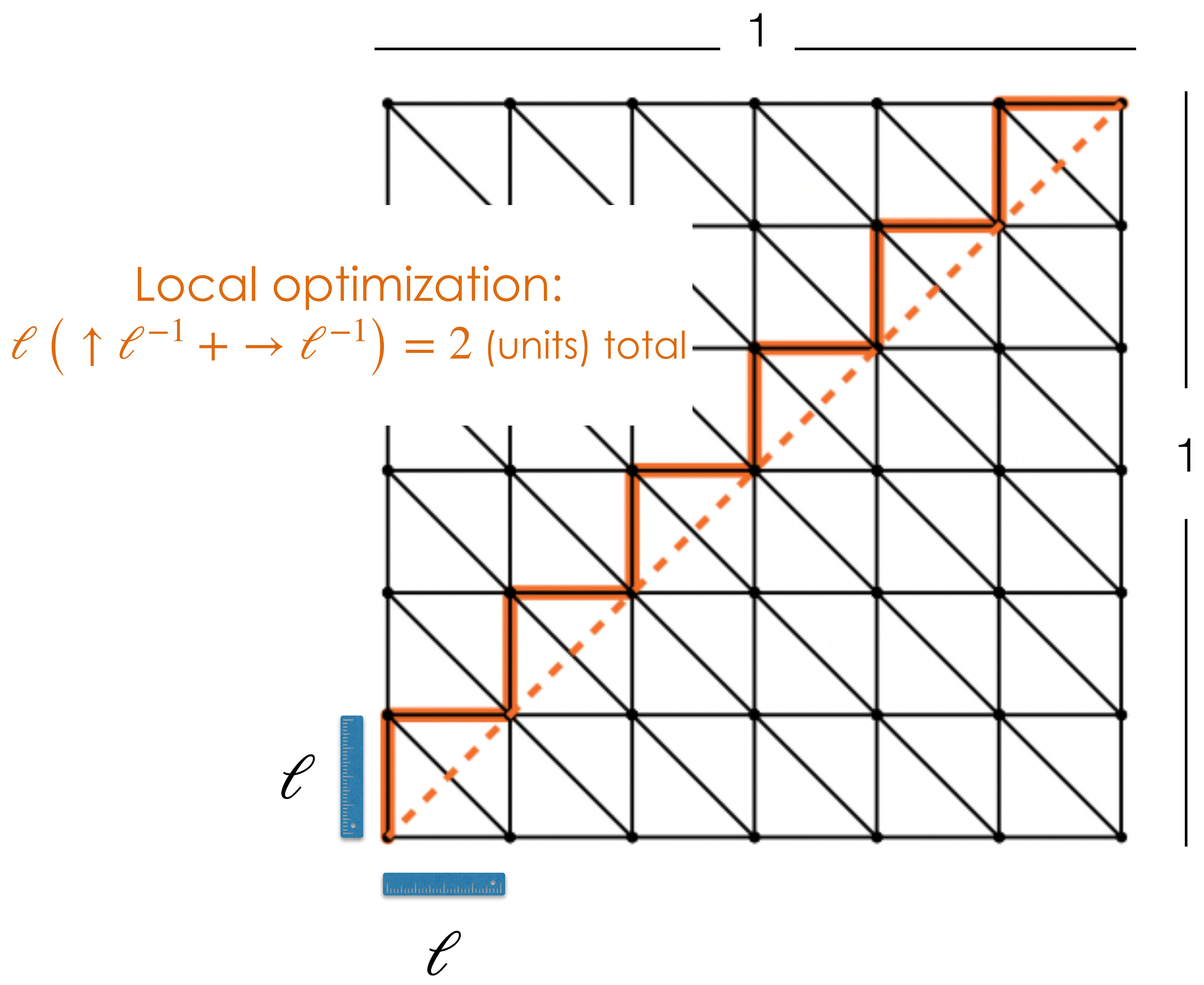

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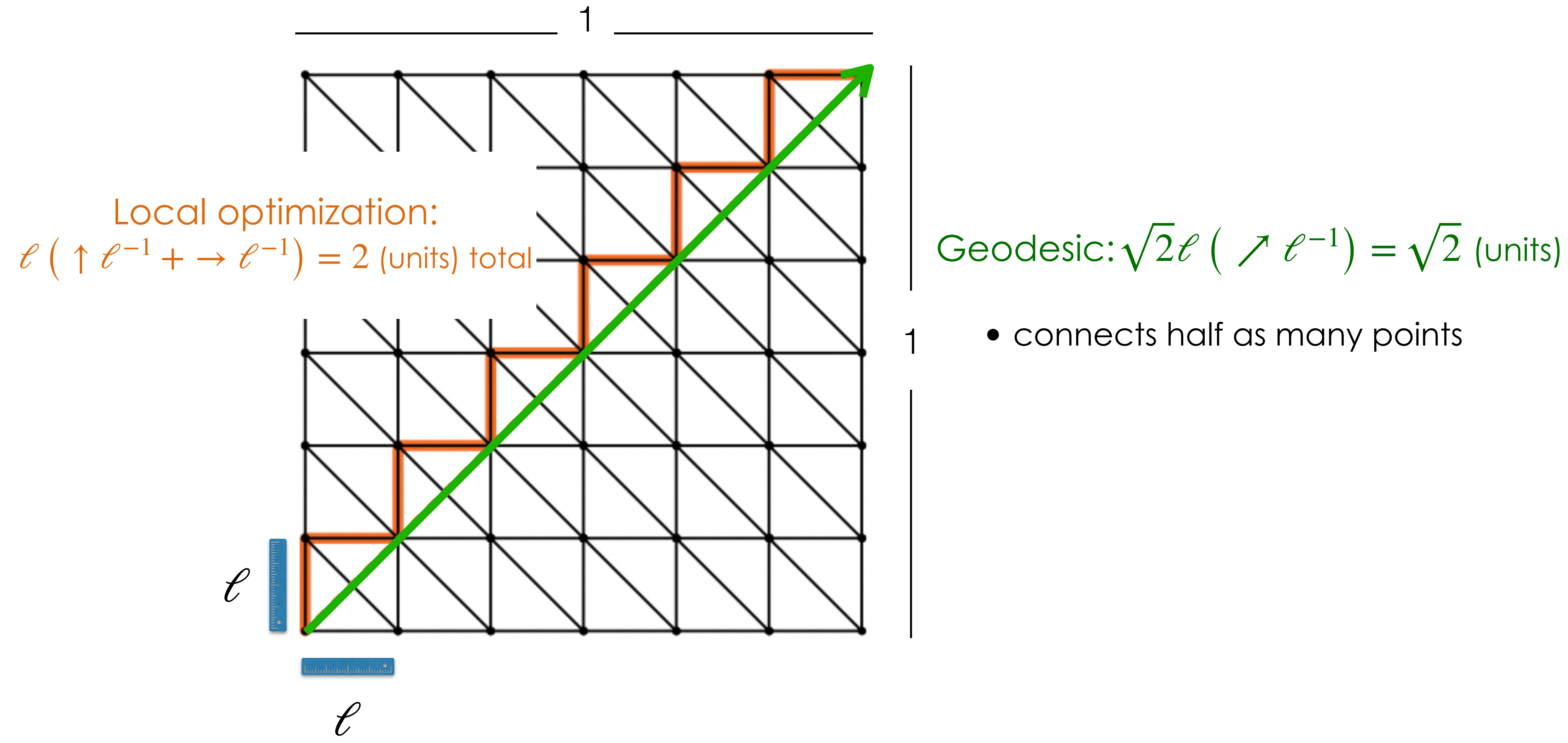


- connects half as many points

intrinsic local and **non-local** relationships between data points

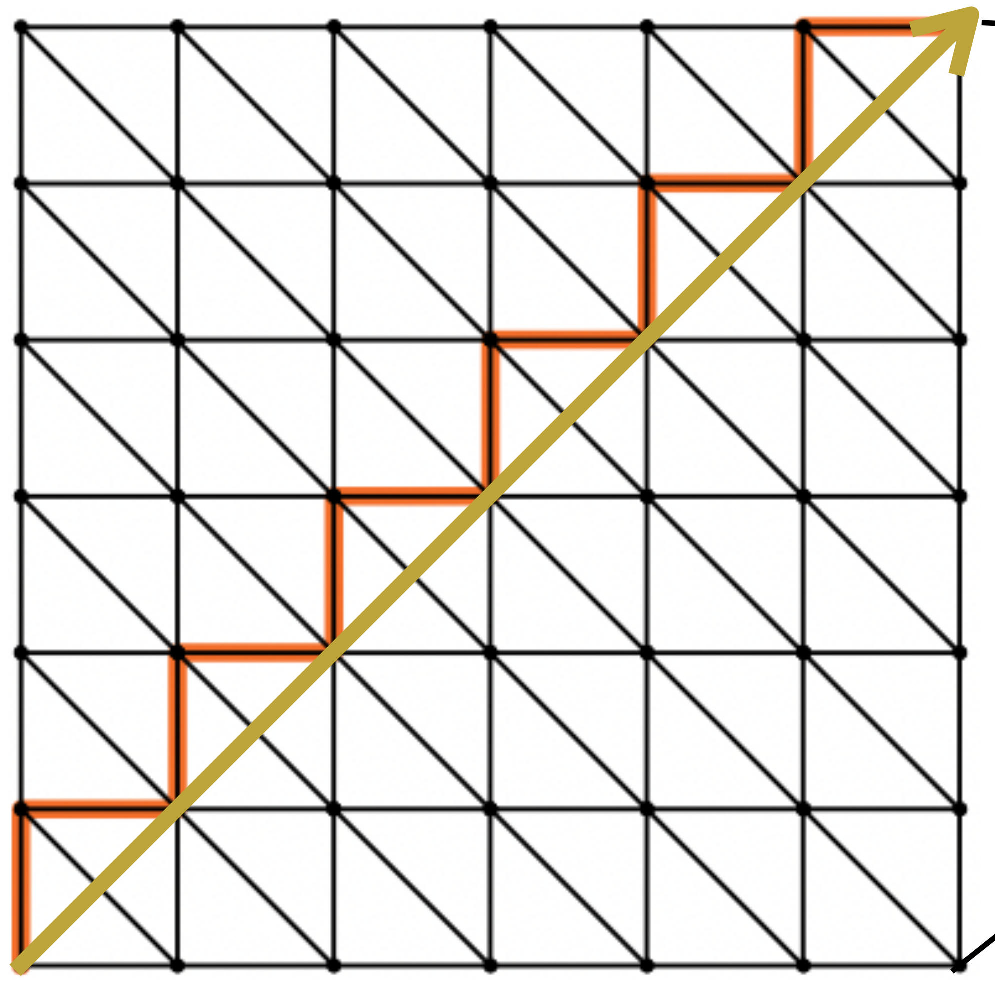
How is data **interconnected**?

Geodesic flow:

nonlinear in x, ξ

$$\begin{cases} \partial_t x_j &= \partial_{\xi_j} H = (g^{-1}(x) \cdot \xi)_j \\ \partial_t \xi_j &= -\partial_{x_j} H = \langle \partial_{x_j} g^{-1}(x) \cdot \xi, \xi \rangle \end{cases}$$

$$\hookrightarrow \Gamma^t(x_0, \xi_0)$$



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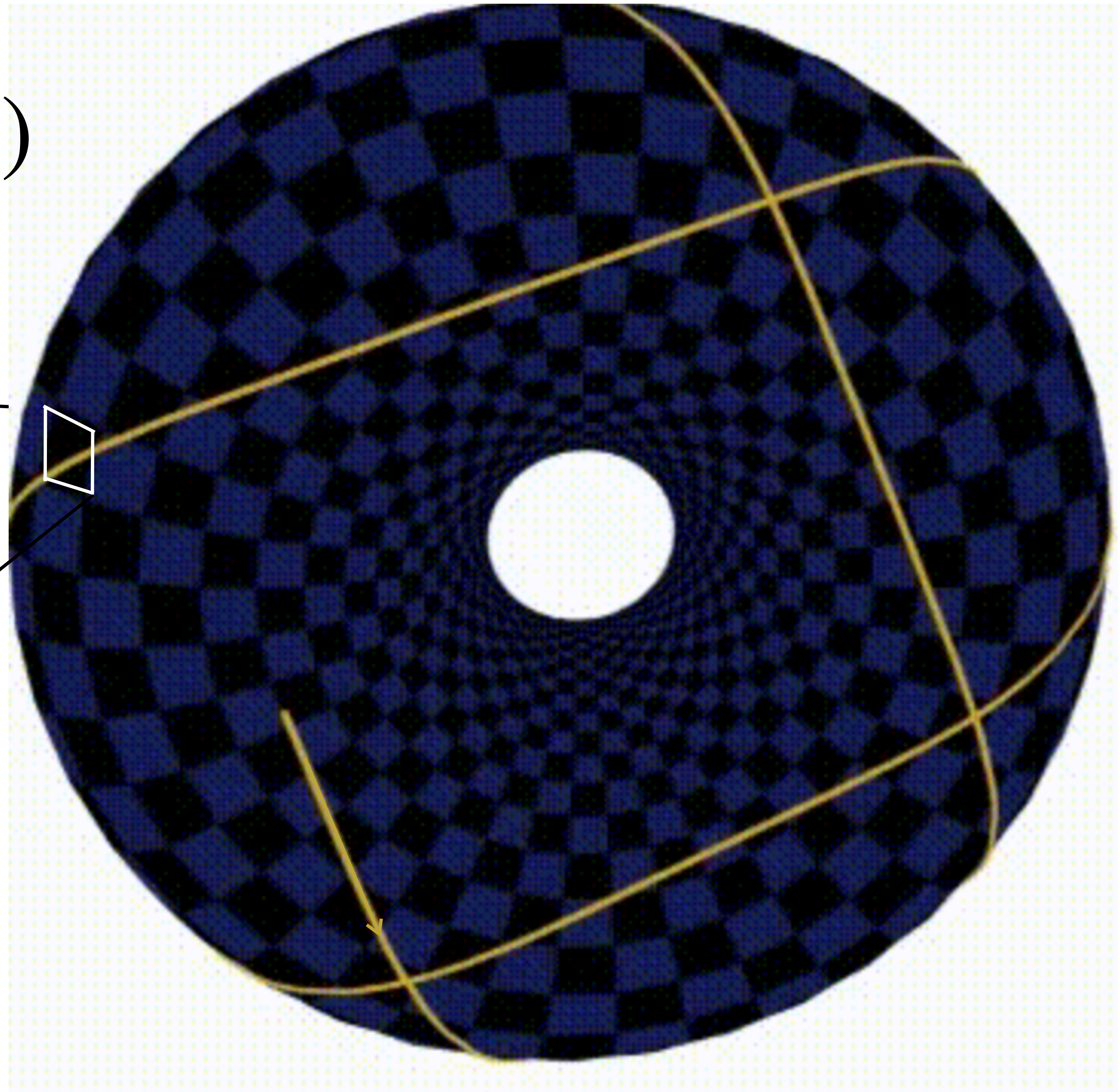

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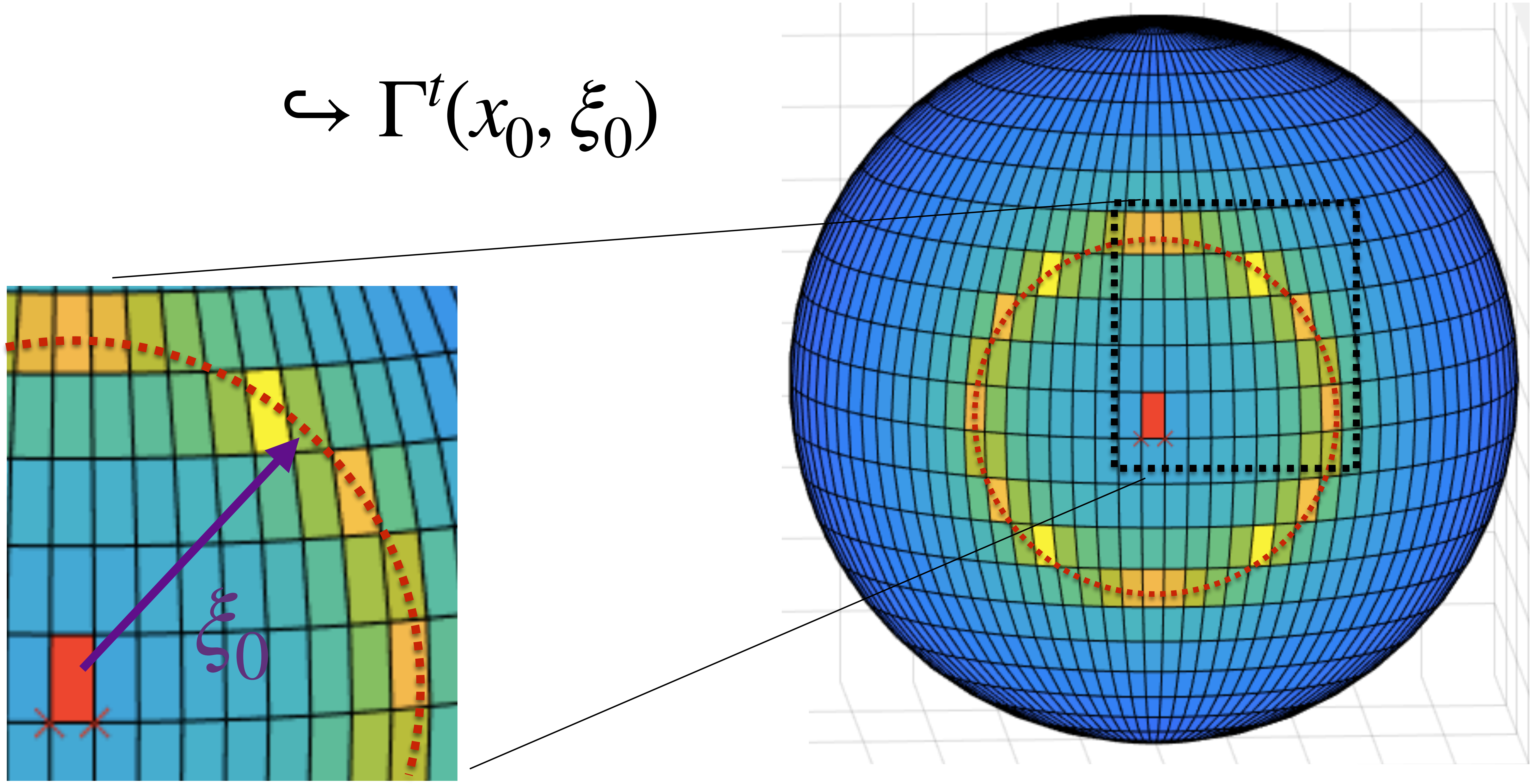
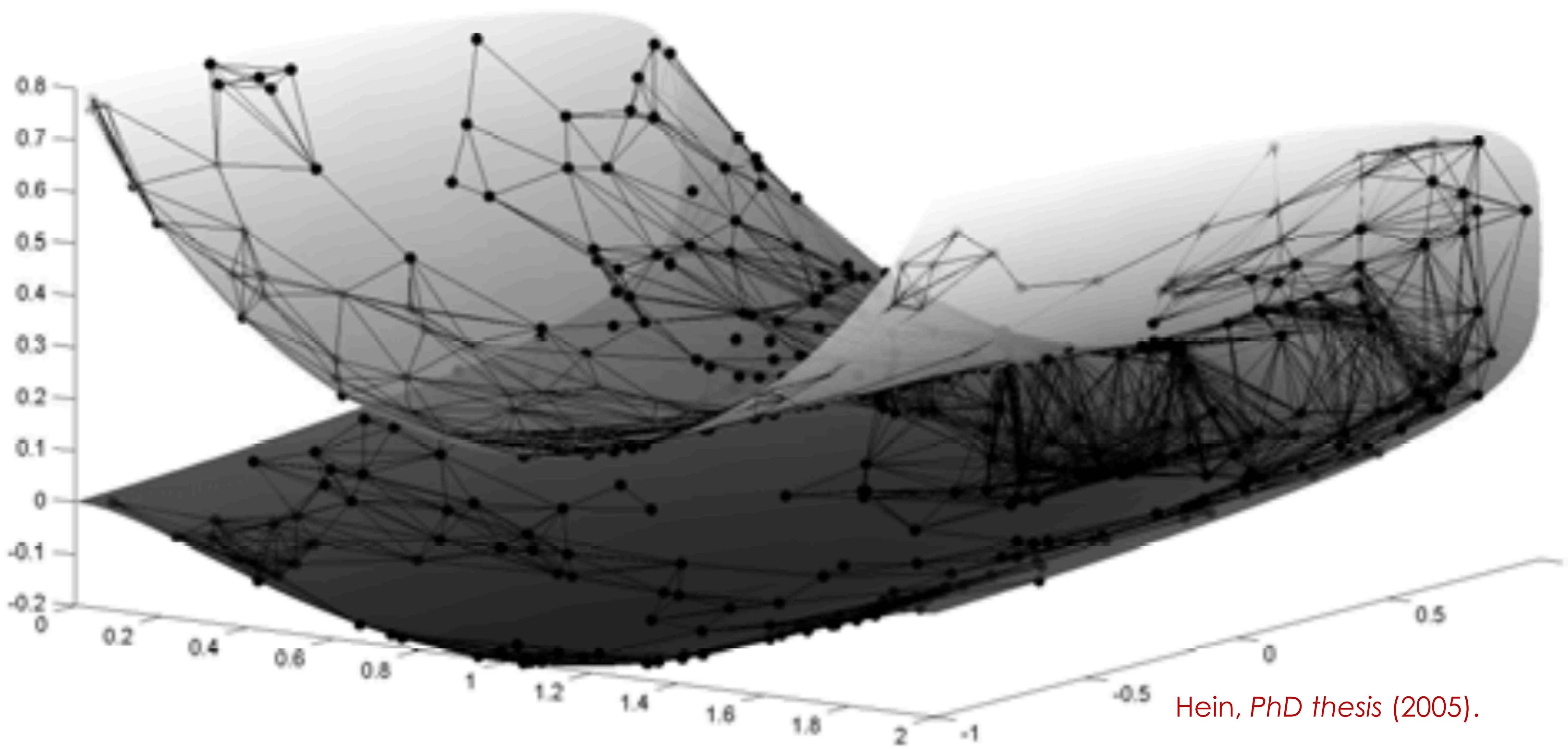


The flow moves the point x_0 to the geodesic neighbour $x(t)$ at distance t in the direction ξ_0

The problem

Given *data* $X_N := \{v_1, \ldots, v_N\} \subset \mathcal{M} \subset \mathbb{R}^D$ sampled from a probability distribution confined to a Riemannian submanifold $\mathcal{M}$, **recover** the *intrinsic geometry* of $\mathcal{M}$



Hein, *PhD thesis* (2005).

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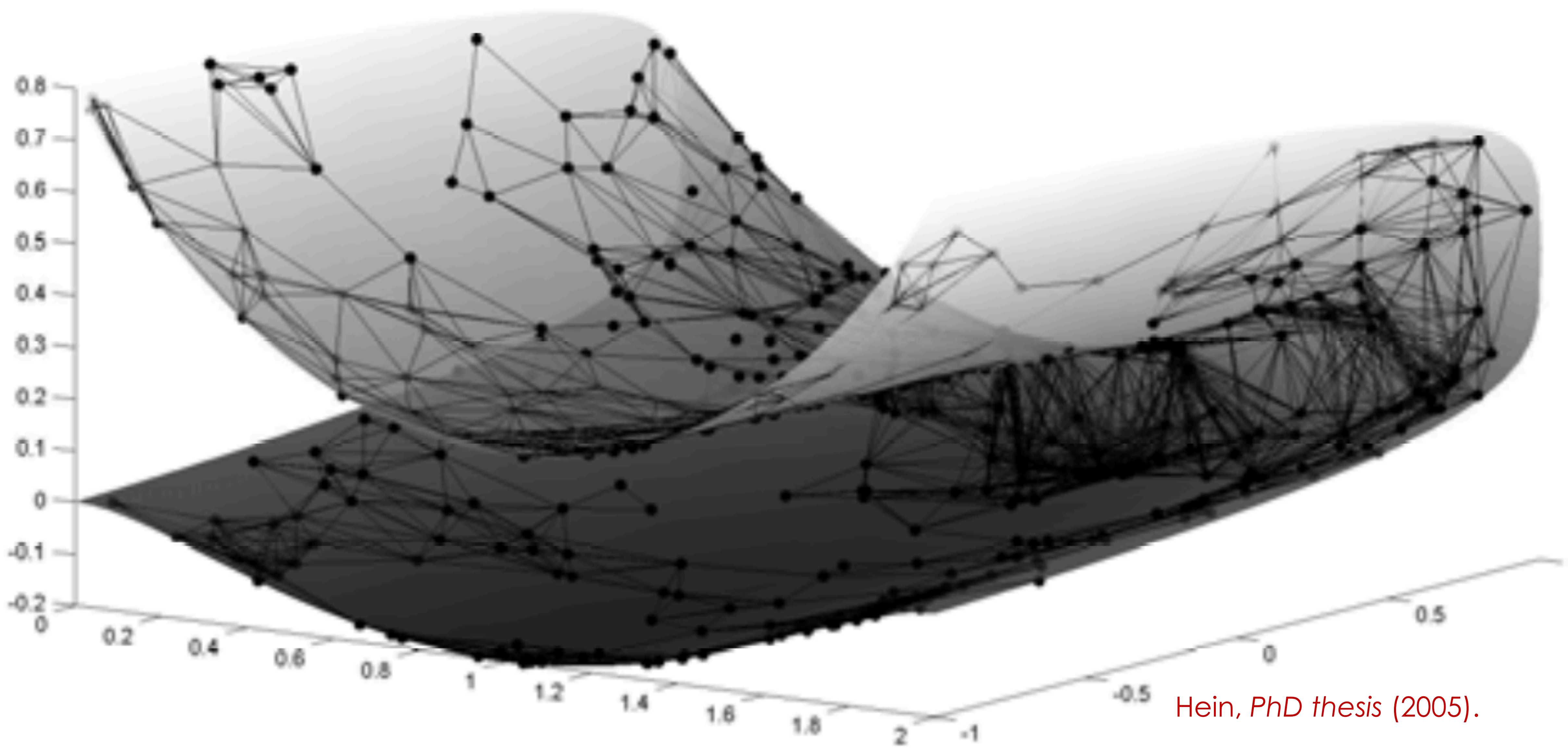
- intrinsic dimension, Riemannian metric, low-dimensional isometric embedding, heat diffusion, $\Delta_{\mathscr{M}}$

Belkin-Niyogi (2003), Hein (2005), Hein-Audibert-von Luxburg (2007), Trillos-Gerlach-Hein-Slepčev (2020)



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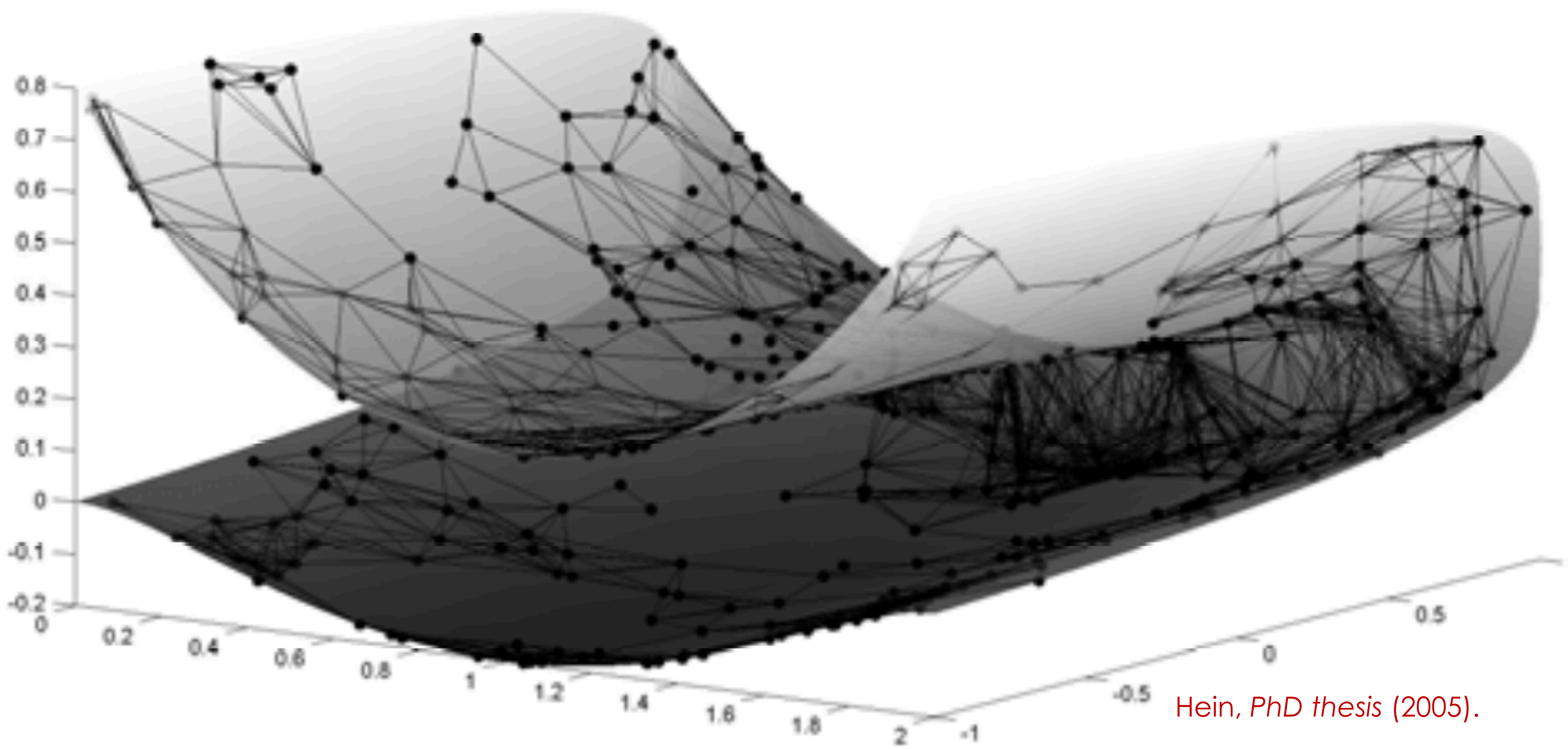
- **This talk:** Geometry-encoding quantum dynamics, geodesics ✦ Much harder!



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Luxburg (2007),
n-Slepčev (2020)

State of the art is to deploy *local* dynamics on data, *i.e.*, Markov processes

These yield *low-frequency* features of data, but miss *high-frequency* components (*i.e.*, geodesics)

Hein, *PhD thesis* (2005).

The solution

Graph Laplacian construction

Discrete quantized propagation: taking the position mean gives $O(h)$ approx. to $\iota(t)$

Inputs: $X_N = \{v_1, ..., v_N\}, v^*, \epsilon > 0, \alpha \geq 1, t > 0$
Output: Propagated state $[\psi_h^\zeta](t)$
procedure PROPAGATE
  for $i, j = 1 : N$ **do** $[T_\epsilon]_{i,j} \leftarrow k(||v_i - v_j||^2/\epsilon)$
  $D_\epsilon \leftarrow$ diagonal matrix $\left(\sum_{j=1}^N [T_\epsilon]_{i,j}\right)_{1 \leq i \leq N}$
  $\Delta_{\epsilon,N} \leftarrow \frac{4(I_N - D_\epsilon^{-1} T_\epsilon)}{\epsilon}$
  $U_{\epsilon,N}^t \leftarrow \exp(-it\sqrt{\Delta_{\epsilon,N}})$
  $h \leftarrow \epsilon^{\frac{1}{(2+\alpha)}}$
  $p_0 \leftarrow v_j - v^*$ for v_j closest to point v^*
  while $1 \leq \ell \leq N$ **do**
    $[\psi_h^\zeta]_\ell \leftarrow e^{-||v_\ell - v^*||^2/2h} e^{\frac{i}{h}(v_\ell - v^*)^\top p_0/||p_0||}$
  return $[\psi_h^\zeta](t) = U_{\epsilon,N}^t [\psi_h^\zeta]$

The solution

Inputs: $X_N = \{v_1, ..., v_N\}, v^*, \epsilon > 0, \alpha \geq 1, t > 0$

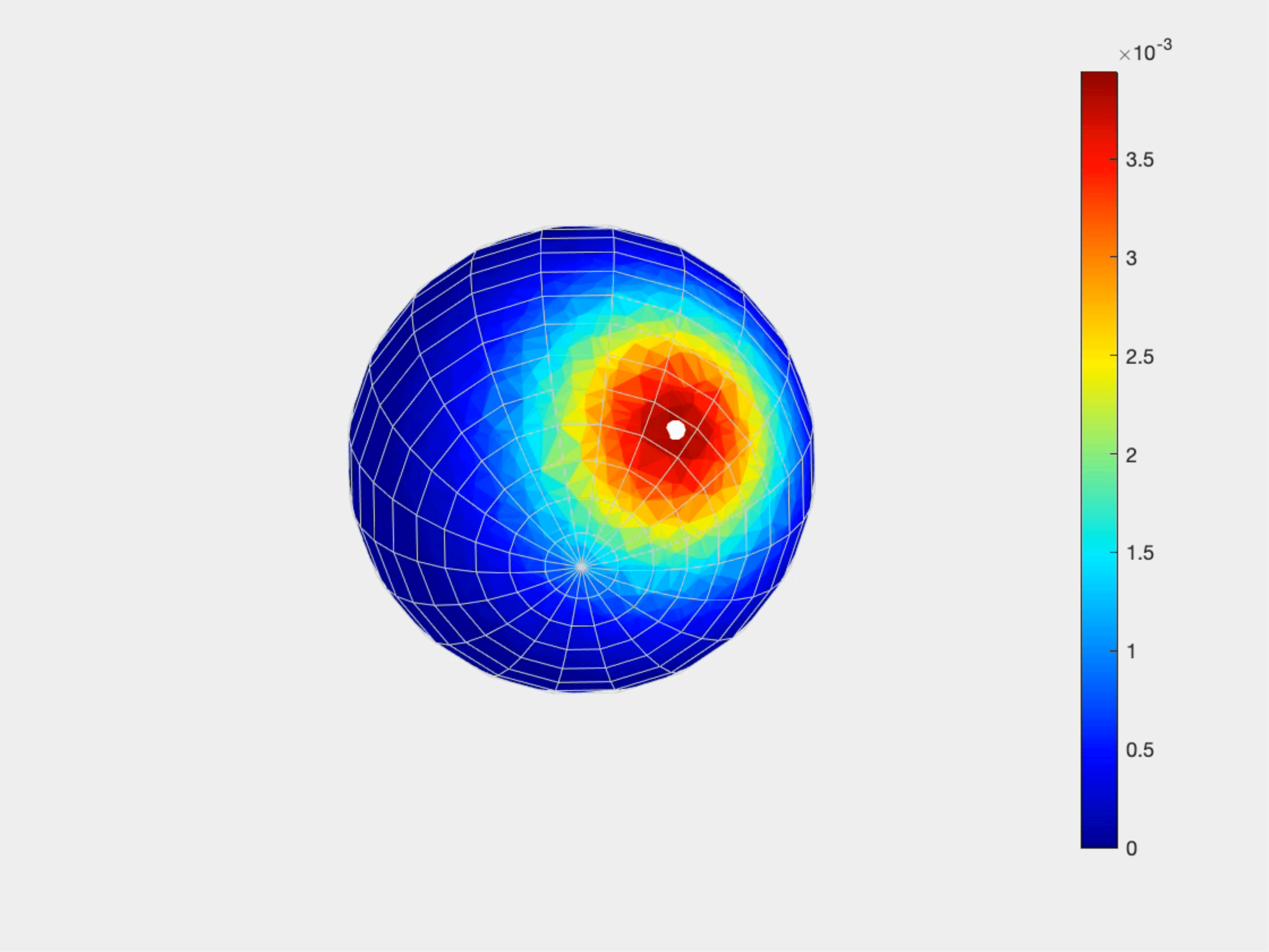
Matrix dynamics $e^{it\sqrt{\Delta}_{\epsilon,N}} \left| \psi_h^{(x_0,\xi_0)} \right\rangle$ on dataset

follows geodesic flow starting at x_0 in direction ξ_0 for distance t

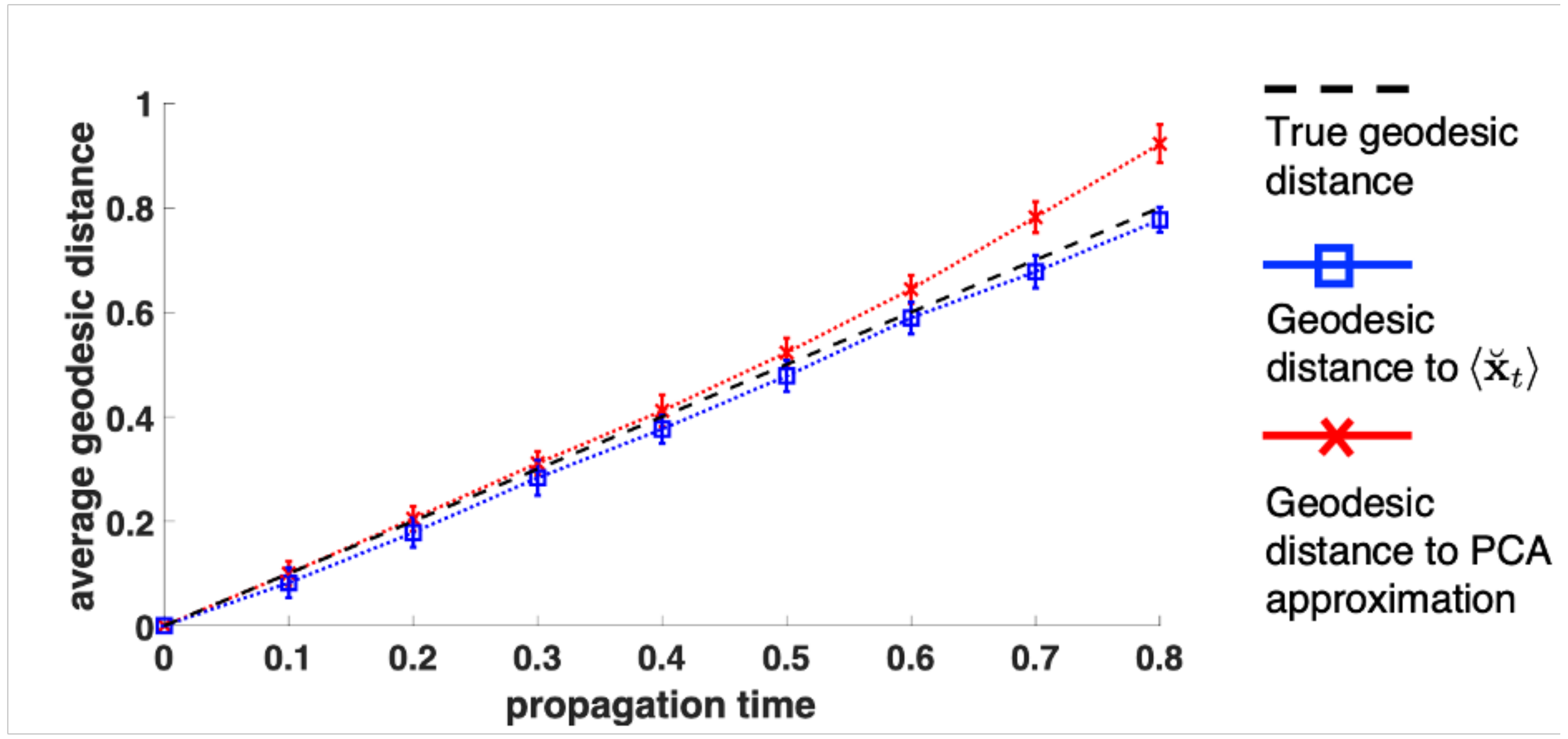
with $O(h)$ error

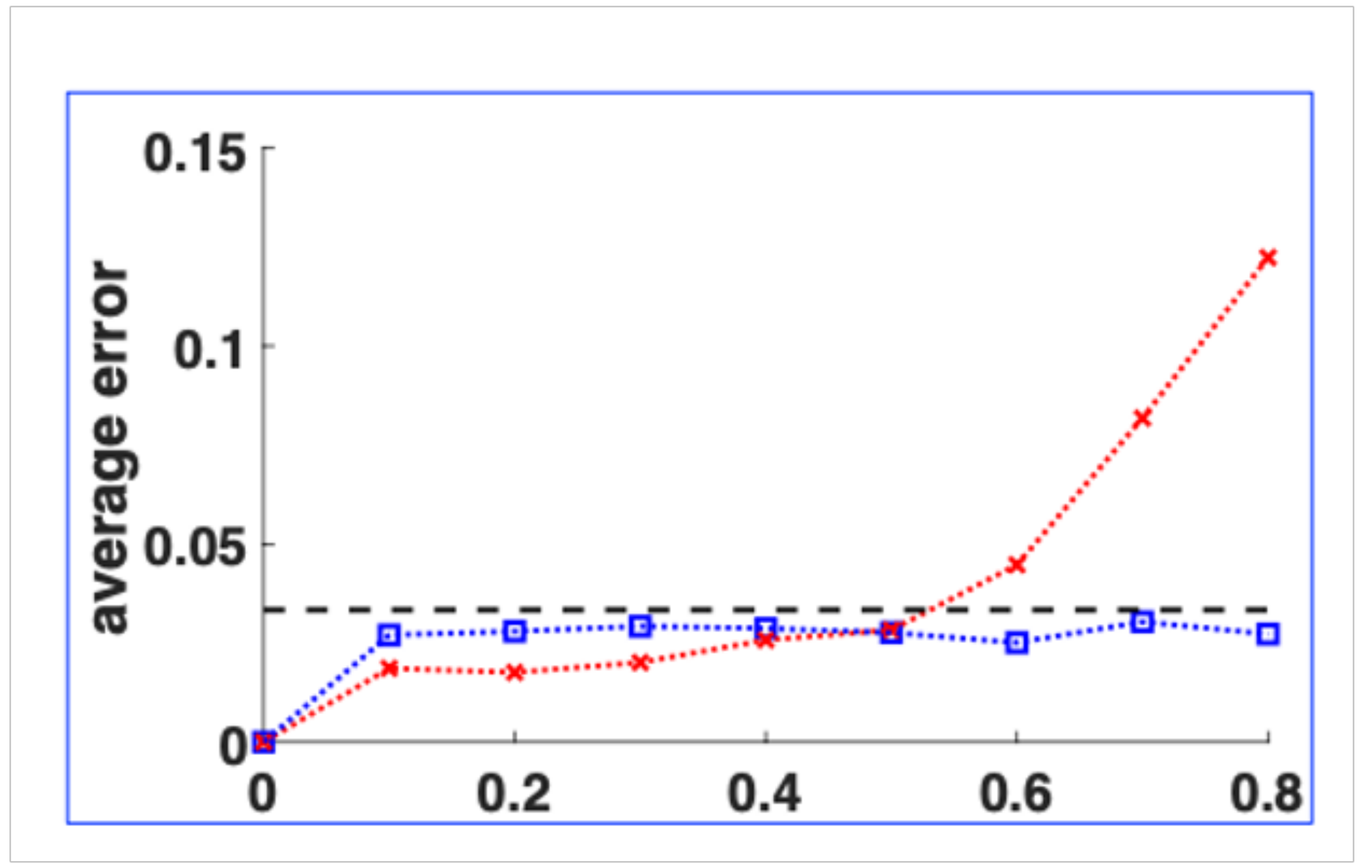
return $[\psi_h^{\zeta}](t) = U^t_{\epsilon,N}[\psi_h^{\zeta}]$

