GraSP23 || GravityShapePisa 2023

Gravitational Waves Observables From Scattering Amplitudes

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GravityShapePisa 2023, Pisa, 25th October 2023

INFN

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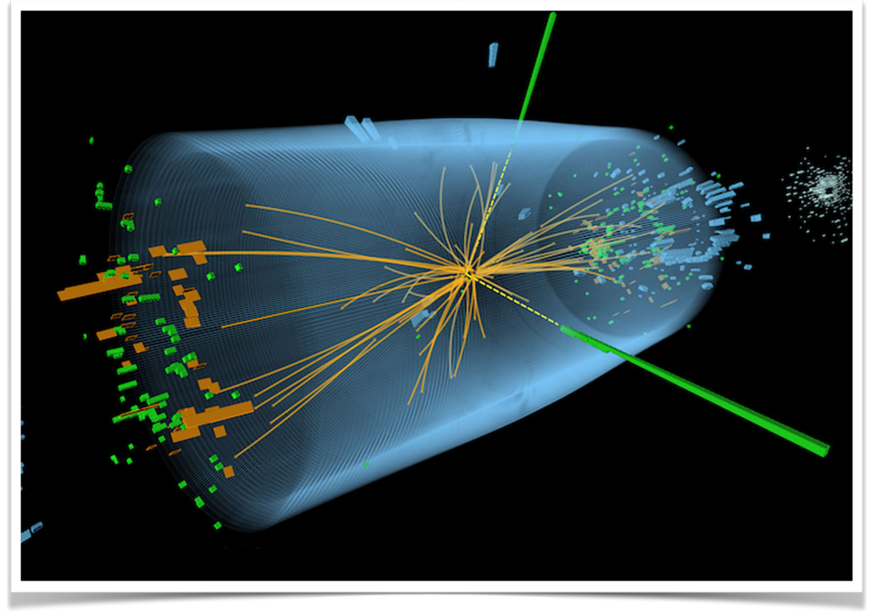
Message:

Scattering amplitudes techniques, born in particle physics to study scattering at colliders, can be applied to study **coalescing binary systems,** to make predictions for **gravitational waves observables**



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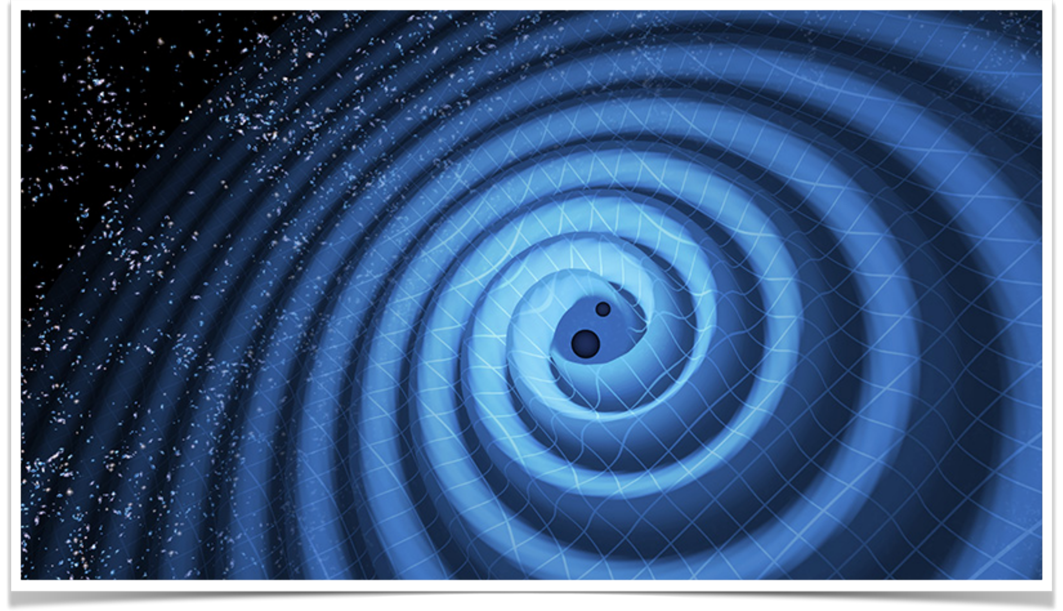
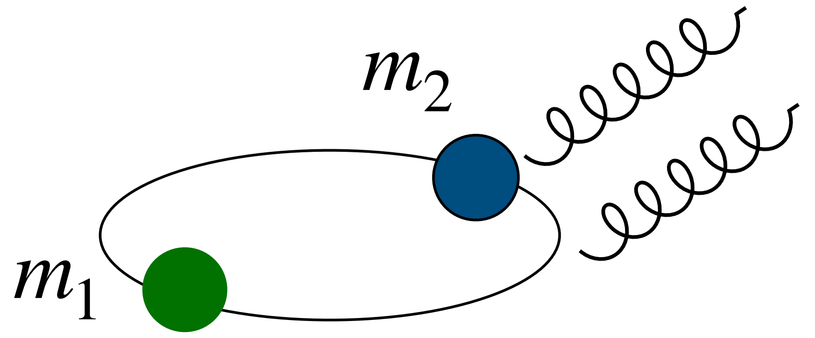
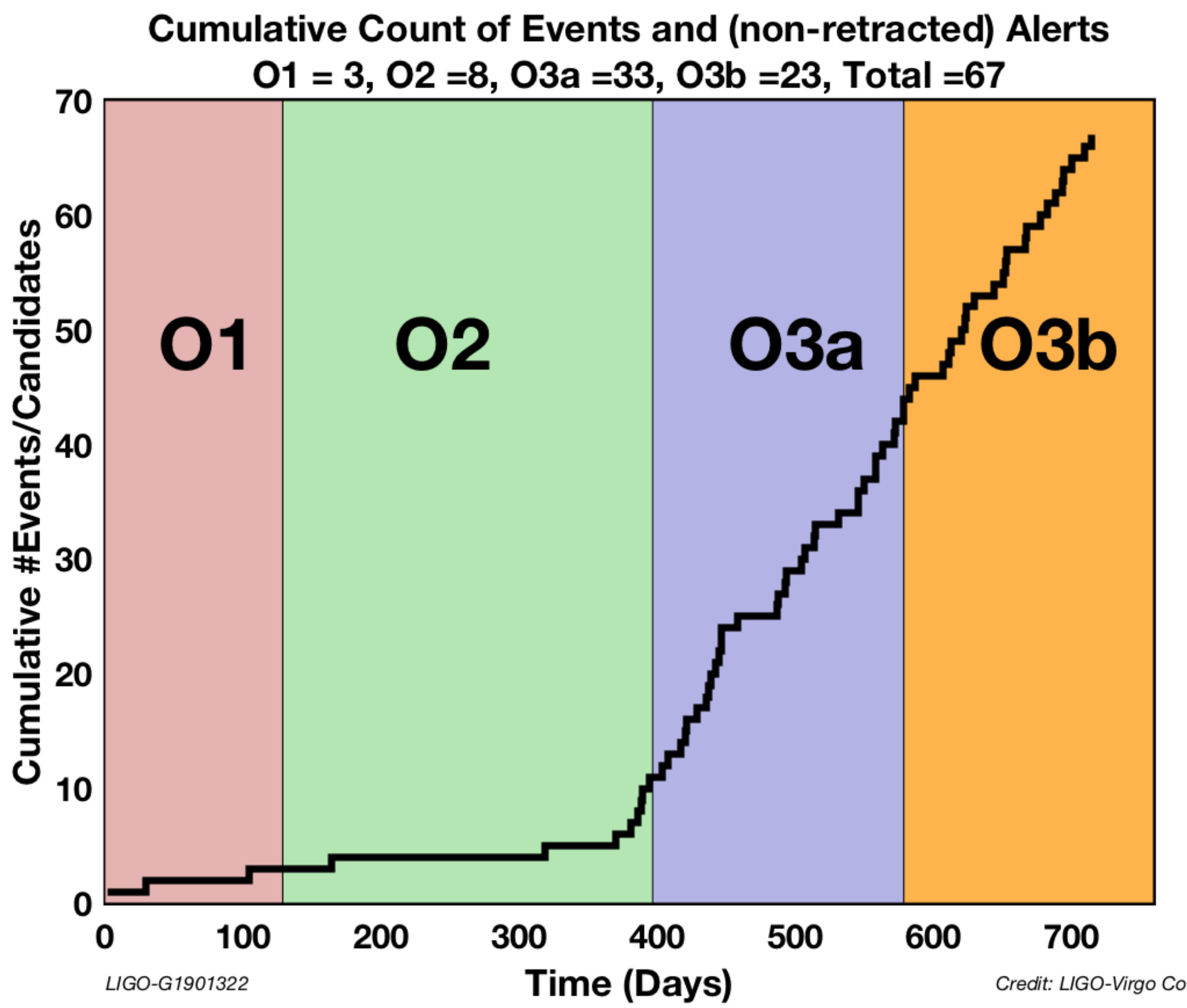


1. Motivation: Gravitational Waves
2. Post-Minkoswskian approach to General Relativity
   - Physical scales
   - Exponential representation of the S-matrix
   - Traditional amplitude computation
   - Optimisations
3. Observable-based approach
   - Impulse from amplitudes
   - Gravitational waveform from amplitudes
4. Outlooks

Based on collaborations with:
S. De Angelis, D. Kosower,
M. Mandal, P. Mastrolia, R. Patil