The solution

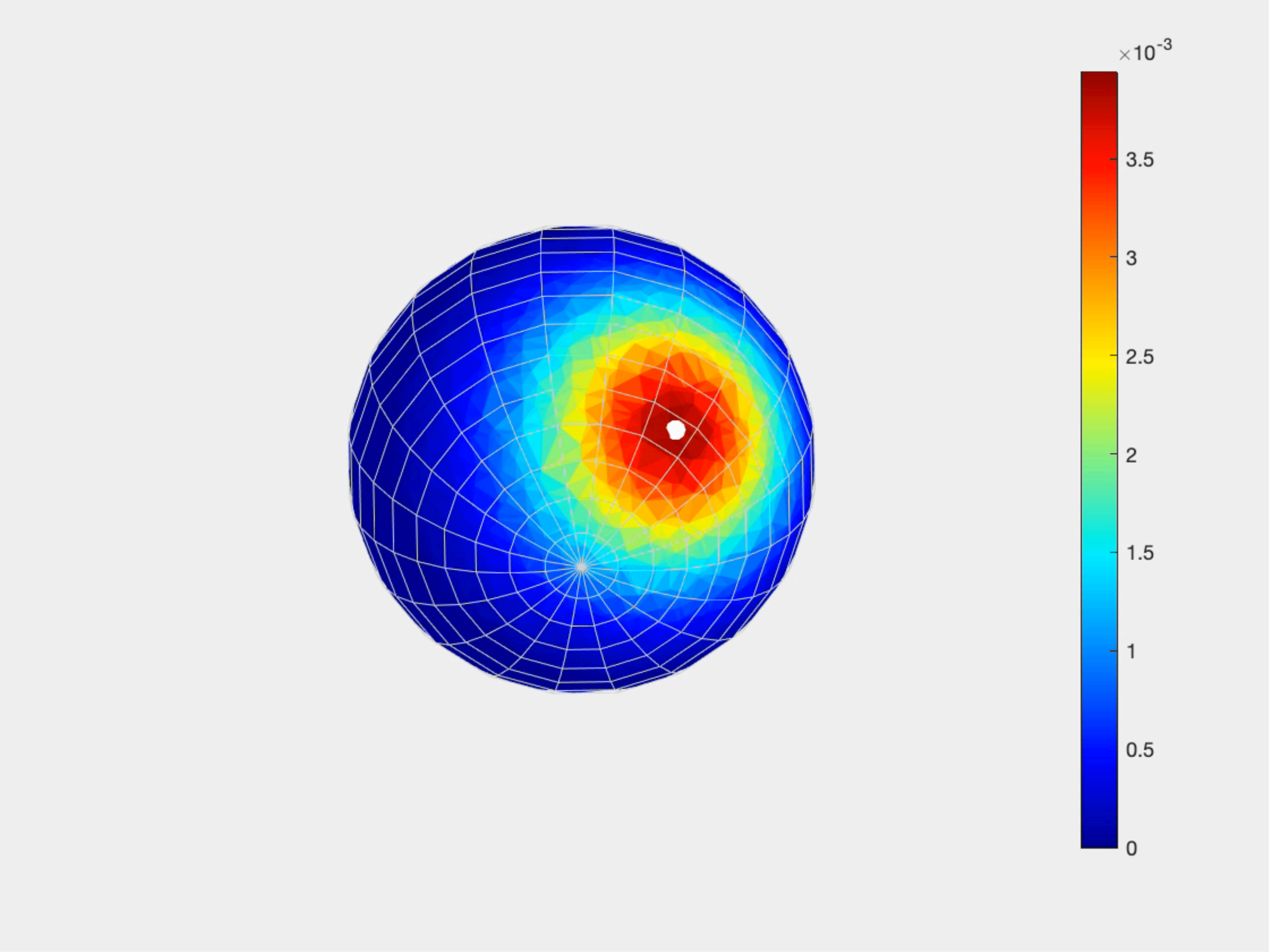


$N = 3000$ points
uniformly sampled on unit sphere





The solution



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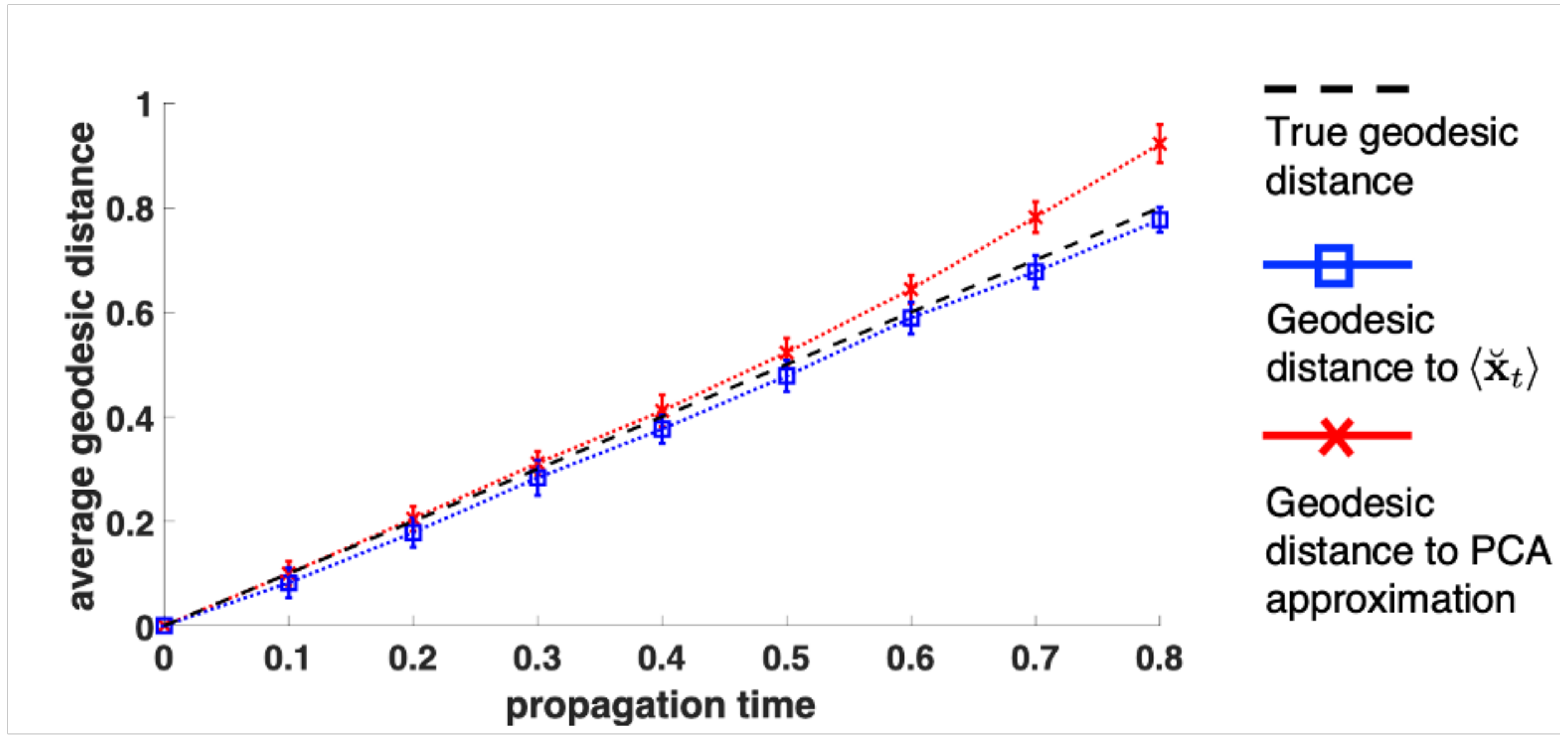
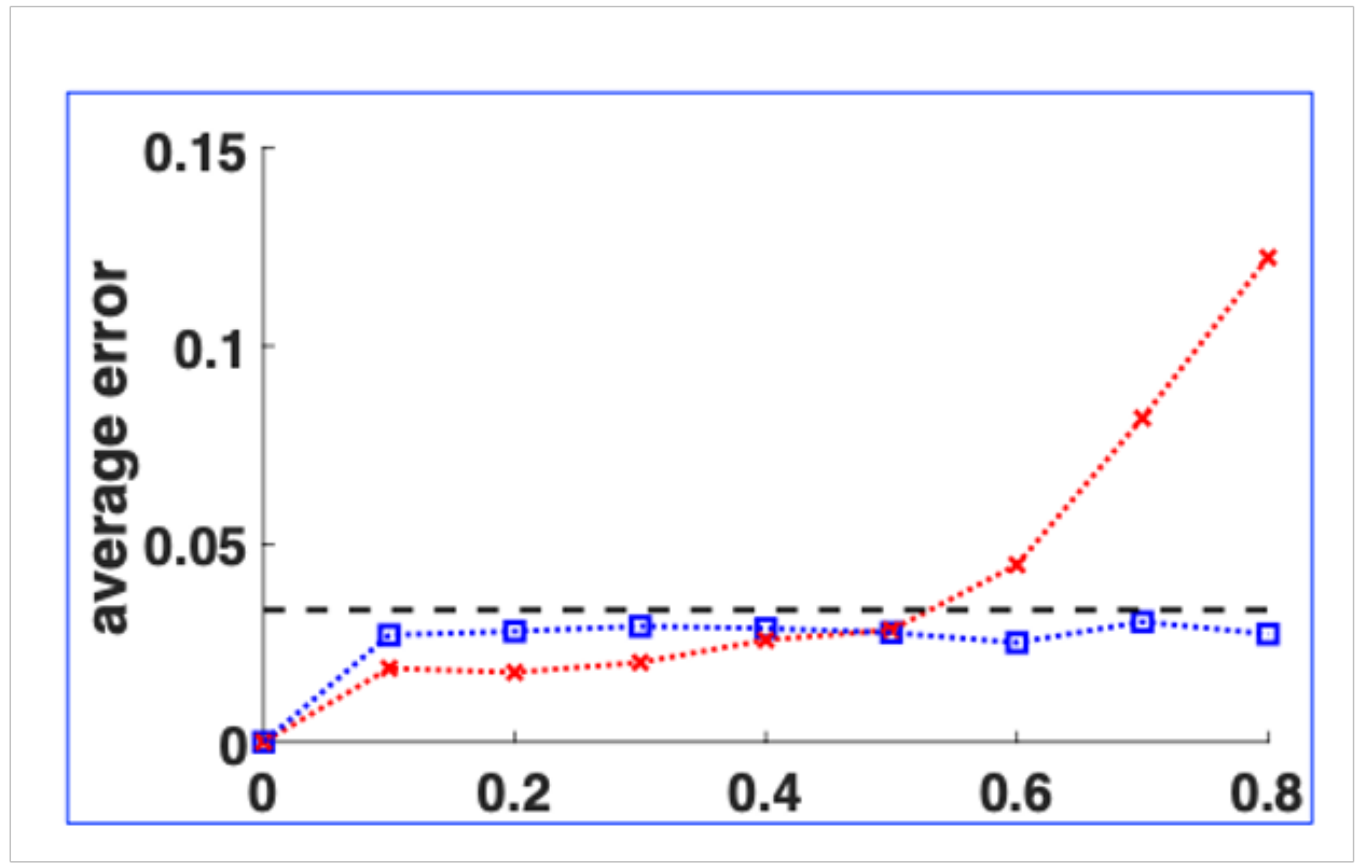
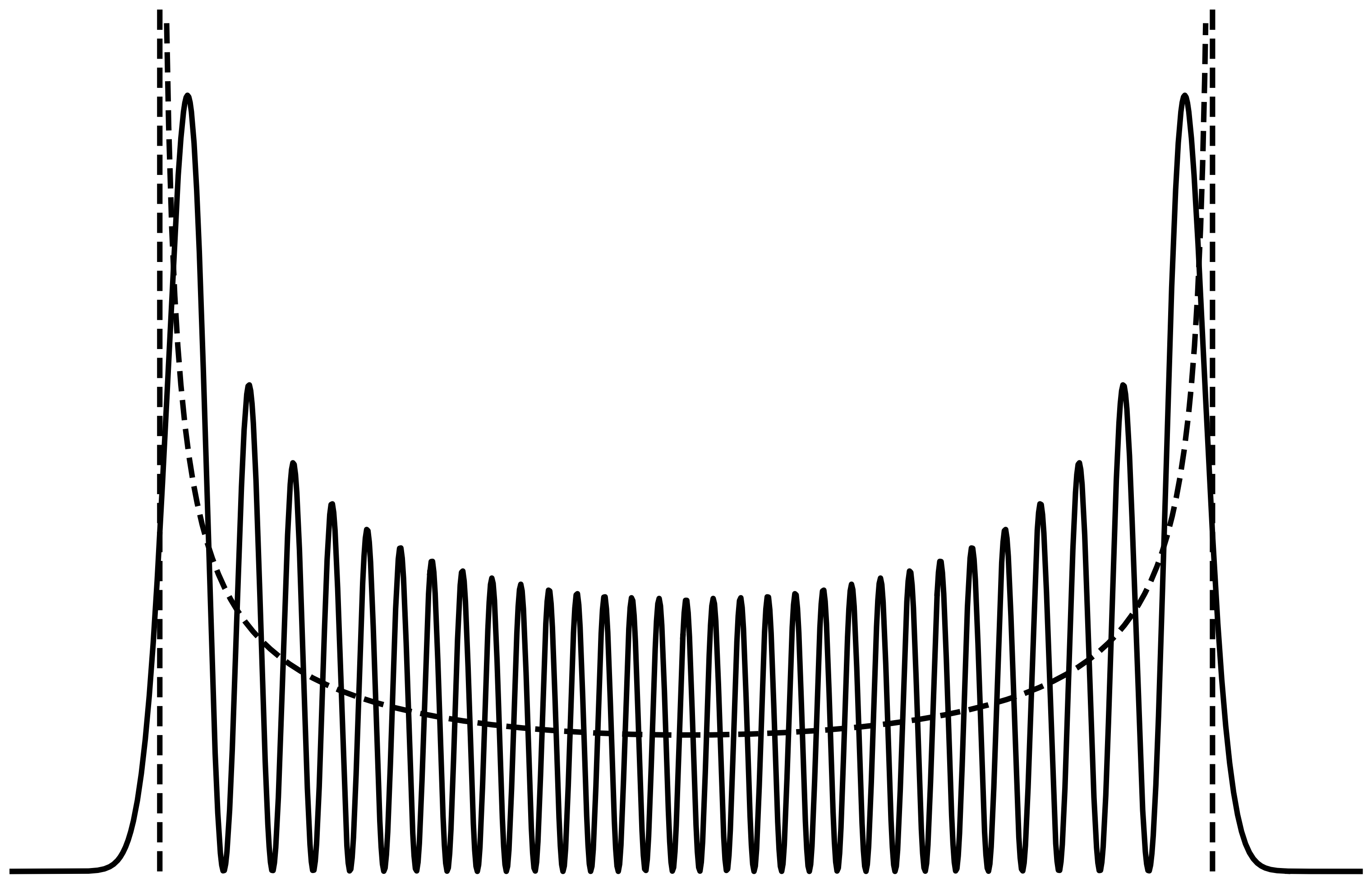


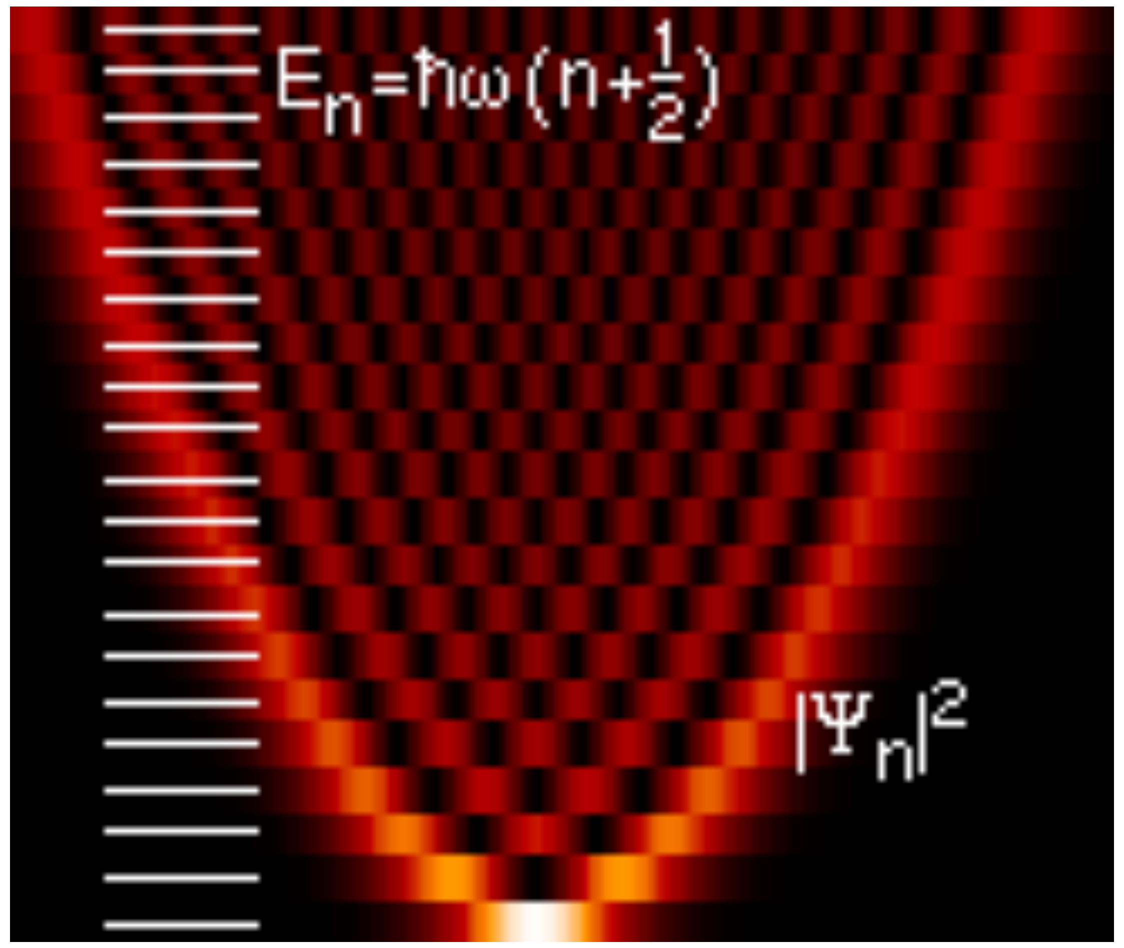


Quantum-classical correspondence

"classical mechanics emerges not directly, but only after averaging over phase-scrambling effects that can be ascribed to the environment […] — in modern parlance, decoherence effects."

Berry, *Quantum Mechanics: Scientific perspectives on divine action*, 41 (2001)





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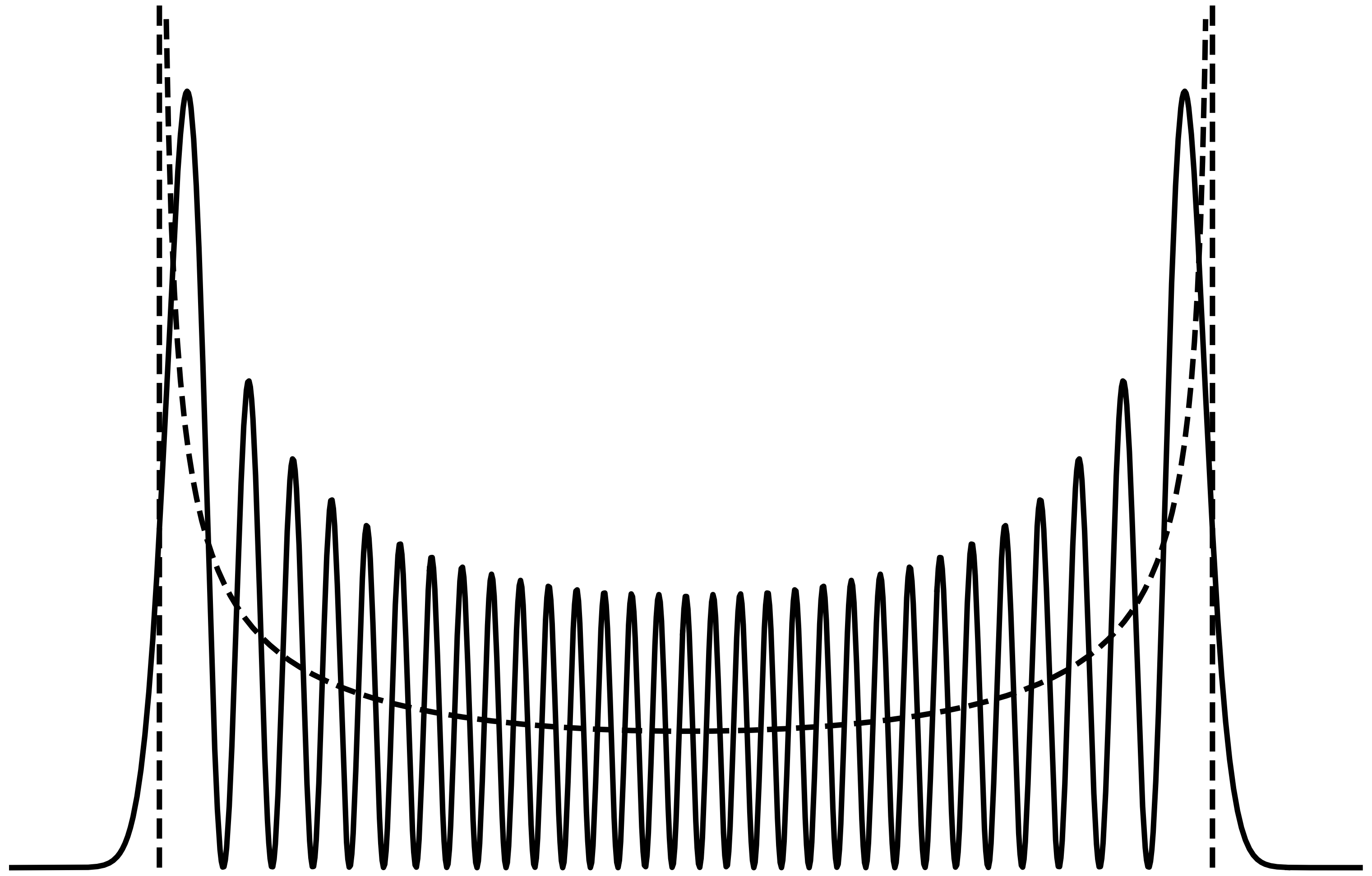

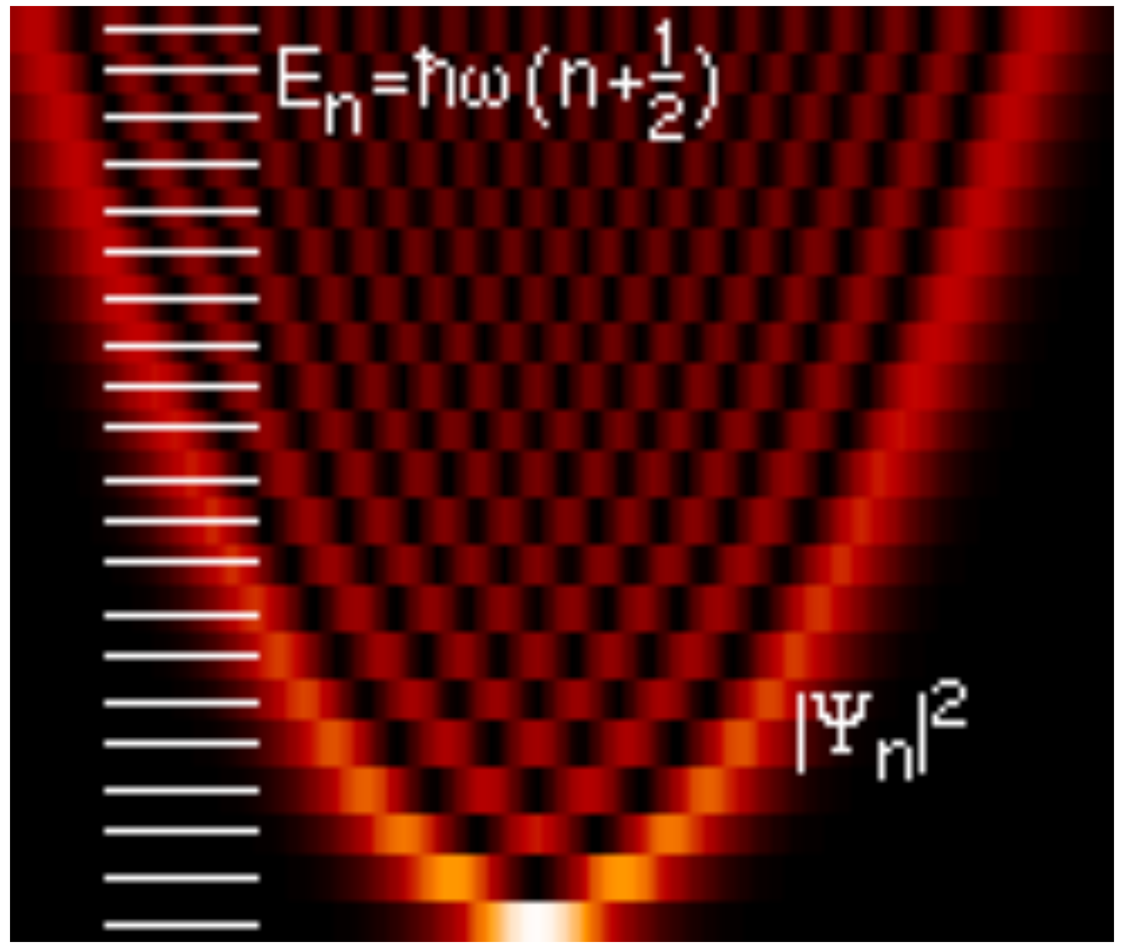

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Quantum dynamics:

linear in $|\psi_h\rangle$

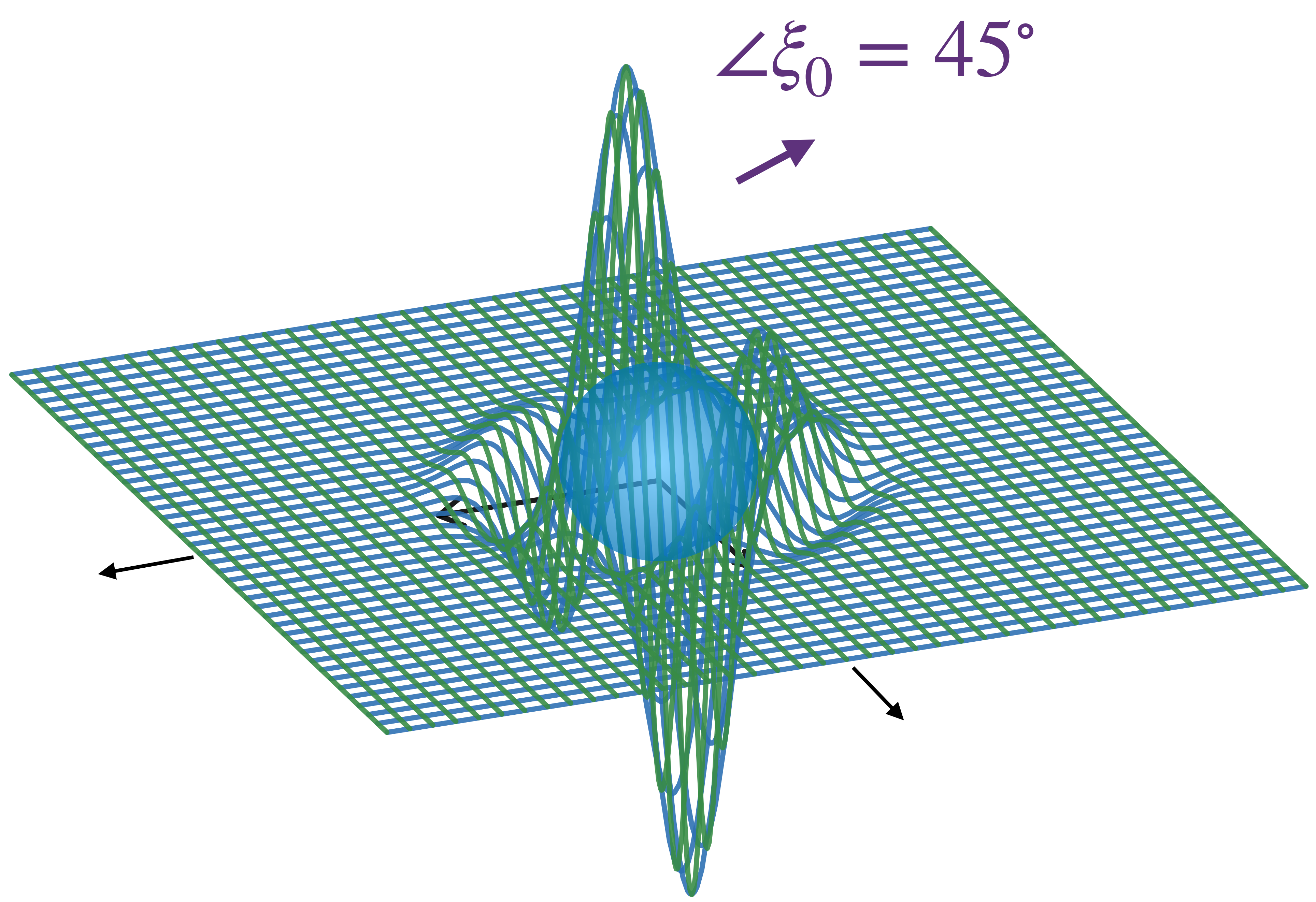
$$ih\partial_t|\psi_h\rangle = \sqrt{\hat{H}_h}\,|\psi_h\rangle$$

Geodesic flow:

nonlinear in x, ξ

$$\begin{cases} \partial_t x_j &= \partial_{\xi_j} H = (g^{-1}(x)\cdot\xi)_j \\ \partial_t \xi_j &= -\partial_{x_j} H = \langle \partial_{x_j} g^{-1}(x)\cdot\xi, \xi\rangle \end{cases}$$

Quantizing particles



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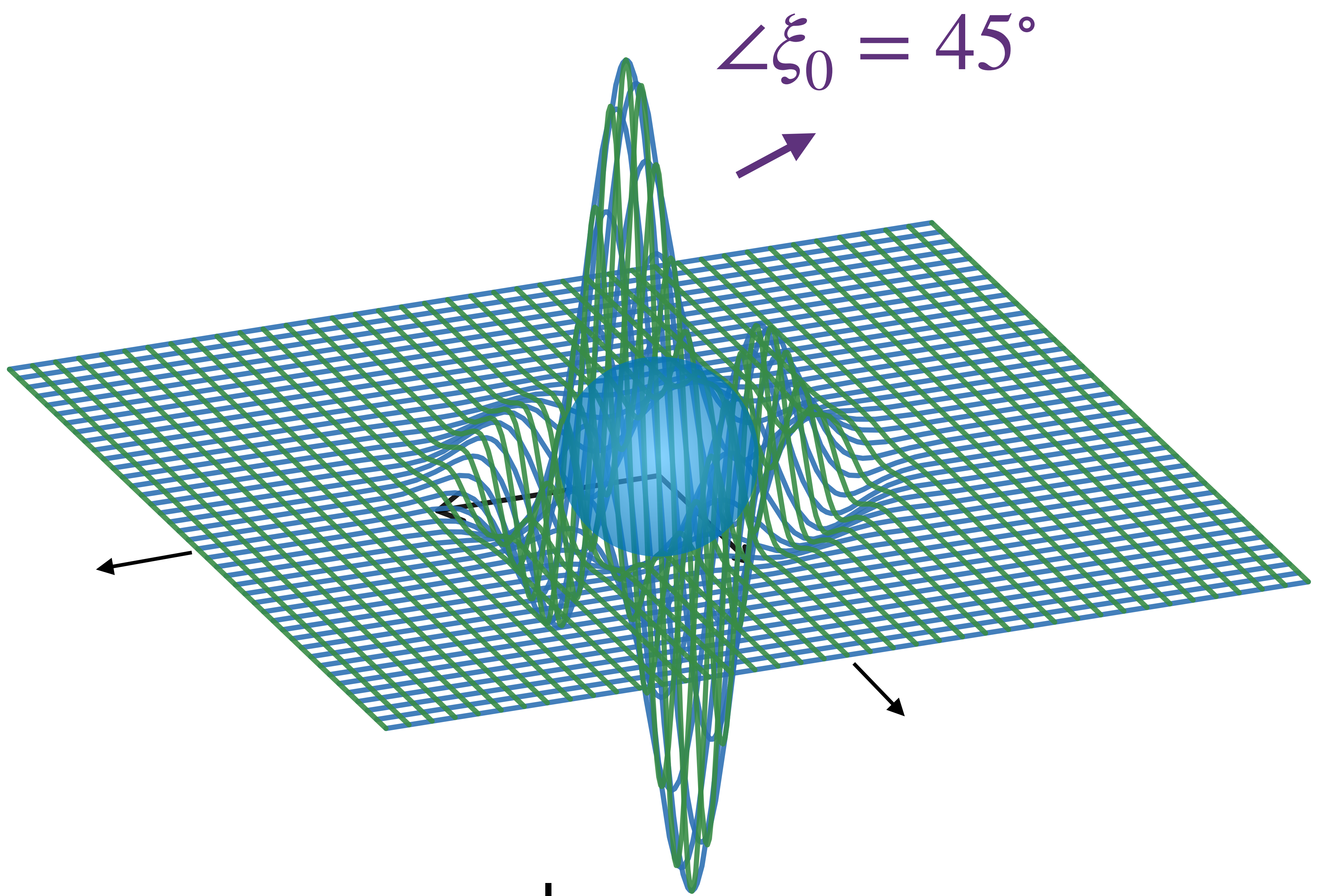
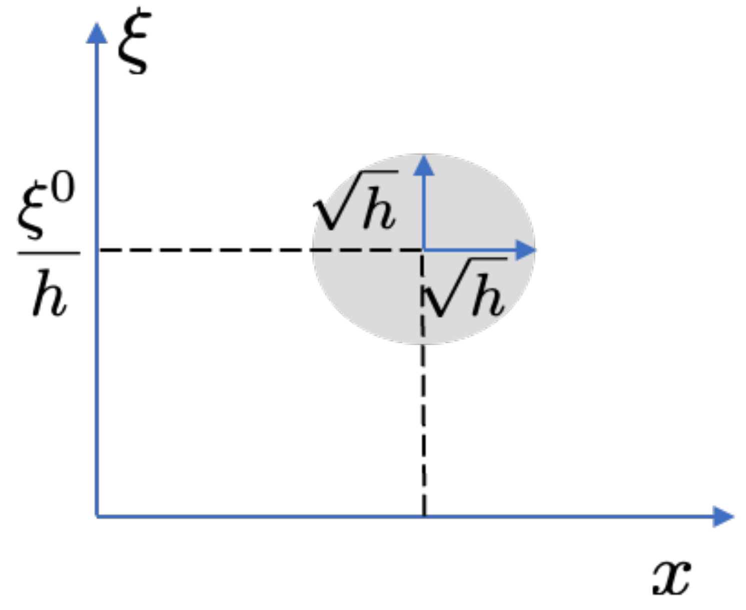
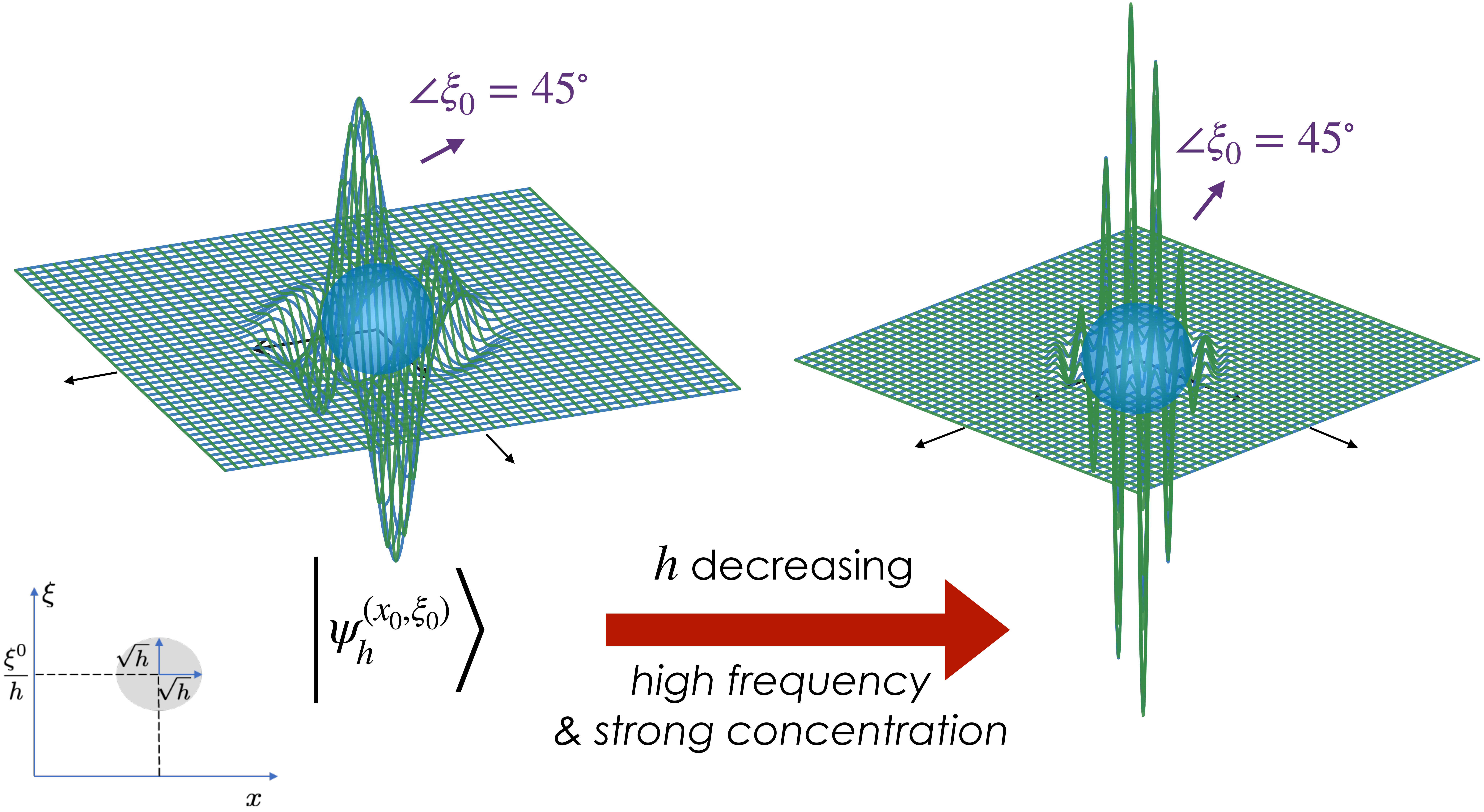




$$\left| \psi_h^{(x_0,\xi_0)} \right\rangle$$

$$\psi_h^{(x_0,\xi_0)}(x) \propto e^{-\frac{1}{2h}||x-x_0||^2-\frac{i}{h}(x-x_0)\cdot\xi_0}$$

Quantizing particles



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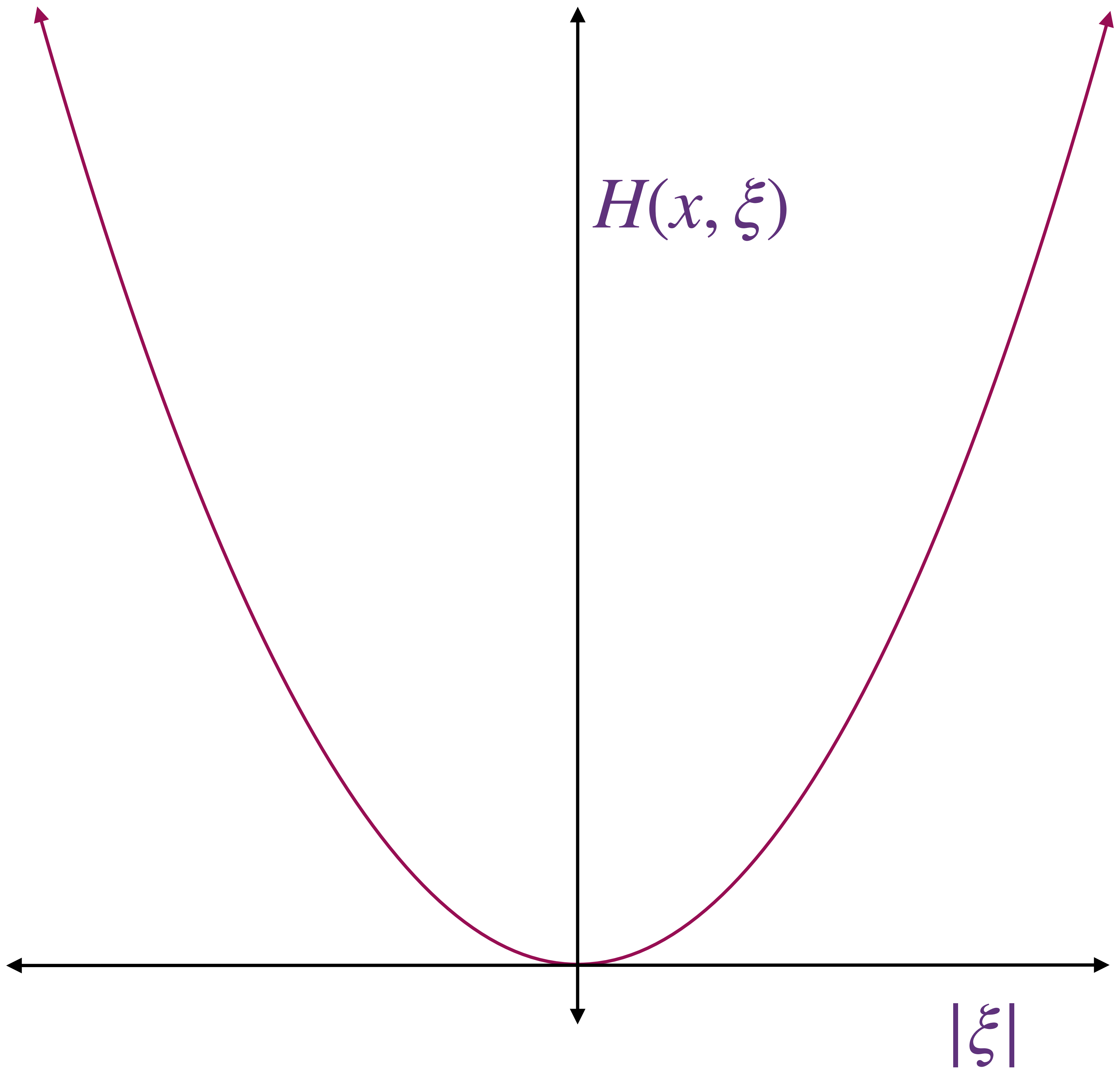
Quantizing particles

$$\psi_h^{(x_0,\xi_0)}(x) \propto e^{-\frac{1}{2h}||x-x_0||^2-\frac{i}{h}(x-x_0)\cdot\xi_0}$$

Quantizing the geodesic flow

Hamiltonian flow governed by $H(x, \xi) := |\xi|^2_{g(x)}$

$\angle\xi = \pm 45^\circ$



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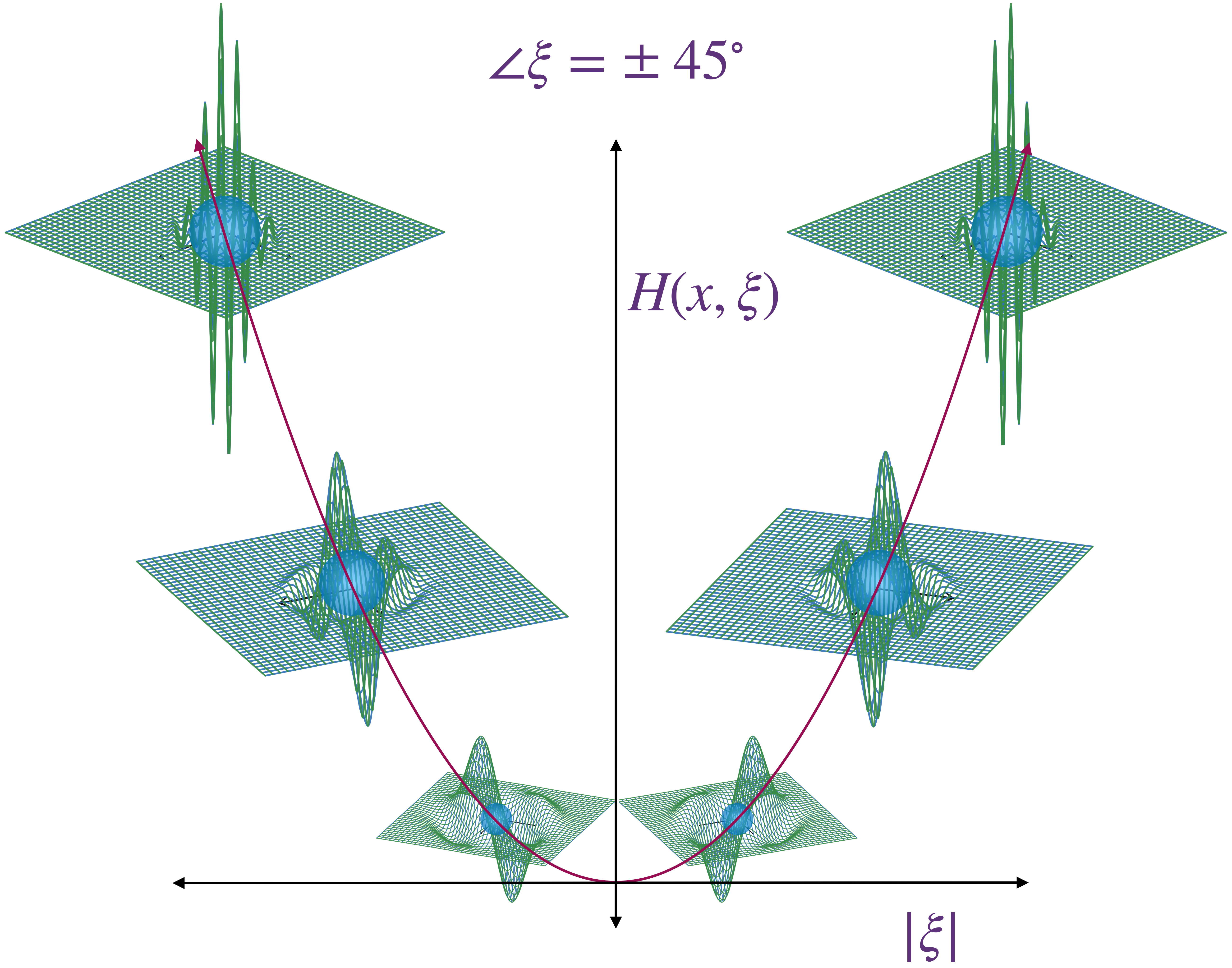

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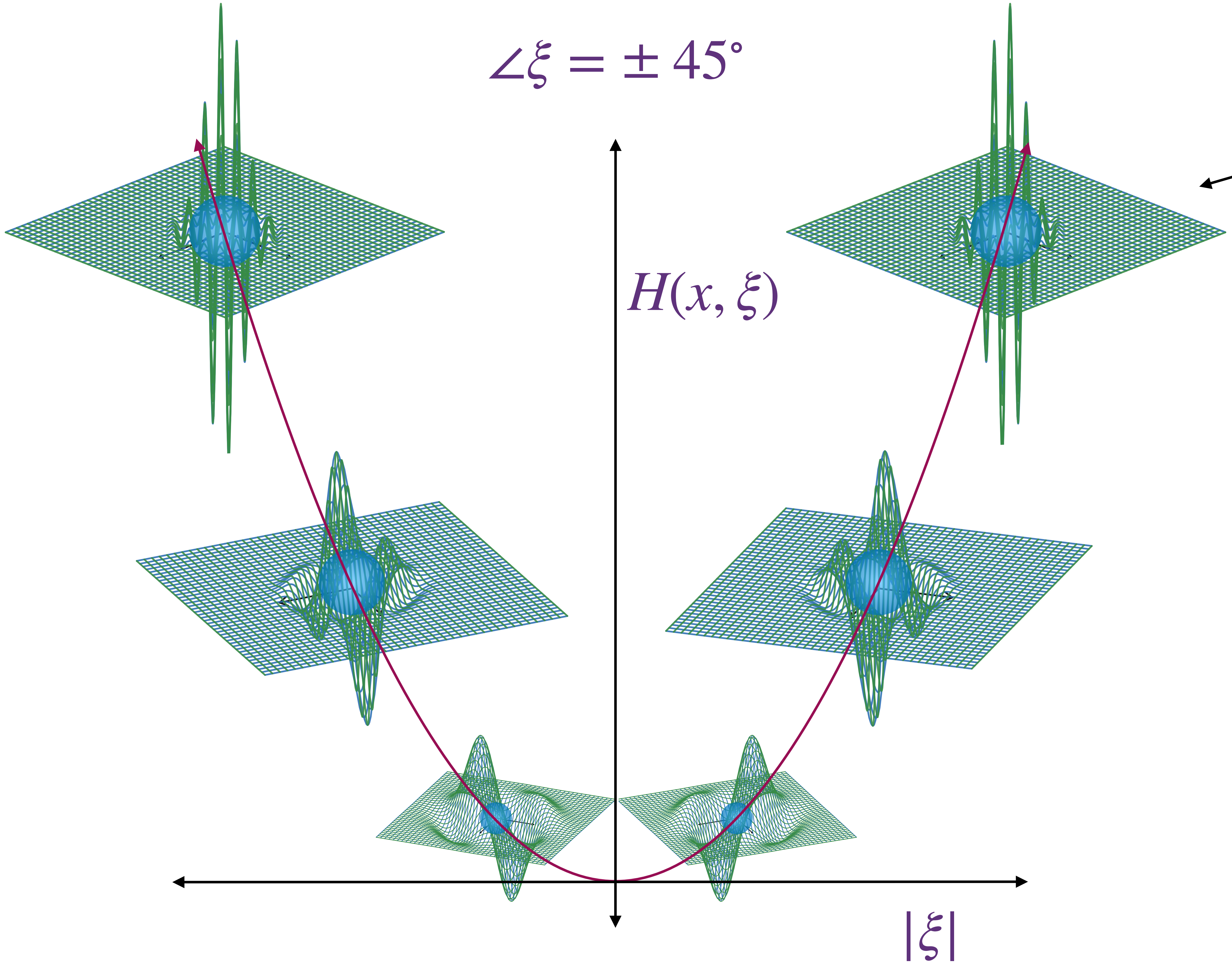

$$\hat{H}_h = \sum_{x,\xi} H(x,\xi) \left|\psi_h^{(x,\xi)}\right\rangle\left\langle\psi_h^{(x,\xi)}\right|$$

$$\langle\psi_h^{(x,\xi)}|\psi_h^{(x,\xi)}\rangle = 1 + O(h^\infty)$$

$$\implies \langle\psi_h^{(x,\xi)}|\hat{H}_h|\psi_h^{(x,\xi)}\rangle = H(x,\xi) + O(h)$$

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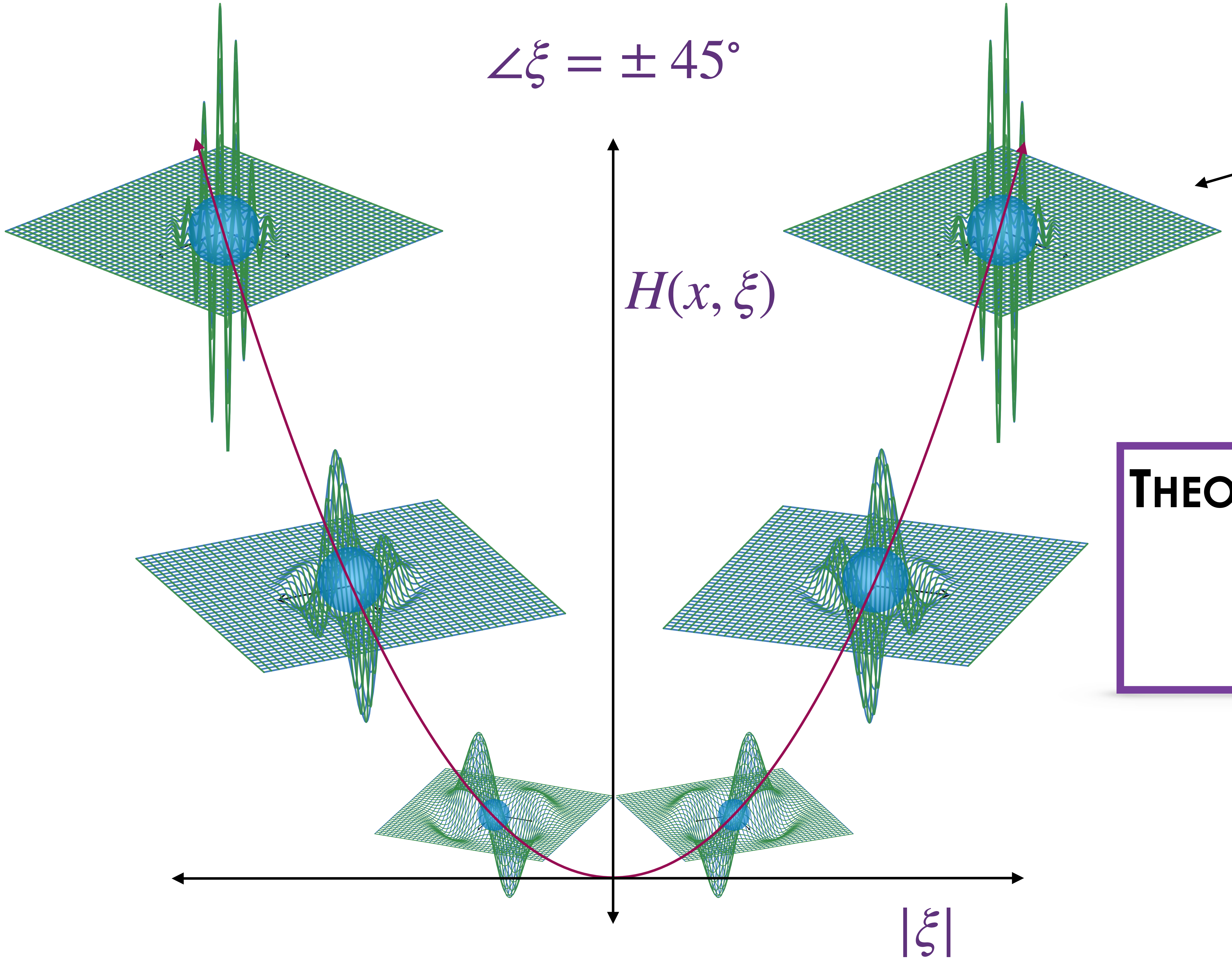

$$\hat{H}_h = \sum_{x,\xi} H(x,\xi) \left| \psi_h^{(x,\xi)} \right\rangle \left\langle \psi_h^{(x,\xi)} \right|$$

Theorem.

$$\hat{H}_h = h^2 \Delta_{\mathscr{M}} + \text{lower oder terms}$$

$$\langle \psi_h^{(x,\xi)} | \psi_h^{(x,\xi)} \rangle = 1 + O(h^\infty)$$

$$\implies \langle \psi_h^{(x,\xi)} | \hat{H}_h | \psi_h^{(x,\xi)} \rangle = H(x,\xi) + O(h)$$

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Schrödinger equation $ih\partial_t |\psi_h\rangle = \sqrt{\hat{H}_h}\,|\psi_h\rangle$

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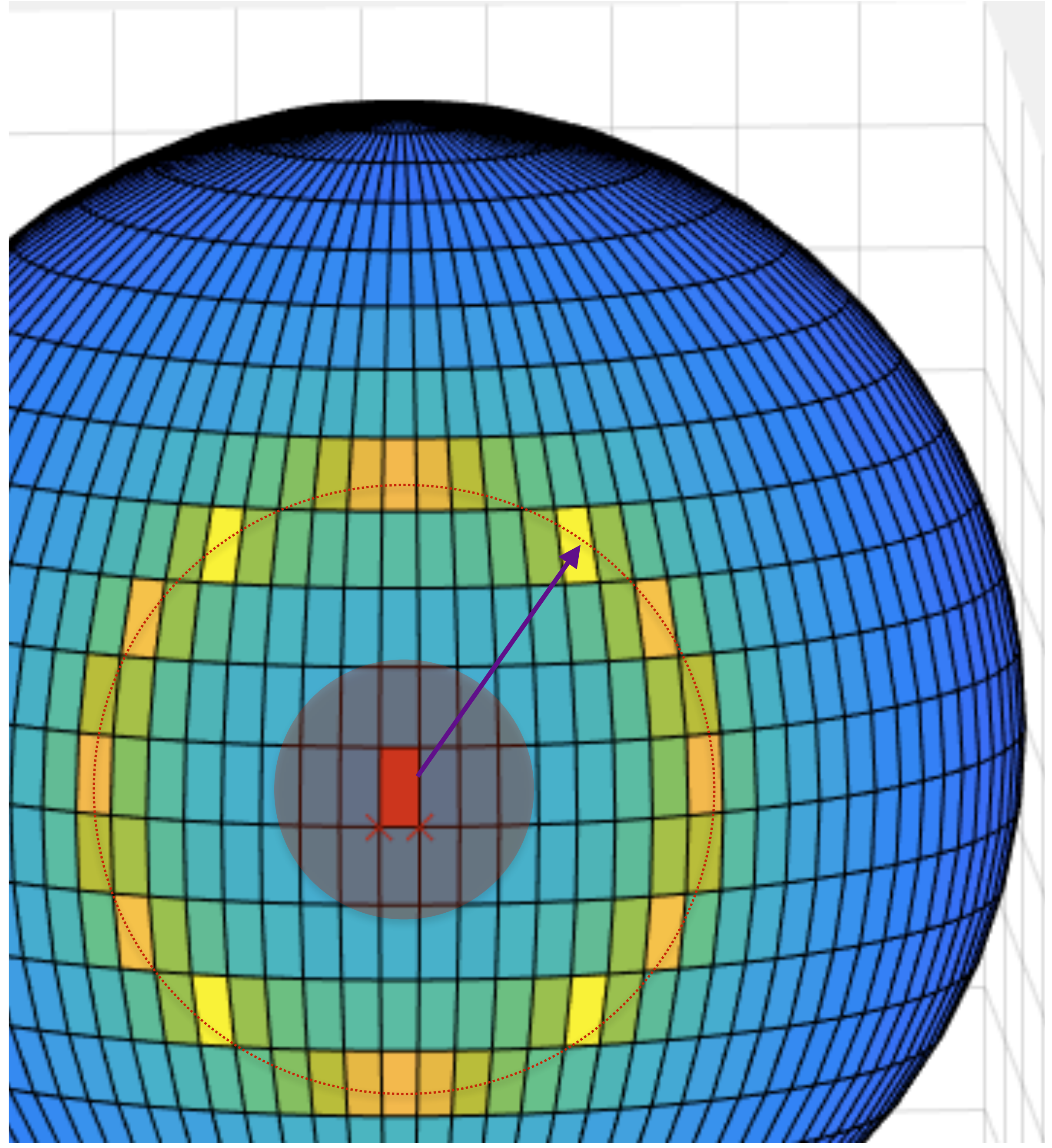
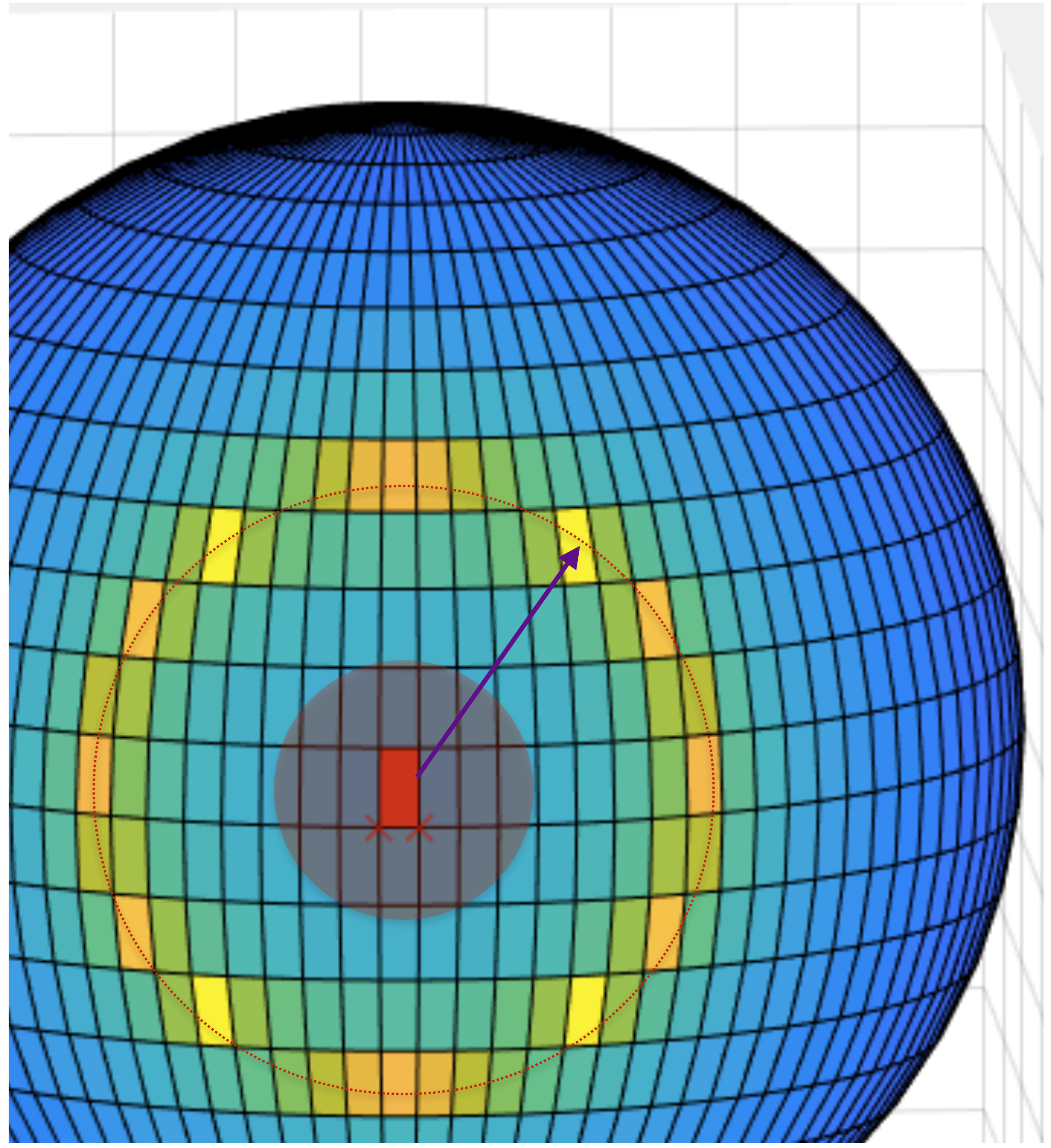


Initial state

$$\left|\psi_h^{(x_0,\xi_0)}\right\rangle$$

Quantizing the geodesic flow

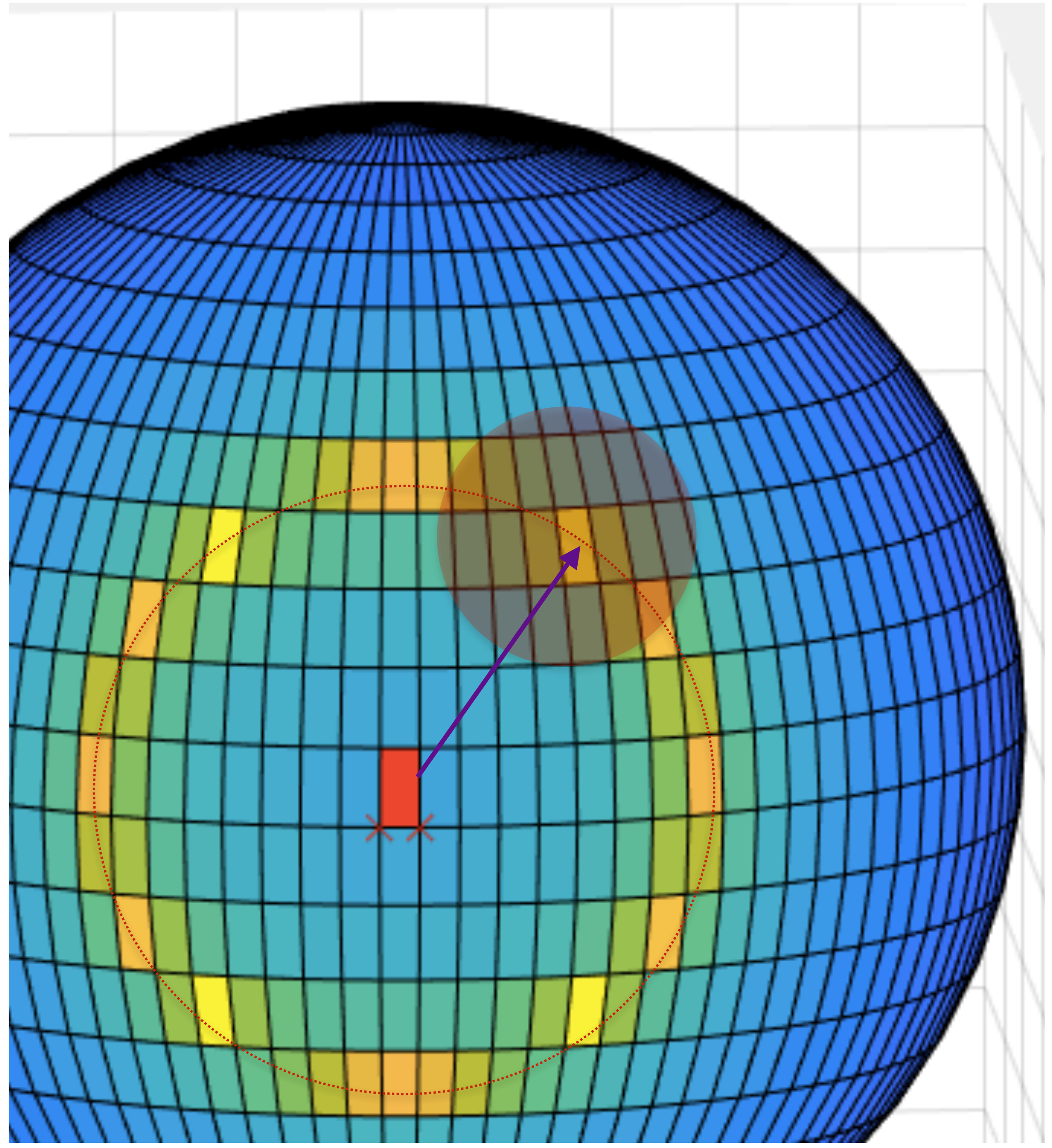
Schrödinger equation $ih\partial_t |\psi_h\rangle = \sqrt{\hat{H}_h} \left|\psi_h\right\rangle$



Initial state

Propagated state

$$\left|\psi_h^{(x_0,\xi_0)}\right\rangle \longrightarrow \left|\psi_h(t)\right\rangle := e^{\frac{i}{h}t\sqrt{\hat{H}_h}} \left|\psi_h^{(x_0,\xi_0)}\right\rangle$$



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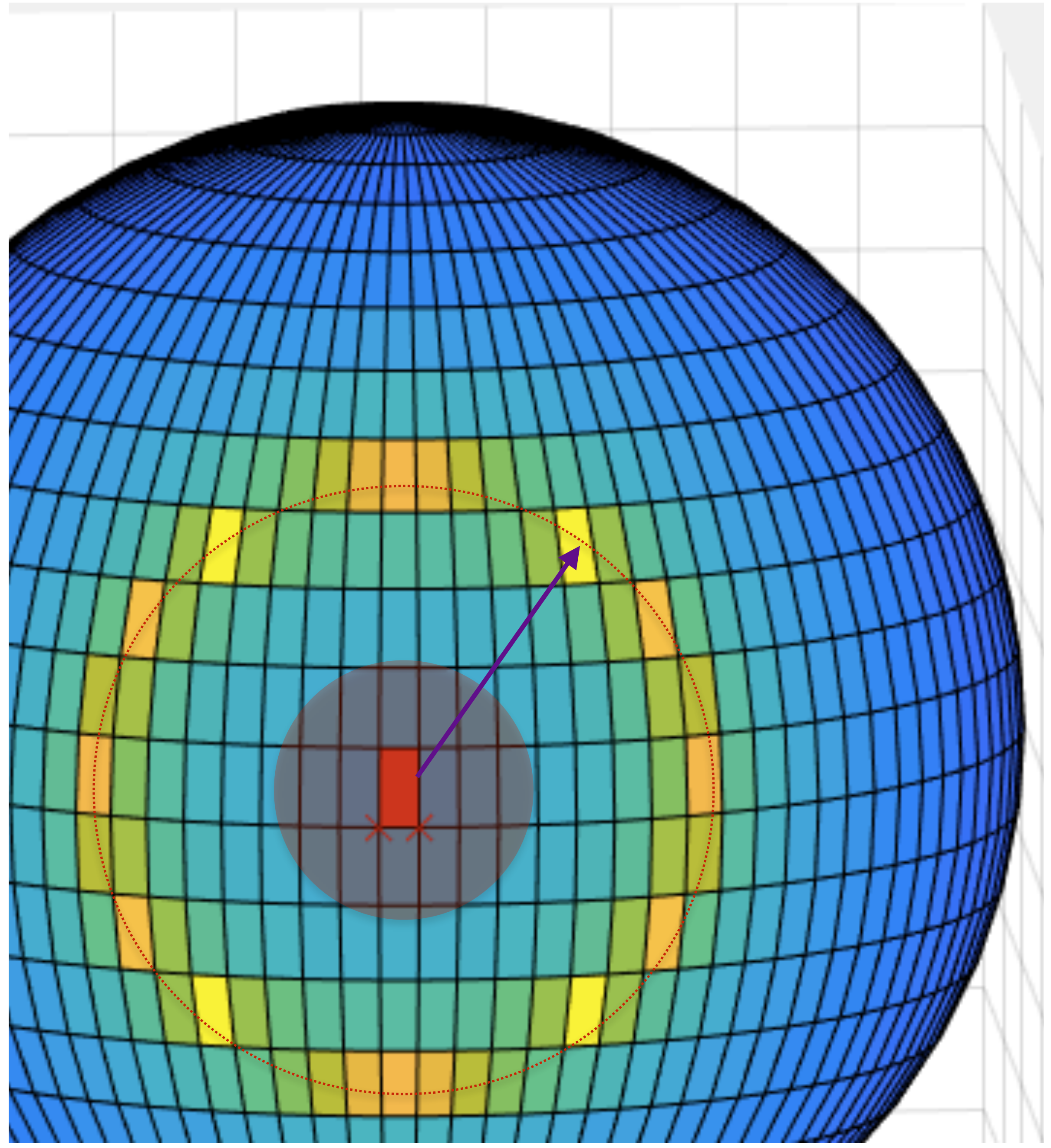

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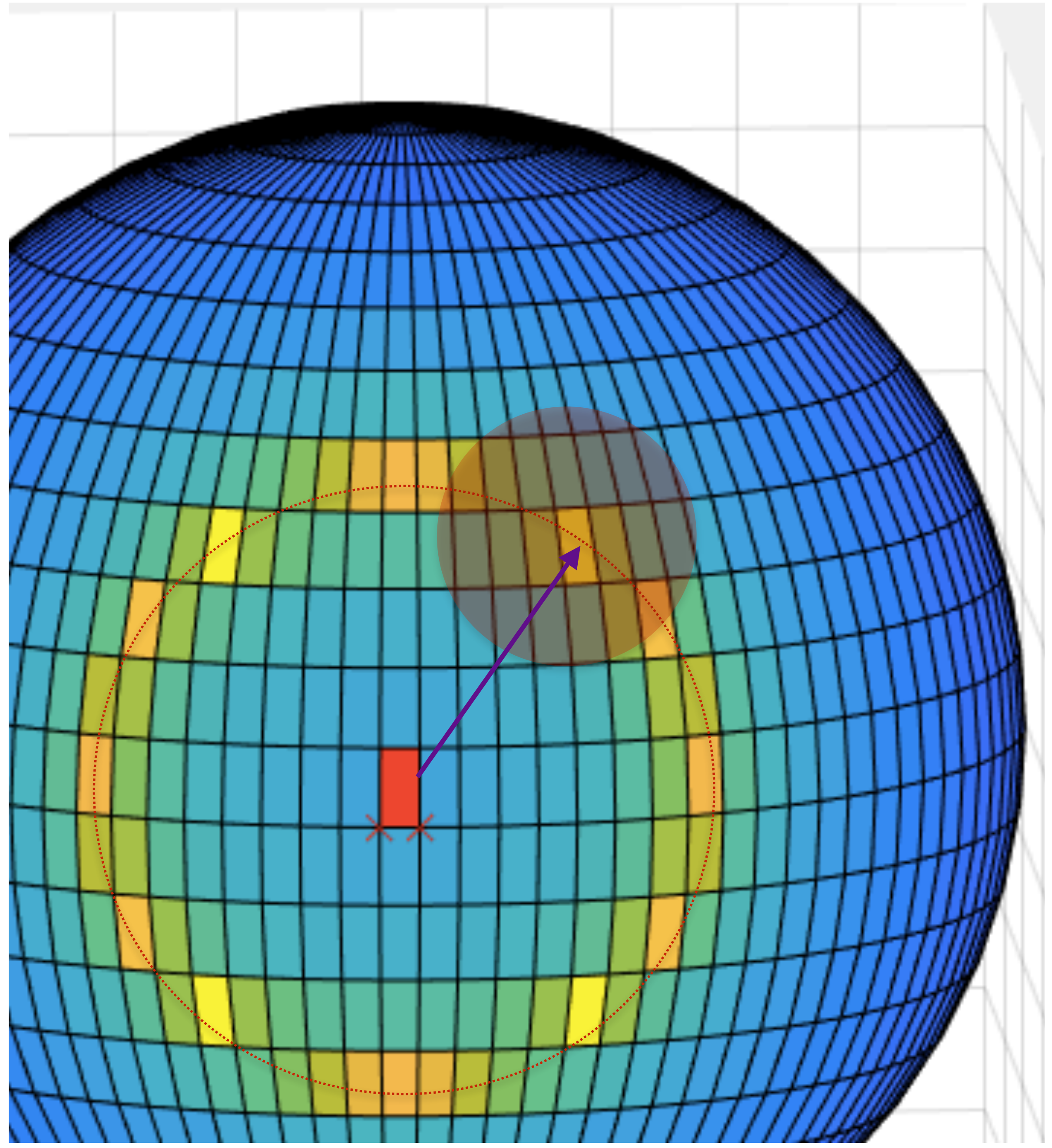
Propagated state



THEOREM.