Motivation: Gravitational Waves

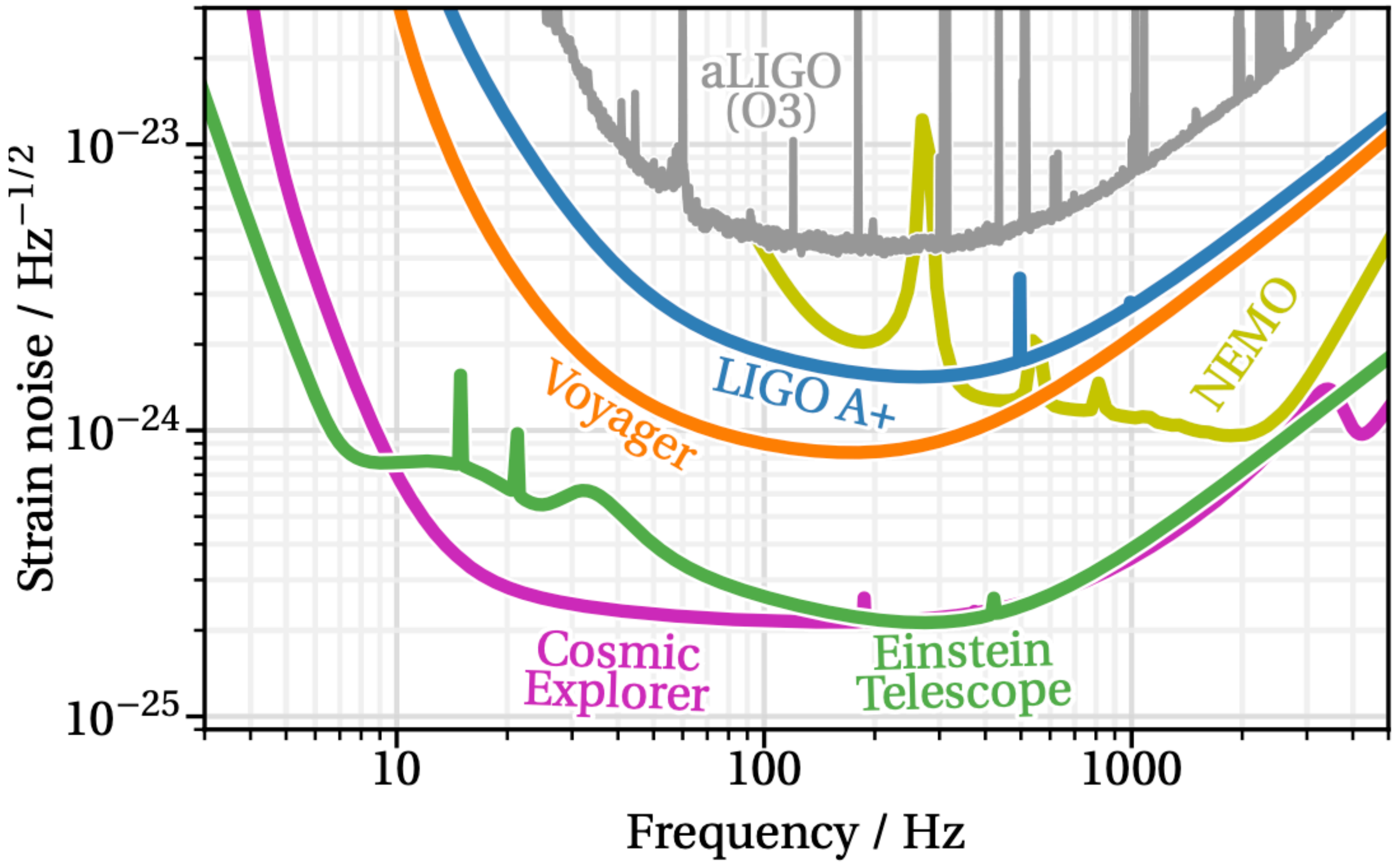




LIGO-G1901322

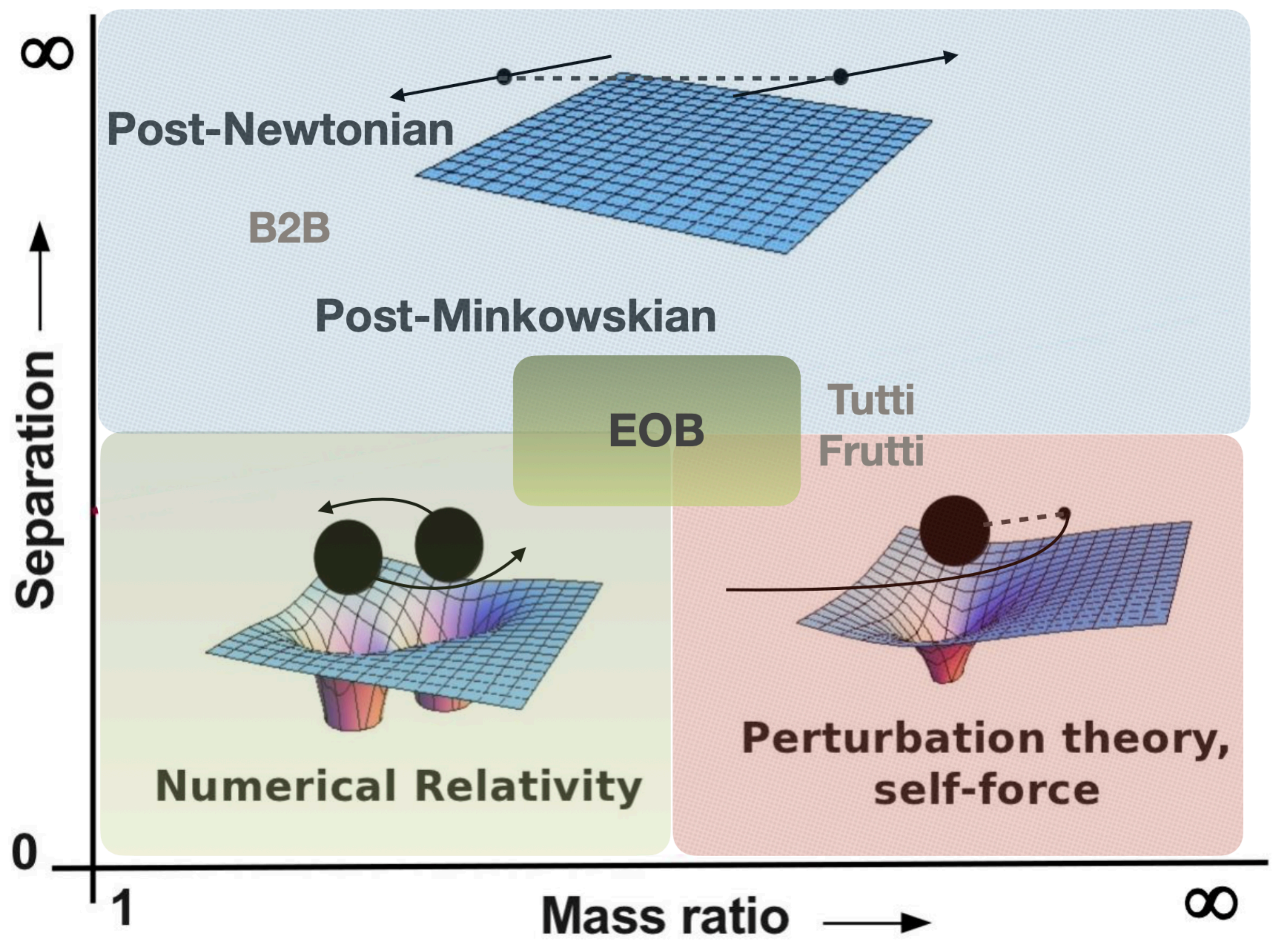
Credit: LIGO-Virgo Collaboration

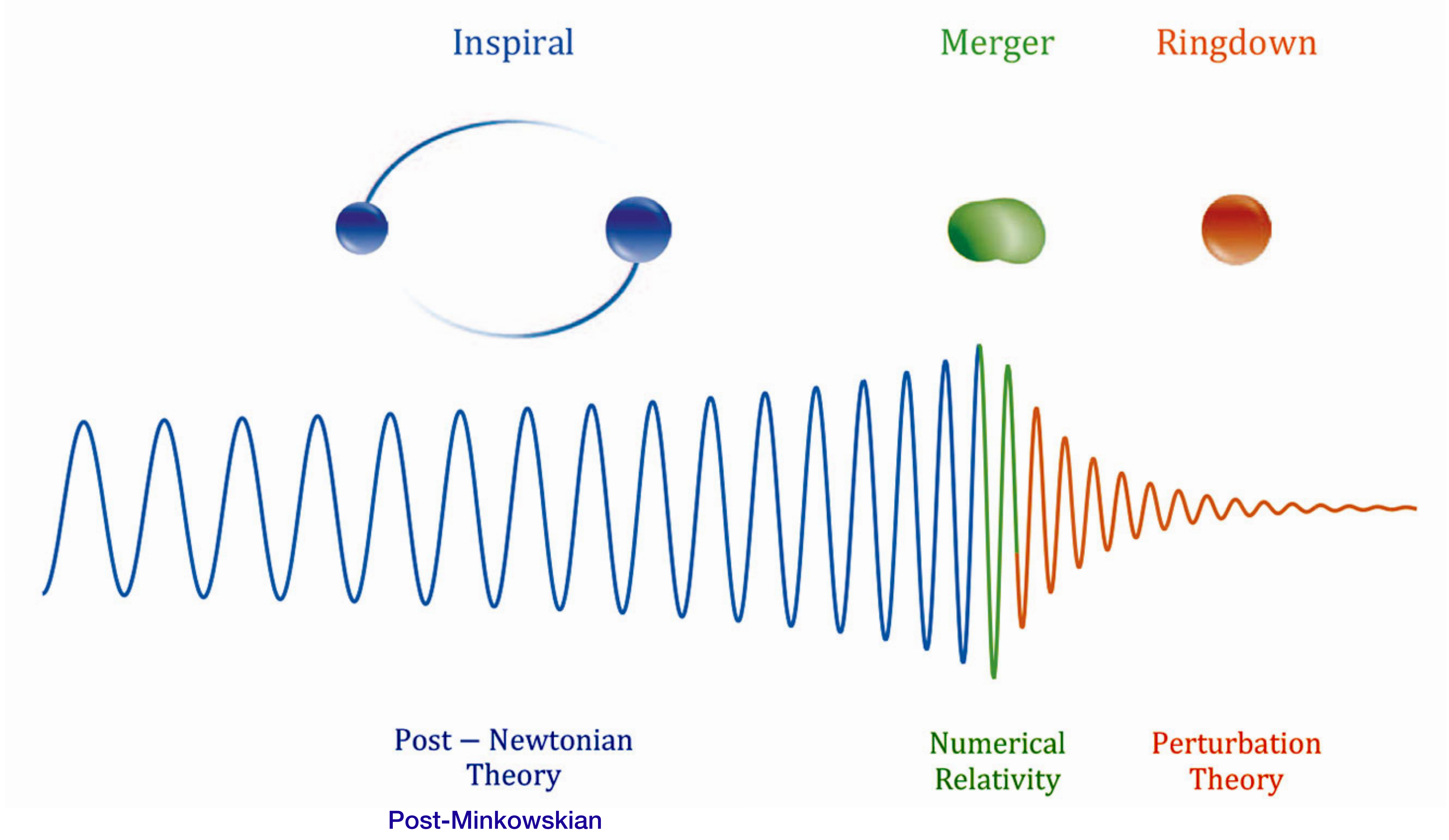
- ▸ Ligo-Virgo-Kagra efficiently detect GWs emitted by **Coalescing Binary Systems.**
- ▸ New instrument to **probe our universe** which allow us to:
  - Testing GR in the strong field regime
  - Cataloging black hole binaries
  - Probe ultra-dense matter (neutron star merging)
  - Multi-messenger astrophysics



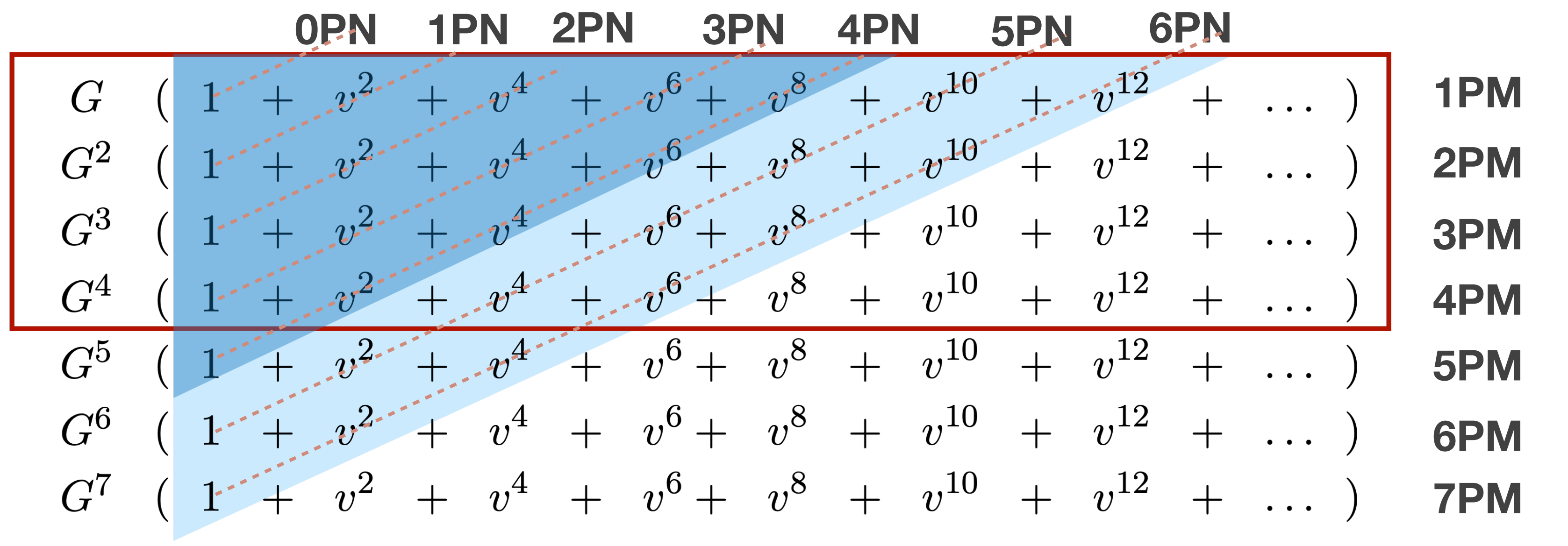
- ▸ More than 90 events during the first 3 operative runs of LIGO/Virgo/Kagra interferometers, expected rate of 1 merger/2-3days during O4.
- ▸ Next generation of gravitational waves interferometers (Einstein Telescope, LISA, …) (2035) will improve SNR of a factor 10-100 with expected $\mathcal{O}(10^6)$ events per year
- ▸ Good handling of experimental uncertainties
- ▸ Extreme need for precise theoretical predictions
- ▸ Scattering Amplitudes can help in reaching this goal

Motivation: Coalescing binary systems





Astrophysicists/Cosmologists' whishlist



0PN 1PN 2PN 3PN 4PN 5PN 6PN

G	$(1+v^2+v^4+v^6+v^8+v^{10}+v^{12}+\dots)$	1PM
G^2	$(1+v^2+v^4+v^6+v^8+v^{10}+v^{12}+\dots)$	2PM
G^3	$(1+v^2+v^4+v^6+v^8+v^{10}+v^{12}+\dots)$	3PM
G^4	$(1+v^2+v^4+v^6+v^8+v^{10}+v^{12}+\dots)$	4PM
G^5	$(1+v^2+v^4+v^6+v^8+v^{10}+v^{12}+\dots)$	5PM
G^6	$(1+v^2+v^4+v^6+v^8+v^{10}+v^{12}+\dots)$	6PM
G^7	$(1+v^2+v^4+v^6+v^8+v^{10}+v^{12}+\dots)$	7PM

[credit: Bern et al.]

- **Post-Newtonian Expansion: [Bound state]**

$$G_N\frac{m}{r}\sim v^2\ll 1$$

Expansion in powers of v/c

- **Post-Minkowskian Expansion: [Scattering]**

$$G_N\frac{m}{r}\ll v^2\sim 1$$

Expansion in powers of G_N

- **BH perturbation theory /Self Force:**

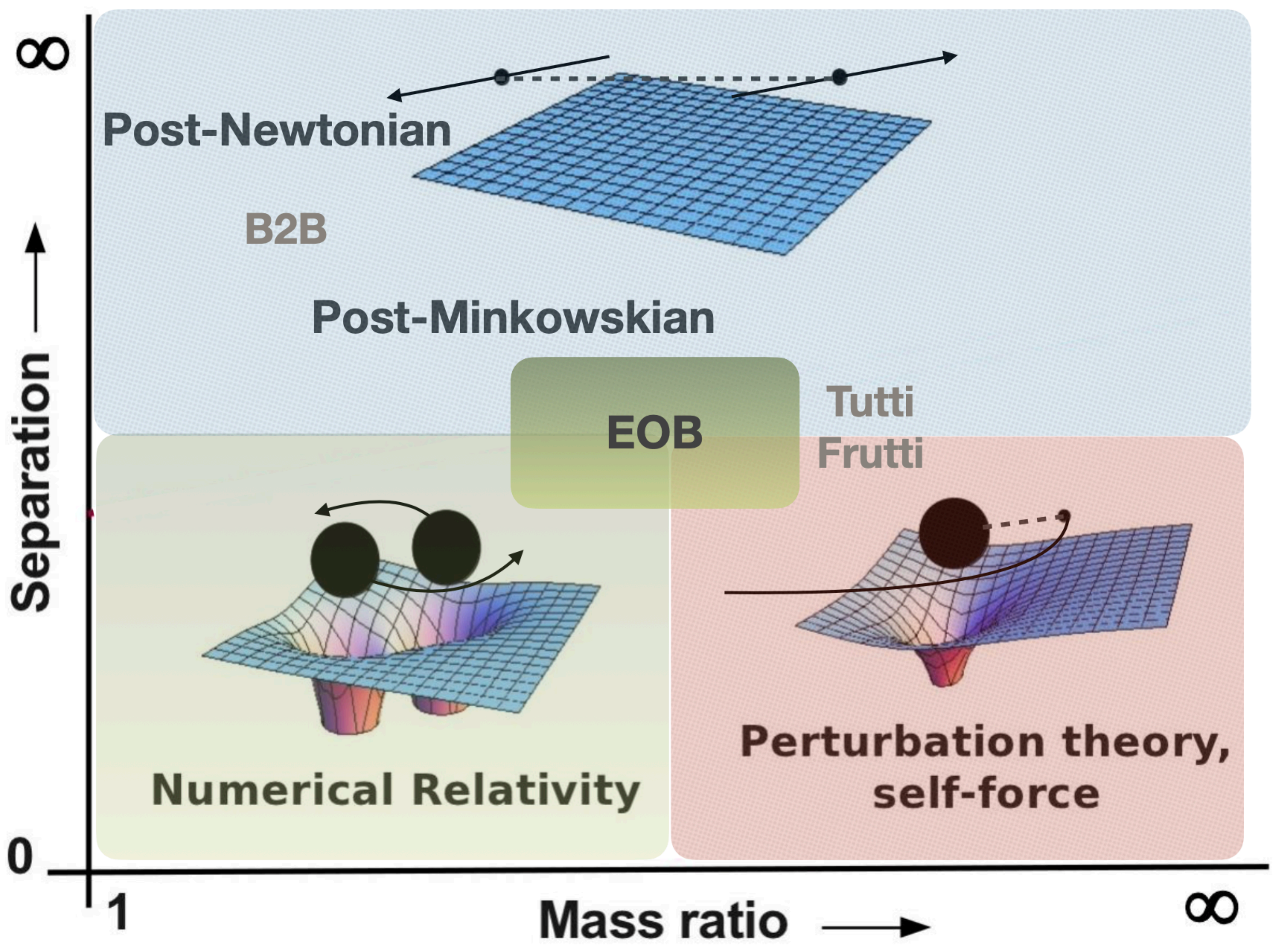
$$G_N\frac{m}{r}\sim v^2\sim 1$$

$$\delta g_{\mu\nu}\sim\epsilon=m_2/m_1\ll 1$$

Expansion in powers of ϵ

Post-Newtonian Expansion

$$G_N \frac{m}{r} \sim v^2 \ll 1$$



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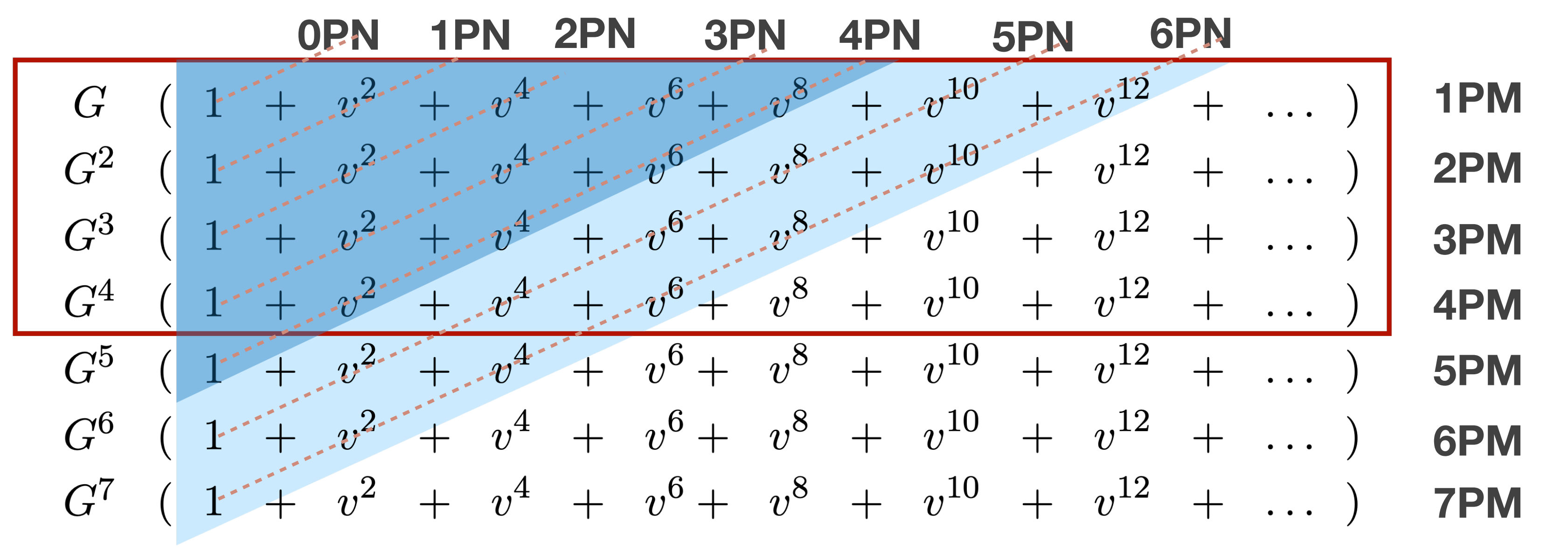

	0PN	1PN	2PN	3PN	4PN	5PN	6PN		
G	$(\ 1$	$+\ v^2$	$+\ v^4$	$+\ v^6$	$+\ v^8$	$+\ v^{10}$	$+\ v^{12}$	$+\ \dots\ )$	1PM
G^2	$(\ 1$	$+\ v^2$	$+\ v^4$	$+\ v^6$	$+\ v^8$	$+\ v^{10}$	$+\ v^{12}$	$+\ \dots\ )$	2PM
G^3	$(\ 1$	$+\ v^2$	$+\ v^4$	$+\ v^6$	$+\ v^8$	$+\ v^{10}$	$+\ v^{12}$	$+\ \dots\ )$	3PM
G^4	$(\ 1$	$+\ v^2$	$+\ v^4$	$+\ v^6$	$+\ v^8$	$+\ v^{10}$	$+\ v^{12}$	$+\ \dots\ )$	4PM
G^5	$(\ 1$	$+\ v^2$	$+\ v^4$	$+\ v^6$	$+\ v^8$	$+\ v^{10}$	$+\ v^{12}$	$+\ \dots\ )$	5PM
G^6	$(\ 1$	$+\ v^2$	$+\ v^4$	$+\ v^6$	$+\ v^8$	$+\ v^{10}$	$+\ v^{12}$	$+\ \dots\ )$	6PM
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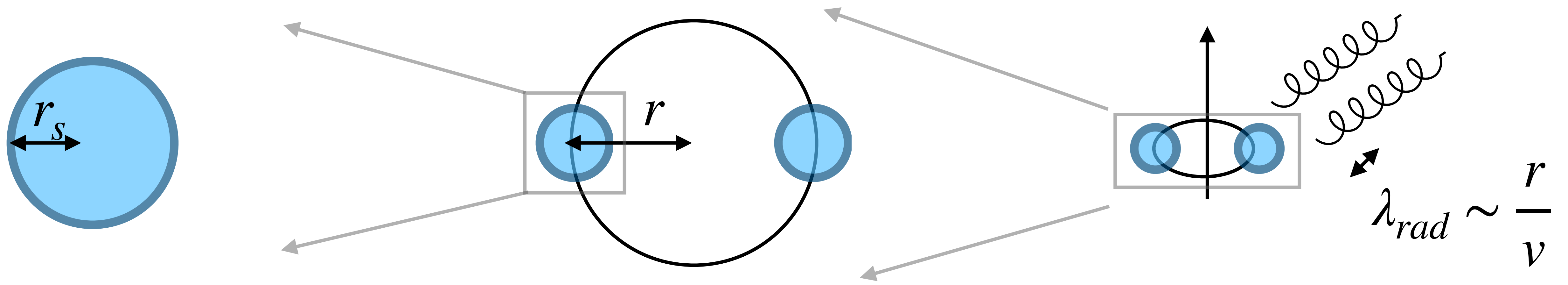
[credit: Bern et al.]

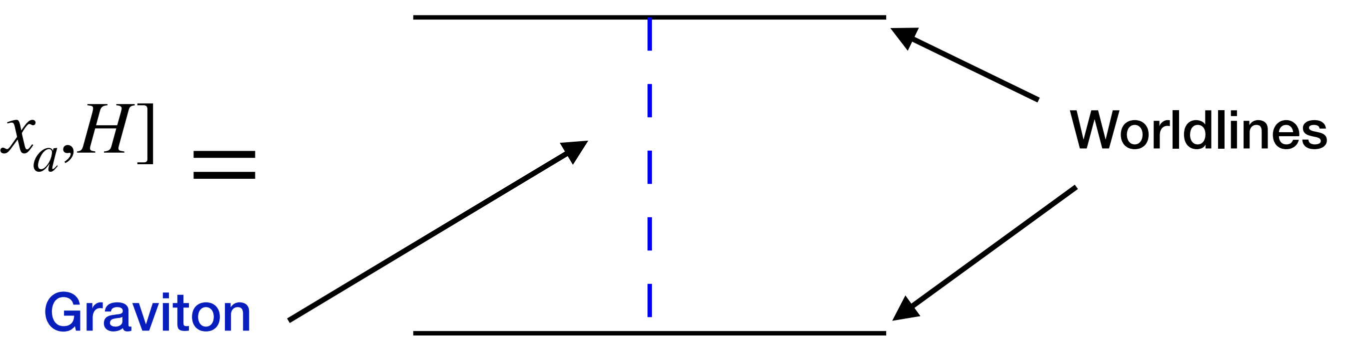
- Systematic study of coalescing compact objects (bound system)
- Effective Field Theory Approach to General Relativity: $r_s \ll r \ll \lambda_{rad}$



- Deviations from Newtonian dynamics via diagrammatic approach:

Goldberger, Rothstein, (2004)

$$e^{iS_{eff}[x_a]} = \int DH\ e^{iS_{tot}[x_a,H]} =$$



$+ \dots$

- 1PN: Einstein, Infeld, Hoffman (1938)
- 2PN: Ohta-Okamura-Kimura-Hiida (1974)
Gilmore, Ross (2008)
- 3PN: Jaranowski,Schaefer (1997); Damour, Jaranowski, Schaefer (1997); Blachę, Faye (2000); Damour, Jaranowski Schaefer (2001); Foffa Sturani (2011)
- 4PN: Damour, Jaranowski, Schaefer (2014);
Bernard, Blanchet, Bohe, Faye, Marsa (2016);
Foffa, Sturani, Mastrolia, Sturm (2016);
Galley, Leibovich, Porto, Ross (2016);
Foffa, Porto, Rothstein, Sturani (2019);
Blumlein, Maier, Marquard,Schaefer (2020);
- 5PN: Bini, Damour, Geralico (2019); Foffa, Mastrolia, Sturani, Sturm, Torres Bobadilla (2019); Blumlein, Maier, Marquard, Schaefer (2020); Almeida, Foffa, Sturani (2021); Edison, Levi(2022); Almeida, Foffa, Sturani (2021); Almeida, Foffa, Sturani (2023);
- 6PN: Blumlein, Maier, marquard, Schafer (2021)

- The 5PN sector requires some understanding: G.B, Master Thesis (2022)

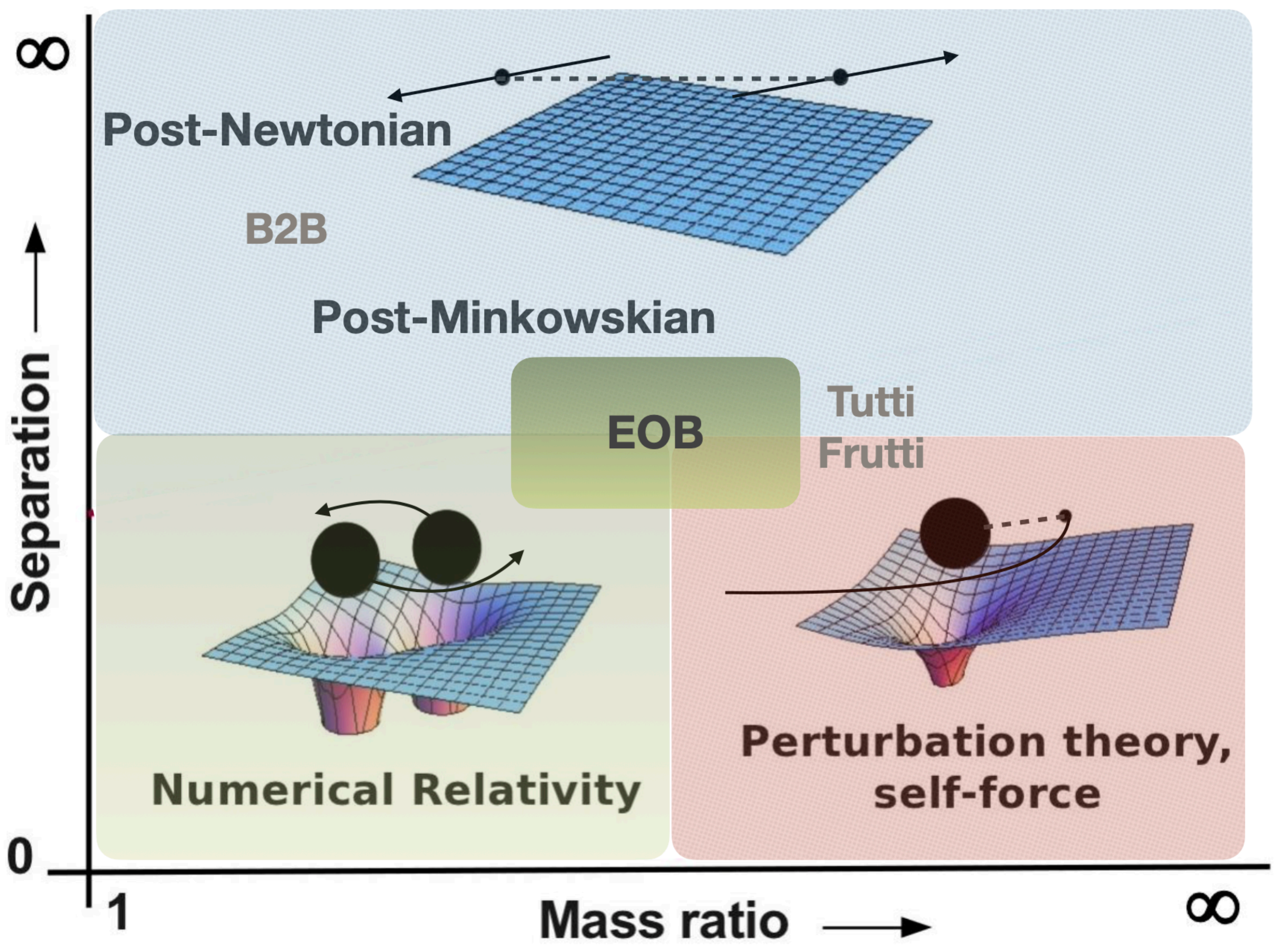
G.B, M.K. Mandal, P. Mastrolia, R. Patil, (in progress)

- Straightforward inclusion of Spin Effects, Tidal forces:

Porto (2013), Levi-Steinoff (2015) ….
Kim, Levi, Yin (2022),
Mandal, Mastrolia Patil, Steinhoff (2022)
Levi, Morales, Yin (2022), Levi, Yin (2022)
Mandal, Mastrolia Patil, Steinhoff. (2023)

Post-Minkowskian Expansion

$$G_N \frac{m}{r} \ll v^2 \sim 1$$



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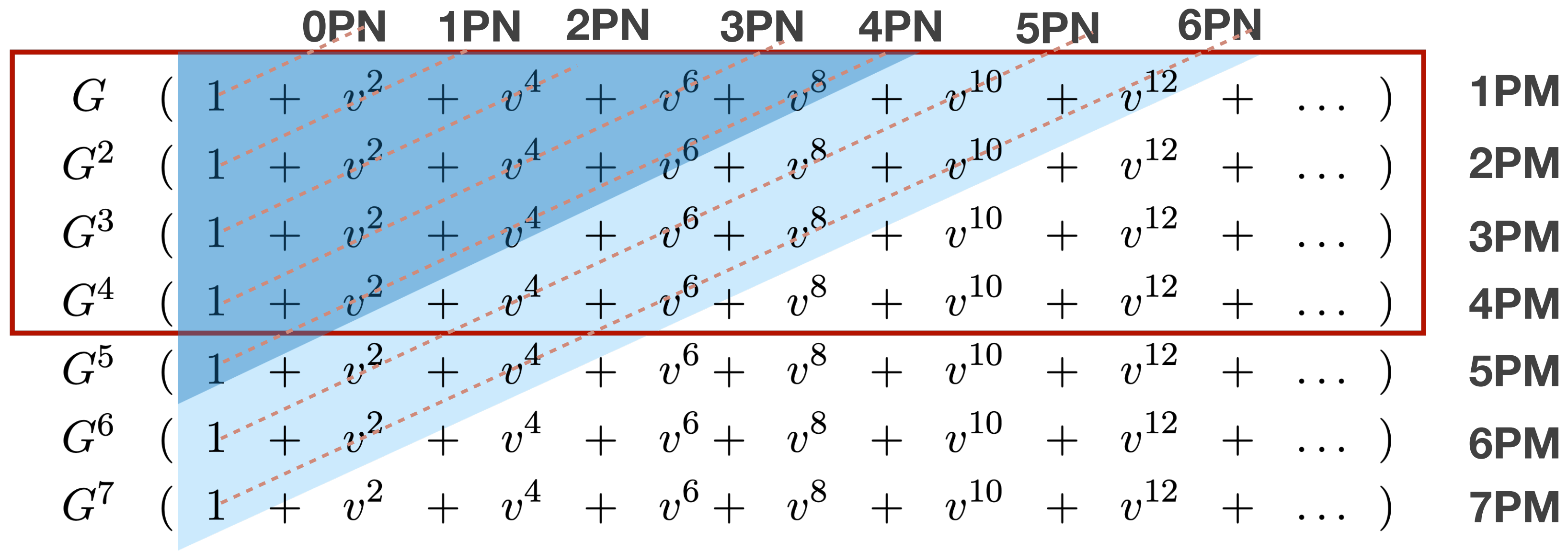