The density $|\psi_h(t)|^2$ is concentrated in an $O(h)$ nbd. of $x(t) := \pi_{\mathscr{M}}\Gamma^t(x_0,\xi_0)$.

Discrete quantum-classical correspondence

“If you want to see something, you send waves in its general direction, you don’t throw heat at it.”

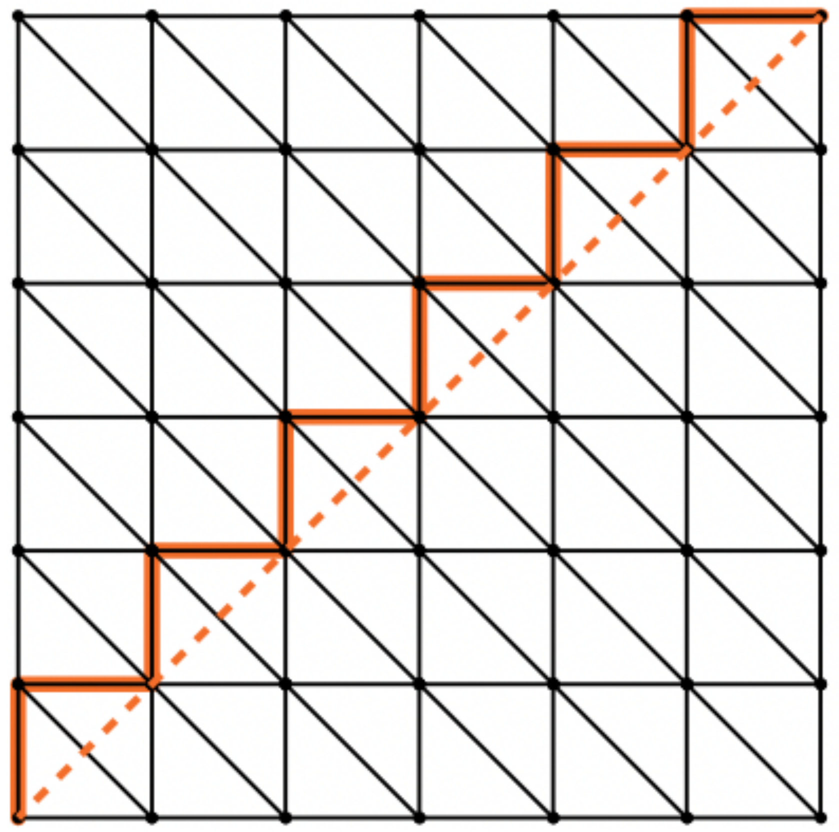
Attributed to Peter Lax (Cloninger & Steinerberger, *Applied & Computational Harmonic Analysis* (2017)

Random walks on data lead to diffusion

Random walk:

1. Nbd. graph:

$$\left[\mathscr{A}_{\epsilon,N}\right]_{i,j} : \begin{cases} \approx 1 & \text{if } |v_j - v_i| \ll \sqrt{\epsilon} \\ \approx 0 & \text{if } |v_j - v_i| \gg \sqrt{\epsilon} \end{cases}$$



Crane et al., Comm. ACM (2017)

2. Markov chain: $A_{\epsilon,N} := \frac{1}{\mathscr{A}_{\epsilon,N}[1]}\mathscr{A}_{\epsilon,N}$

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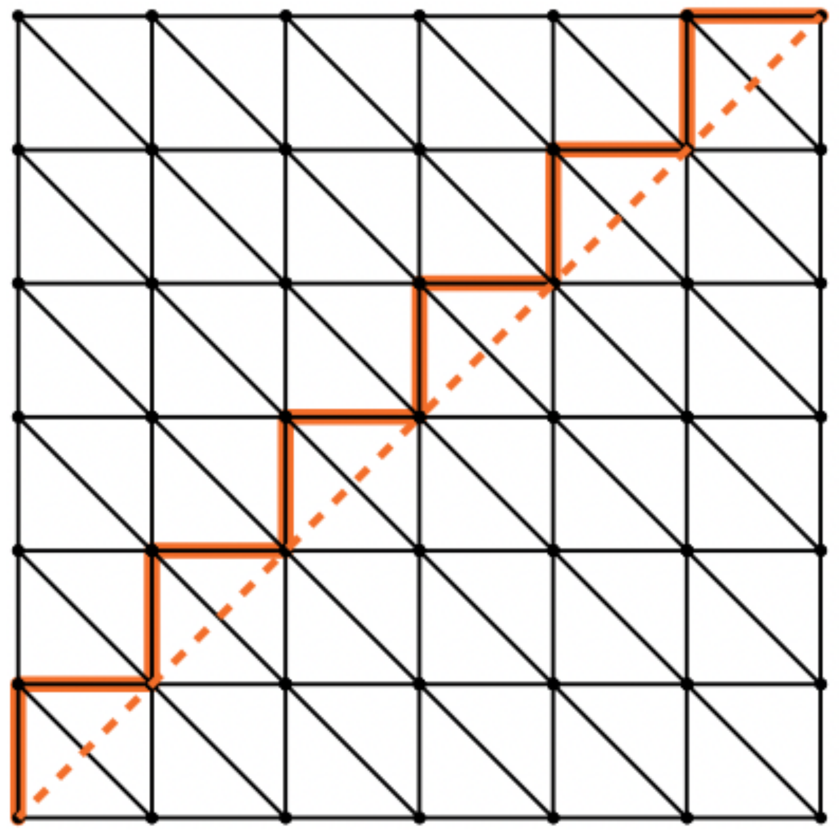


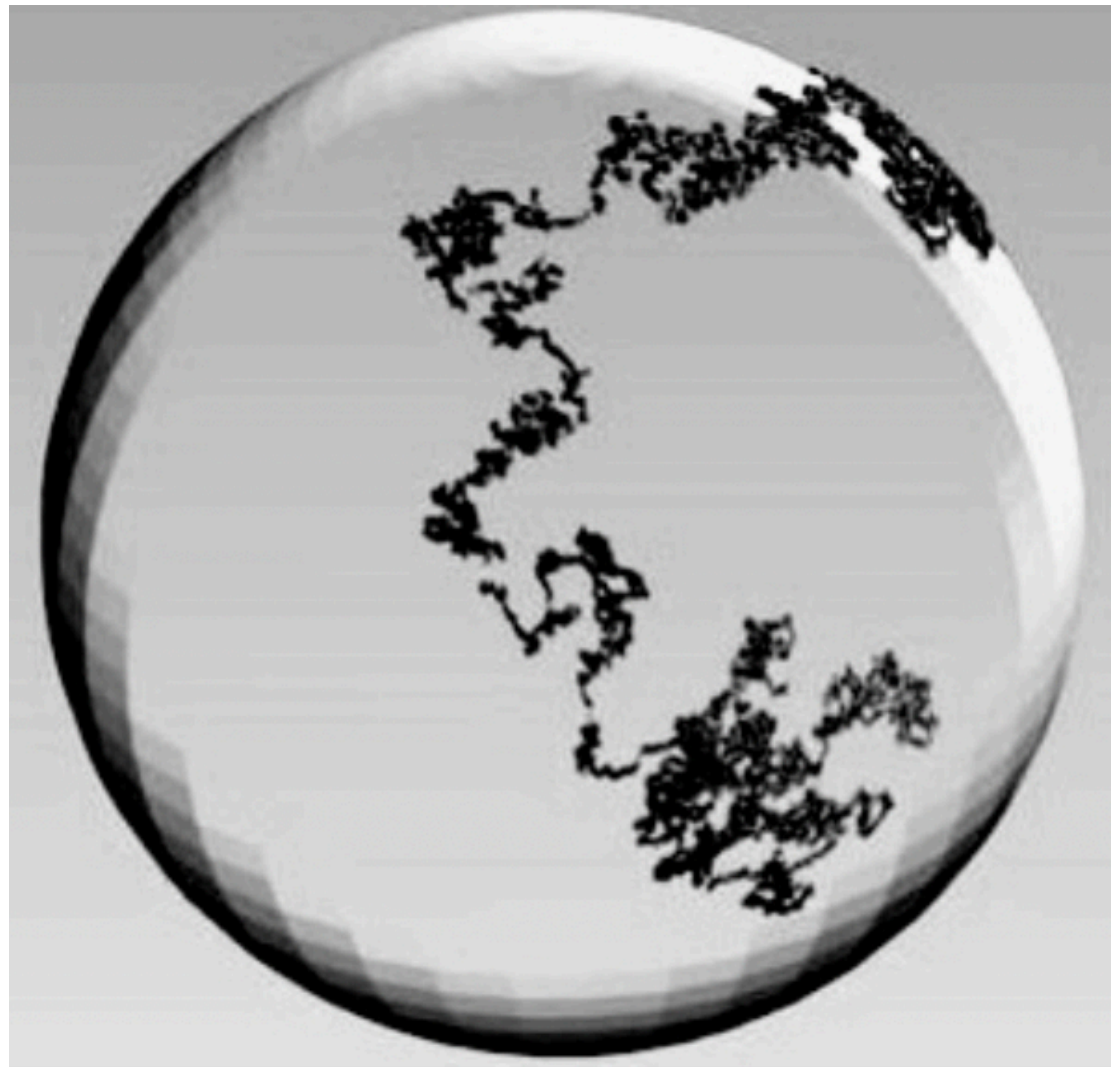
Crane et al., Comm. ACM (2017)

large N

with high prob.,

$\sqrt{\epsilon} \ll 1$

Geodesic random walk



Ginkel & Redig, *Journal of Statistical Physics* (2020).

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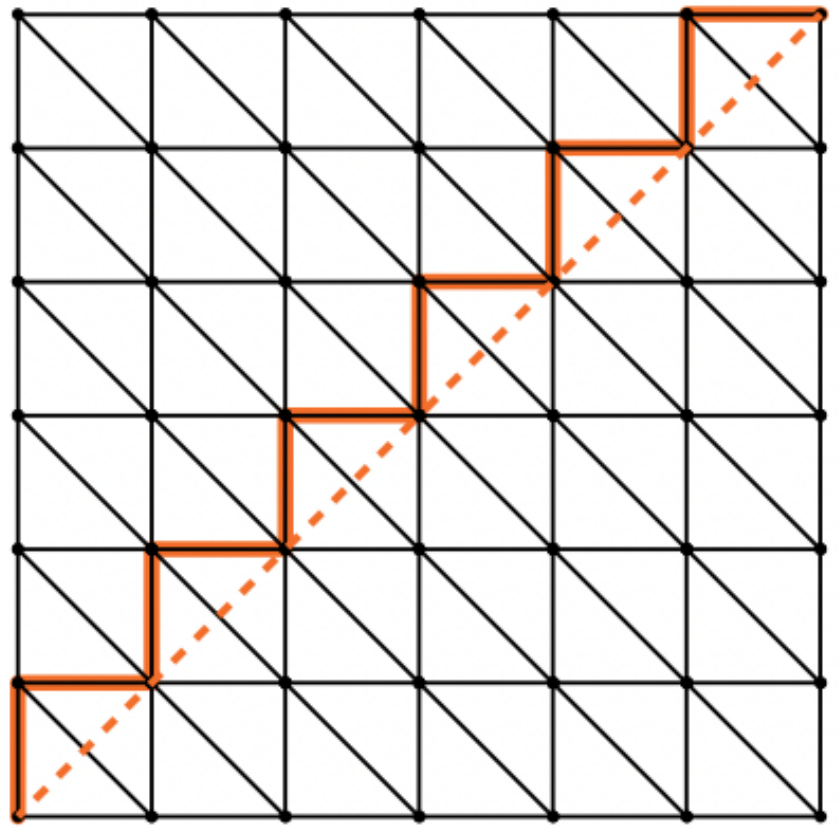

Crane et al., Comm. ACM (2017)

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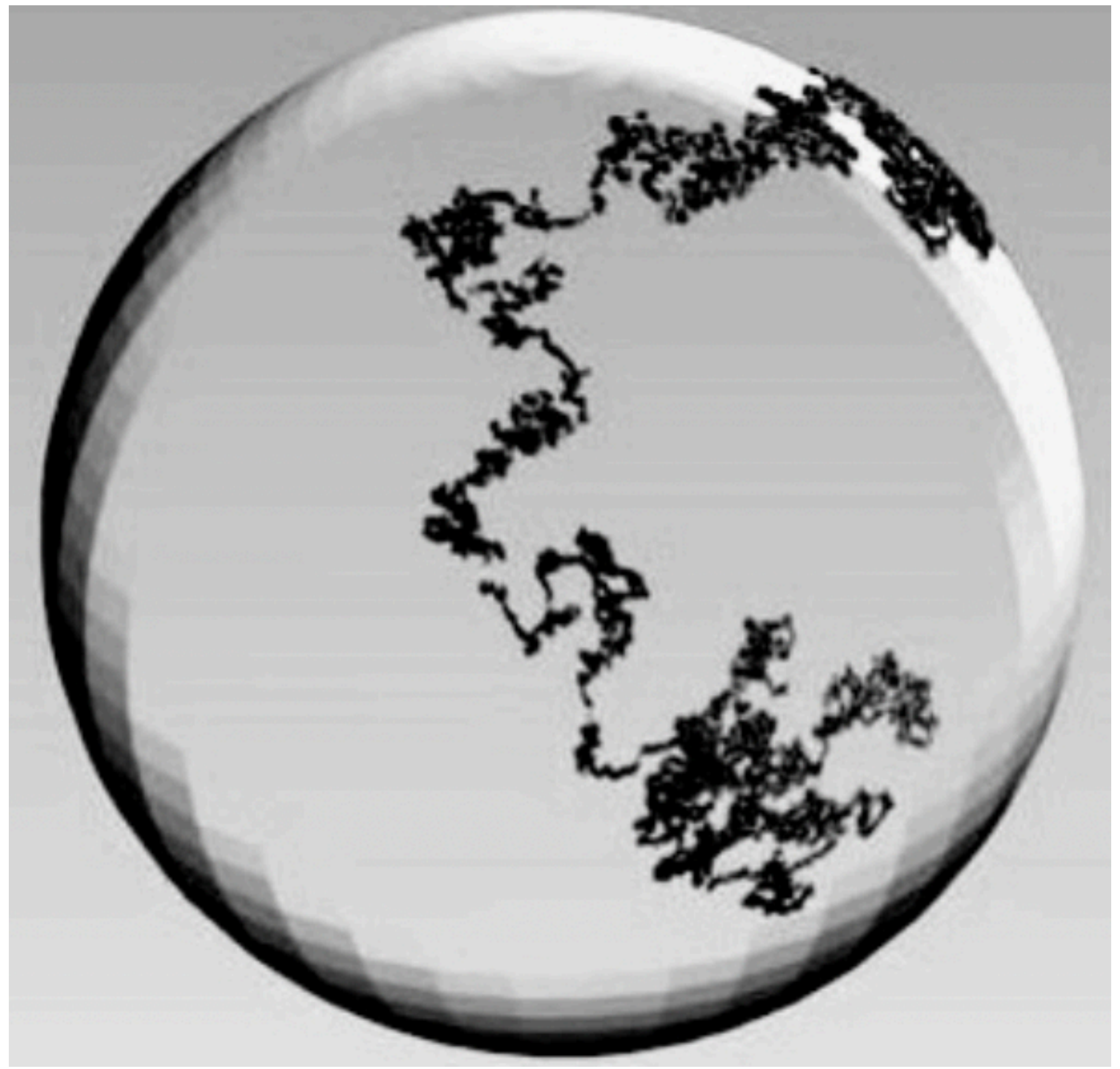


Ginkel & Redig, *Journal of Statistical Physics* (2020).

DISCRETE GENERATOR: $\Delta_{\epsilon,N} := c(I - A_{\epsilon,N})/\epsilon$ IS THE *RANDOM WALK GRAPH* LAPLACIAN.

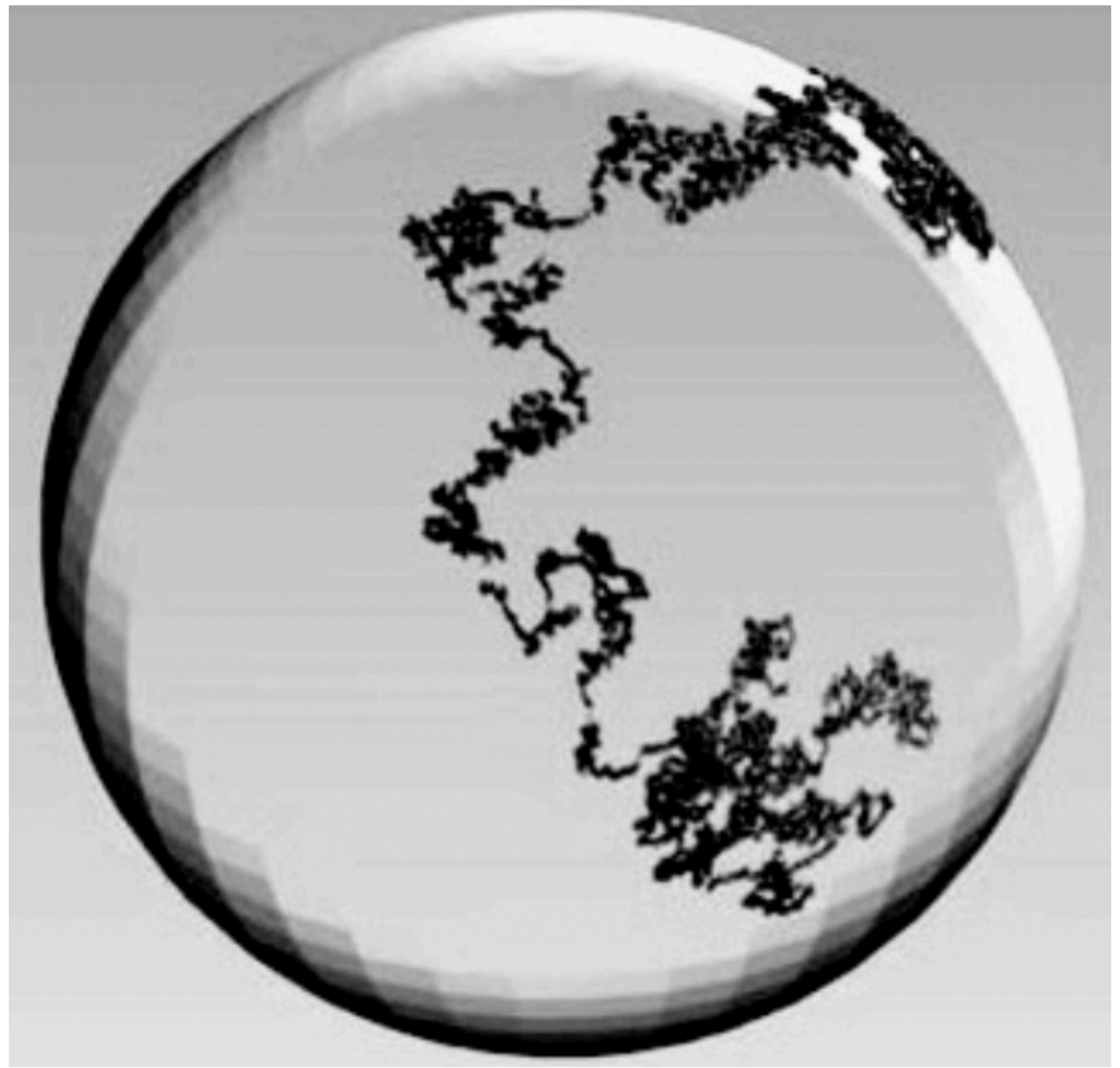
With high probability,

Hein, *PhD thesis* (2005).

$$\Delta_{\epsilon,N}|\psi\rangle = \Delta_{\mathscr{M}}|\psi\rangle + \underbrace{O(\partial^1)}_{\text{sampling density}}|\psi\rangle + \underbrace{O(\epsilon)}_{\textit{diffusion terms}}$$

From random walks to quantum dynamics on data

Schrödinger equation $ih\partial_t|\psi_h\rangle = \sqrt{h^2\Delta_{\epsilon,N}}\,|\psi_h\rangle$



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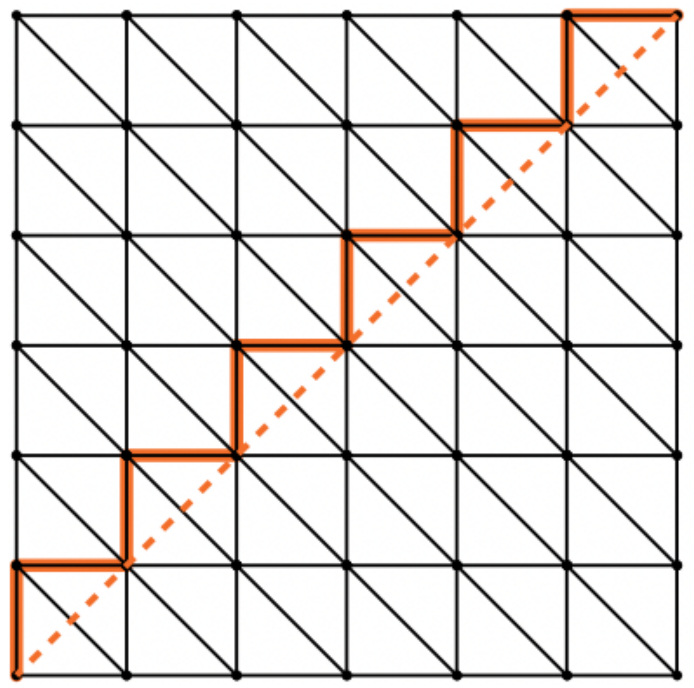
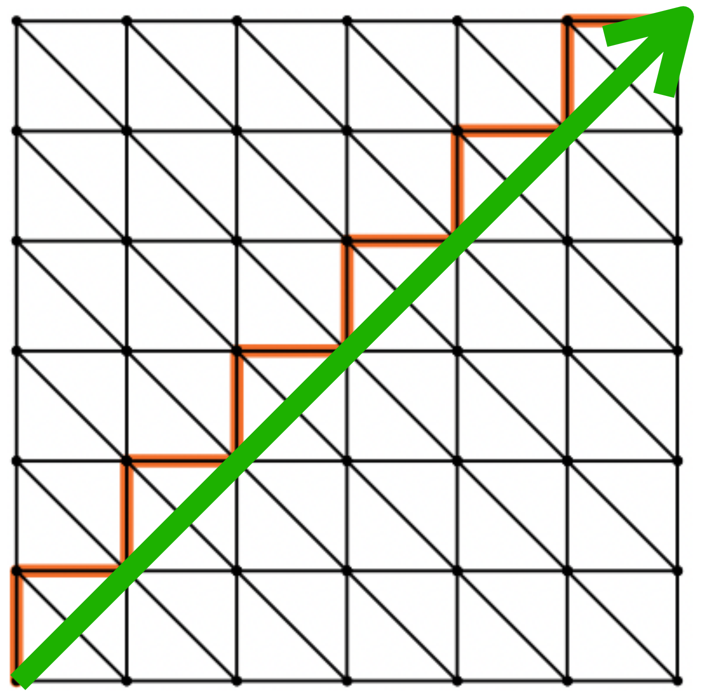
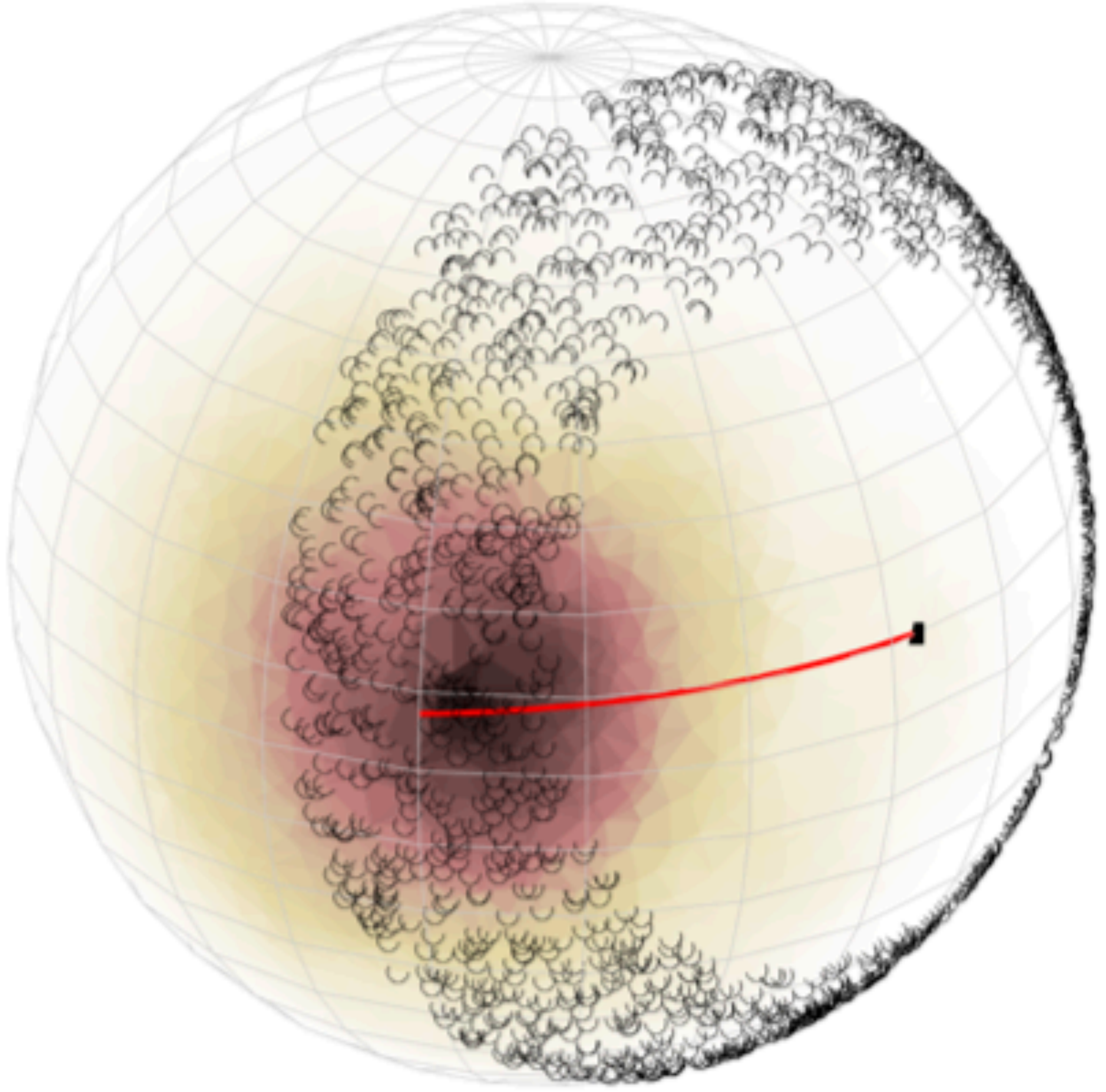


Crane et al., Comm. ACM (2017)

with high prob.,

$h \gg \sqrt{\epsilon}$





Kumar, arXiv:2112.10748 (2022)

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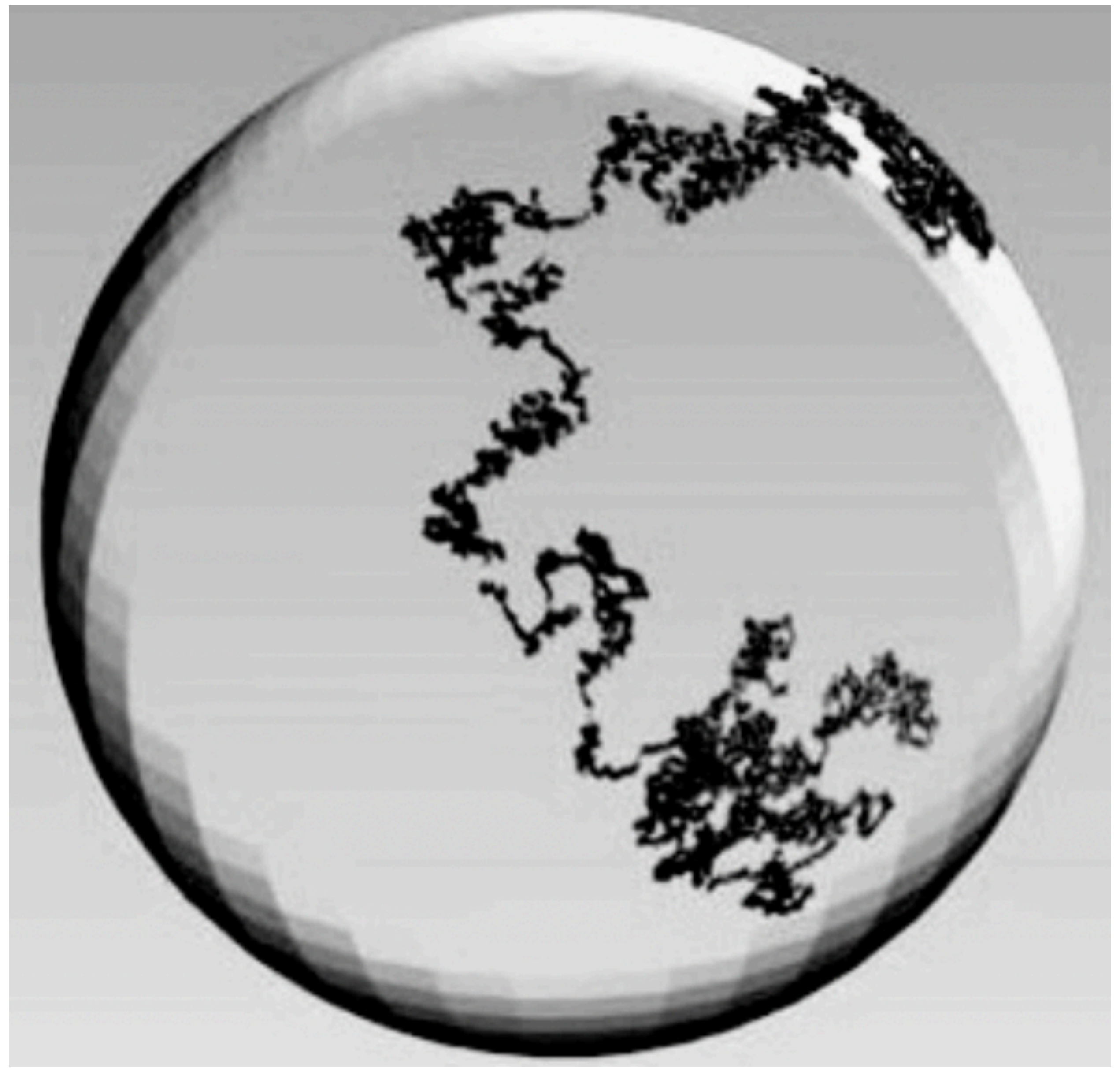

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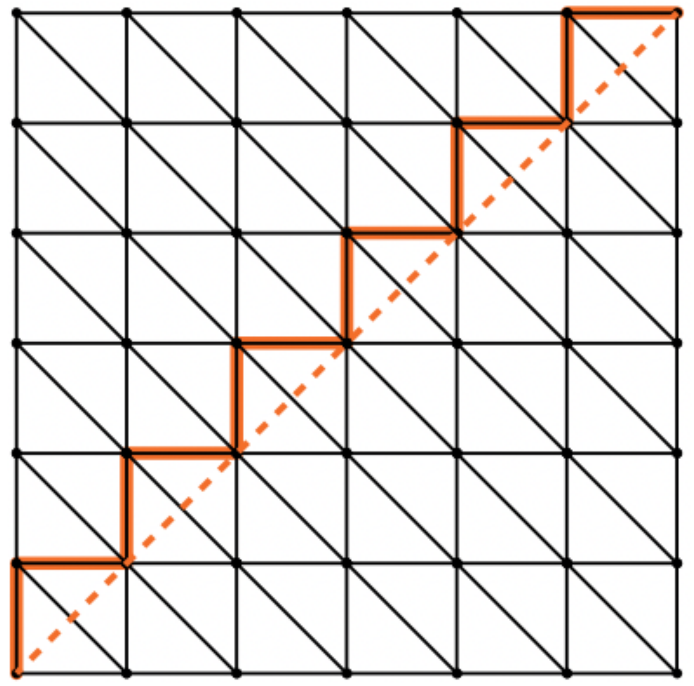

Crane et al., Comm. ACM (2017)

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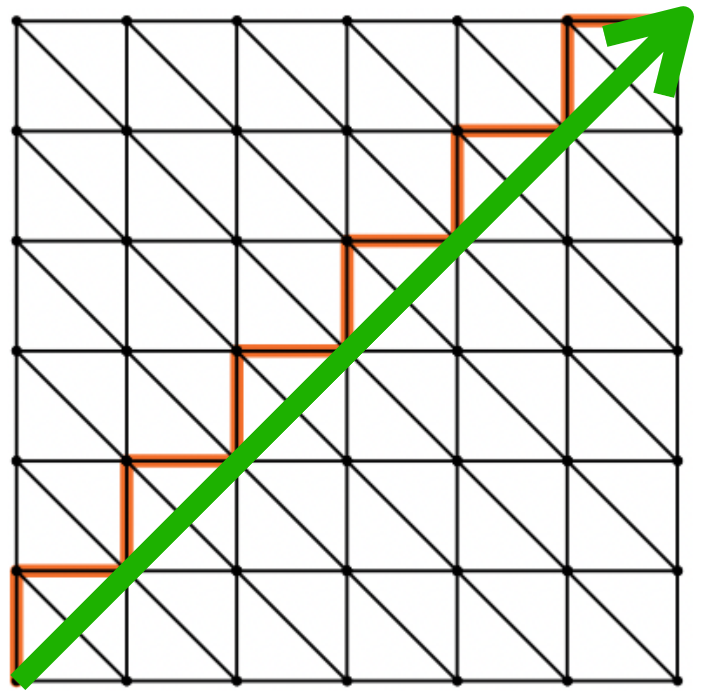
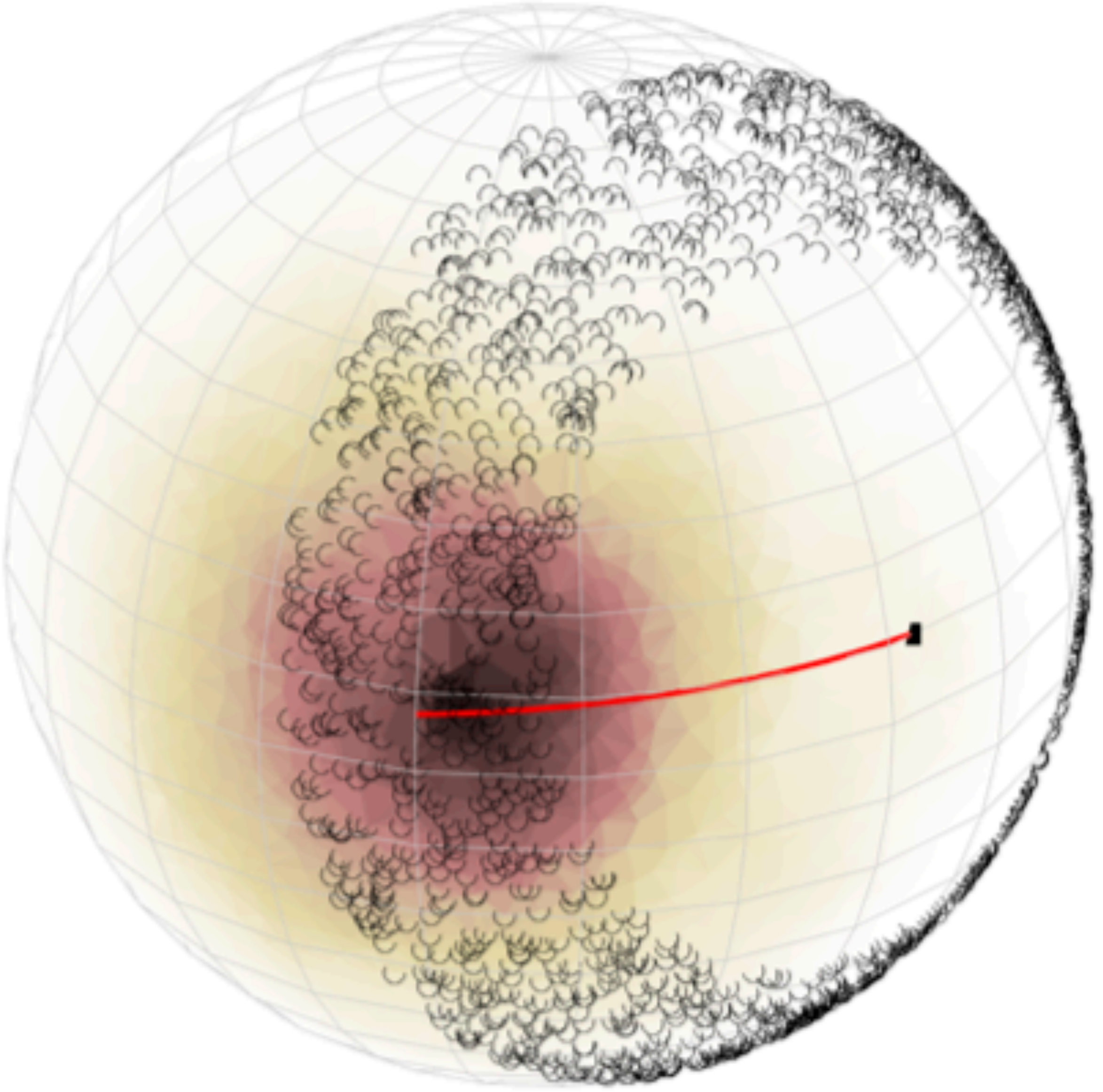