0PN 1PN 2PN 3PN 4PN 5PN 6PN

$$G \, (1 + v^2 + v^4 + v^6 + v^8 + v^{10} + v^{12} + \dots) \quad \text{1PM}$$
$$G^2 (1 + v^2 + v^4 + v^6 + v^8 + v^{10} + v^{12} + \dots) \quad \text{2PM}$$
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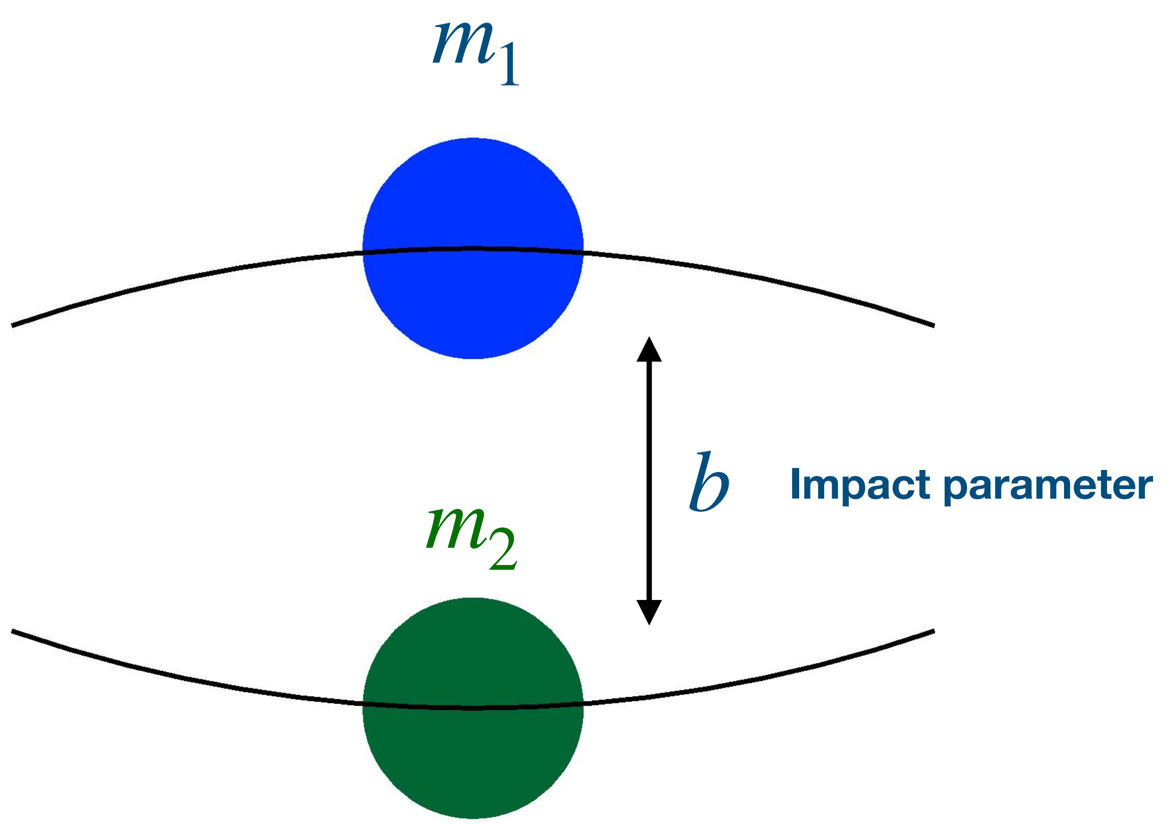
$$G^6 (1 + v^2 + v^4 + v^6 + v^8 + v^{10} + v^{12} + \dots) \quad \text{6PM}$$
$$G^7 (1 + v^2 + v^4 + v^6 + v^8 + v^{10} + v^{12} + \dots) \quad \text{7PM}$$

[credit: Bern et al.]

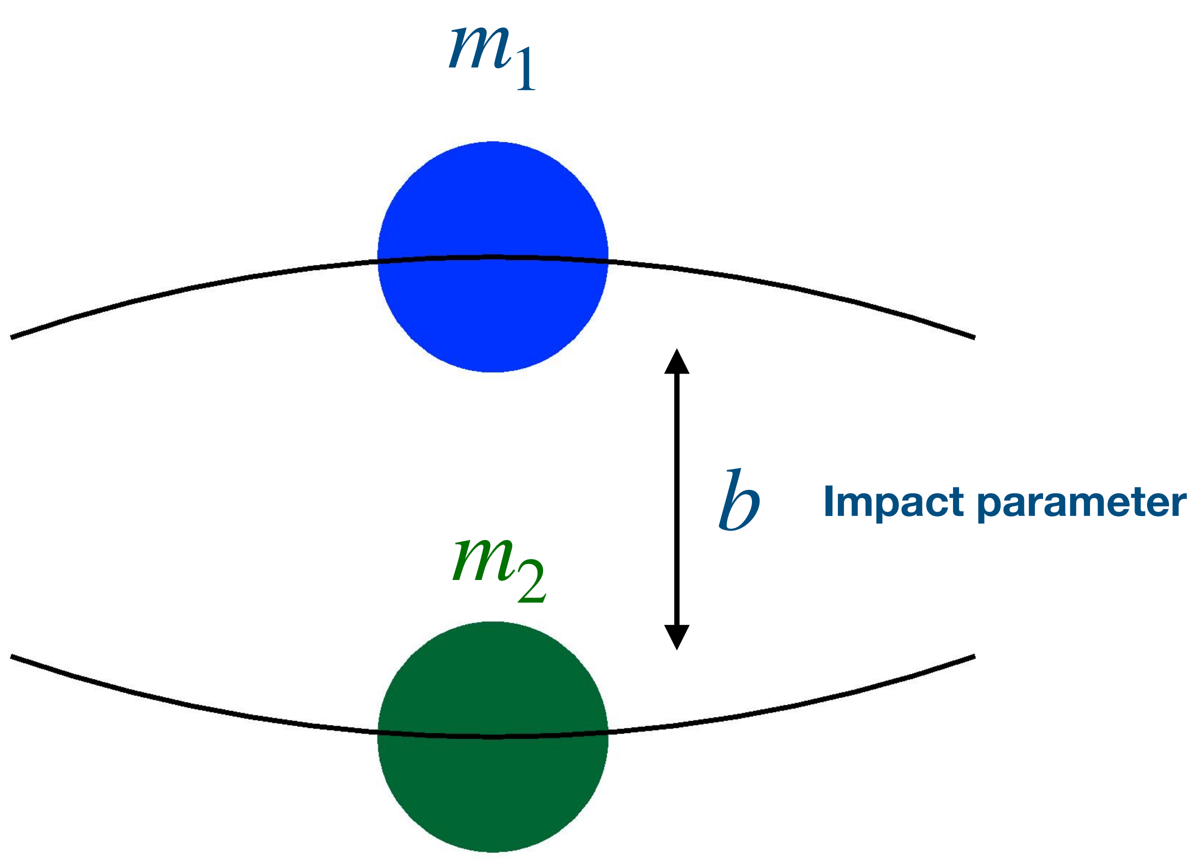
- Systematic study of scattering compact objects:
- Manifestly gauge-coordinate invariant
- Connection to the bound problem via analytic continuations.



- **2PM:** Iwasaki (1971)
  Westphal (1985)
- **3PM:** Bern, Cheung, Roiban, Shen, Solon, Zeng
  Parra-Martinez, Ruf, Zeng (2020)
  Kälin, Liu, Porto (2020)
  Bjerrum-Bohr, Damgaard, Planté and Vanhove (2021)
  Mogull, Plefka, Steinhoff(2021)
  P. Di Vecchia, C. Heissenberg, R. Russo and G. Veneziano
- **4PM:** Bern, Parra-Martinez, Robin, Ruf, Shen, Solon, Zeng (2021)
  Dlapa, Kälin, Liu, Porto (2022)
  Jakobsen, Moguls, Plefka, Sauer, Xu (2023)
  Jakobsen, Moguls, Plefka, Sauer, Xu (2023)
  Damgaard, Hansen, Plante, Vanhove(2023)

- Different approaches: amplitudes based, or worldlines based
- One can include spin effects, tidal effects
- Physical observables through:
  - Observable-based approach Kosower, Maybee, O'Connell
  - EFT matching Cheung, Solon, Rothstein
  - Eikonal approach Di Vecchia, Heissenberg, Russo, Veneziano
  - Radial Action Bern, Parra-Martinez, Roiban, Ruf, Shen, Solon, Zeng
  - Boundary-to-bound Dlapa, Kälin, Liu, Porto

Scales of the problem



- Black-hole scattering: $\frac{G_N m}{b} \ll 1$
- No quantum corrections: $G_N m^2 \gg \hbar$
- Chain of inequalities: $\frac{\hbar}{m} \ll G_N m \ll b$

Compton Wavelength

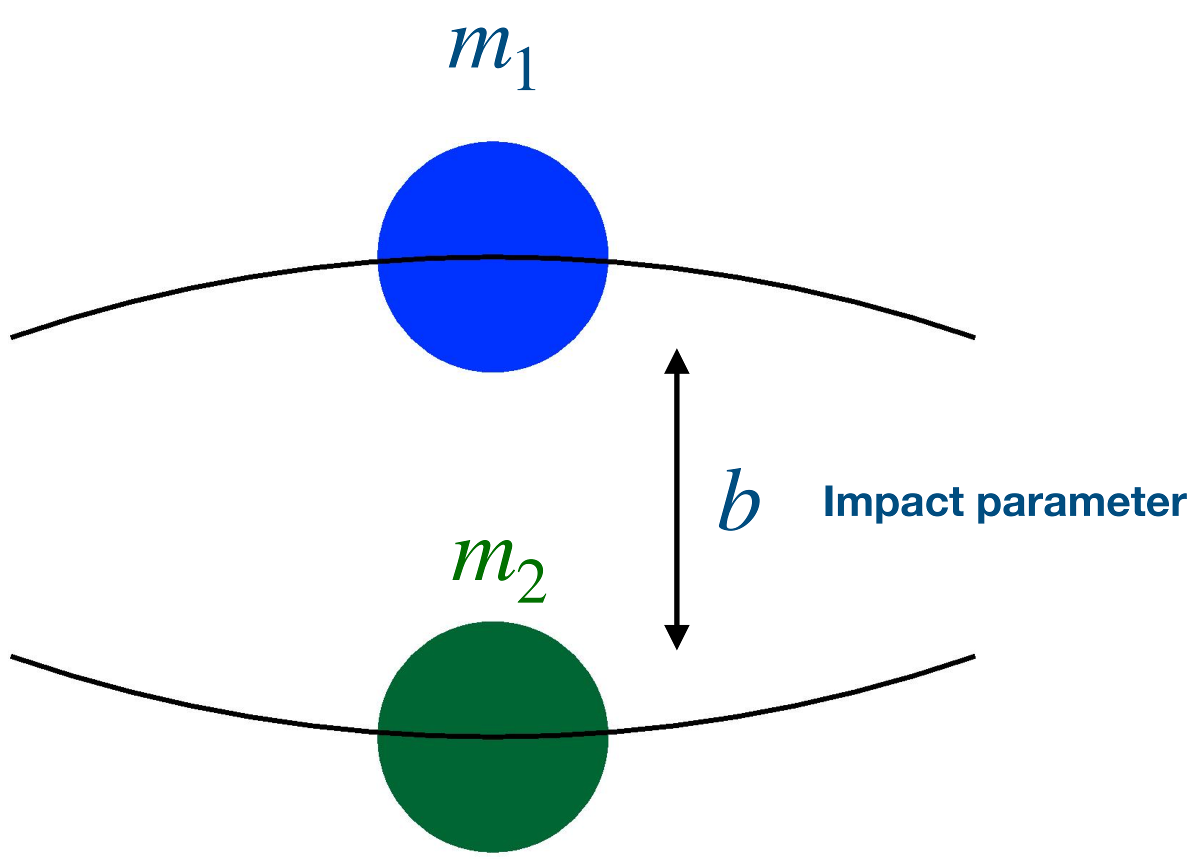
Schwarzschild Radius

Impact Parameter

- In terms of q Transfered Momentum $\frac{q}{m} \ll G_N m q \ll 1$

$b = \frac{\hbar}{q}$

Scales of the problem



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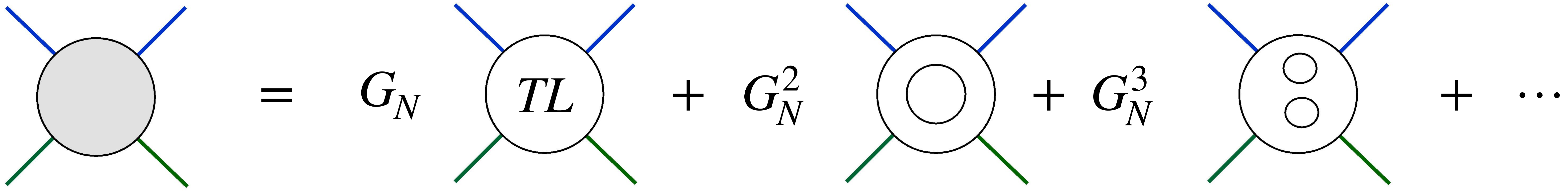
Compton Wavelength — Schwarzschild Radius — Impact Parameter

- In terms of q Transfered Momentum $\frac{q}{m} \ll G_N m q \ll 1$ $b = \frac{\hbar}{q}$
- Naively Black-hole scattering as a 4-point scattering amplitude, with scalar fields gravitationally interacting:



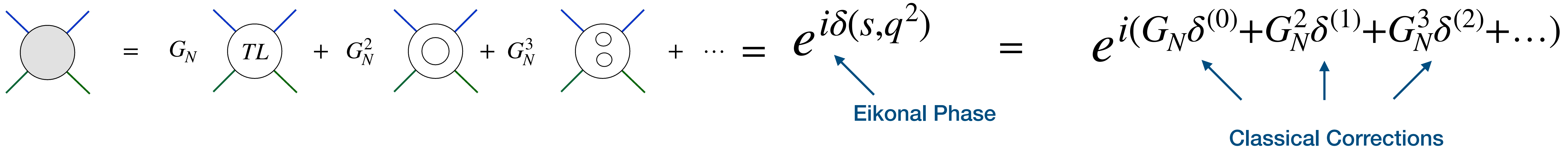
- Then we take the classical limit: $\frac{q}{m} \ll 1$ $q \ll 1$ $m \gg 1$
- But ordinary perturbation theory breaks down: $G_N m^2 \gg 1$

Exponential Representation of the S-matrix

[Di vecchia, Heisenberg, Russo, Veneziano]

[Damgaard, Plante, Vanhove]

- In the classical limit the amplitude exponentiates:

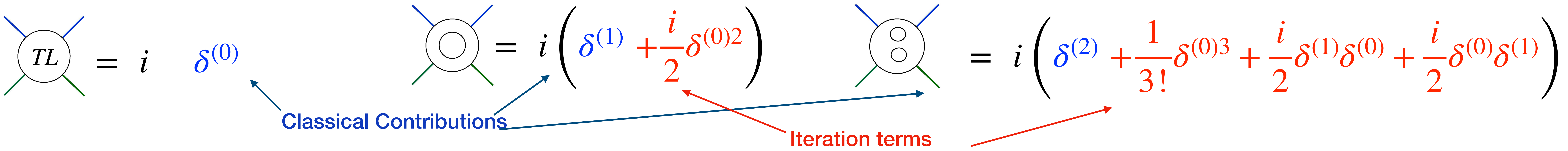


$$= G_N \quad + G_N^2 \quad + G_N^3 \quad + \cdots = e^{i\delta(s,q^2)} = e^{i(G_N\delta^{(0)}+G_N^2\delta^{(1)}+G_N^3\delta^{(2)}+\dots)}$$

Eikonal Phase

Classical Corrections

- Expanding both sides and matching the G_N orders:



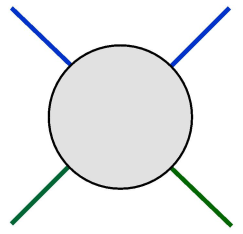
$$= i \quad \delta^{(0)}$$

$$= i\left(\delta^{(1)} + \frac{i}{2}\delta^{(0)2}\right)$$

$$= i\left(\delta^{(2)} + \frac{1}{3!}\delta^{(0)3} + \frac{i}{2}\delta^{(1)}\delta^{(0)} + \frac{i}{2}\delta^{(0)}\delta^{(1)}\right)$$

Classical Contributions

Iteration terms

- Prescription: Recover $\delta^{(i)}$ computing  and subtracting iterations:

Exponential Representation of the S-matrix

[Di vecchia, Heisenberg, Russo, Veneziano]
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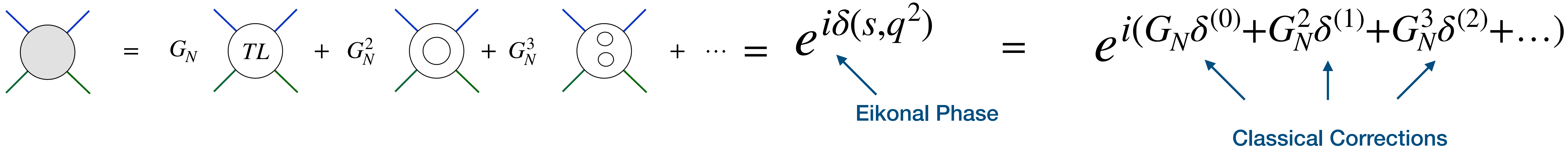

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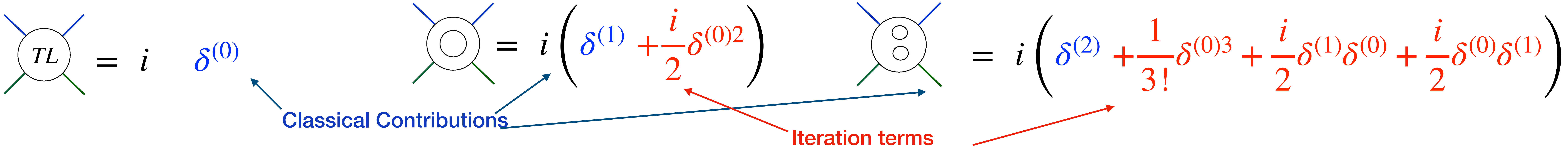

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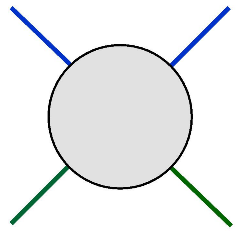
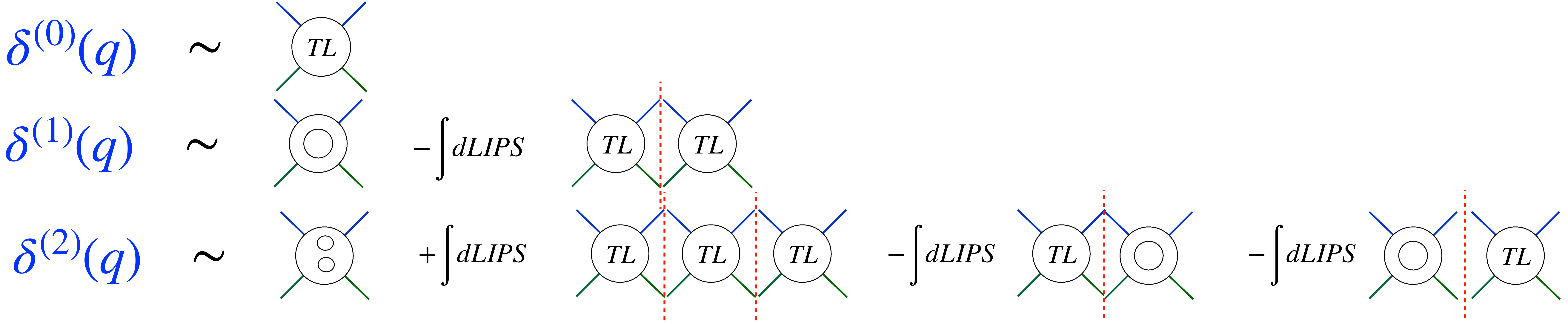




$$\delta^{(0)}(q) \sim$$

$$\delta^{(1)}(q) \sim \qquad - \int dLIPS$$

$$\delta^{(2)}(q) \sim \qquad + \int dLIPS \qquad - \int dLIPS \qquad - \int dLIPS$$

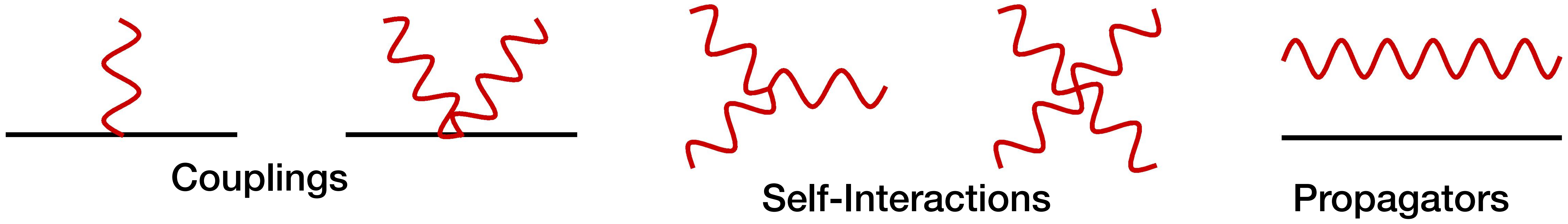
[see Claudio's talk]

Warmup: 1PM Computation:

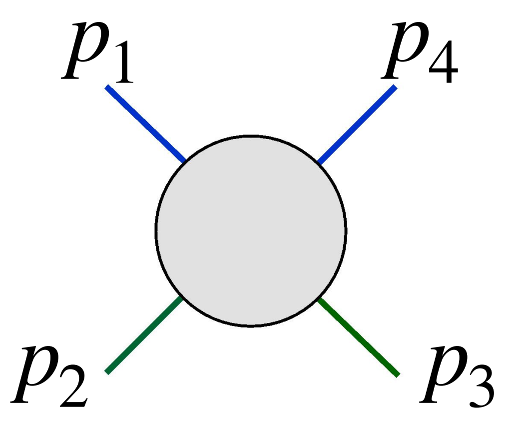
[G.B. Master Thesis]

- **Action of the theory**: $S_{TOT} \; = \; S_{EH} + S_{GF} + S_\phi$ $\quad S_{EH} = \frac{1}{16\pi G_d}\int d^dx\sqrt{-g}\; R$ $\quad S_\phi = -\frac{1}{2}\int d^dx\sqrt{-g}(\partial_\mu\phi\partial^\mu\phi - m^2\phi^2)$ $\quad d = 4 - 2\epsilon$

- Expand $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ to get the **Feynman Rules**:



- **Kinematics**



$$p_1^2 = p_4^2 = m_1^2 \qquad s = (p_1 + p_2)^2$$

$$p_2^2 = p_3^2 = m_2^2 \qquad t = (p_1 - p_4)^2 = q^2$$

Transfered Momentum

$$\gamma = \frac{p_1 \cdot p_2}{m_1 m_2}$$

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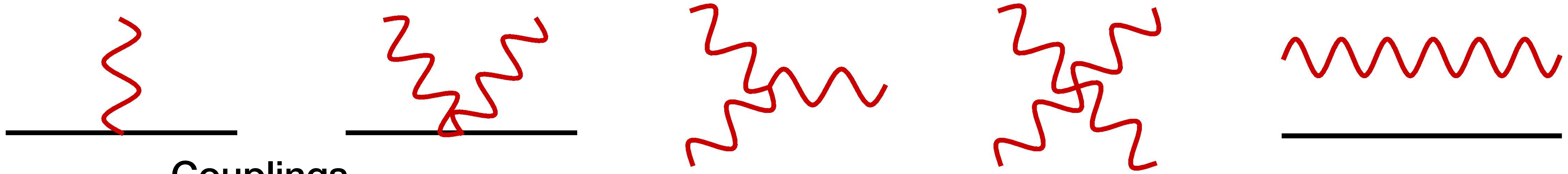
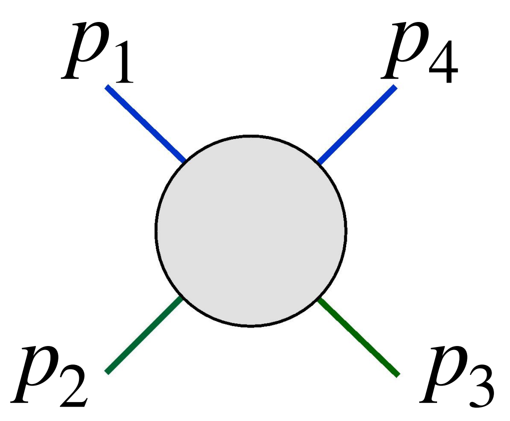


Couplings

Self-Interactions

Propagators

- **Kinematics**



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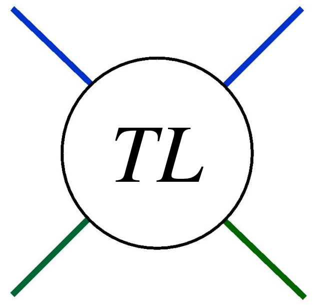
Transfered Momentum

$$\gamma = \frac{p_1 \cdot p_2}{m_1 m_2}$$

- **Tree level calculation:** $\delta^{(0)} =$



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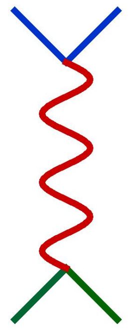
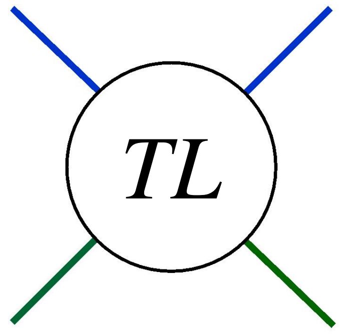


$$= \frac{16\pi G}{q^2(\epsilon-1)} m_1 m_2\big(q^2 y(\epsilon-1) + m_1 m_2(2y^2(\epsilon-1)-1)\big)$$
$$= \frac{16\pi G}{q^2(\epsilon-1)} m_1^2 m_2^2\big(2y^2(\epsilon-1)-1\big) + \mathcal{O}(q^0)$$

Classical Limit

- **FT to Impact parameter space:** $\delta^{(0)}(\mathbf{b}) = \frac{1}{4m_1m_2\sqrt{y^2-1}}\int \frac{d^{d-2}}{(2\pi)^{d-2}} e^{-iq\cdot b}$



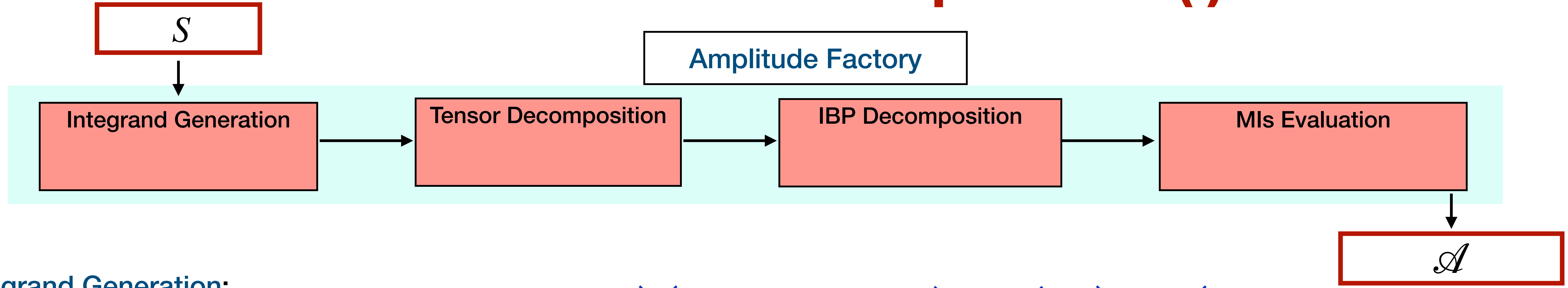
$$= -\frac{G_N J^{2\epsilon}}{\epsilon} m_1 m_2 \frac{2y^2-1}{\sqrt{y^2-1}}$$

- **Scattering Angle:**

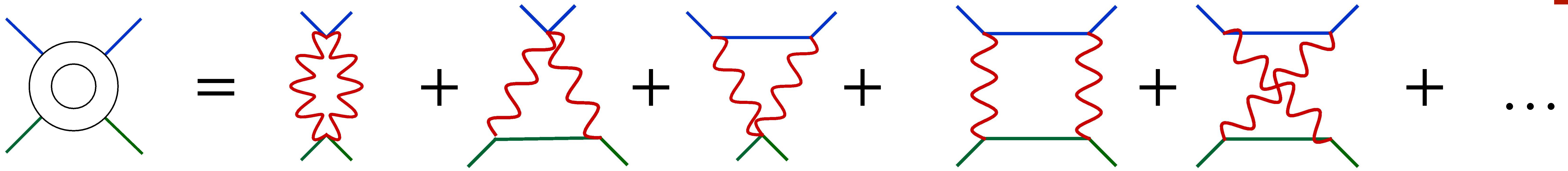
$$\boxed{\chi = -\frac{\partial}{\partial J} Re[\delta^{(0)}(b)] = \frac{G_N}{J}\frac{2m_1m_2(2y^2-1)}{\sqrt{y^2-1}}}$$

2PM: Traditional Computation (I):

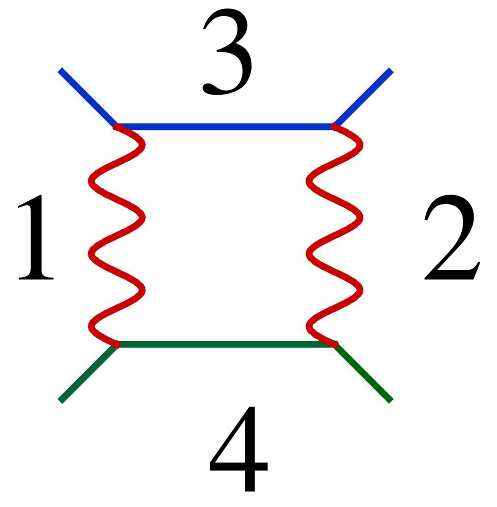
[G.B. Master Thesis]



- **Integrand Generation**:



- **IBP Decomposition**:
  - 1-loop scalar integrals belong to the same family of Feynman Integrals:



$$I_{a_1a_2a_3a_4} = \int \frac{d^d\ell}{(2\pi)^d}\frac{1}{D_1^{a_1}D_2^{a_2}D_3^{a_3}D_4^{a_4}}$$

Loop momentum

Denominators

$$D_1 = \ell^2 \qquad D_3 = (\ell + p_1)^2 - m_1^2$$
$$D_2 = (\ell - q)^2 \qquad D_4 = (\ell - p_2)^2 - m_2^2$$

  - Hundreds of integrals $I_{a_1a_2a_3a_4}$ appearing, but they are not all independent

2PM: Traditional Computation (II):

[G.B. Master Thesis]

- **IBP identities**:

  - Integrals of the same family are related by IBP identities.