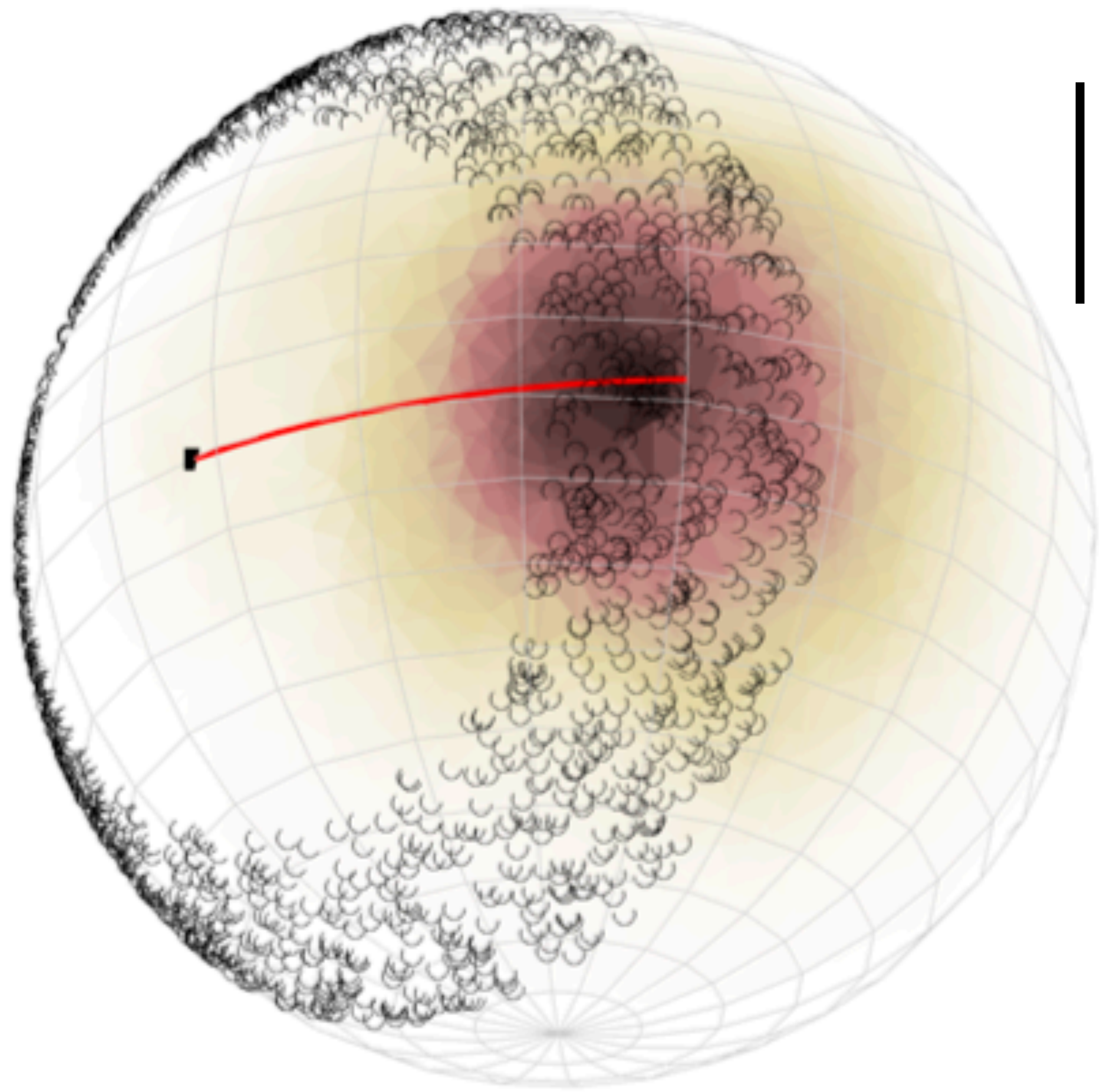


Theorem. If $h \gg \sqrt{\epsilon}$, then $\left| e^{it\sqrt{\Delta_{\epsilon,N}}}\,|\psi_h^{(x_0,\xi_0)}\rangle \right|^2 = \left| e^{\frac{i}{h}t\sqrt{\hat{H}_h}}\,|\psi_h^{(x_0,\xi_0)}\rangle \right|^2 + O(h)$

with probability at least $1 - e^{-\Omega(Nh^{\beta})}$ ($\beta > 0$ a constant). Kumar, arXiv:2112.10748 (2022)

From random walks to quantum dynamics of observables on data

Heisenberg equation $i\partial_t \operatorname{diag}(a|_{X_N}) = \left[\sqrt{\Delta}_{\epsilon,N}, \operatorname{diag}(a|_{X_N})\right]$



$\left|\psi_h(t)\right\rangle$

with high prob.,

$h \gg \sqrt{\epsilon}$

$$a \in C^\infty(\mathcal{M})$$

$$\left\langle \psi_h(t) \middle| \operatorname{diag}(a|_{X_N}) \middle| \psi_h(t) \right\rangle = a(x(t)) + O(h)$$

Kumar, arXiv:2112.10748 (2022)

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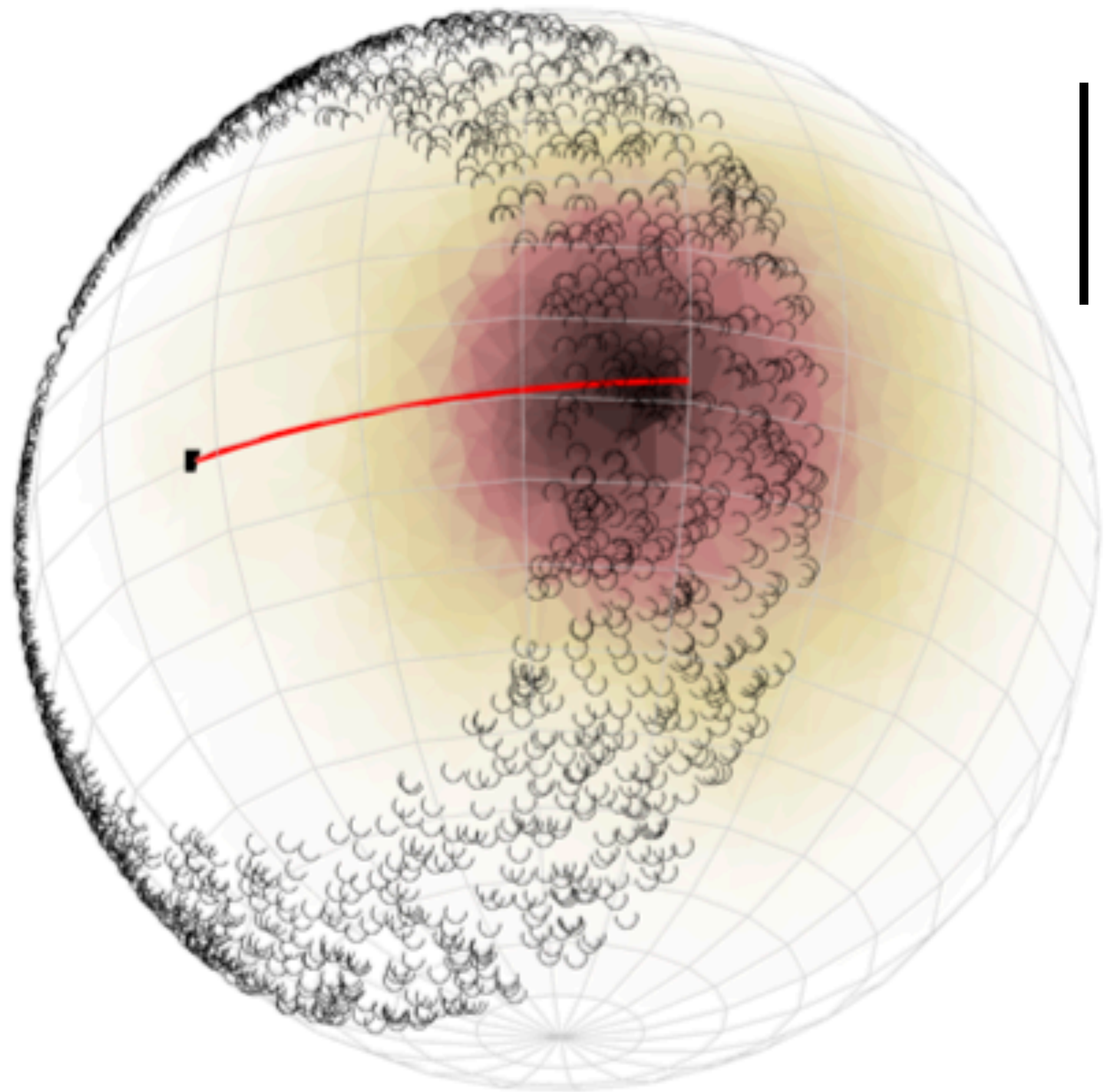

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Theorem. If $a \in C^\infty(\mathcal{M})$ and $h \gg \sqrt{\epsilon}$, then $\left\langle \psi_h(t) \middle| \operatorname{diag}(a|_{X_N}) \middle| \psi_h(t) \right\rangle = a(x(t)) + O(h)$ with probability at least $1 - e^{-\Omega(Nh^\beta)}$ ($\beta > 0$ a constant).

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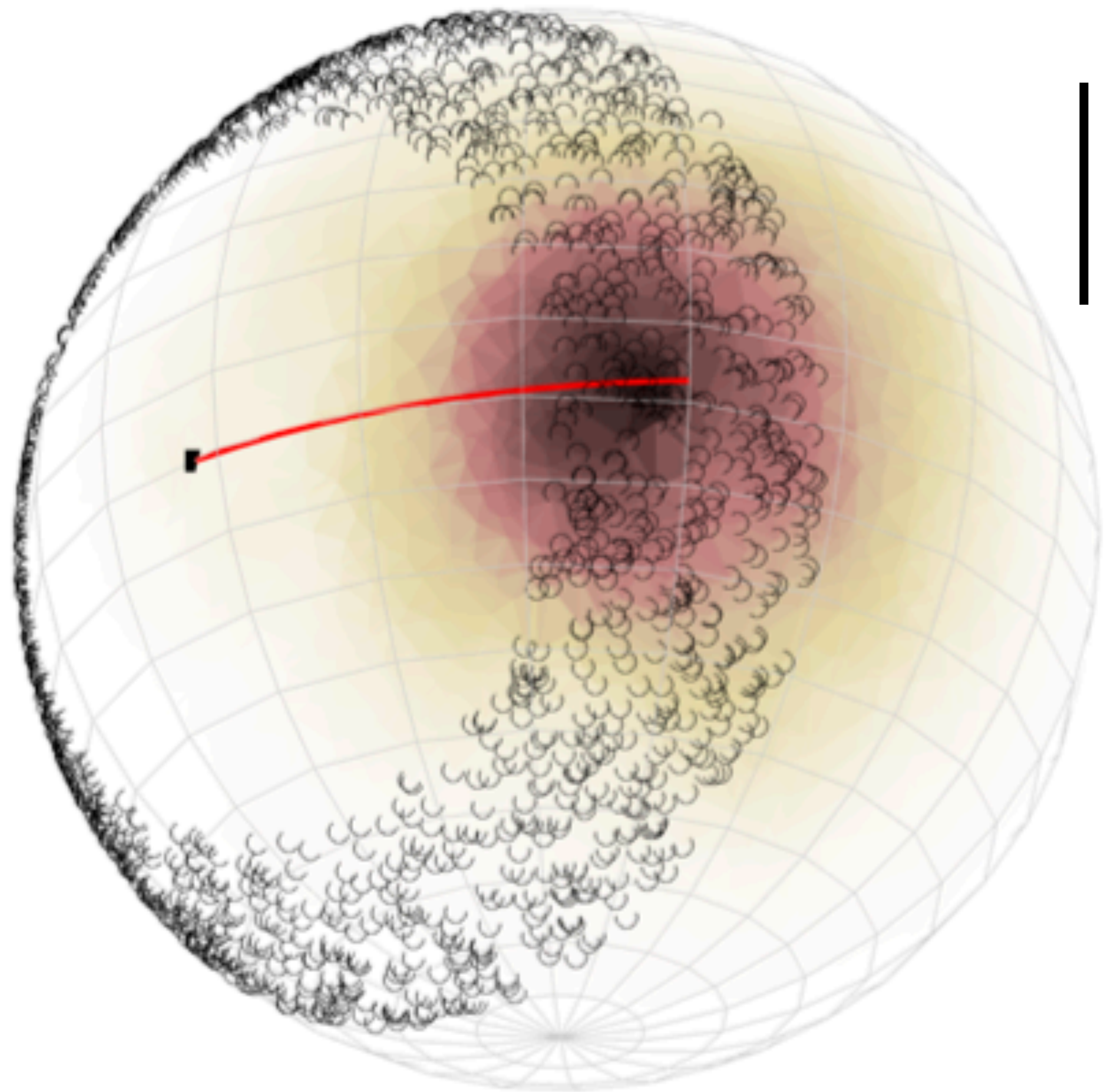

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Microlocal analysis ∩ geometry ∩ markov processes ∩ probability/statistics ∩ quantum dynamics

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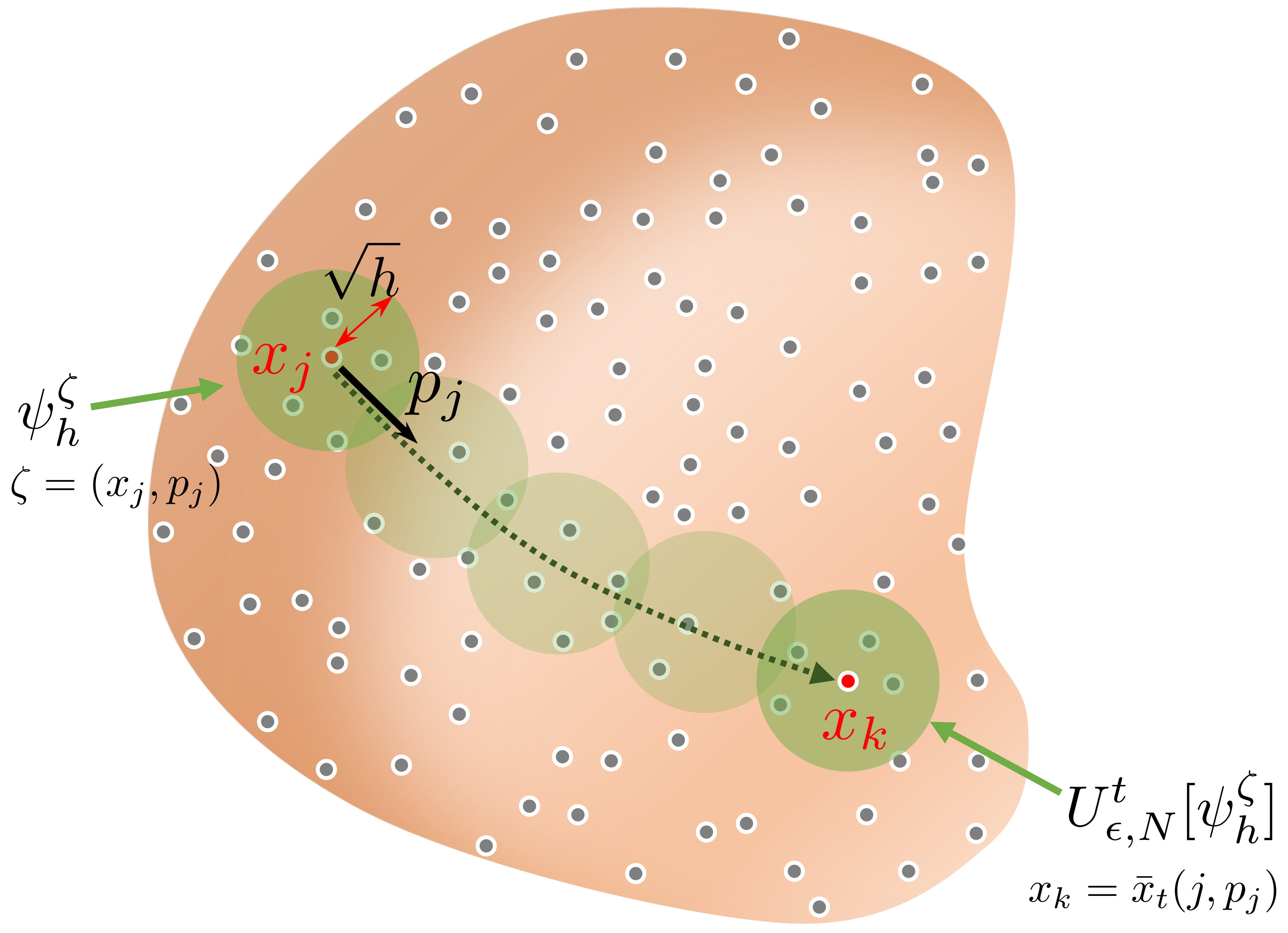
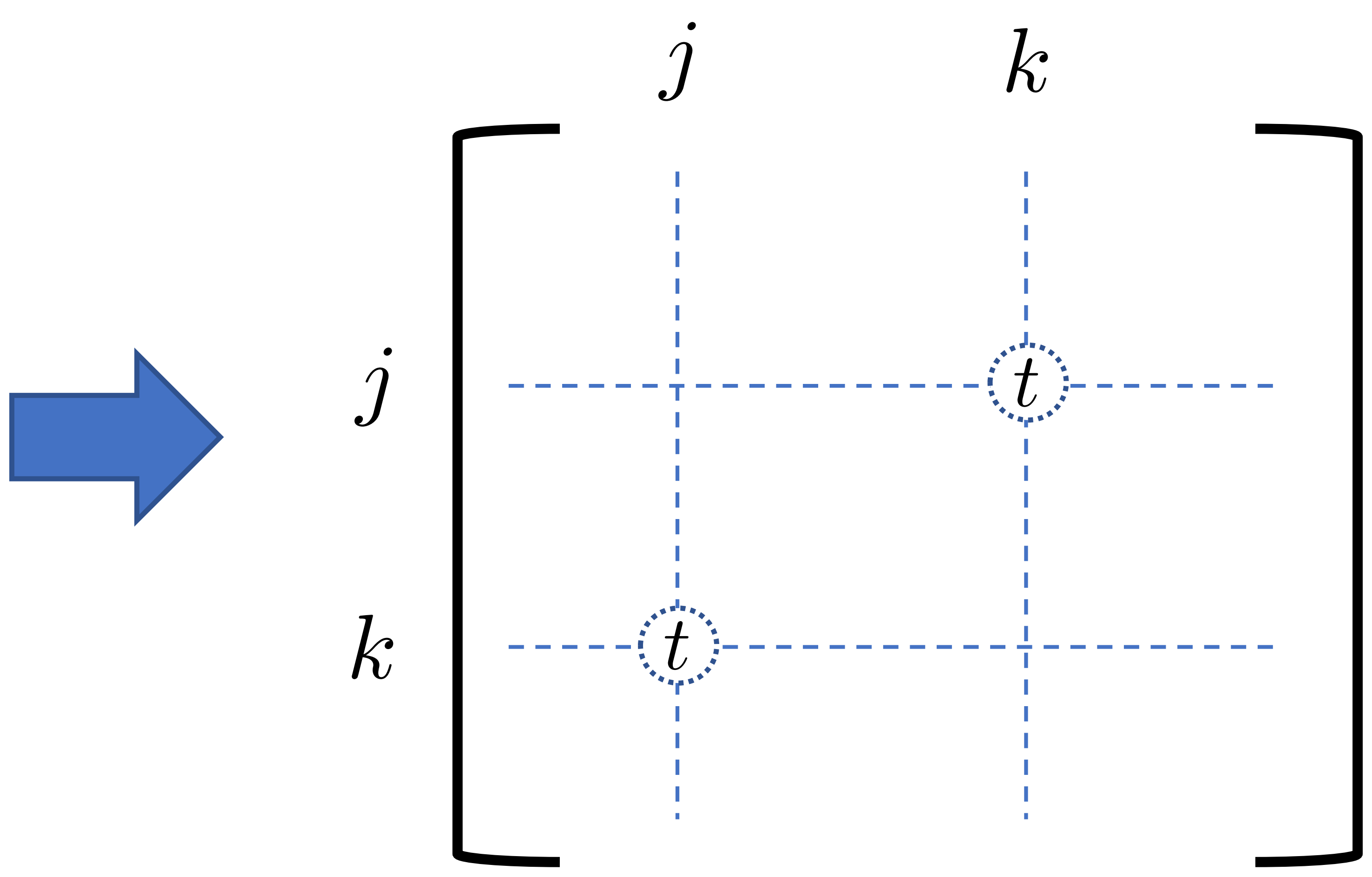
Kumar, arXiv:2112.10748 (2022)

Applications

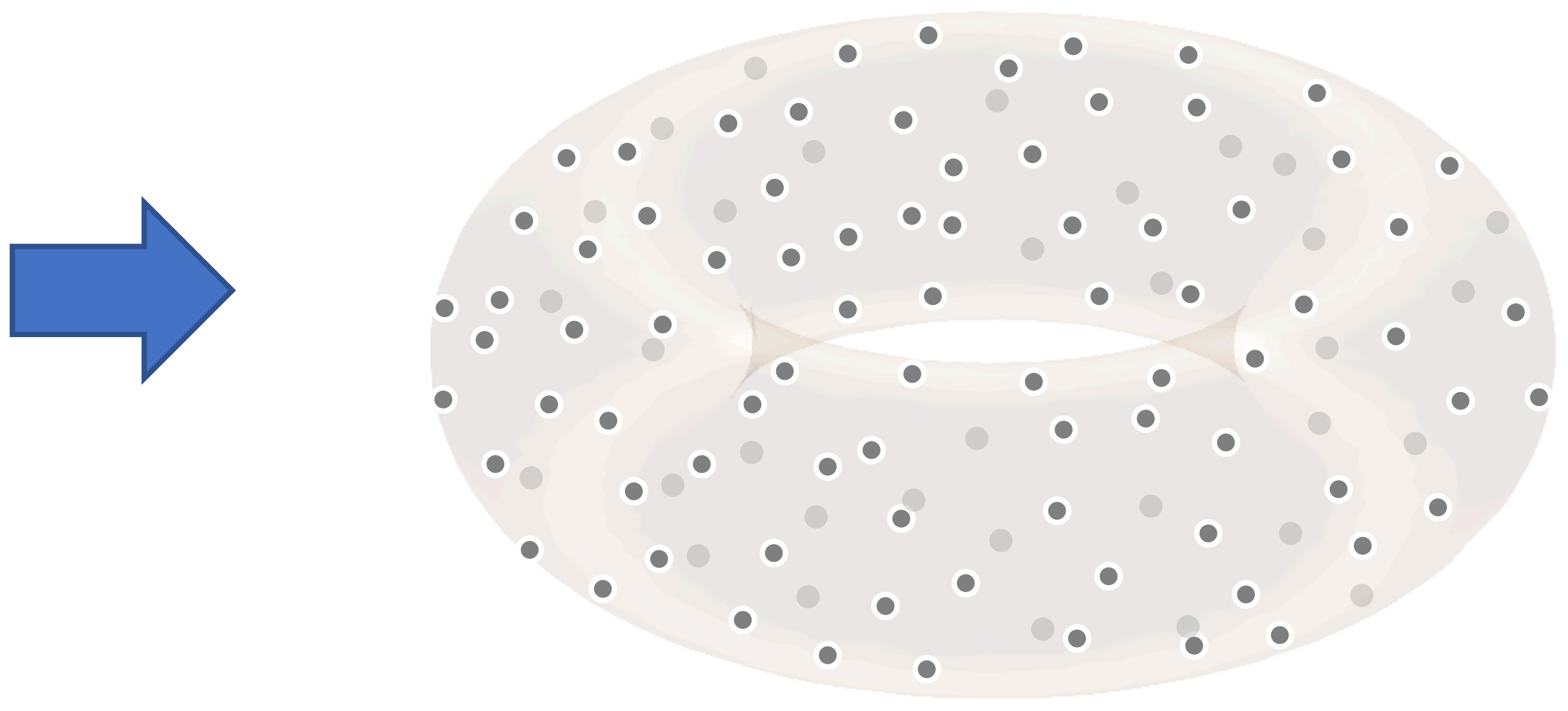
Quantum dynamical *reach-time* embedding of datasets



Cross-section of dataset of size N in $\mathbb{R}^n$



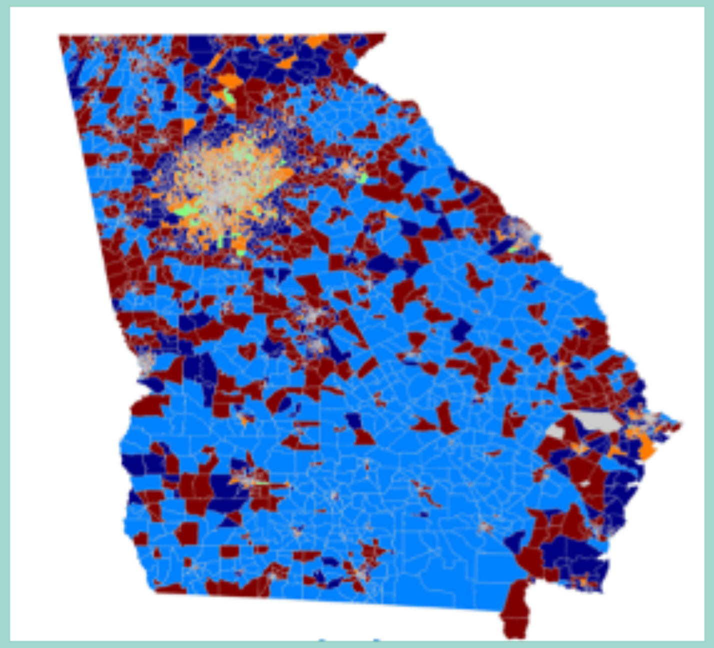
$\mathcal{G}$ $(N \times N \text{ matrix})$



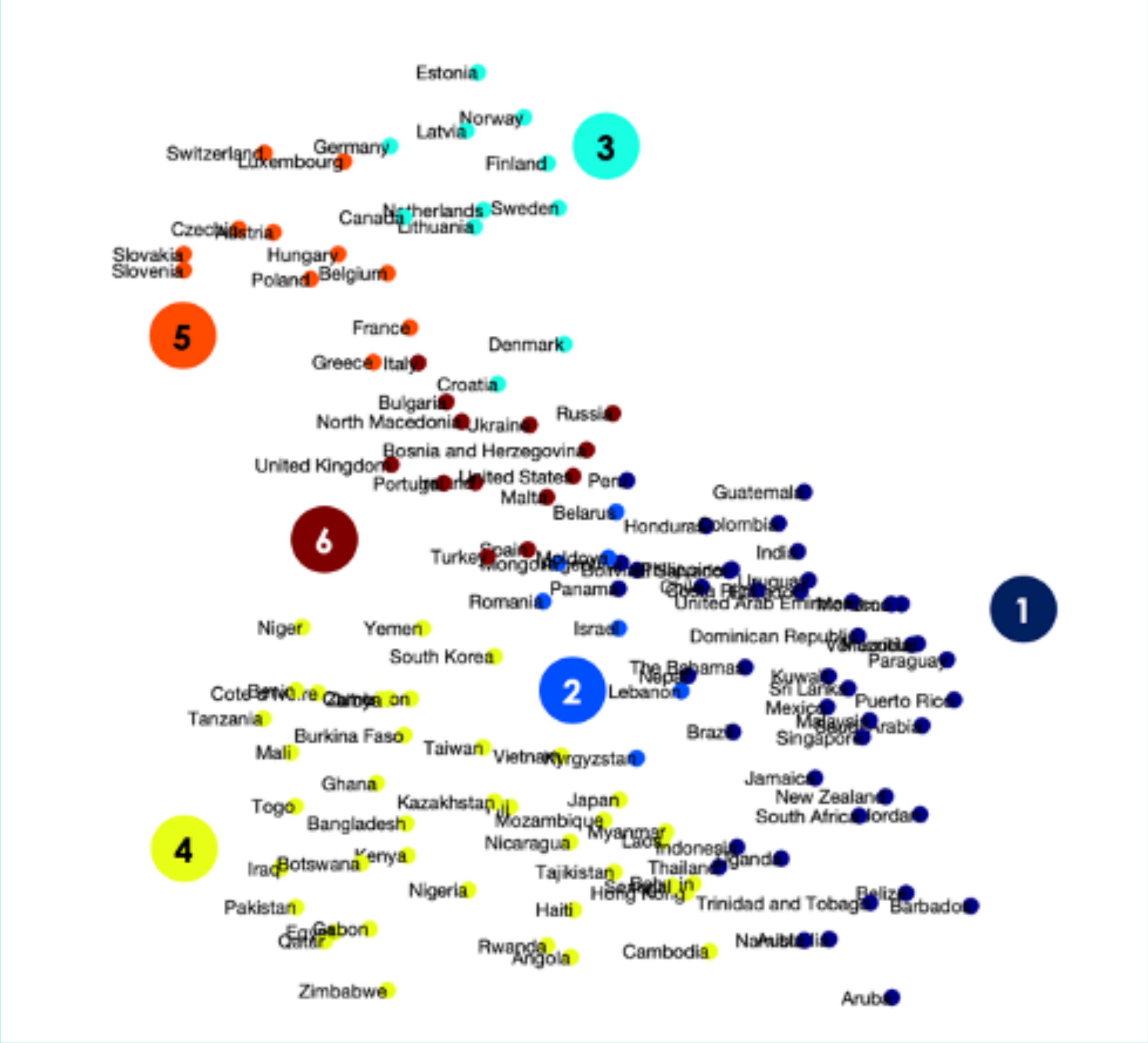
Graph embedding of dataset in $\mathbb{R}^3$ based on weighted adjacency matrix $\mathcal{G}$

Example applications

Clustering of geographic regions according to COVID-19 social distancing behavior

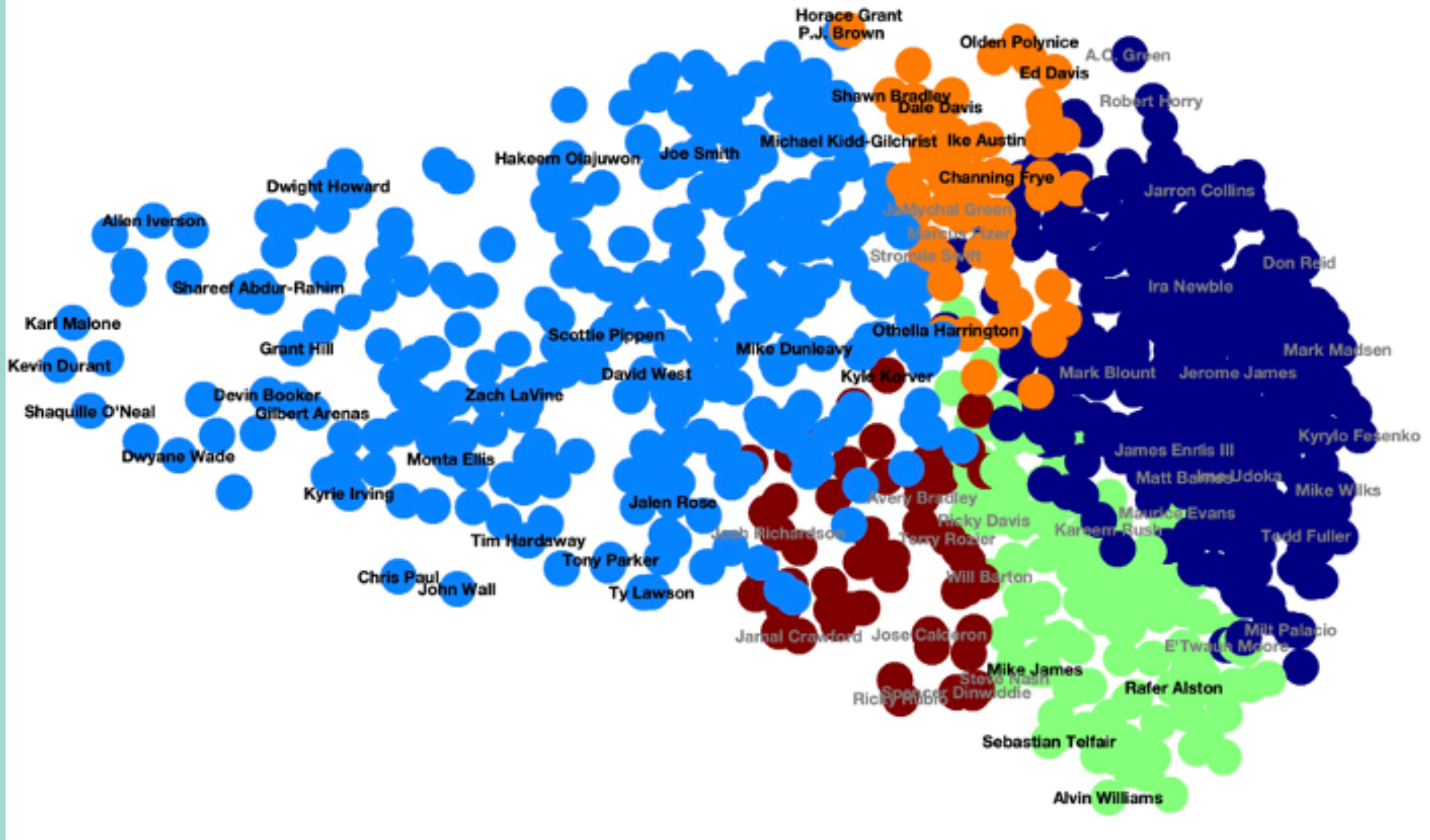


SafeGraph dataset for state of GA



Google dataset for worldwide mobility patterns

Clustering of NBA players based on performance



NOTES:

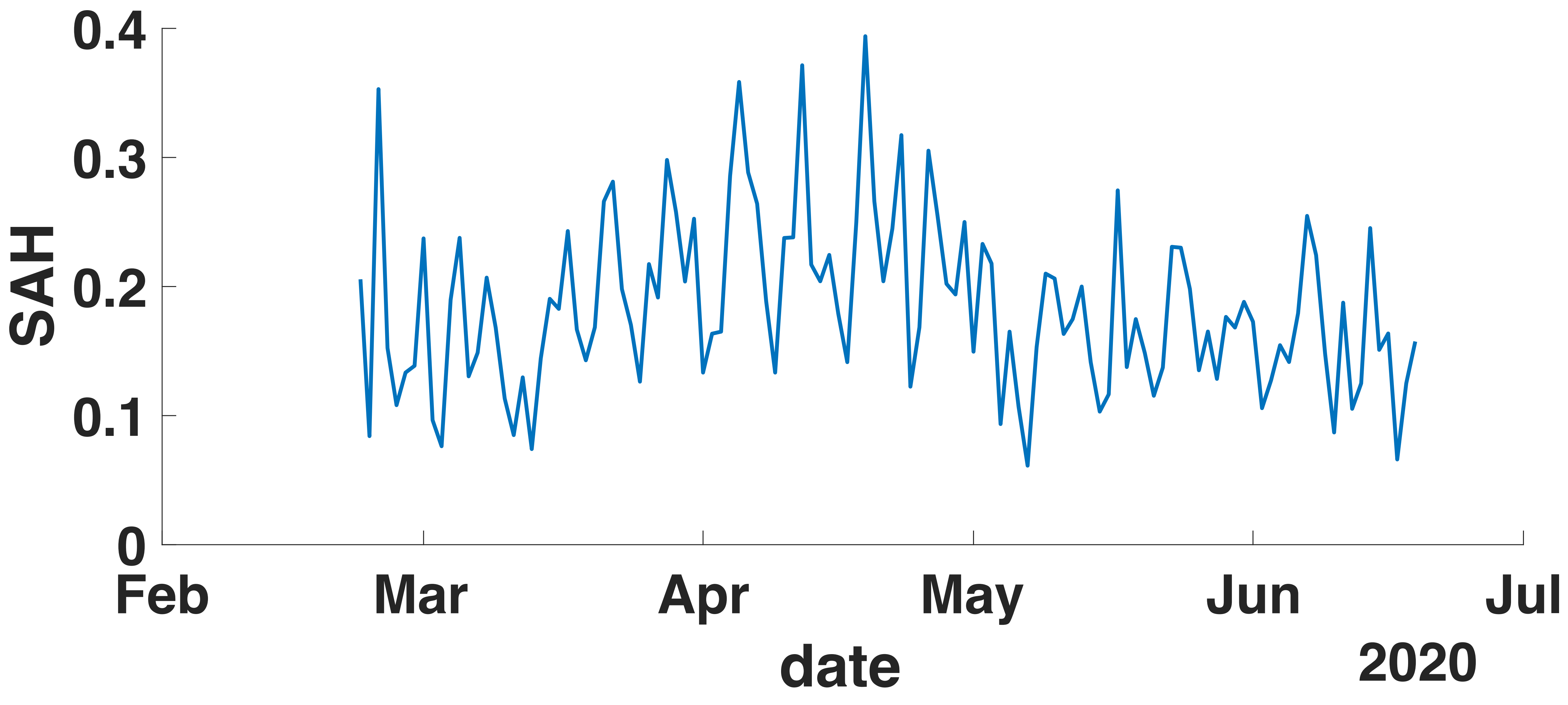
1. Manifold hypothesis not necessary for utility
2. Some of these datasets are small (e.g., $N = 126$)

See "Manifold learning via quantum dynamics." A. Kumar & M. Sarovar. arXiv:2112.11161 (2021)

Example: COVID-19 mobility data

Social Distancing Metric dataset from SafeGraph Inc.
https://docs.safegraph.com/docs/social-distancing-metrics

- Dataset collects user location information (from cellphone GPS data) over the course of the initial ~3 months of the COVID-19 pandemic (Feb 23, 2020 – June 19, 2020: 117 days).
- Data aggregated at the census block group (CBG) level.
- We compute a "stay-at-home" fraction for each CBG, which represents the fraction of devices that stayed at their home location on a day.
- We concentrate on the data for Georgia (GA), which has 5509 CBGs.
- Dataset: 5509 x 117



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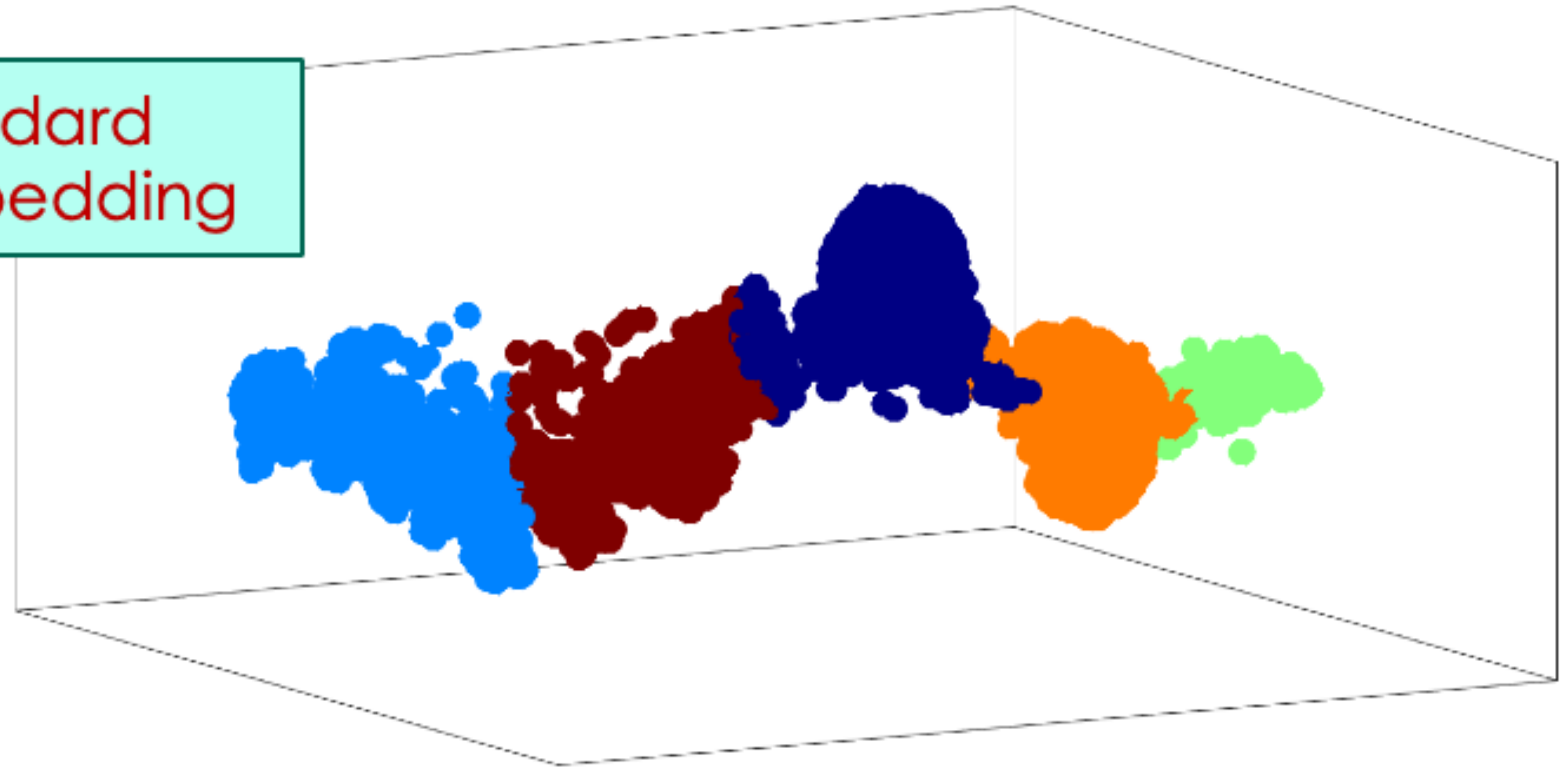
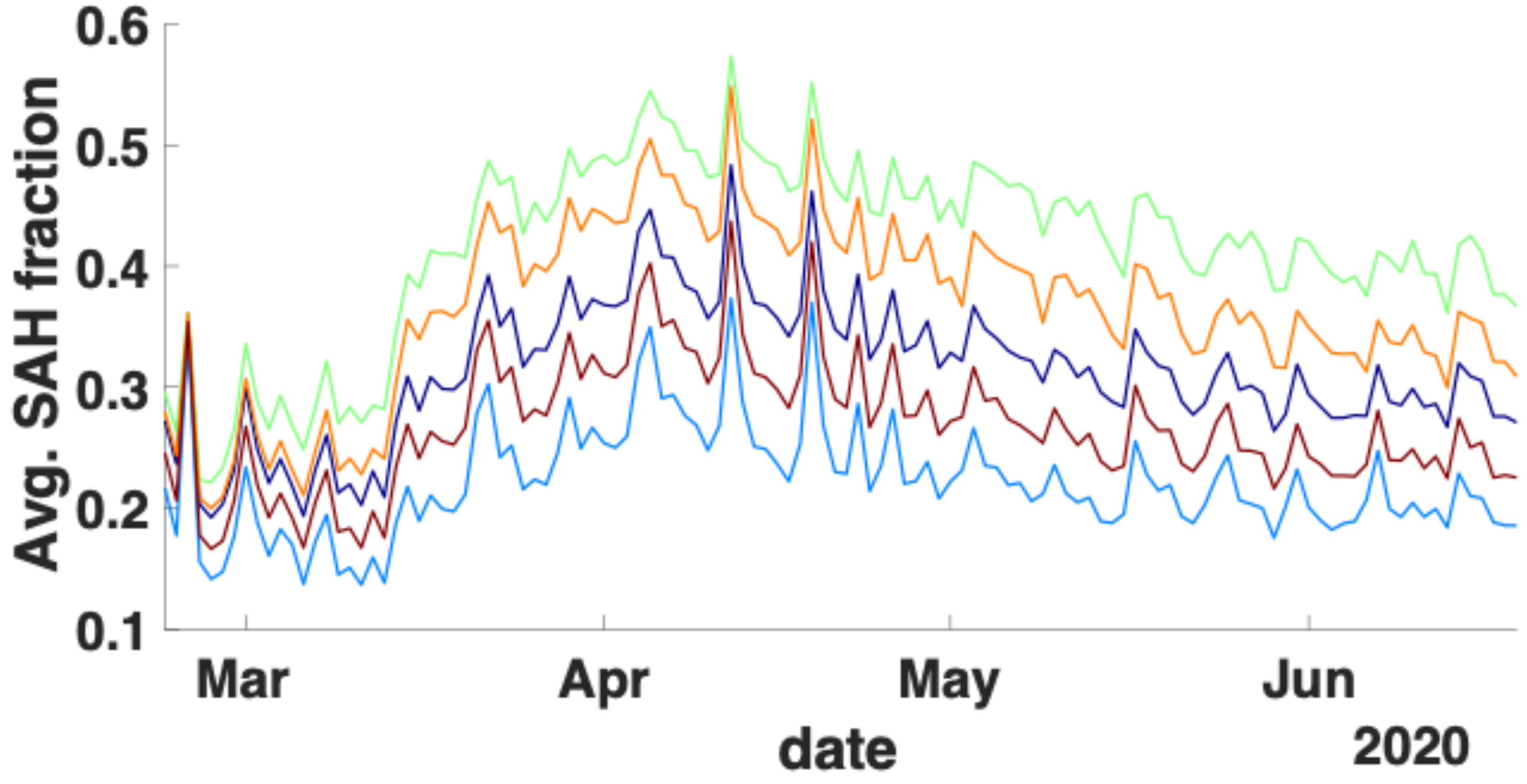
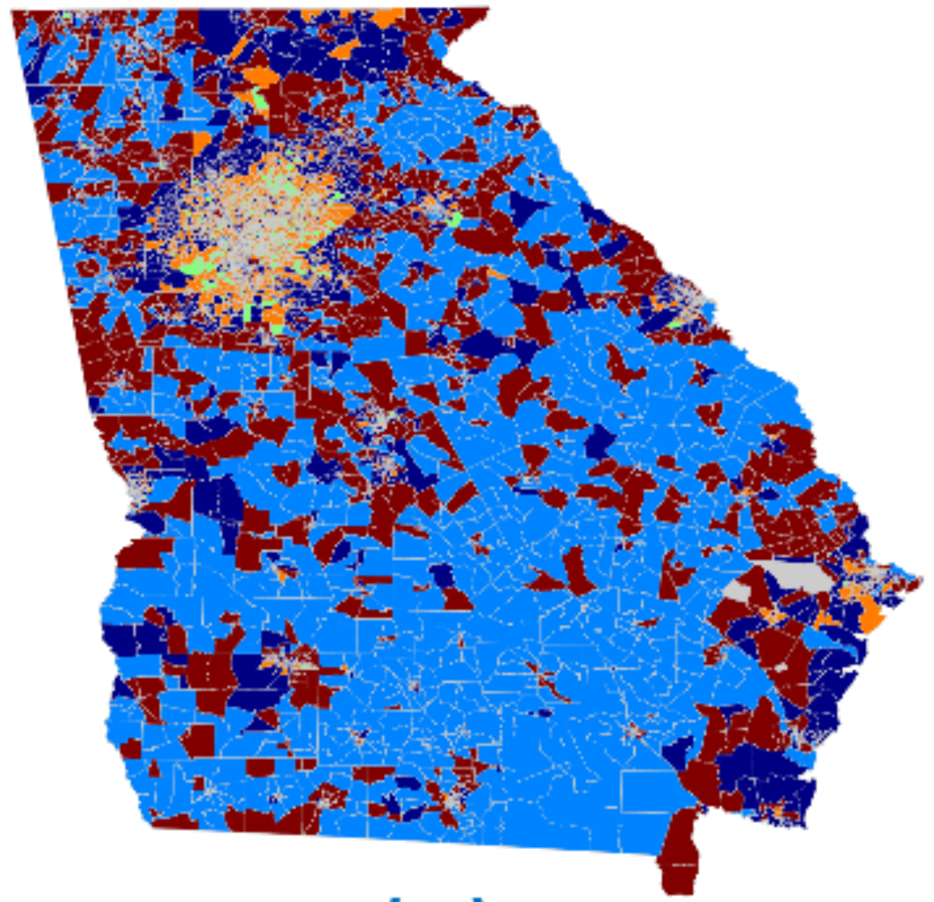
Standard embedding



Embedding in 3D and k-means using these 3D coordinates

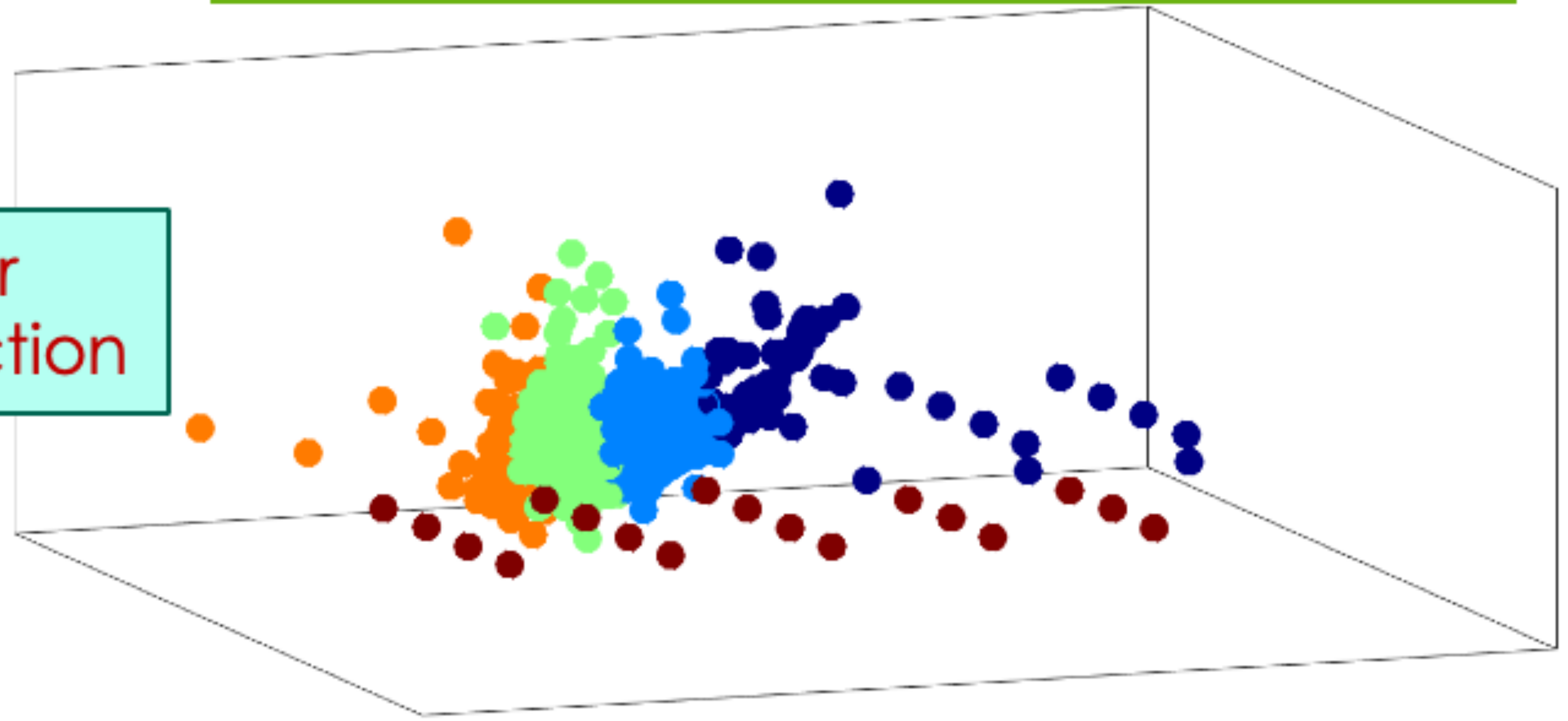


Average SAH fraction time series for each cluster

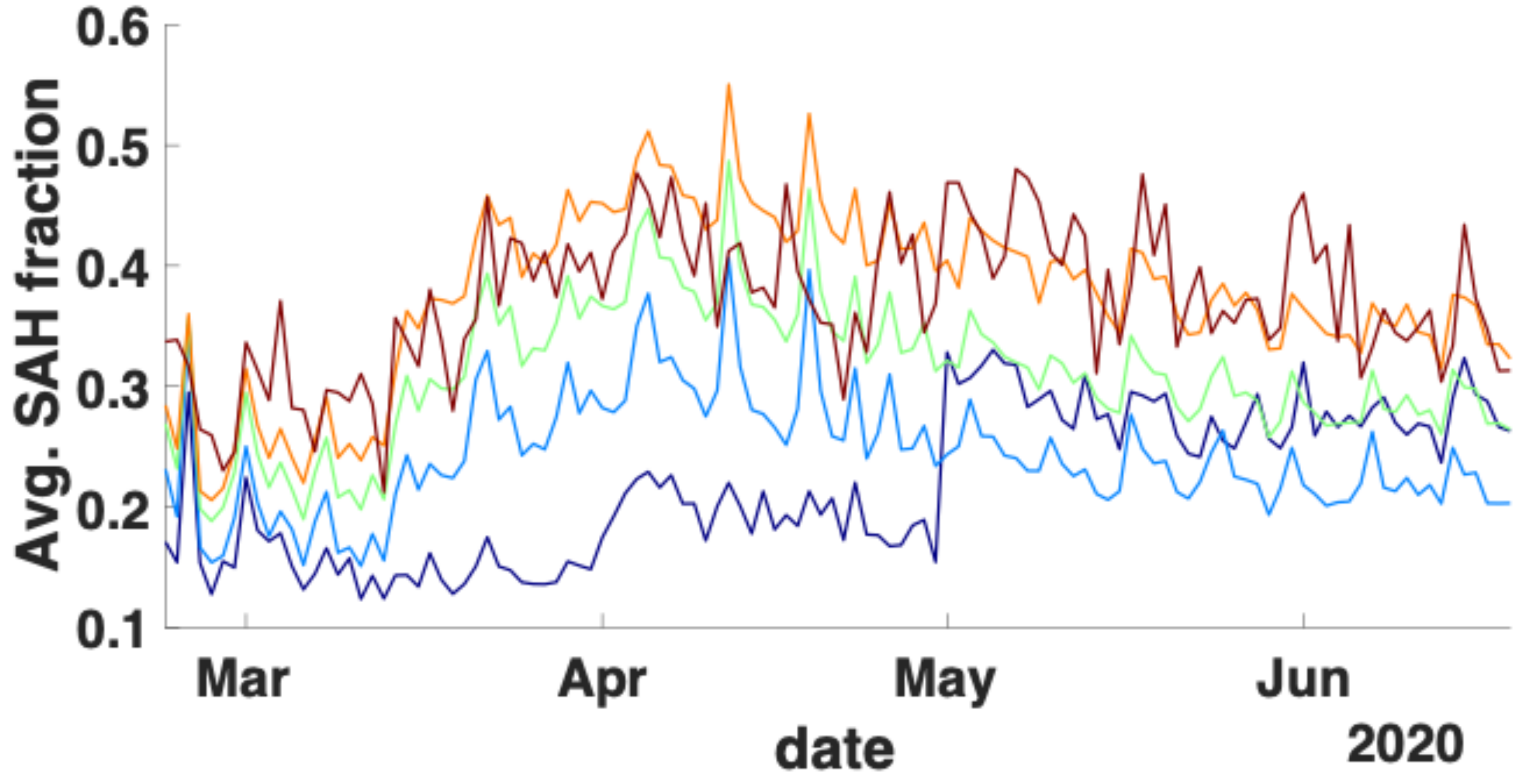


(c)

Outlier detection

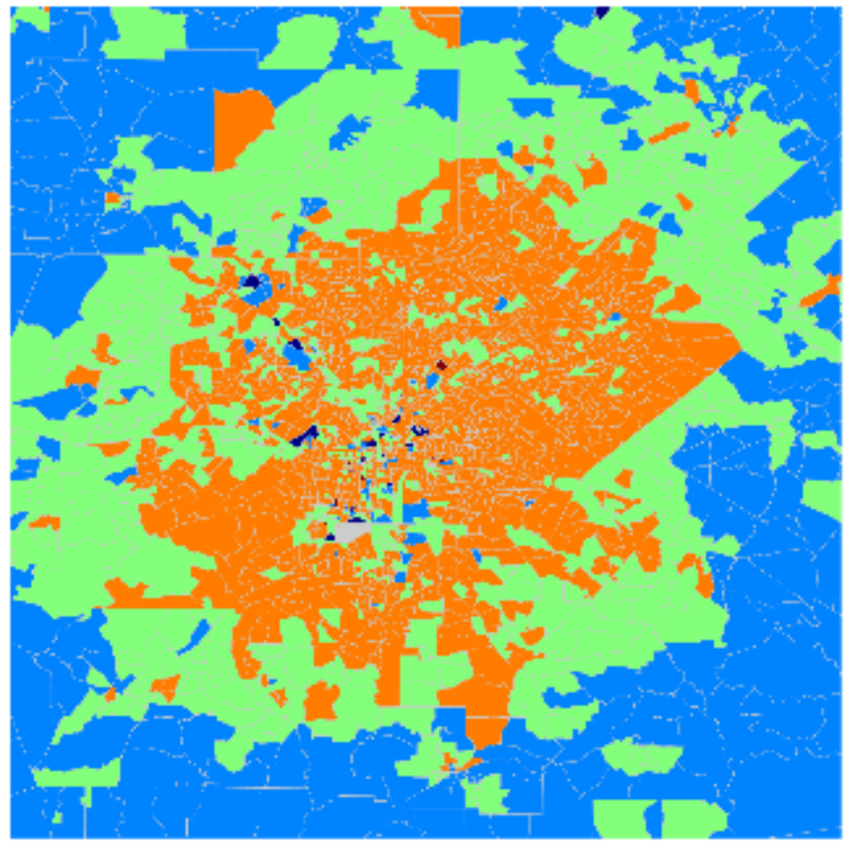


(d)



(e)

Outlier (1.2% of data)



(f)

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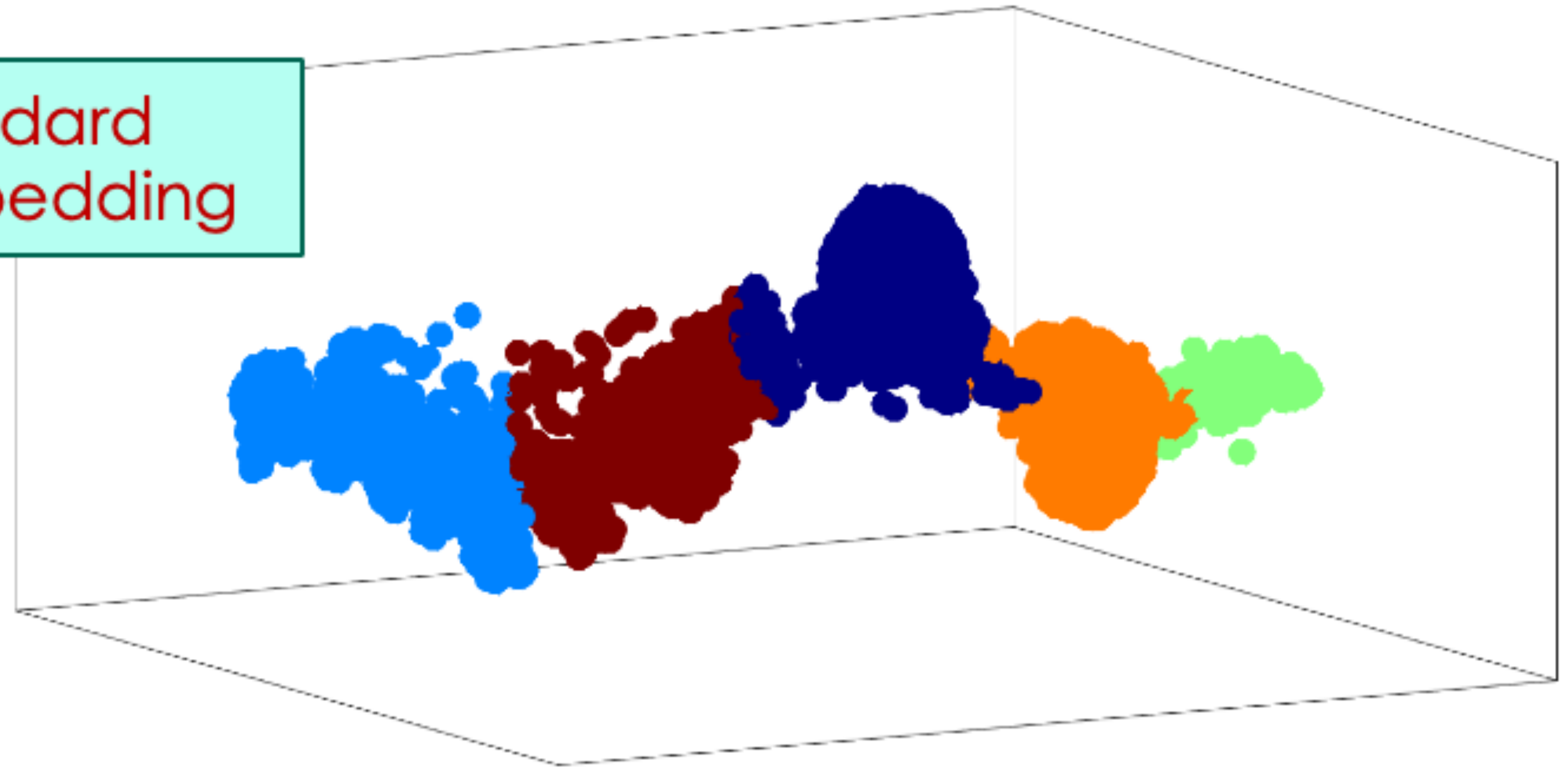
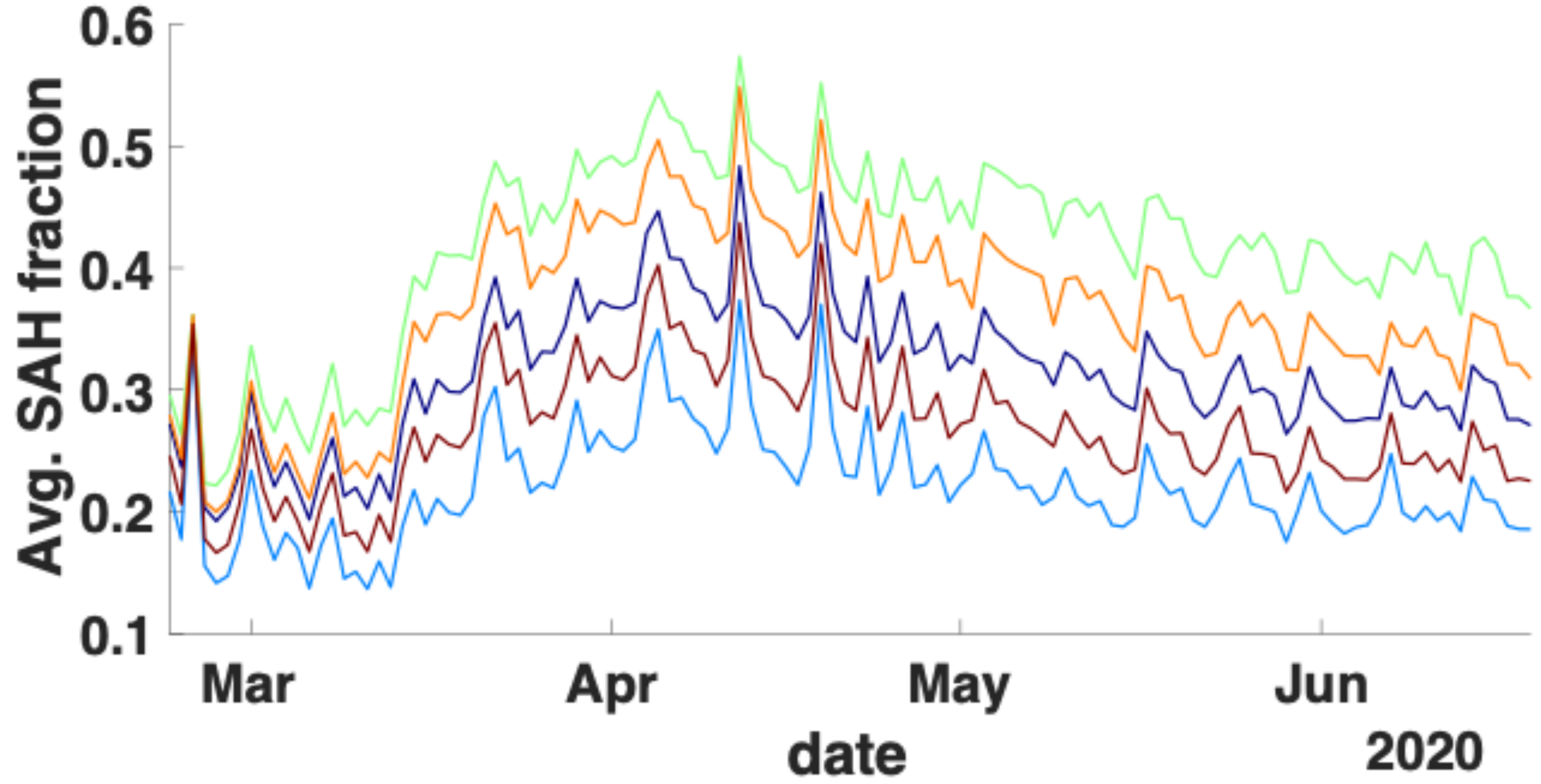
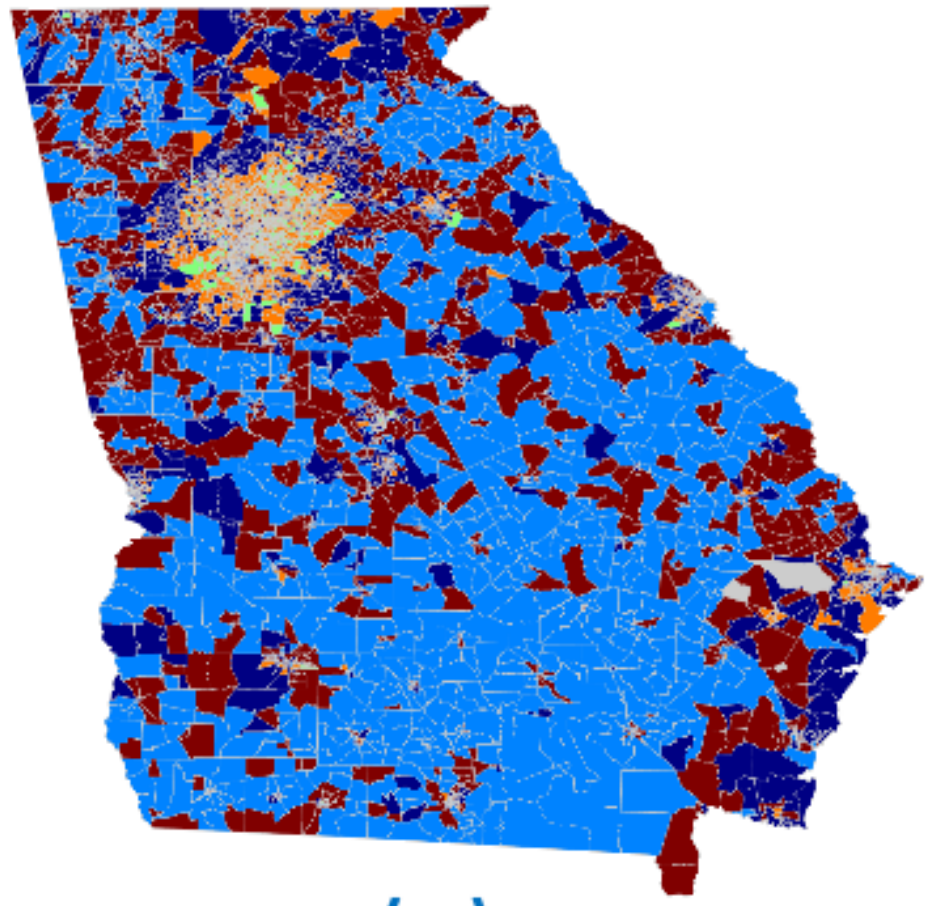
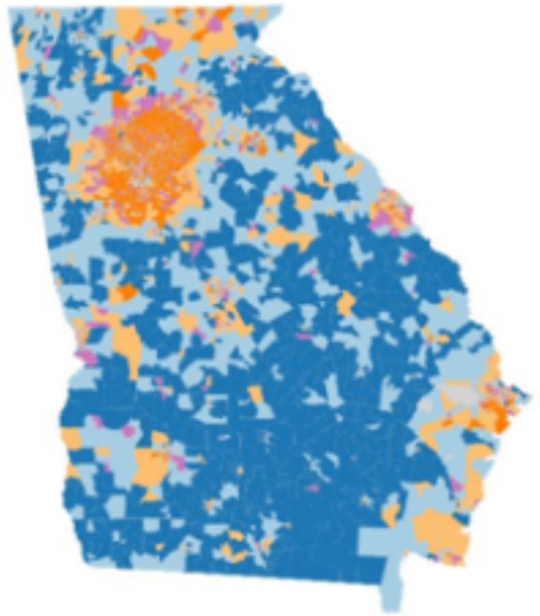






c.f. Levin et al., "Cell Phone Mobility Data and Manifold Learning." https://doi.org/10.1101/2020.10.31.20223776



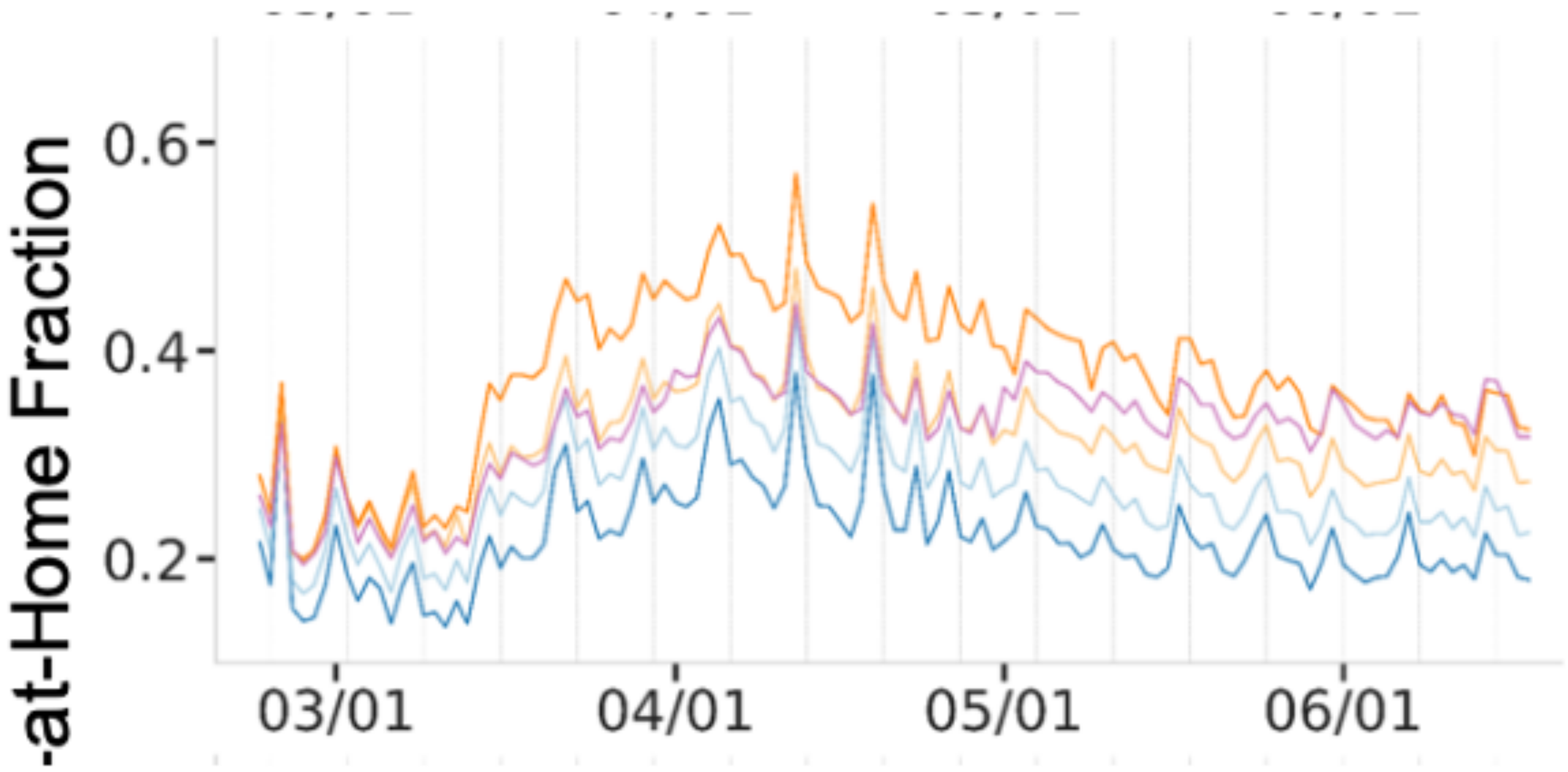


Figure 2: Laplacian eigenmaps, clustering done after embedding in **14 dimensions**

Example: COVID-19 mobility data

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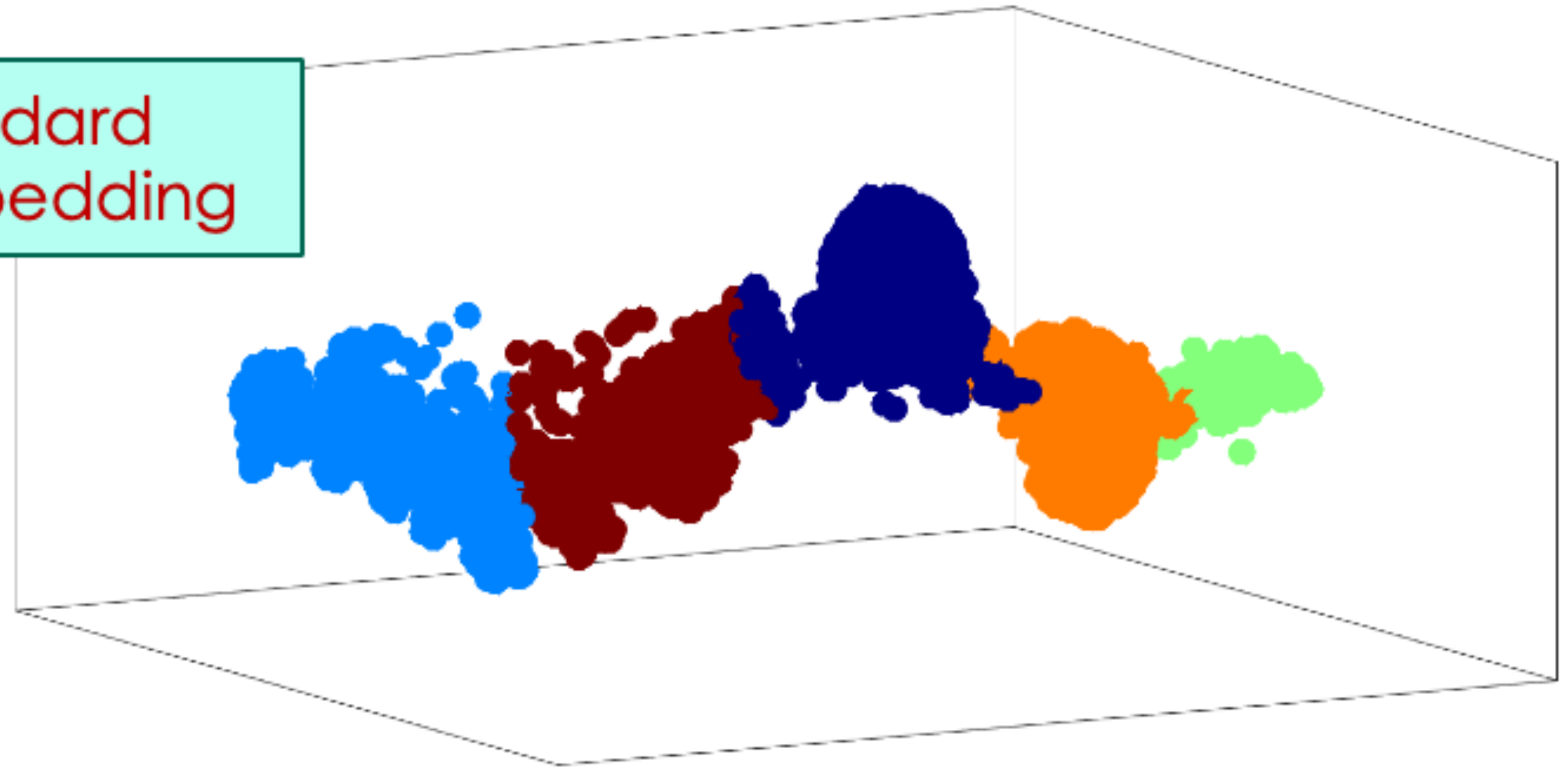
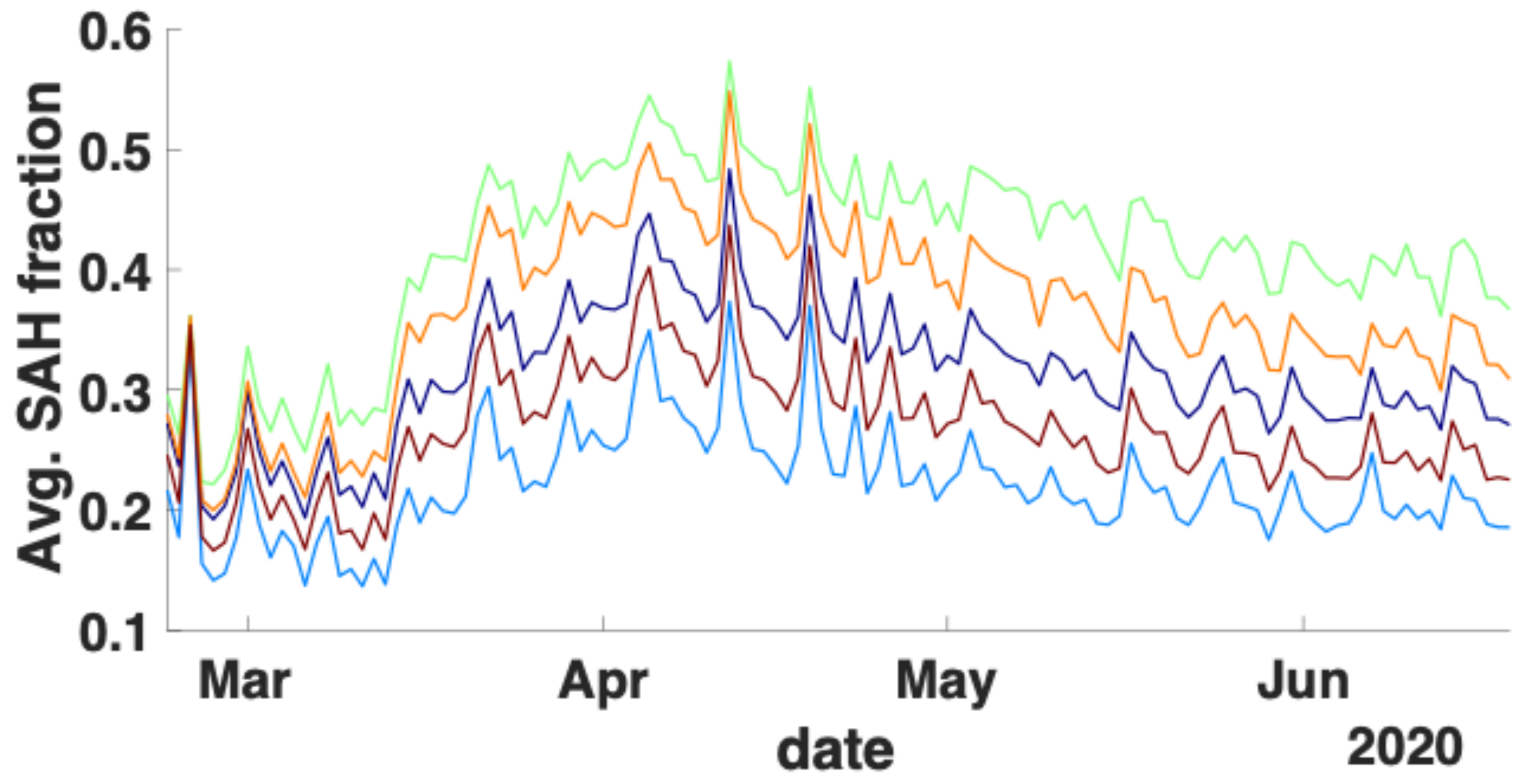
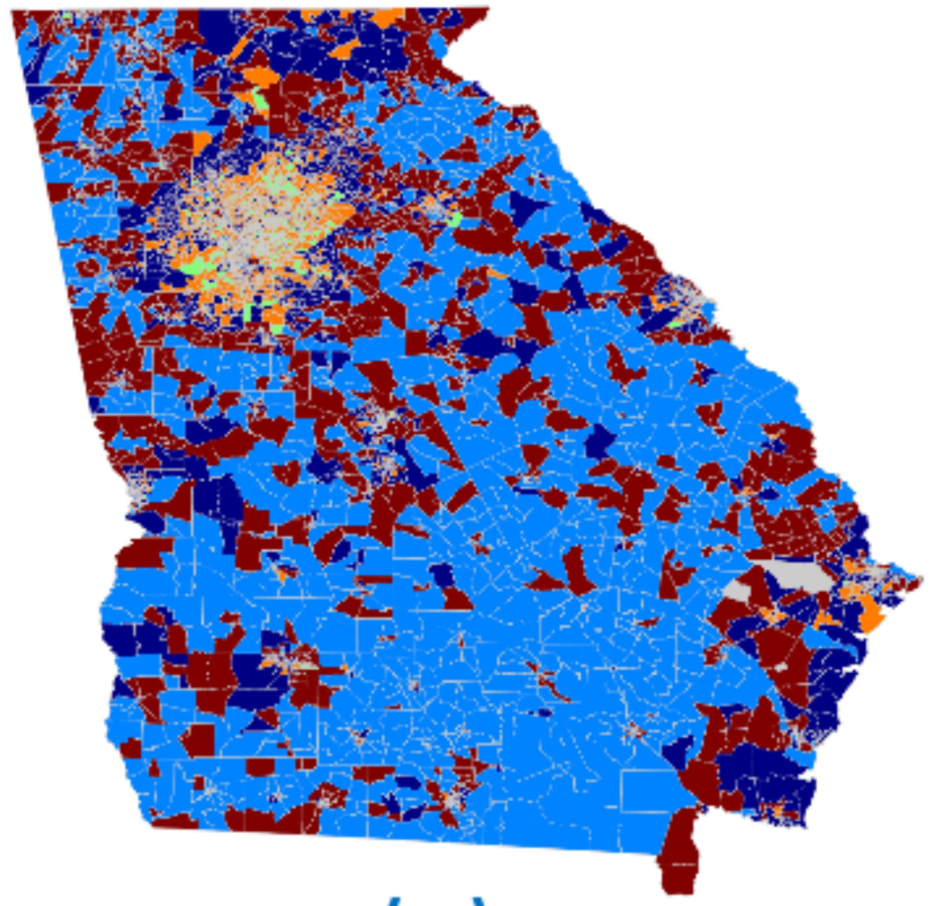
Standard embedding