Kotikov, Remiddi, Gehrmann; Laporta,...

$$\int \frac{d^d\ell}{(2\pi)^d}\frac{\partial}{\partial\ell^\mu}\left(\frac{v^\mu}{D_1^{a_1}D_2^{a_2}D_3^{a_3}D_4^{a_4}}\right)=0 \quad \Rightarrow \quad \sum_i c_i I_i = 0$$

Linear relation

$$v^\mu = v^\mu(\ell, p_i, q)$$

  - One can generate thousands of identities and solve them by Gauss elimination

  - Every integral can be decomposed in terms of a finite basis of master integrals $\{\mathcal{I}_i\}_{i=1}^{\nu}$:

$$I = \sum_i c_i \, \mathcal{I}_i^{MI}$$

Coefficients

Master Integrals

2PM: Traditional Computation (II):

[G.B. Master Thesis]

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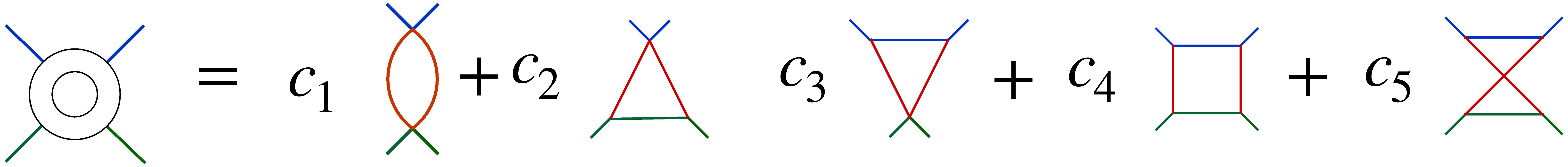
Coefficients

Master Integrals

- **IBP Decomposition**:
  - 1-loop amplitude decomposed in terms of 5 Master Integrals (MIs):



  - How can we evaluate these integrals?

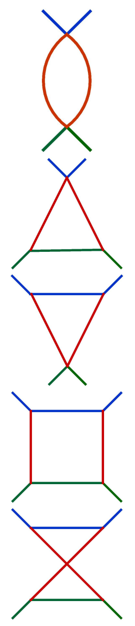
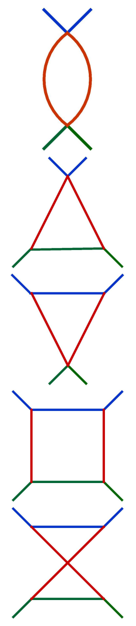
2PM: Traditional Computation (III):

- **MIs evaluation:**
  - The derivative of a MI w.r.t. a kinematic variable can be decomposed in MIs:

$x \in \{s, t, m_1, m_2\}$

$$\partial_x \mathcal{I}_i^{MI} = \sum_j c_j \mathcal{I}_j^{MI}$$

  - One can derive homogeneous DEQs for MIs:

$\partial_x \Bigg($  $\Bigg) = \mathscr{A}_x \Bigg($  $\Bigg)$

←

Can be solved perturbatively

Henn,
Argeri, Di Vita, Mastrolia, Mirabella, Schlenk,
,…

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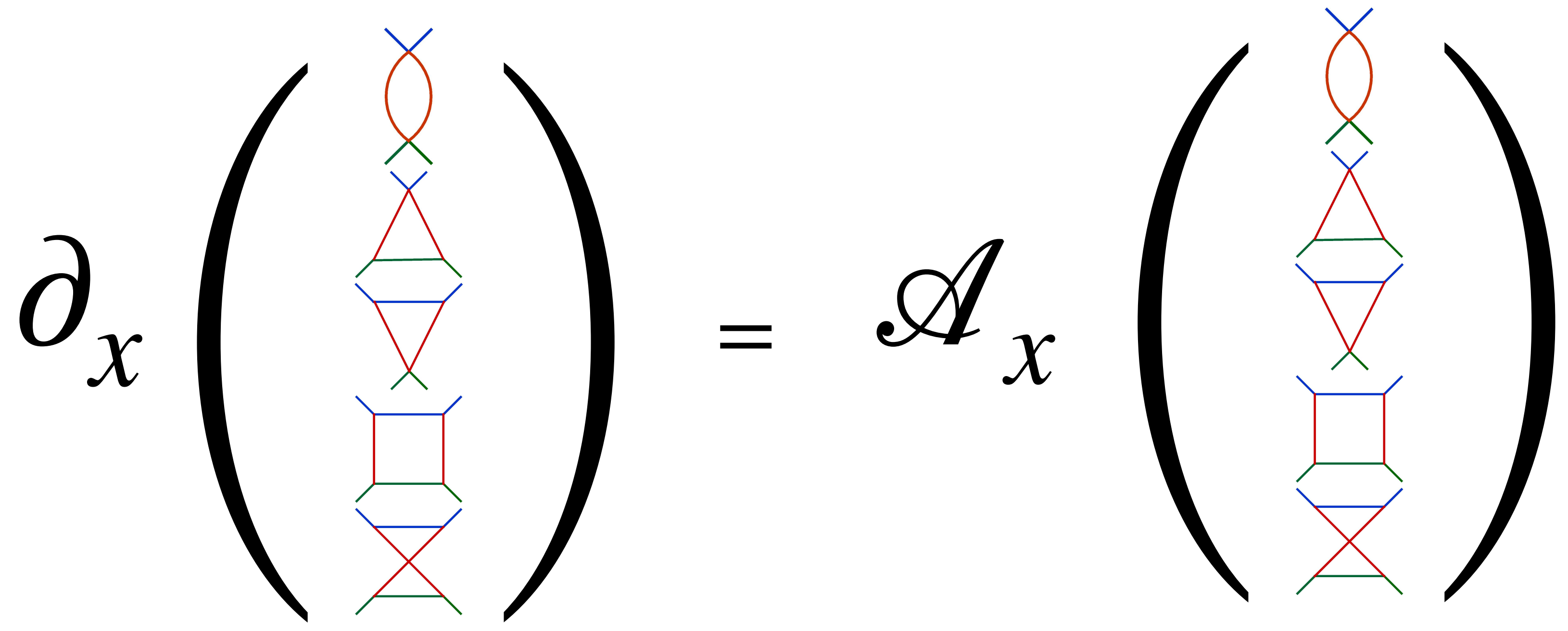


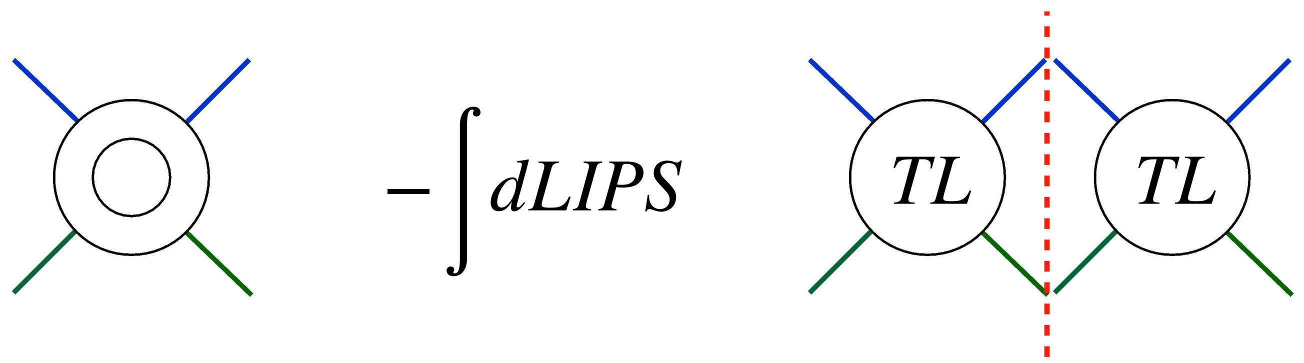
Can be solved perturbatively

Henn,
Argeri, Di Vita, Mastrolia, Mirabella, Schlenk,
,…

- **Classical limit:** $\mathscr{A}(s, t, m_1, m_2) \xrightarrow{q \ll 1} \mathscr{A}(y, q, m_1, m_2)$

- **Subtraction of iteration term:**

$$\delta^{(1)}(q) =$$



$$= G_N^2 m_1^2 m_2^2 (m_1 + m_2) \frac{6\pi^2(5y^2 - 1)}{\sqrt{-q^2}}$$

- **Scattering Angle**

$$\boxed{\chi^{(1)} = \frac{G_N^2}{J^2} \frac{3\pi}{4\sqrt{s}} m_1^2 m_2^2 (m_1 + m_2)(5y^2 - 1)}$$

Simple Result

Optimisations

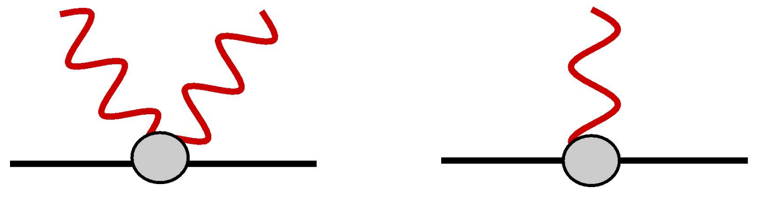
- **Bottlenecks:** ○ Many diagrams ○ Difficult reduction ○ Difficult Integrals ○ Classical Limit ○ Iteration terms

- **Solution:** **Take the classical limit as early as possible in the computation**

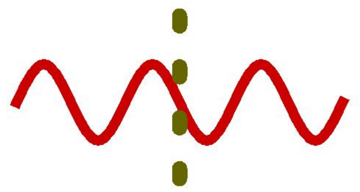
○ **Generalised Unitarity for Integrand Generation:**

Bern, Dixon, Dunbar Kosower, Britto Cachazo, Feng, Mastrolia, Forde, Travaglini, Buchbinder, Anastasiou, Badger, Bjerrum-Bohr, Ossola, Papadopoulous, Pittau, …

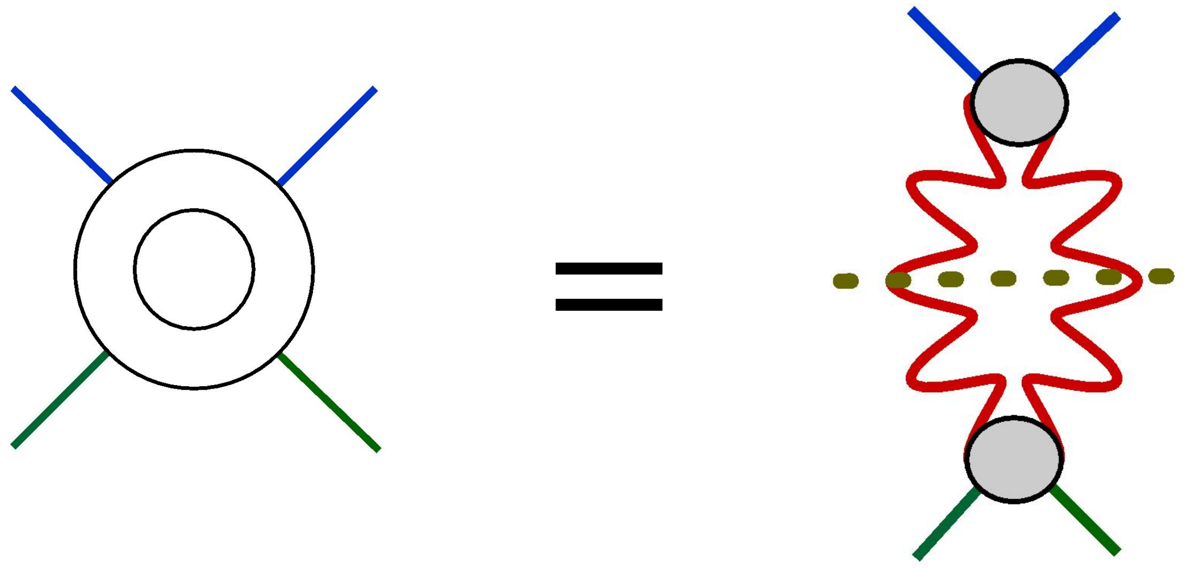
On-shell tree level amplitudes:



Cuts:



$$\frac{1}{D_i} \to \delta(D_i) \quad \Rightarrow$$



✓Only few cuts

○ **Method of regions for expansion under the integral sign** [Beneke, Smirnov]

Hard region: $\ell \sim \mathcal{O}(m) \sim \mathcal{O}(p_i)$

Soft region: $\ell \sim \mathcal{O}(q)$

Classical Physics ⇔ Soft region

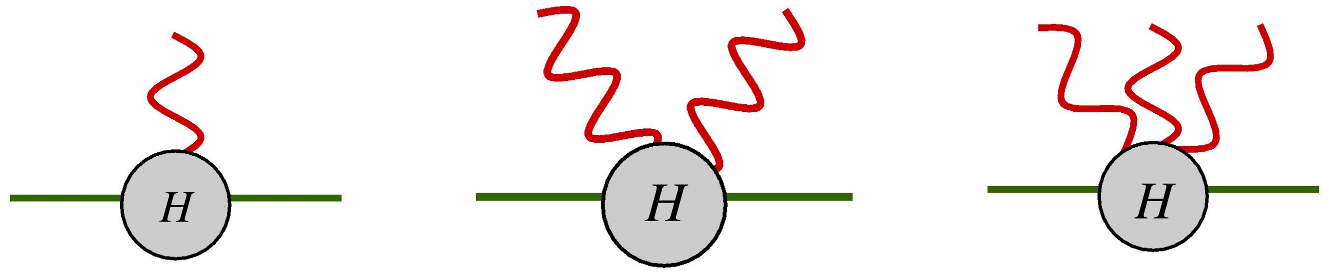
[Bern, Cheung, Roiban, Shen, Solon, Zeng]

✓Only classical physics

✓ Easy reduction to Master

✓Only single scale integrals

○ **Effective Field Theory Approach (HEFT) use tree level amplitude for computing only classical physics**

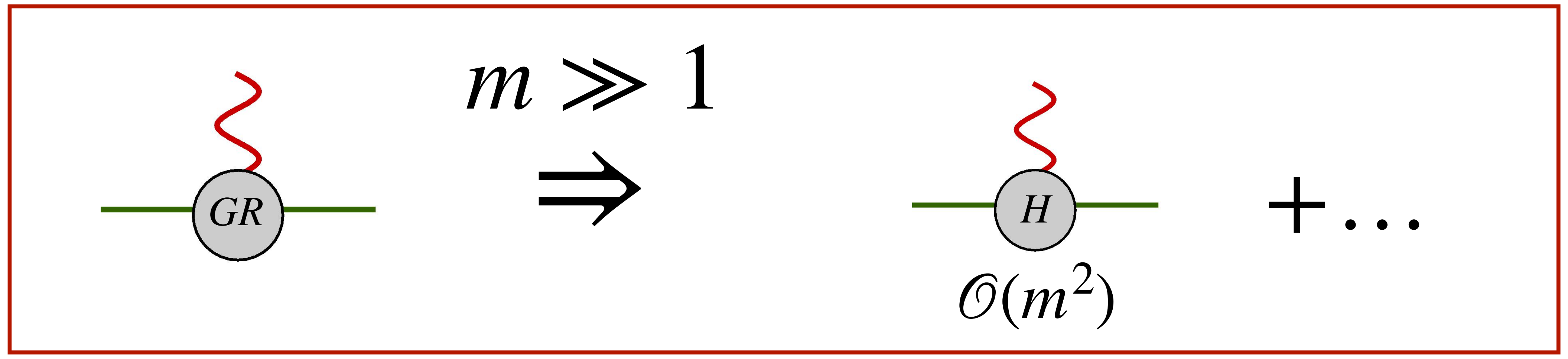


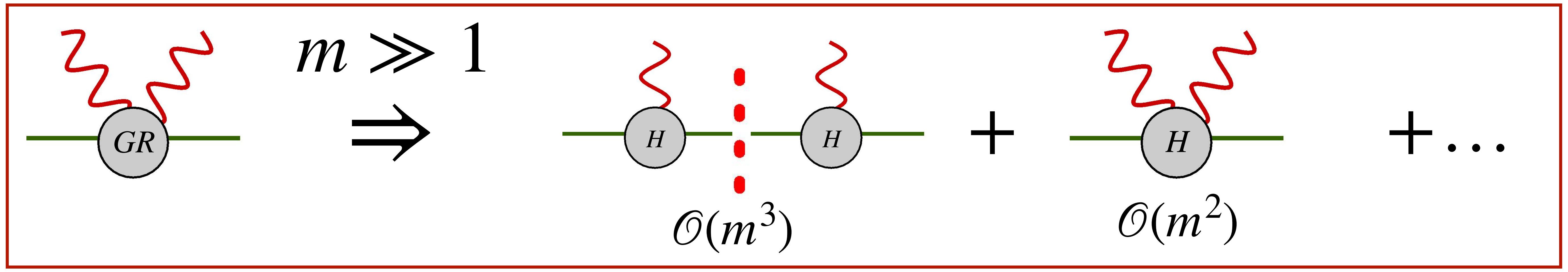
[Damgaard, Haddad, Helset]
[Aoude, Haddad, Helset]
[Brandhuber, Chen, Travaglini, Wen]