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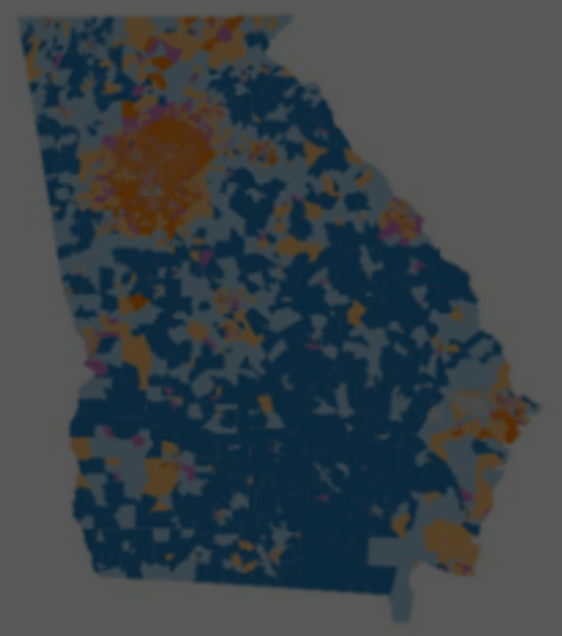


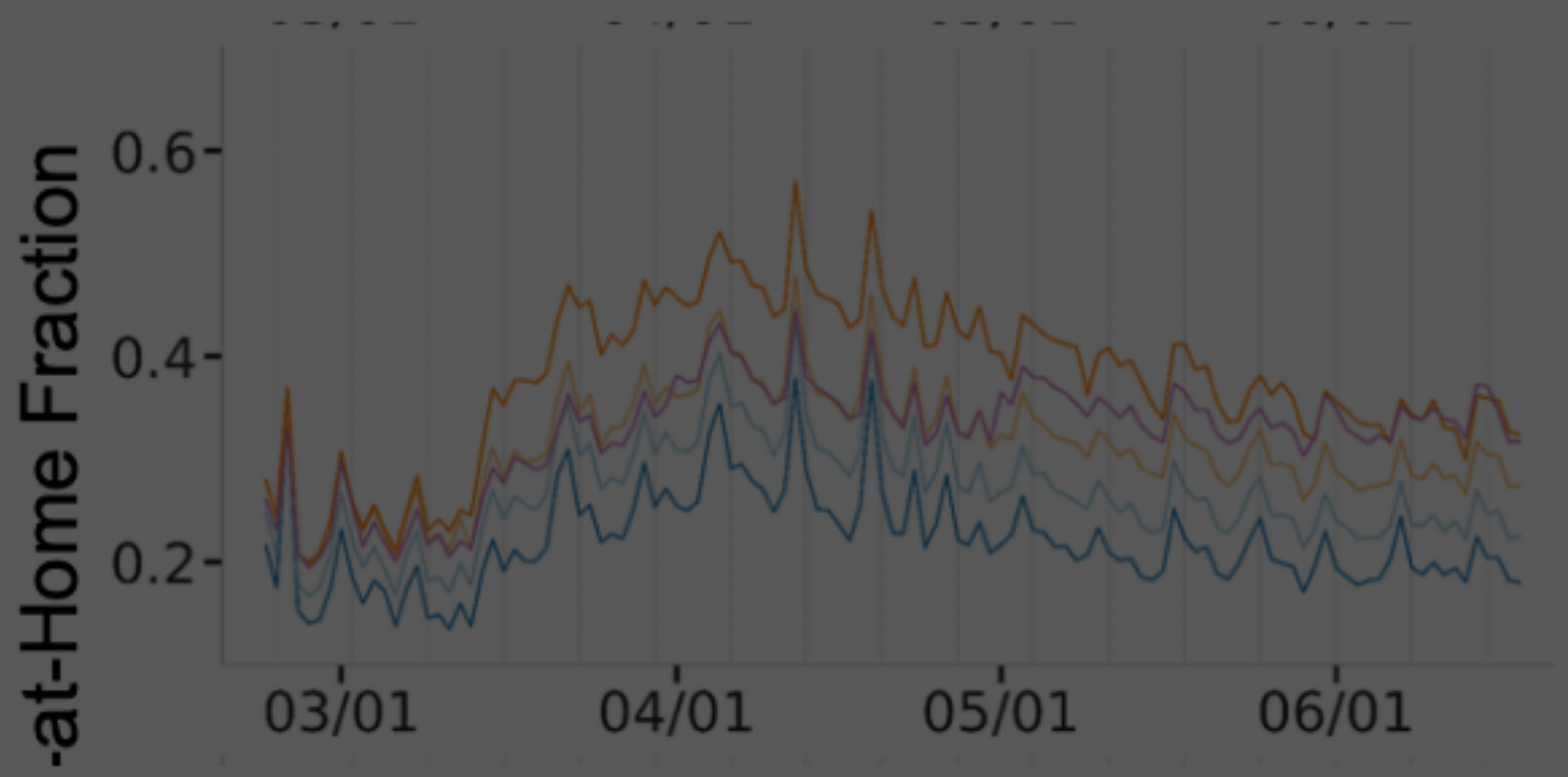


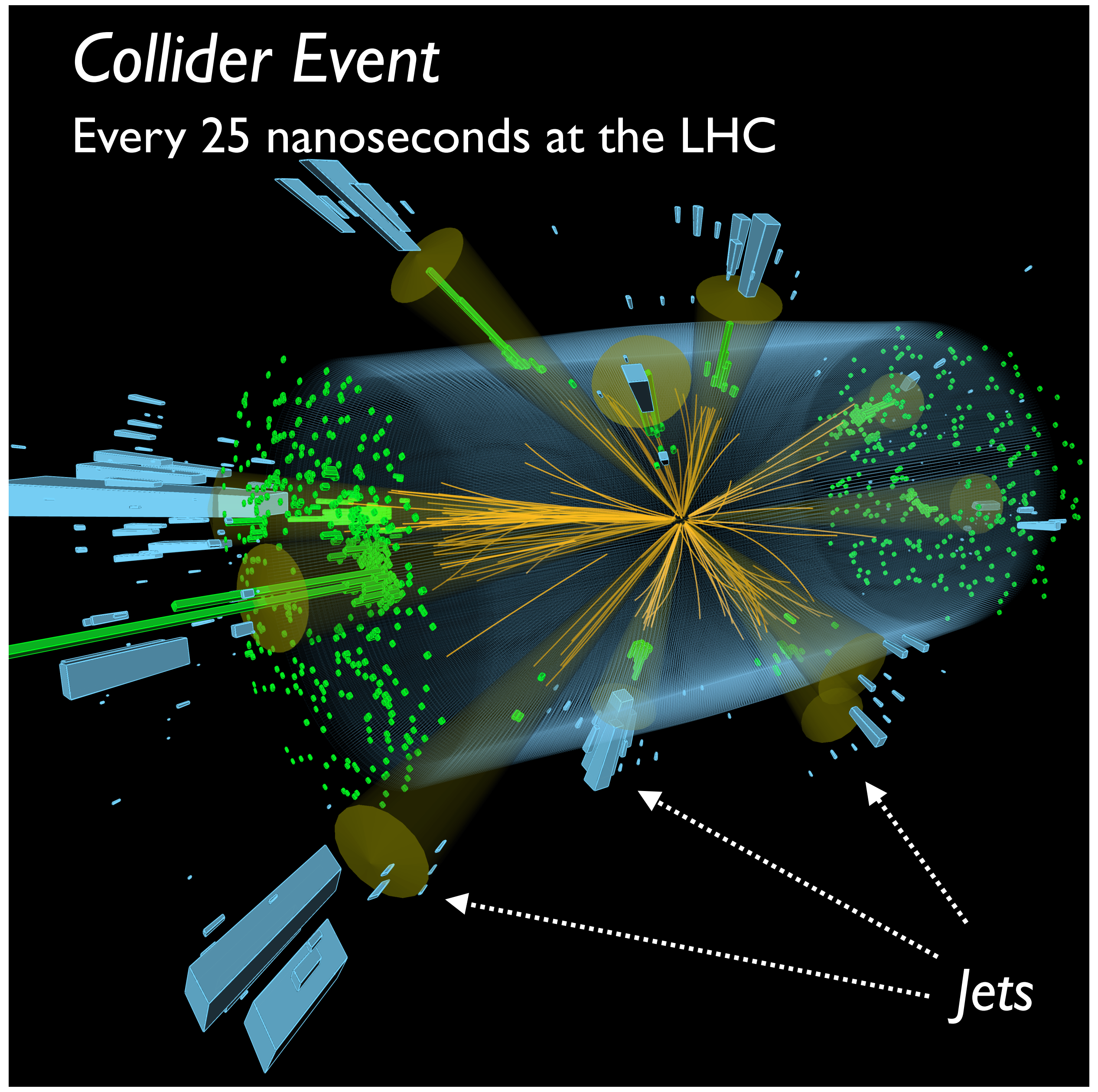
Figure 2: Laplacian eigenmaps, clustering done after embedding in **14 dimensions**

We achieve better quality clustering and are able to identify outliers, even with an embedding into 3 dimensions (**reduction from 117 dimensions**)

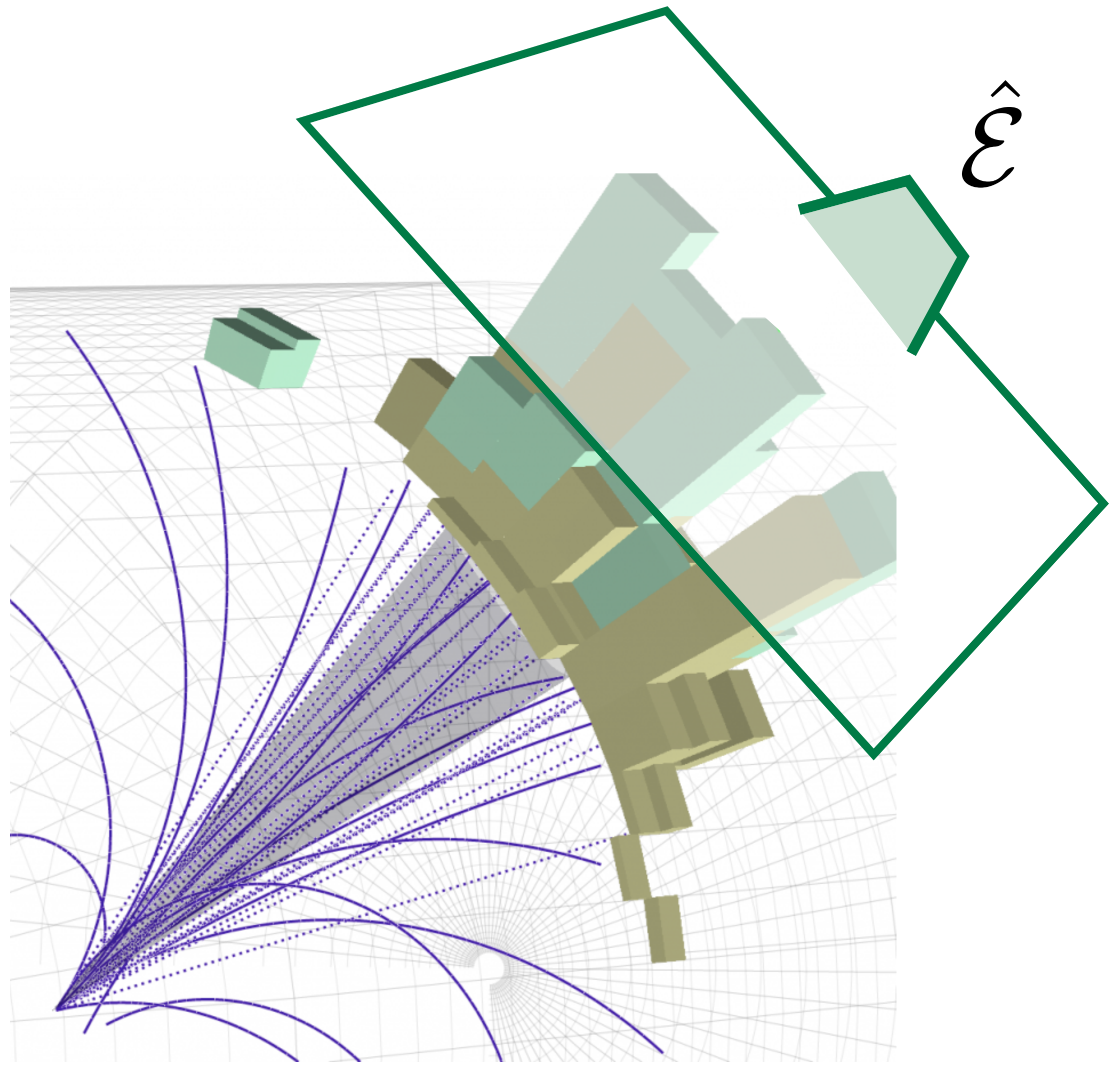
Applications: HEP?

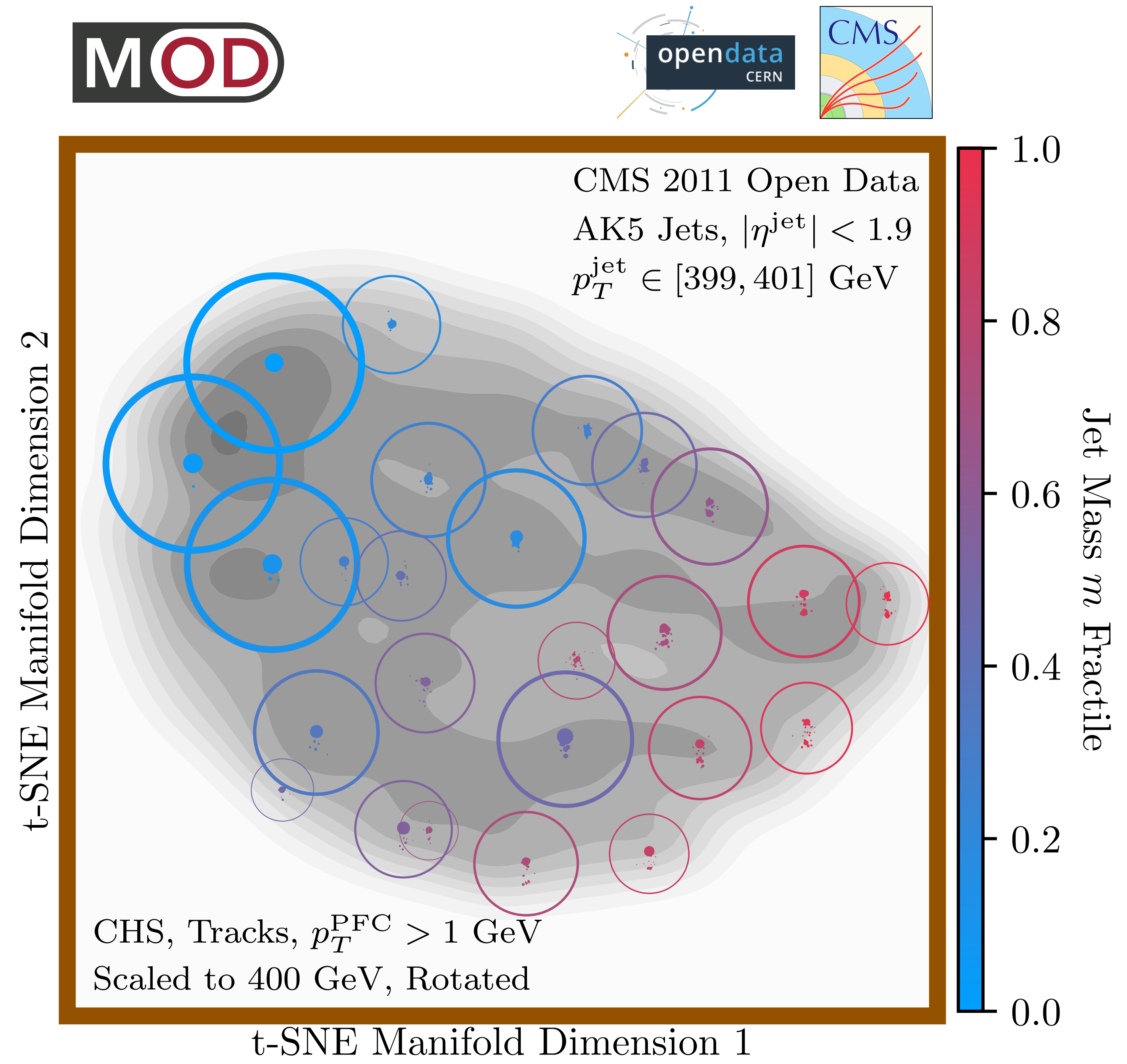
Future directions for HEP applications

Recall: Collider events have a geometry that can be used to organize, categorize and interpret collisions, e.g.,



Events summarized by "energy flow"





Komiske, Metodiev, Thaler,
The hidden geometry of particle collisions
J. HEP, 6 (2020)
The Metric Space of Collider Events
Phys. Rev. Lett. 123, 041801 (2019)

Figures from talk by Jesse Thaler, University of Chicago and Caltech AI+Science: https://www.youtube.com/watch?v=BMBSAWUxBn4

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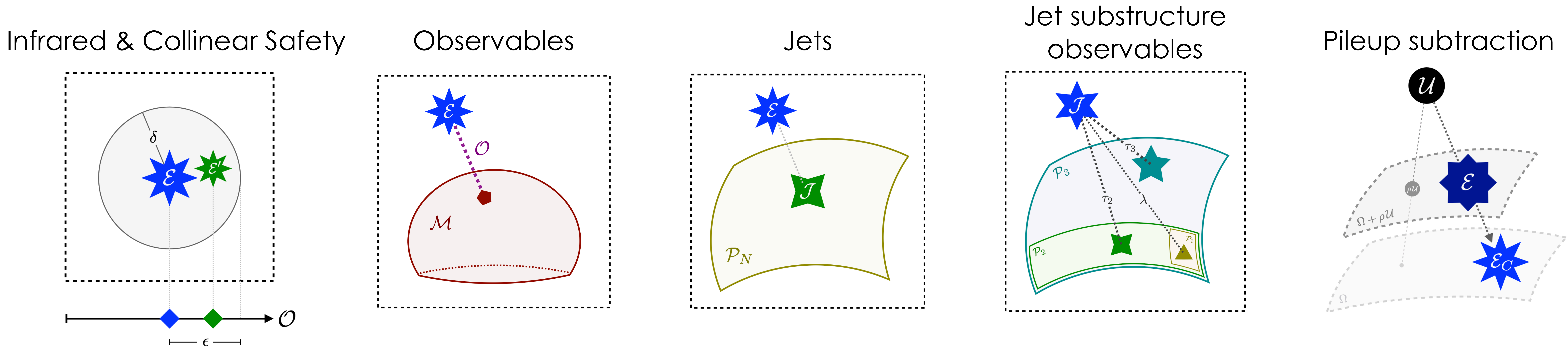

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Future directions for HEP applications

Does quantum dynamics make sense on this metric space?

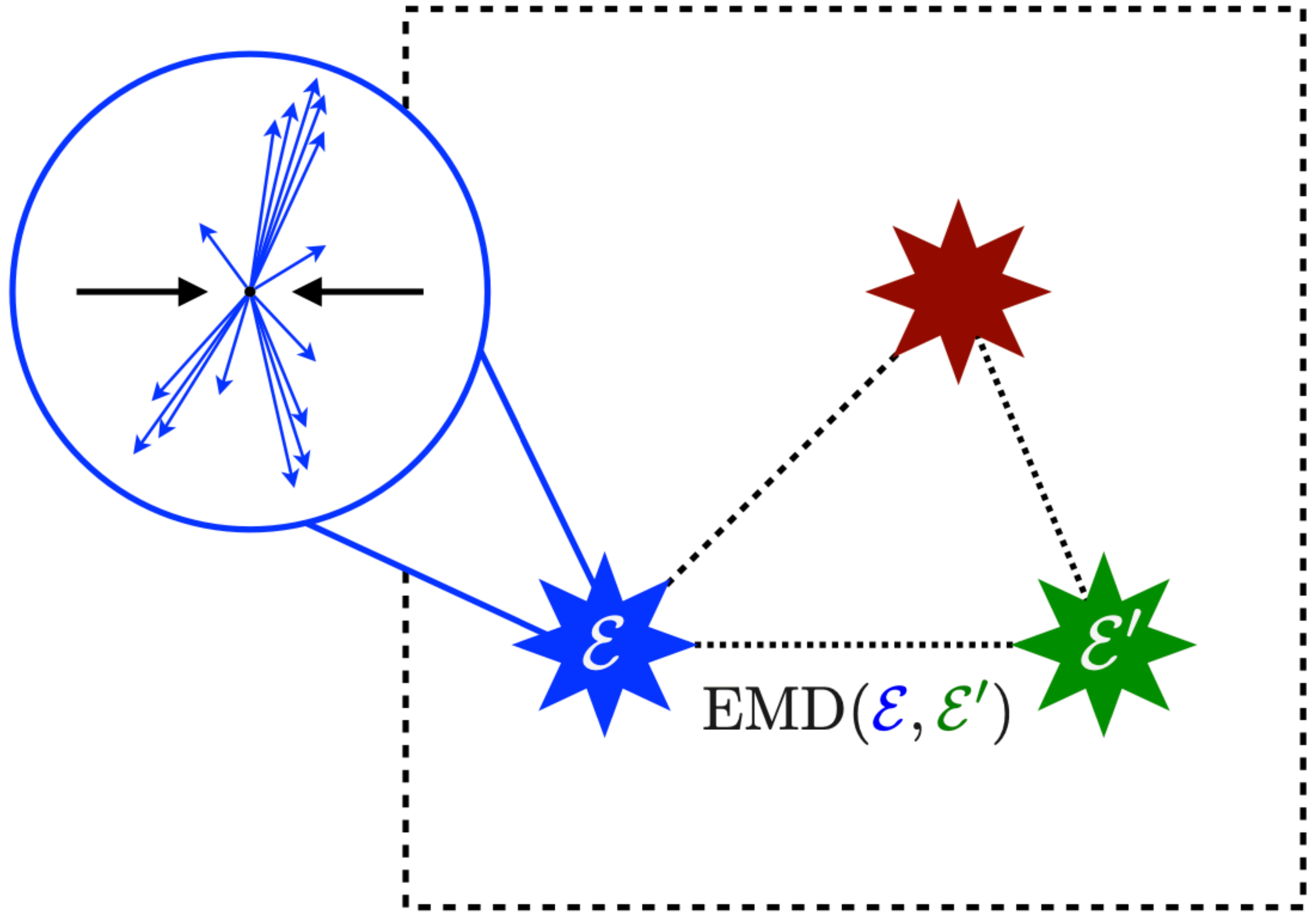
energy (GeV)

cost to move energy

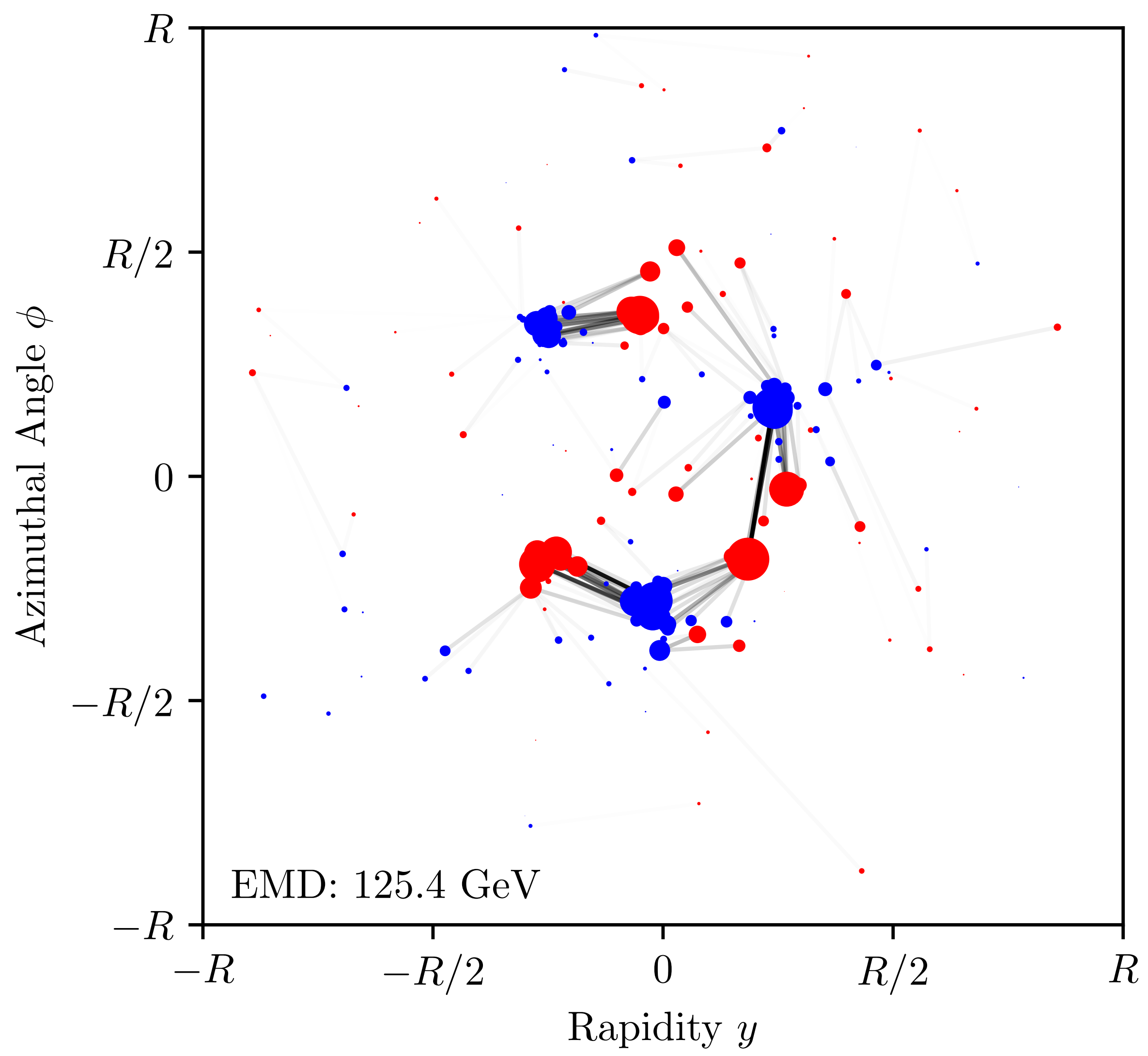
cost to create energy

$$\mathrm{EMD}(\mathcal{E},\mathcal{E}') := \min_{\{f_{i,j}\geq 0\}} \sum_{i=1}^{N}\sum_{j=1}^{N'} f_{i,j}\,\frac{\mathrm{dist}(\hat{n}_i,\hat{n}_j)}{R} + \left|\sum_{i=1}^{N} E_i + \sum_{j=1}^{N'} E_j\right|$$

Metric* space of possible events



$\mathcal{E} \longrightarrow \mathcal{E}'$



* for R sufficiently large

i.e., $R \geq$ jet radius for conical jets

Figures from talk by Jesse Thaler, University of Chicago and Caltech AI+Science: https://www.youtube.com/watch?v=BMBSAWUxBn4

Future directions for HEP applications

Does quantum dynamics make sense on this metric space?

- *Not a Riemannian manifold!*

energy (GeV) — cost to move energy — cost to create energy

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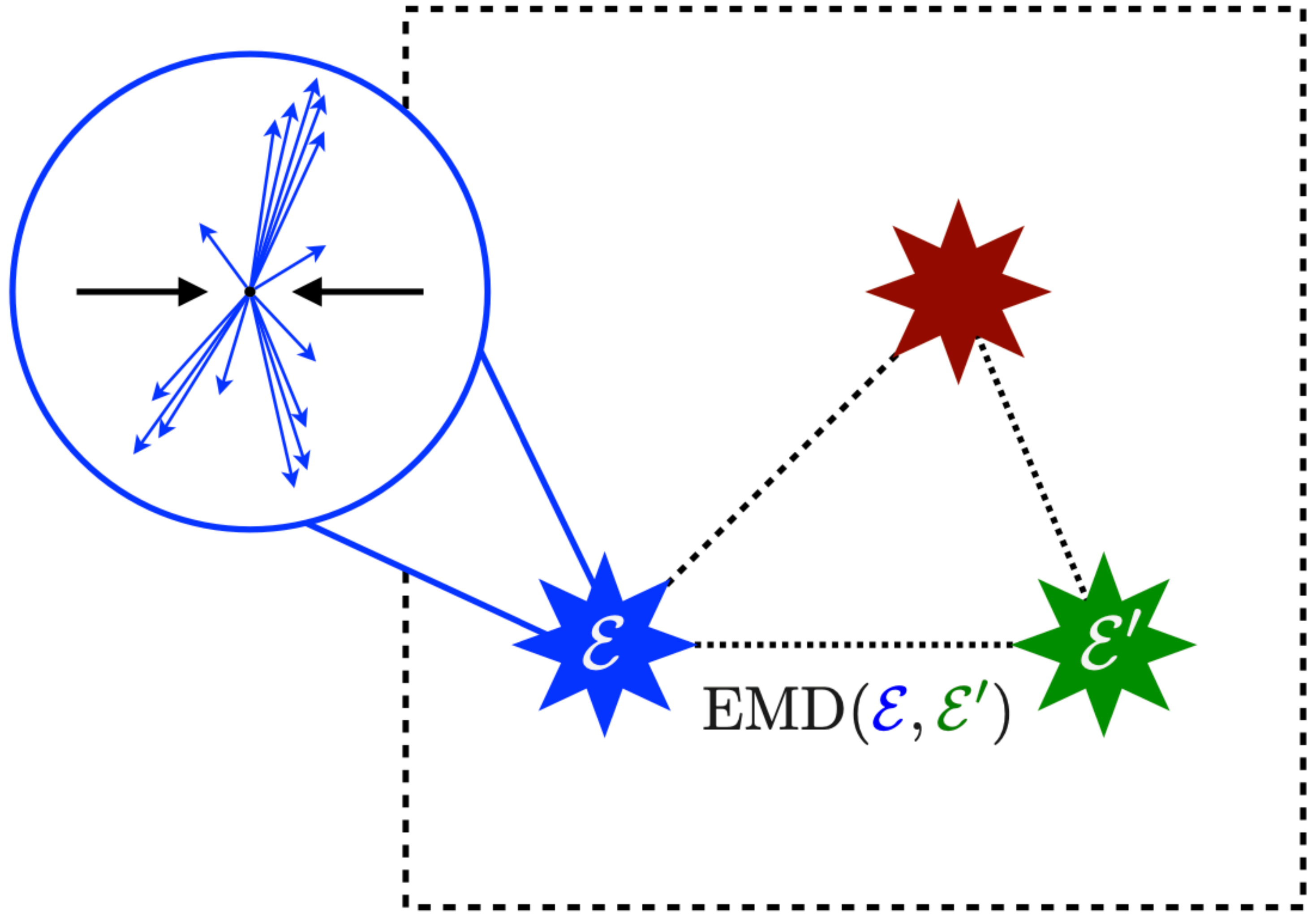

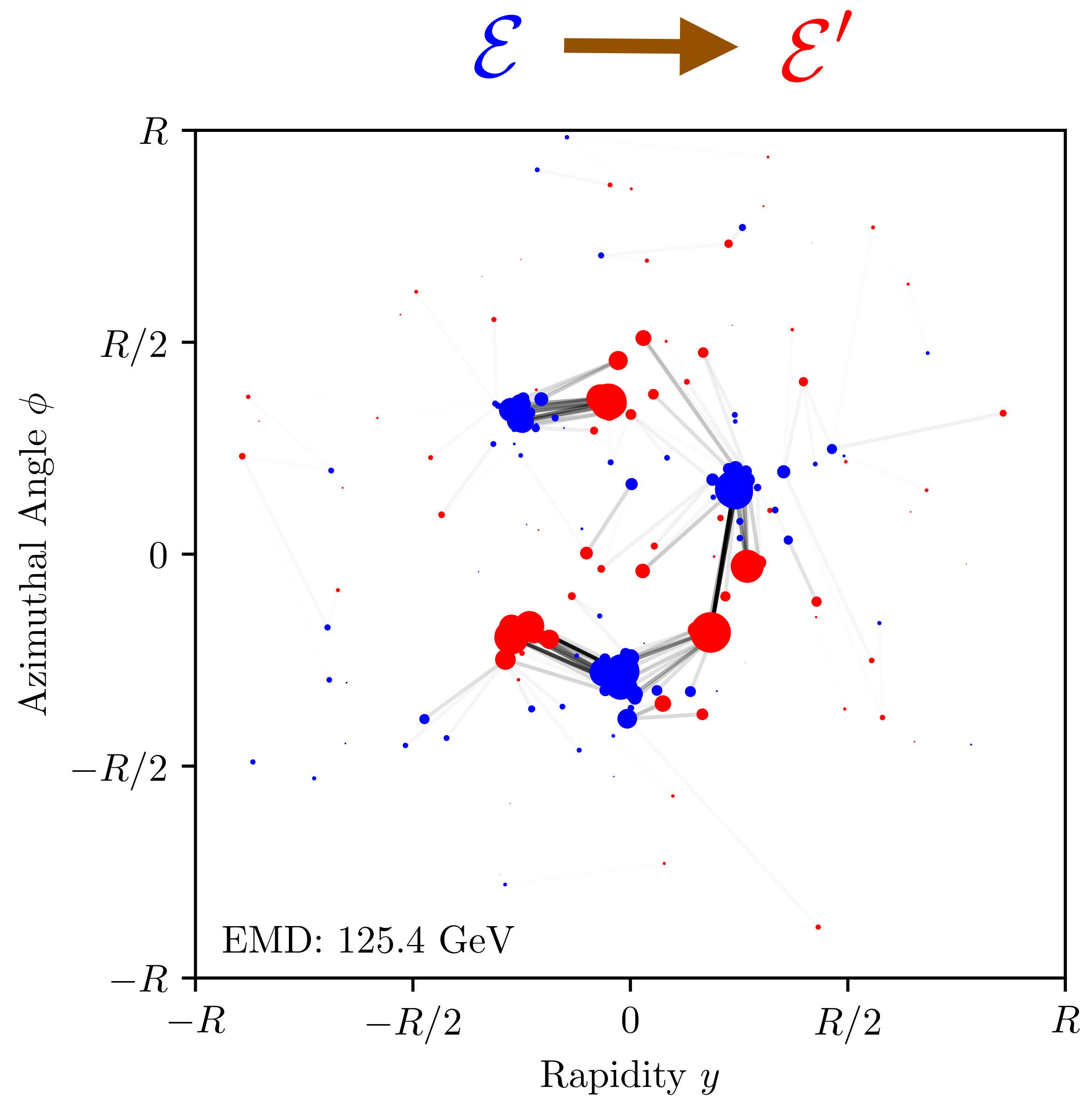

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i.e., $R \geq$ jet radius for conical jets

Figures from talk by Jesse Thaler, University of Chicago and Caltech AI+Science: https://www.youtube.com/watch?v=BMBSAWUxBn4

Future directions for HEP applications

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- *Not a Riemannian manifold!*
    - but maybe *formally* it is?

energy (GeV)

cost to move energy

cost to create energy

$$\mathrm{EMD}(\mathcal{E},\mathcal{E}') := \min_{\{f_{i,j}\geq 0\}} \sum_{i=1}^{N}\sum_{j=1}^{N'} f_{i,j}\,\frac{\mathrm{dist}(\hat{n}_i,\hat{n}_j)}{R} + \left|\sum_{i=1}^{N} E_i + \sum_{j=1}^{N'} E_j\right|$$

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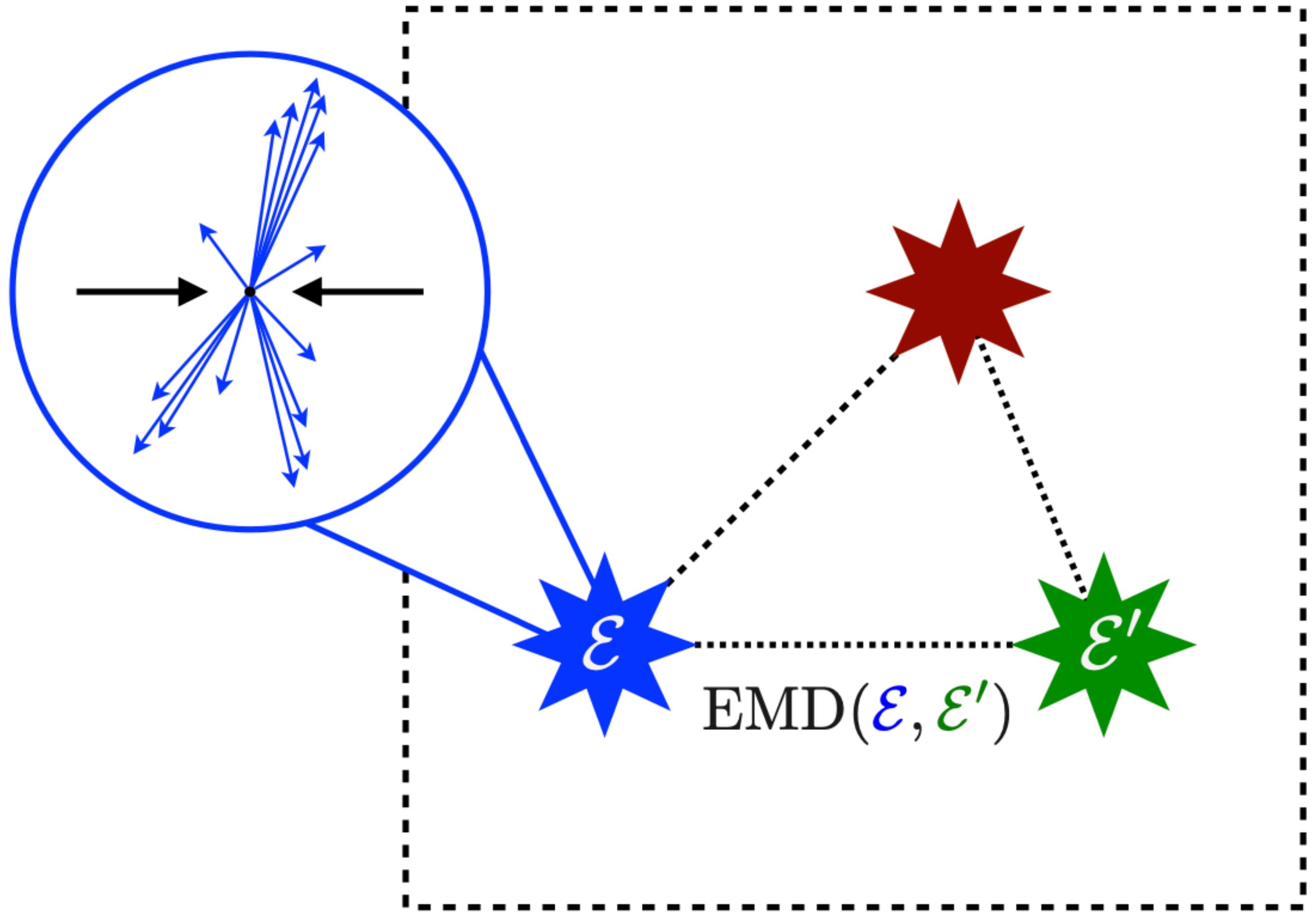

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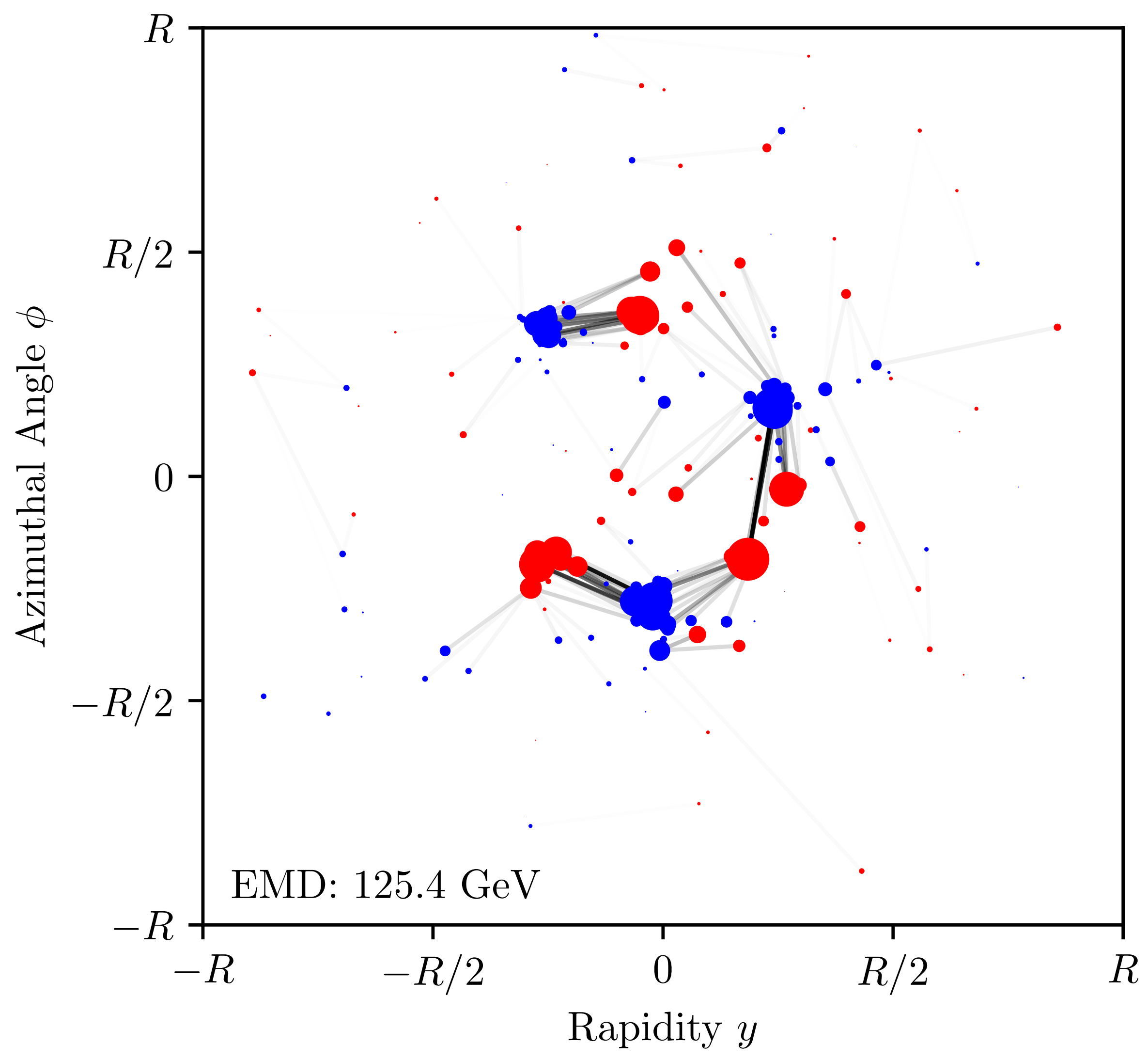

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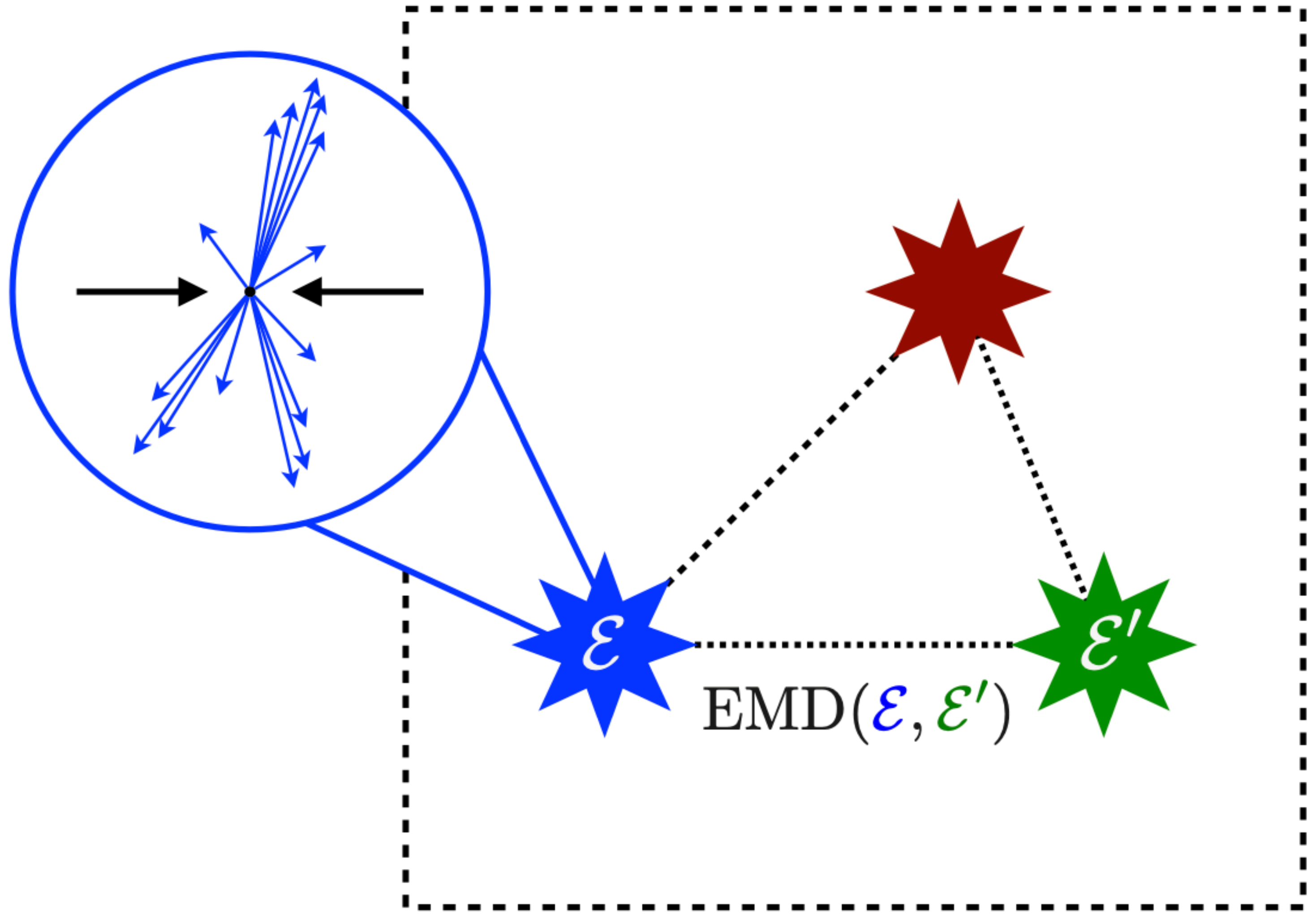

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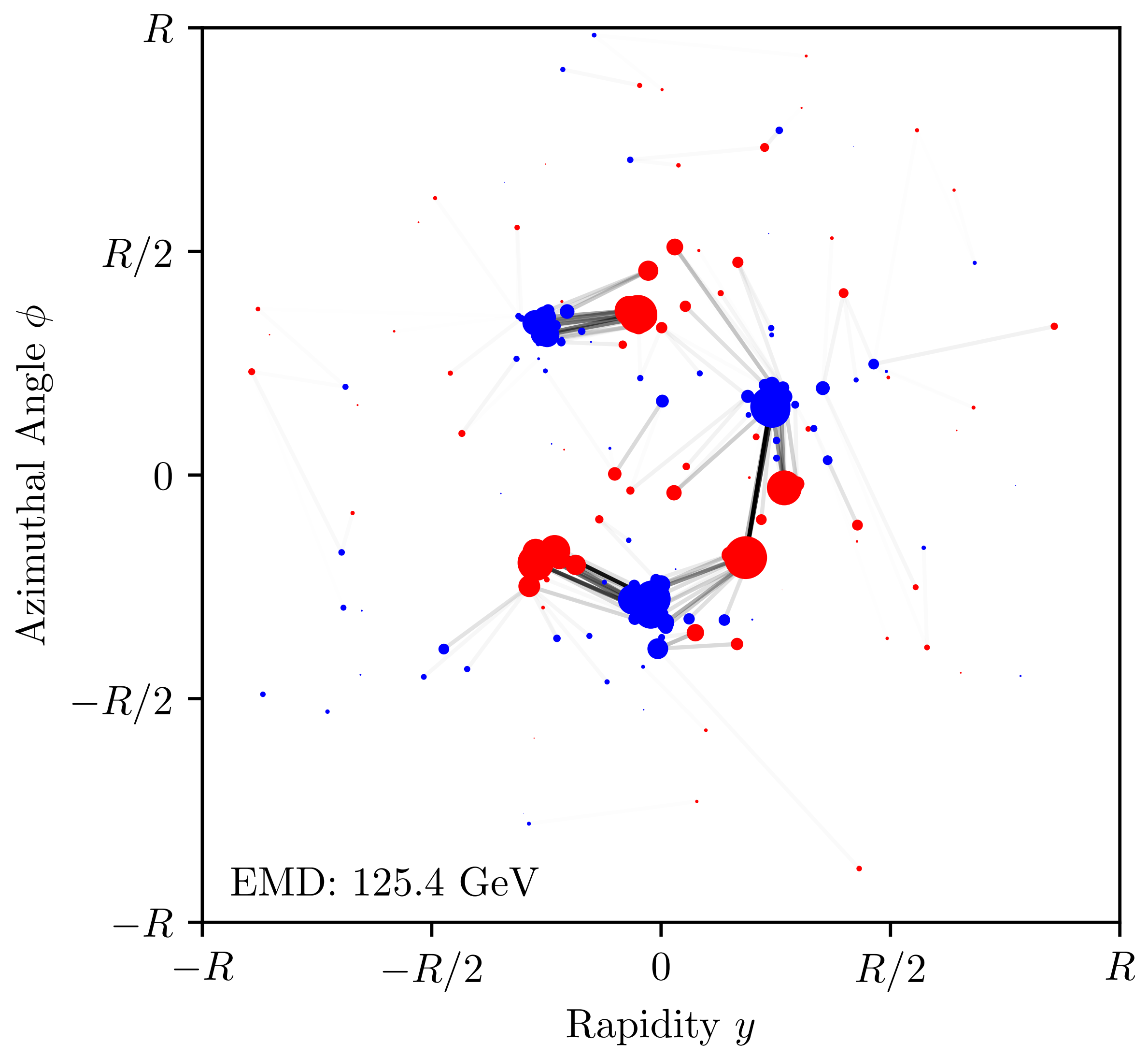

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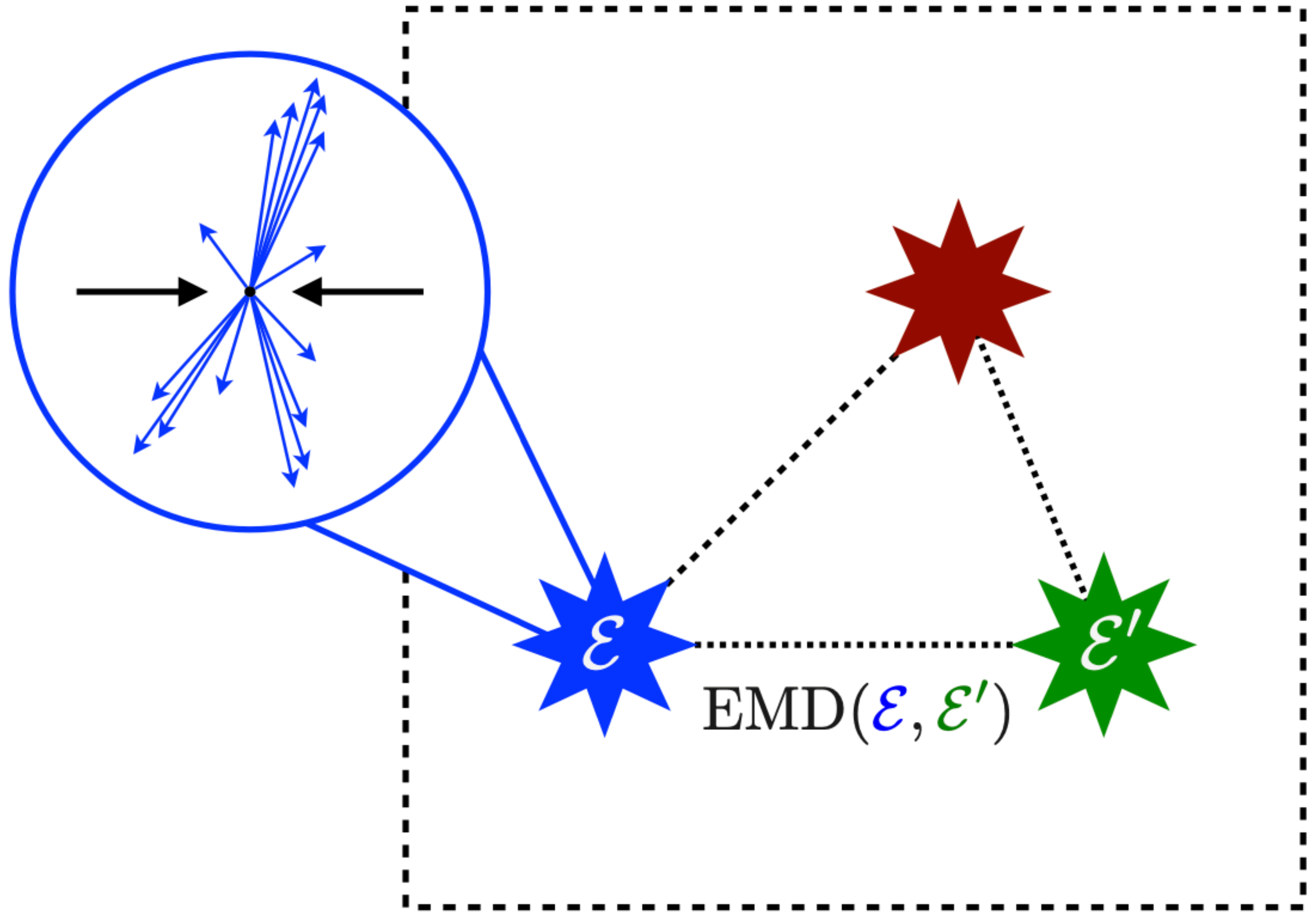

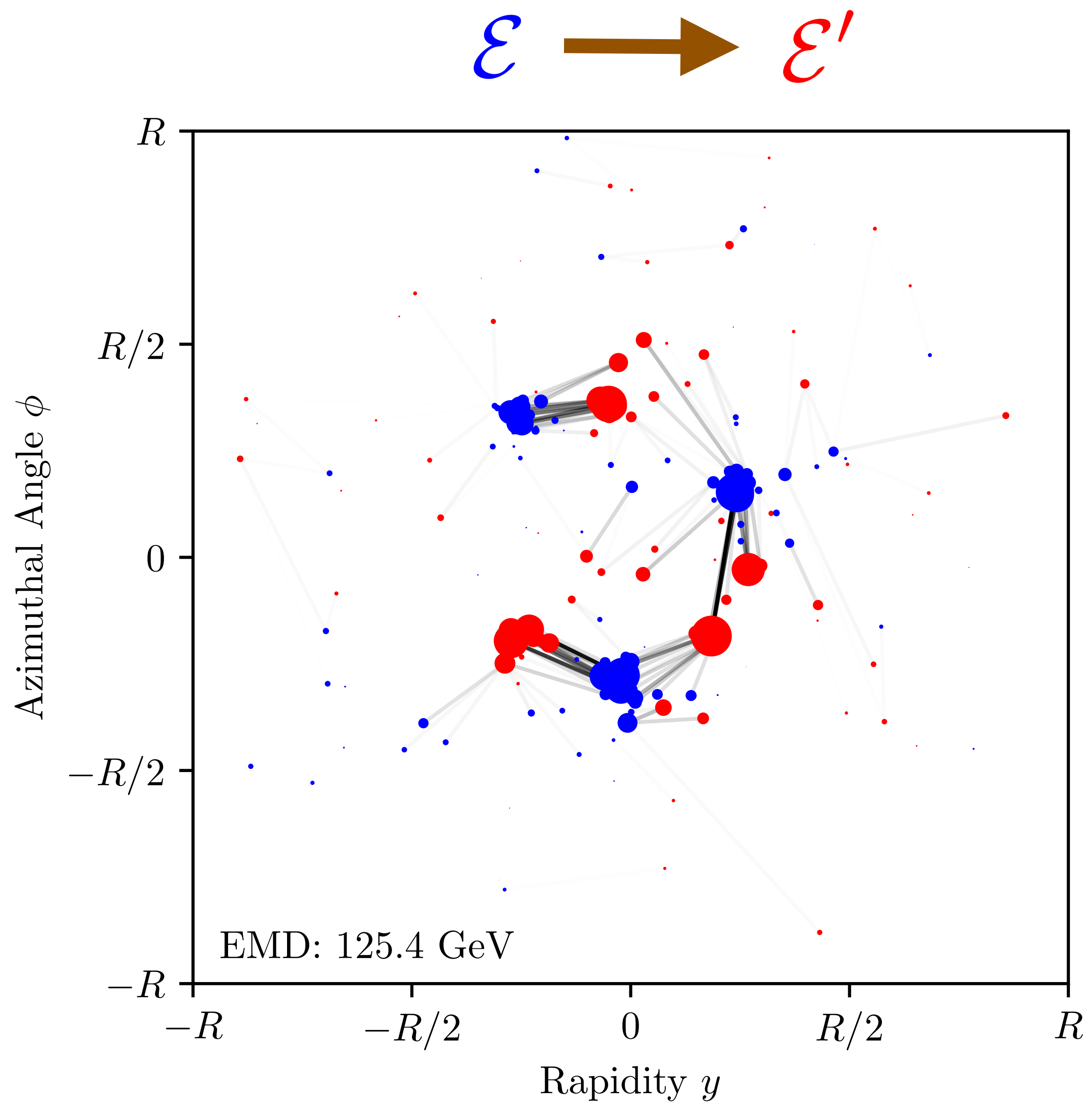

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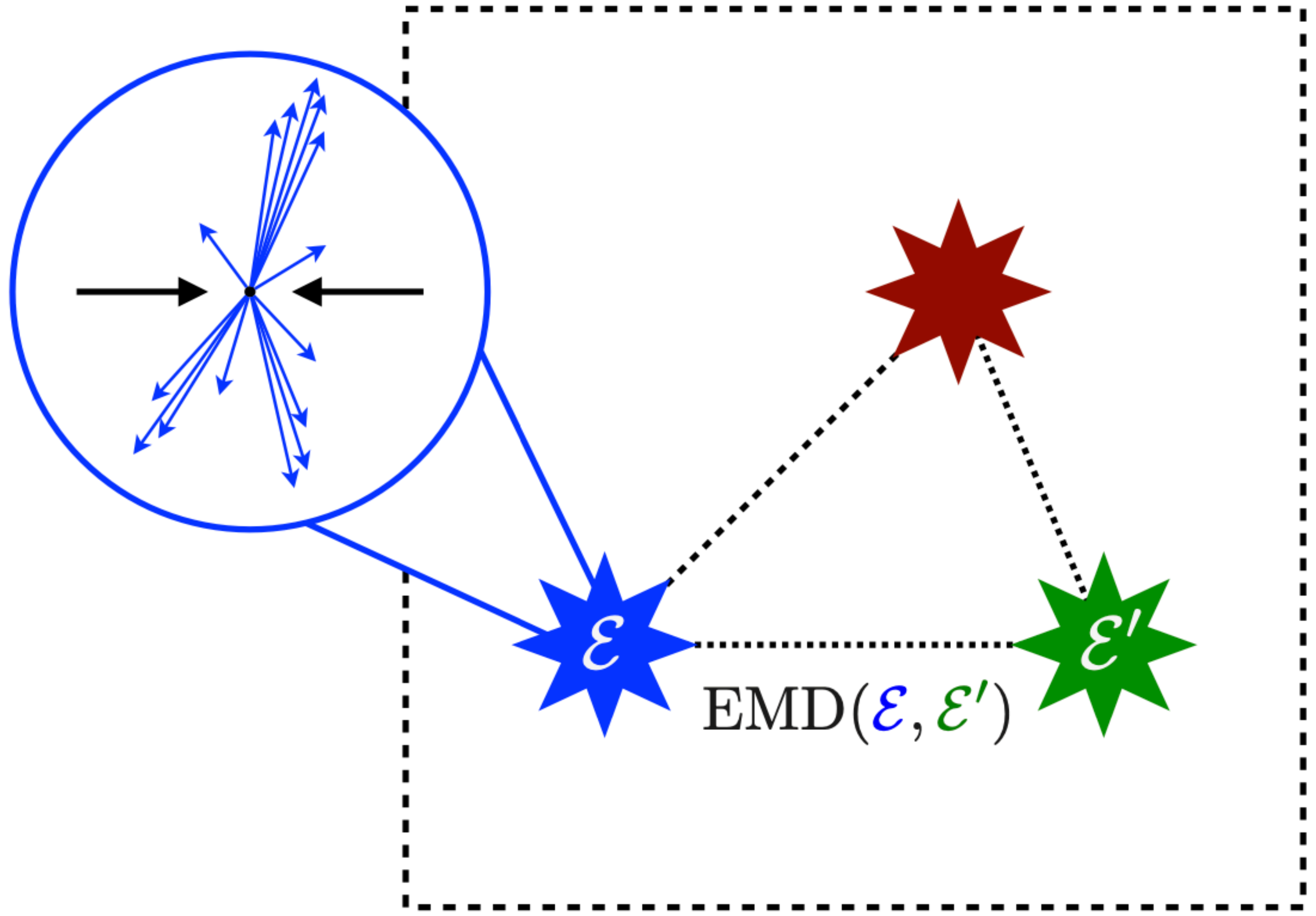

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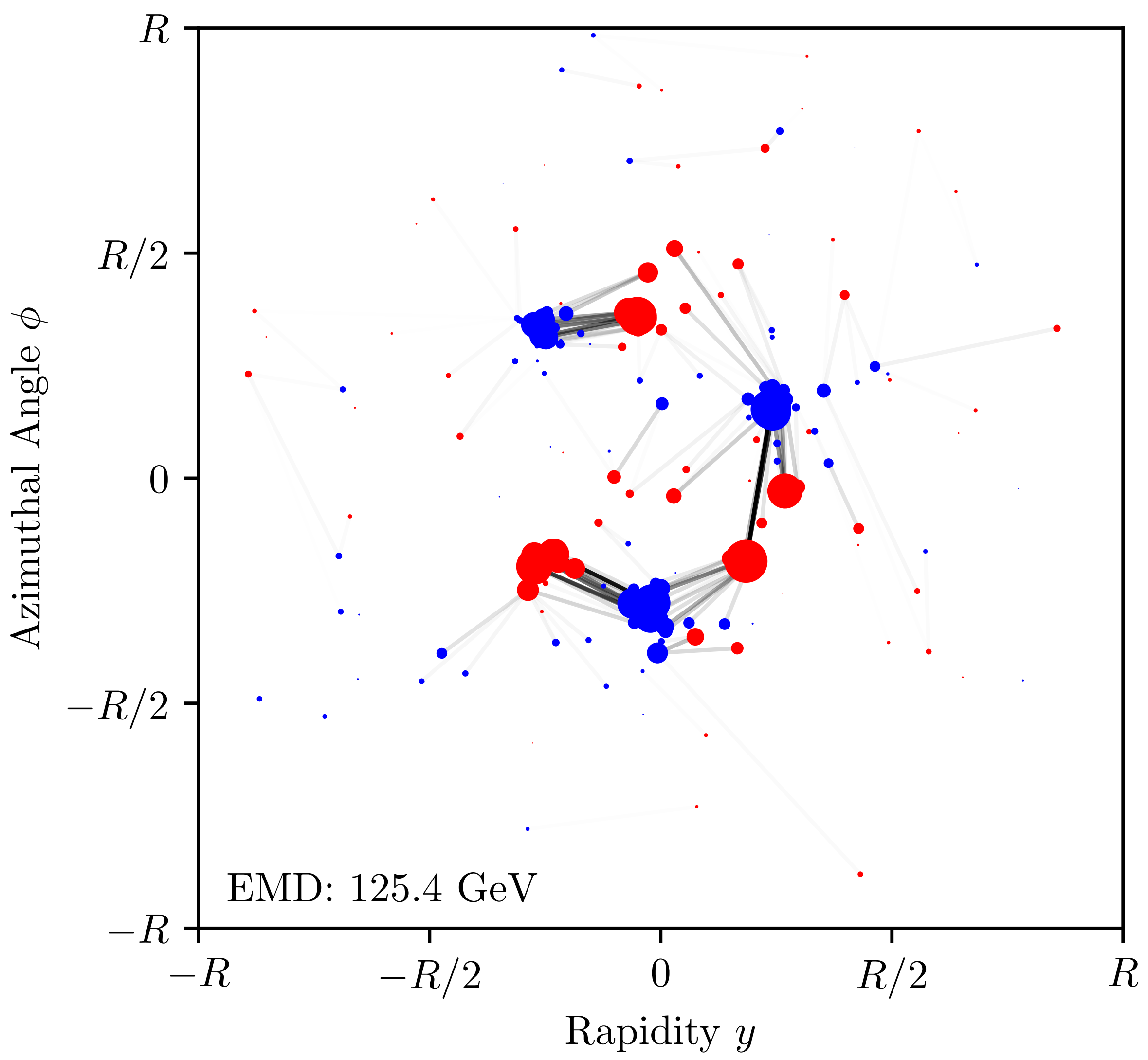

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- The EMD is solvable quickly, but optimal transport plans are harder. Could quantization resolve optimal transport plans more rapidly?

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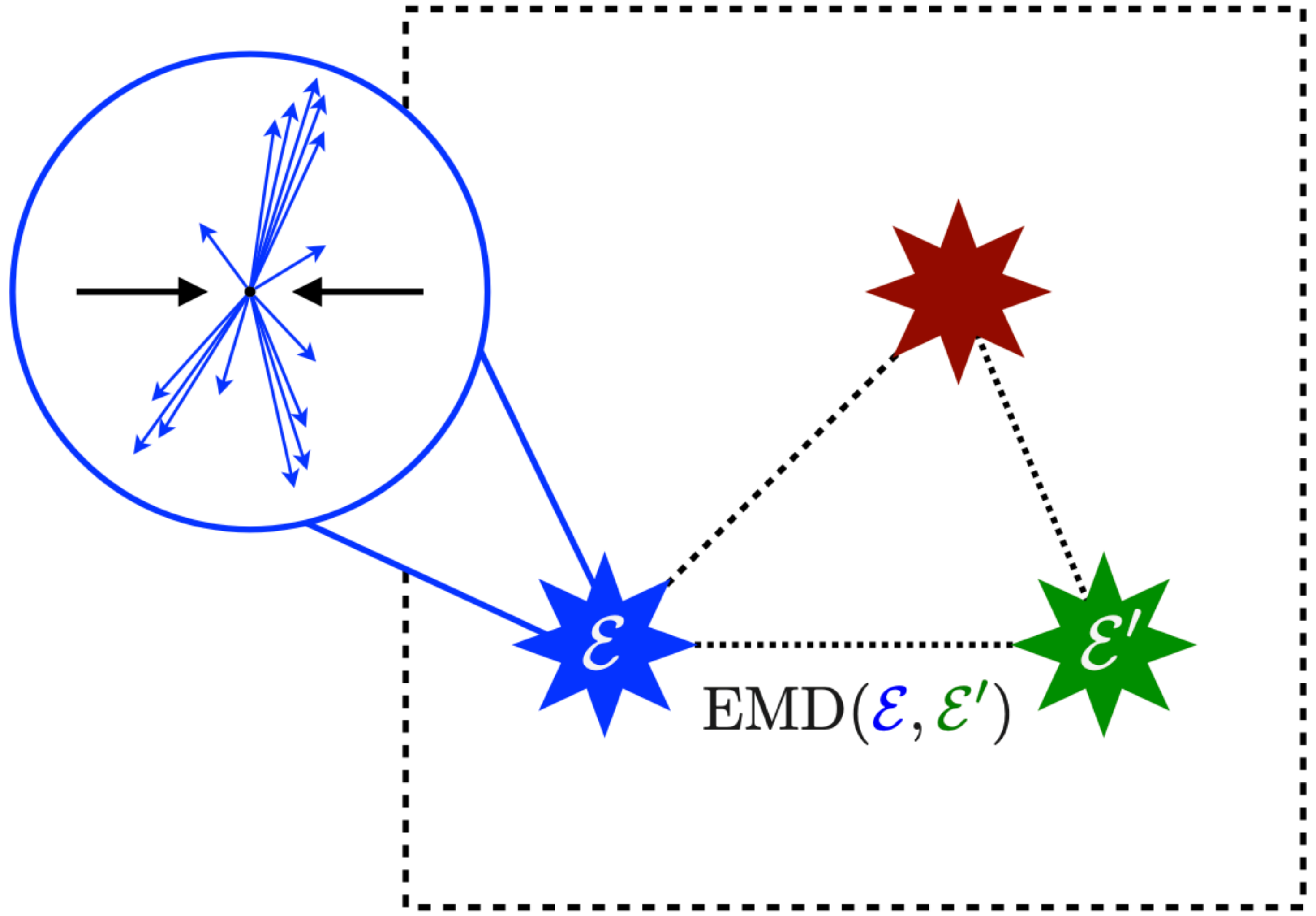

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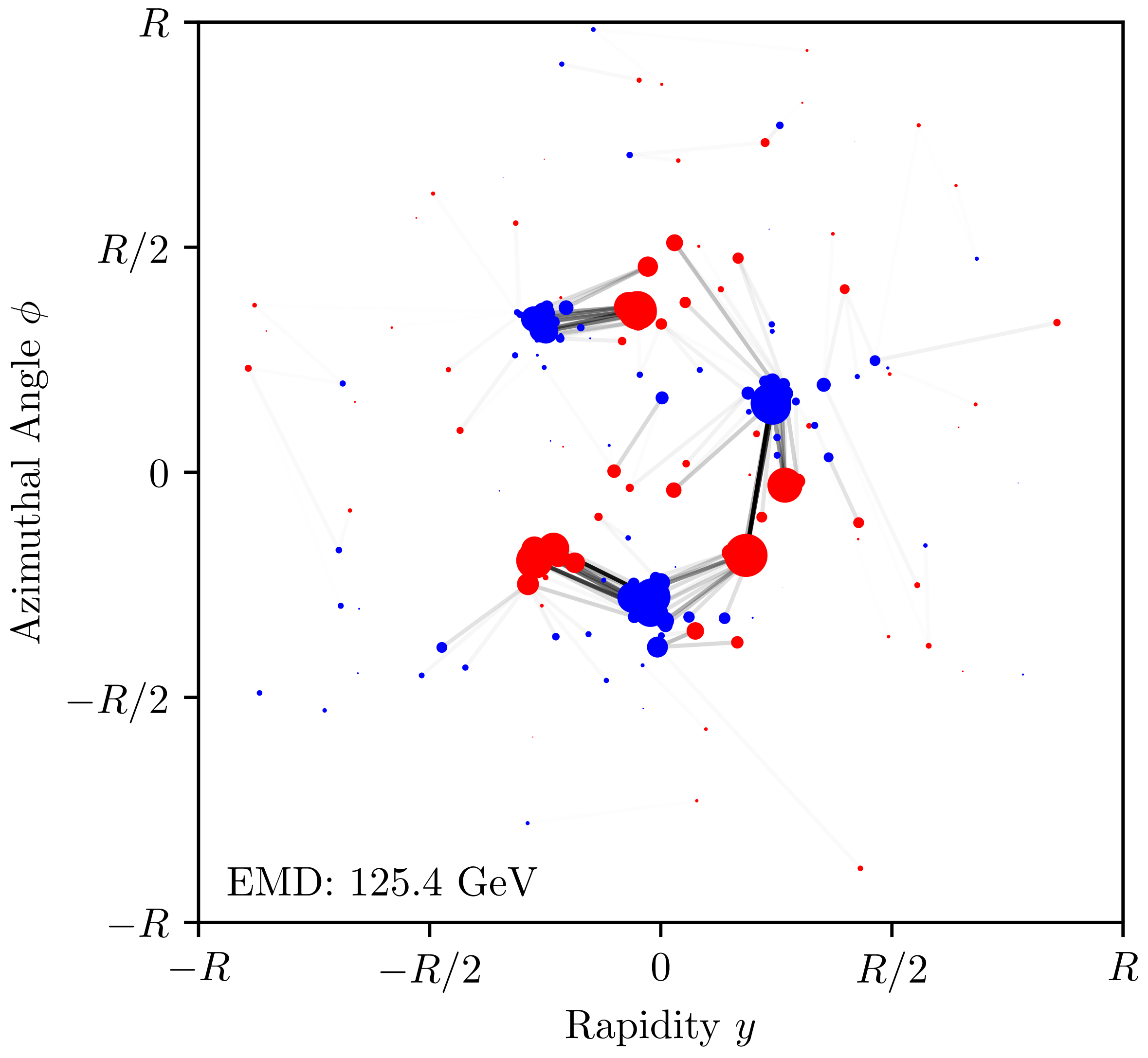

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What about quantum algorithms?

We have a rigorous convergence theory and classical algorithm.

Inputs: $X_N = \{v_1, ..., v_N\}, v^*, \epsilon > 0, \alpha \geq 1, t > 0$
Output: Propagated state $[\psi_h^\zeta](t)$
procedure PROPAGATE
    for $i, j = 1 : N$ **do** $[T_\epsilon]_{i,j} \leftarrow k(||v_i - v_j||^2/\epsilon)$
    $D_\epsilon \leftarrow$ diagonal matrix $\left(\sum_{j=1}^N [T_\epsilon]_{i,j}\right)_{1 \leq i \leq N}$
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- Fast implementation of $U^t_{\epsilon,N}$ has widespread implications for quantum algorithms for manifold learning.

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Thank you

Manifold learning via quantum dynamics

Akshat Kumar[1,2,a] and Mohan Sarovar[3,b]
[1] *Department of Mathematics, Clarkson University, Potsdam, NY 13699 USA*
[2] *Instituto de Telecomunicações, Lisbon, Portugal*
[3] *Sandia National Laboratories, Livermore, California 94550, USA*

arXiv:2112.11161

Shining light on data: Geometric data analysis through quantum dynamics

Akshat Kumar,[1*] Mohan Sarovar[2*]

arXiv:2212.00682

ON A QUANTUM-CLASSICAL CORRESPONDENCE: FROM GRAPHS TO MANIFOLDS

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arXiv:2112.10748