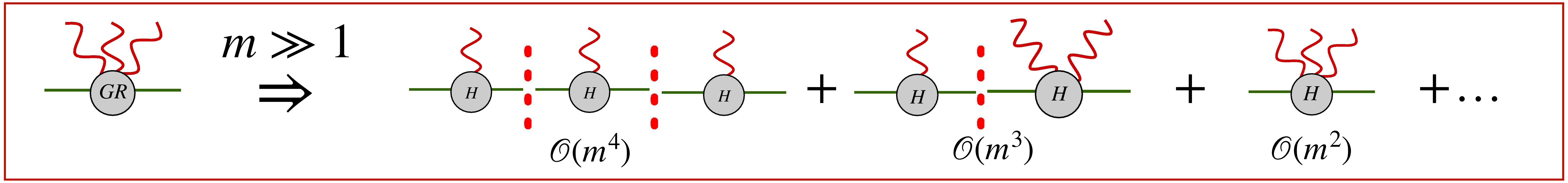
HEFT (Heavy Mass Effective Theory)

[Damgaard, Haddad, Helset]
[Aoude, Haddad, Helset]
[Brandhuber, Chen, Travaglini, Wen]

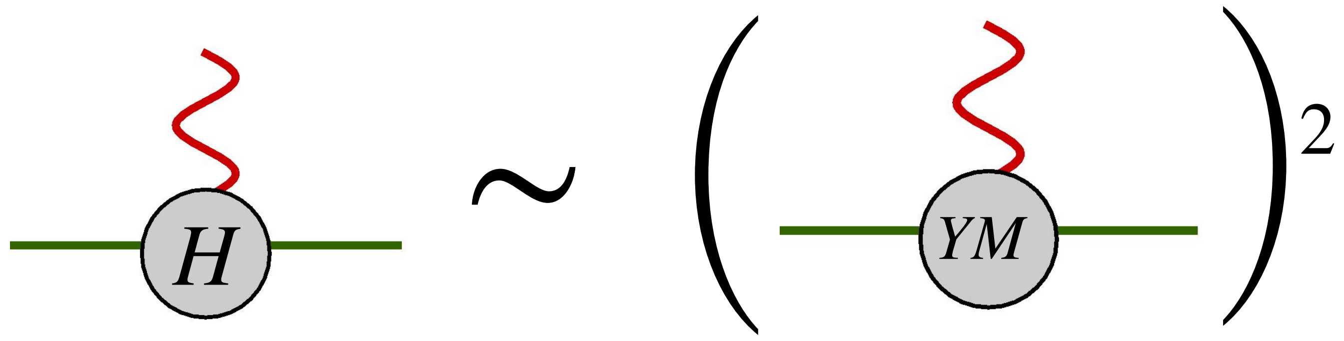
- Classical tree level amplitudes by heavy mass expanding quantum amplitudes:

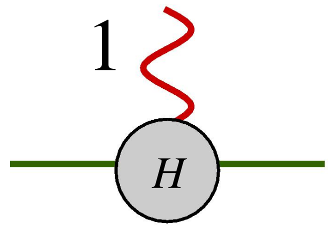




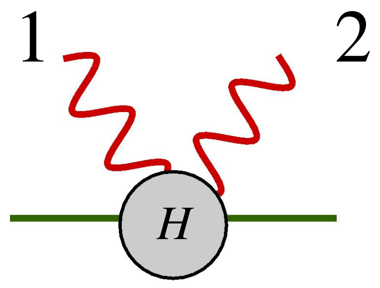


- These amplitudes obey a **double-copy structure:**

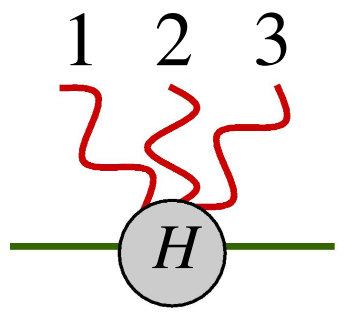




$\sim (\mathcal{N}_3^{YM}[1,v])^2$



$$\sim \frac{(\mathcal{N}_4^{YM}[[1,2],v])^2}{s_{12}}$$



$$\sim \frac{(\mathcal{N}_5^{YM}[[[1,2],3],v])^2}{s_{123}s_{12}} + \frac{(\mathcal{N}_5^{YM}[[[1,3],2],v])^2}{s_{123}s_{13}} + + \frac{(\mathcal{N}_5^{YM}[[[2,3],1],v])^2}{s_{123}s_{23}}$$

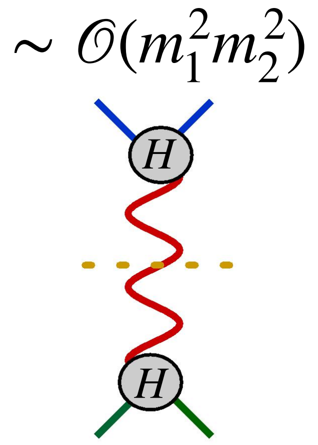
- Generate directly classical amplitudes
- No need for iteration terms
- Manifestly gauge invariant

Classical Phases from HEFT

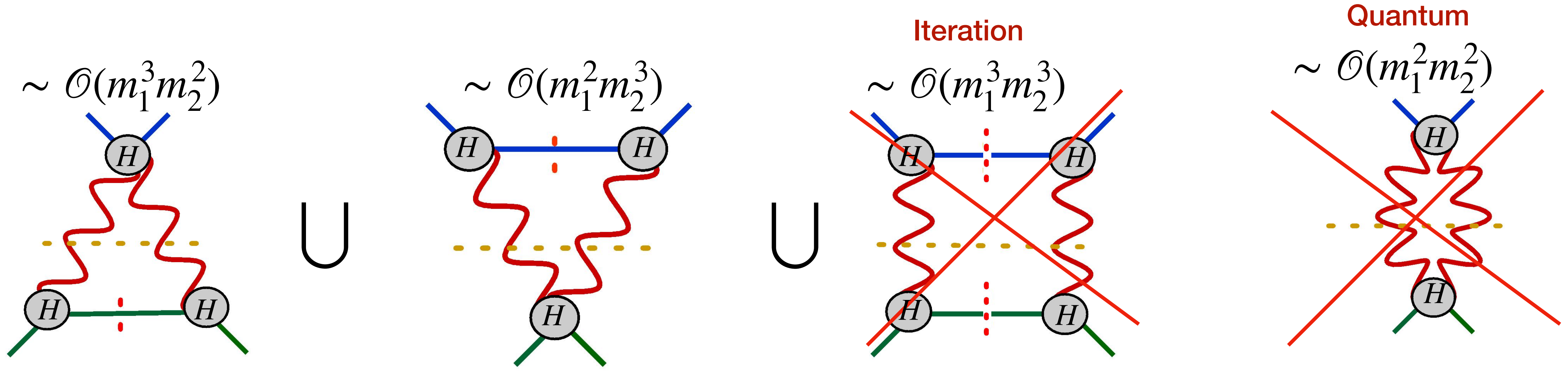
[Brandhuber, Chen, Travaglini, Wen]

- Classical Phases are given by 2MPI amplitudes
- We can eliminate iteration terms and quantum terms at the diagrammatic level

$\delta^{(0)}(q) =$

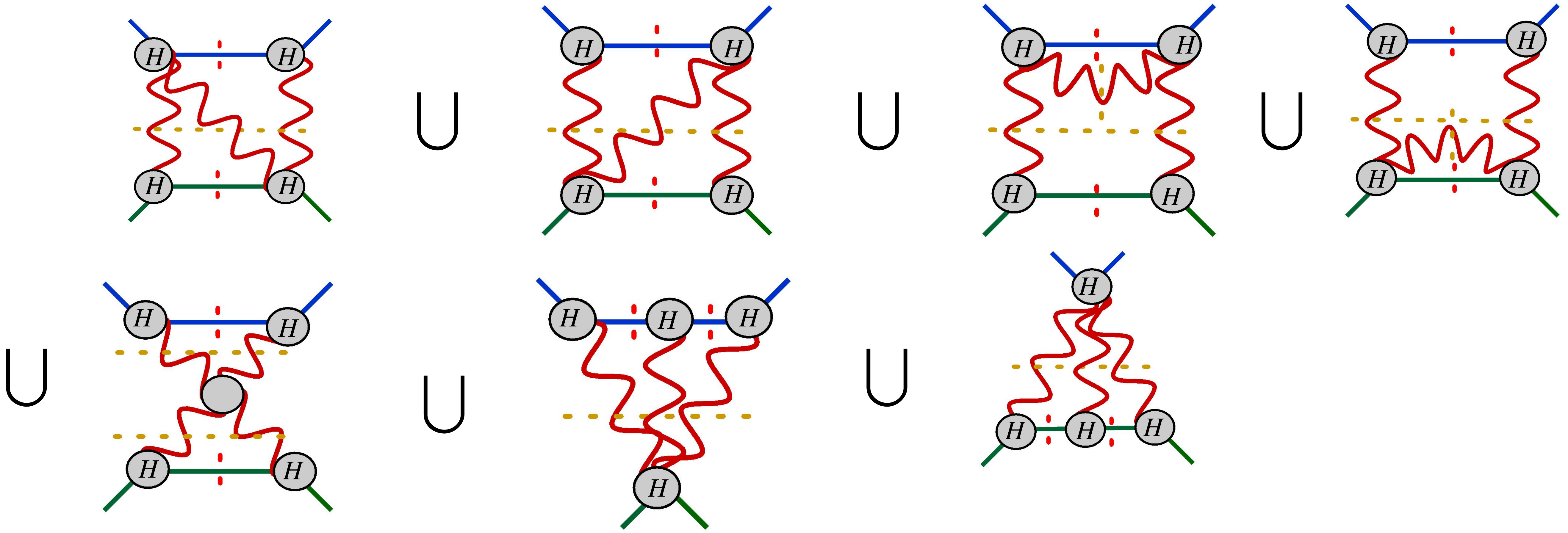


$\delta^{(1)}(q) =$



- Easy to generalise at higher order

$\delta^{(2)}(q) =$



- Easy integrand generations
- Integrals are single-scale (easy to compute using differential equations)
- Explicit power counting rules

- Straightforward to compute the scattering angle:

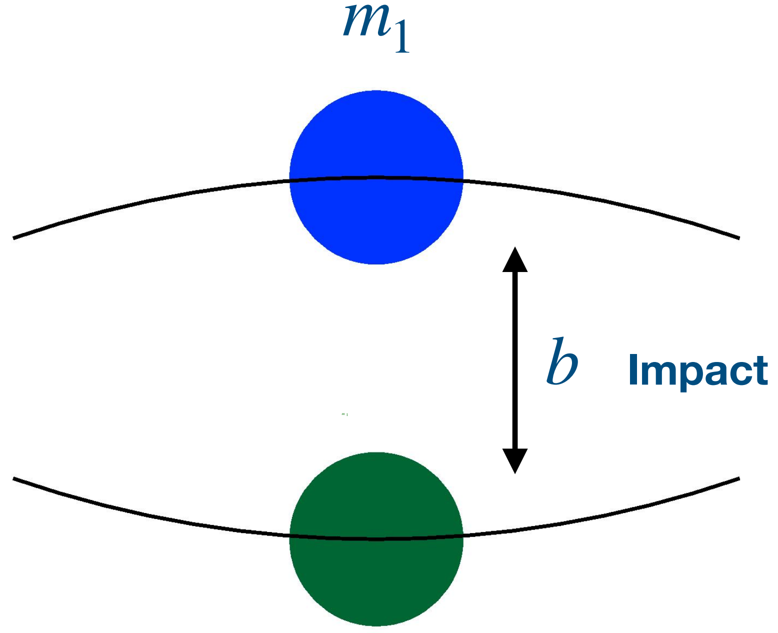
$$\chi = -\frac{\partial}{\partial J} Re[\delta(\mathbf{b})] \qquad \delta(\mathbf{b}) \sim \int \frac{d^{d-2}q}{(2\pi)^{d-2}} e^{iq\cdot b}\delta(q)$$

Angular Momentum

Fourier Transform

Observable-based approach

[Kosower, Maybee, O'Connell, (2018)]



- GWs observables can be computed from amplitudes via an Observable-based approach.
- Consider well defined asymptotic states:

$$\boxed{|\psi\rangle = \int d\phi(p_1p_2) \qquad \phi_1(p_1)\phi_2(p_2)e^{ib\cdot p_1} \qquad |p_1p_2\rangle}$$

On-shell phase space integral

Wavepacket describing particles. Localised with small uncertainty w.r.t masses of the problem

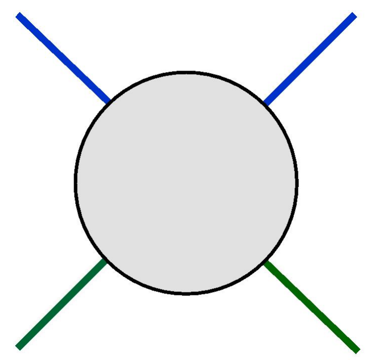
Two different quanta of masses m_1 m_2

- Expectation value of a well defined observable:

$$\Delta P_1^\mu = {}_{out}\langle\psi|\mathbb{P}_1^\mu|\psi\rangle_{out} - {}_{in}\langle\psi|\mathbb{P}_1^\mu|\psi\rangle_{in} \qquad\qquad |\psi\rangle_{out} = S|\psi\rangle_{in} \qquad S = 1 + i\ T$$

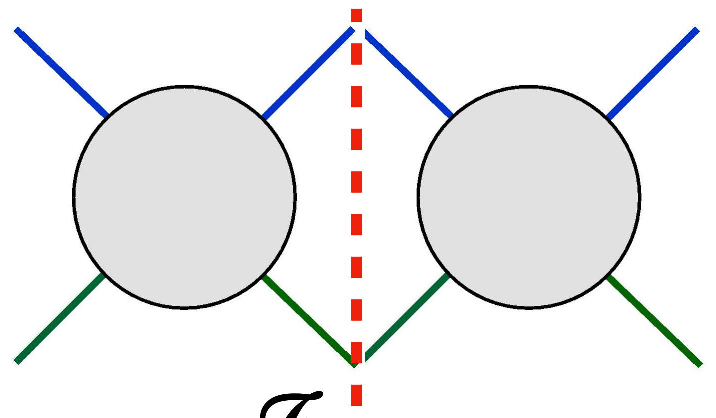
$$= {}_{in}\langle\psi|[i\ \mathbb{P}_1^\mu, T]|\psi\rangle_{in} + {}_{in}\langle\psi|T^\dagger[\mathbb{P}_1^\mu, T]|\psi\rangle_{in}$$

$$= \int \hat{d}q\,\hat{\delta}(2p_1\cdot q)\hat{\delta}(-2p_2\cdot q)e^{-ib\cdot q}\; i\Bigg(\; q^\mu$$



$\mathcal{J}_v$

$$-i\int d(LIPS)l_1^\mu$$



$\mathcal{J}_r$

$$\Bigg)$$

Fourier Transform

Amplitude

Iterations

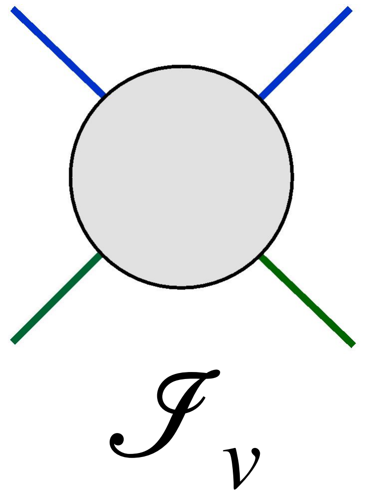
- Can we apply the same techniques also here? Yes

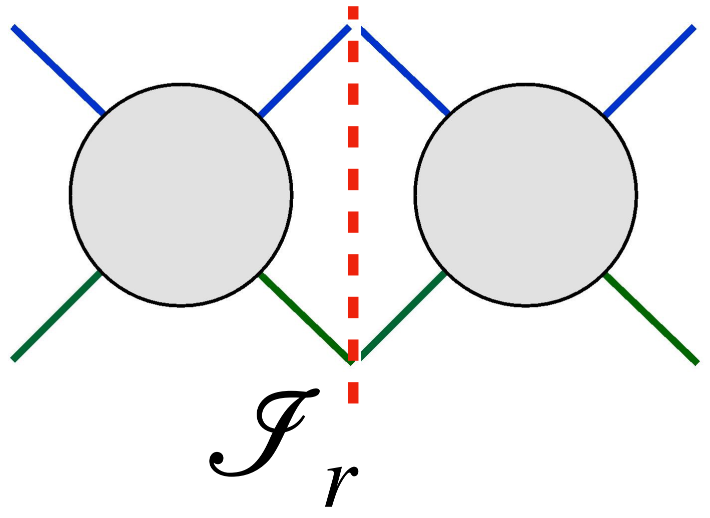
[Herrmann, Parra-Martinez, Ruf, Zeng (2021)]

Impulse from HEFT

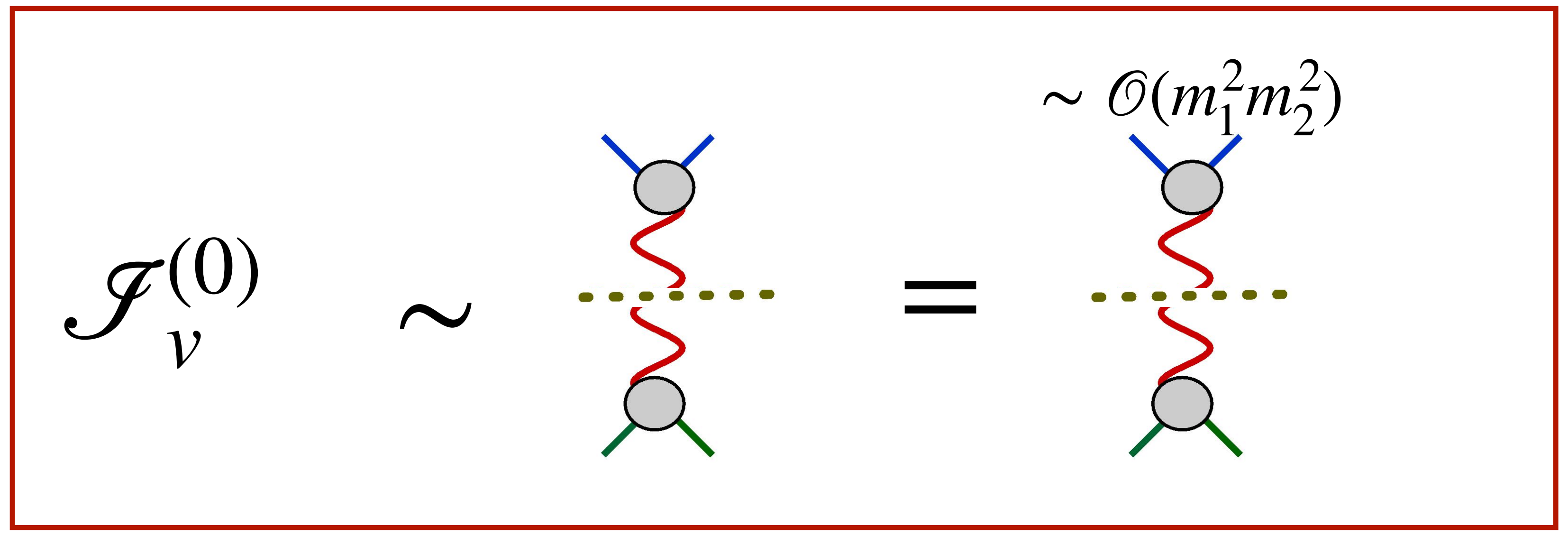
- Classical impulse can be evaluated from HEFT:

$$\Delta P_1^\mu = \int \hat{d}q\,\hat{\delta}(2\bar{p}_1\cdot q)\hat{\delta}(-2\bar{p}_2\cdot q)e^{-ib\cdot q}i\left(q^\mu \quad \mathcal{J}_v \quad -i\int d(LIPS)l_1^\mu \quad \mathcal{J}_r \right)$$

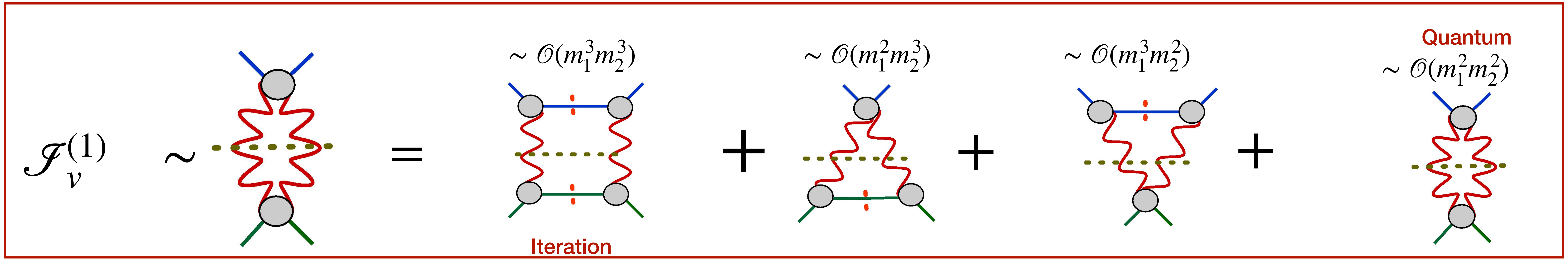


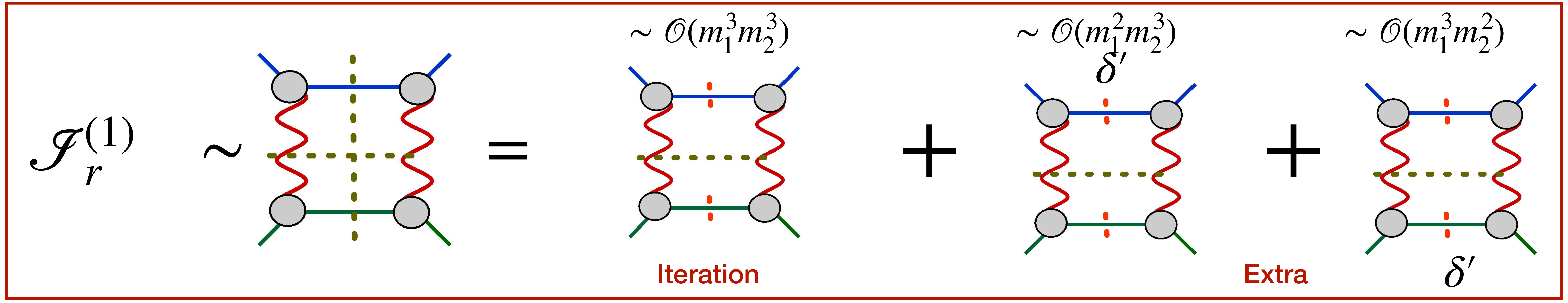


- At 1PM only the virtual contribution appears:



- At 2PM extra terms coming from the cut contribution:





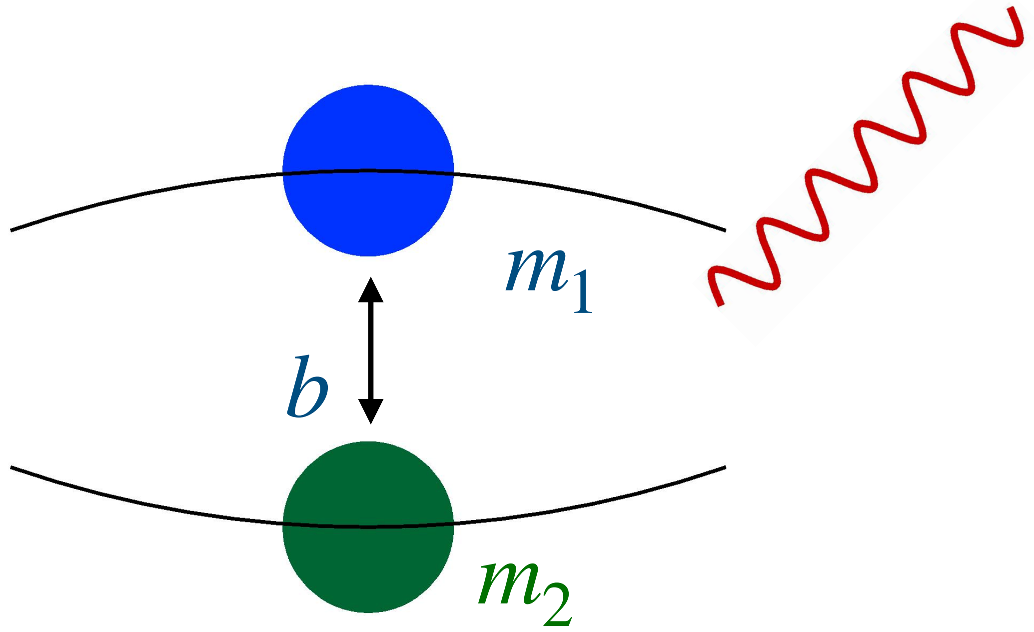
- HEFT tree level can reproduce the amplitude
- Cut contributions need to be computed
- KMOC kernel different from classical amplitude

[Caron-Huot, Giroux, Hannesdottir, Mizera (2023)]

[Cristofoli, Gonzo, Kosower, O'Connell (2021)]

Gravitational Waveform from Amplitudes

- Local observables can be computed from amplitudes, such as the gravitational waveform.



- Expectation value of the Riemann tensor $R_{\mu\nu\rho\sigma}(x)$, or equivalently of the the graviton field $h_{\mu\nu}(x)$:

$$\langle h_{\mu\nu}(x)\rangle = {}_{out}\langle\psi|\mathbb{H}_{\mu\nu}(x)|\psi\rangle_{out} = \langle\psi|S^{\dagger}\mathbb{H}_{\mu\nu}(x)S|\psi\rangle$$

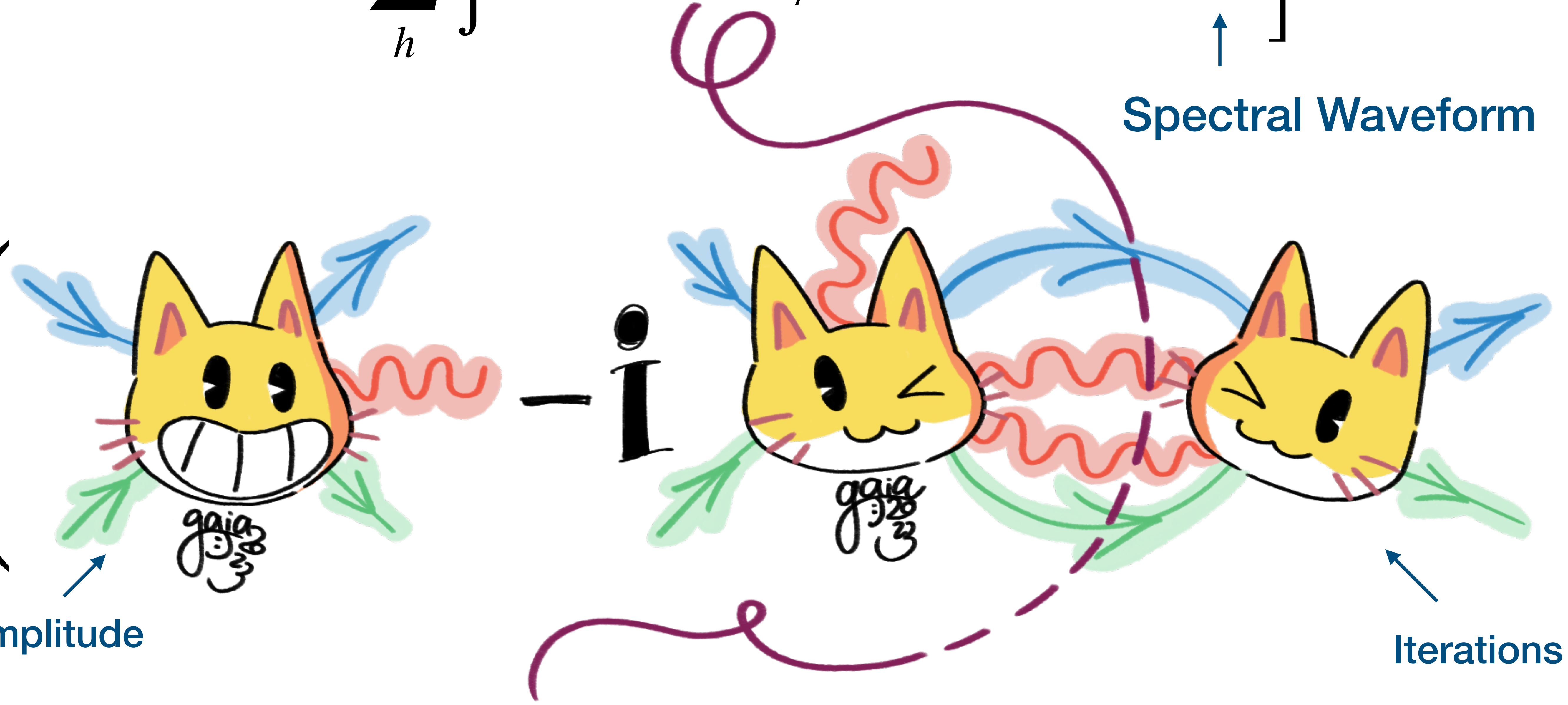
$$= 2\kappa Re\left[\int\prod_{j=1}^{2}\int d\Phi(p_j)|\phi(p_1)|^2|\phi(p_2)|^2\sum_{h}\int d\Phi(k)e^{-ik\cdot x}\epsilon_{\mu}^{(h)*}(\mathbf{k})\epsilon_{\nu}^{(h)*}(\mathbf{k})\ i\ \tilde{W}\right]$$

$|\psi\rangle_{out} = S|\psi\rangle_{in}$

$S = 1 + i\ T$

Spectral Waveform

$$\tilde{W} = \int d\mu^{(d)}e^{i(q_1\cdot b_1+q_2\cdot b_2)}\Big(\qquad - i \qquad \Big)$$



Fourier Transform

Amplitude

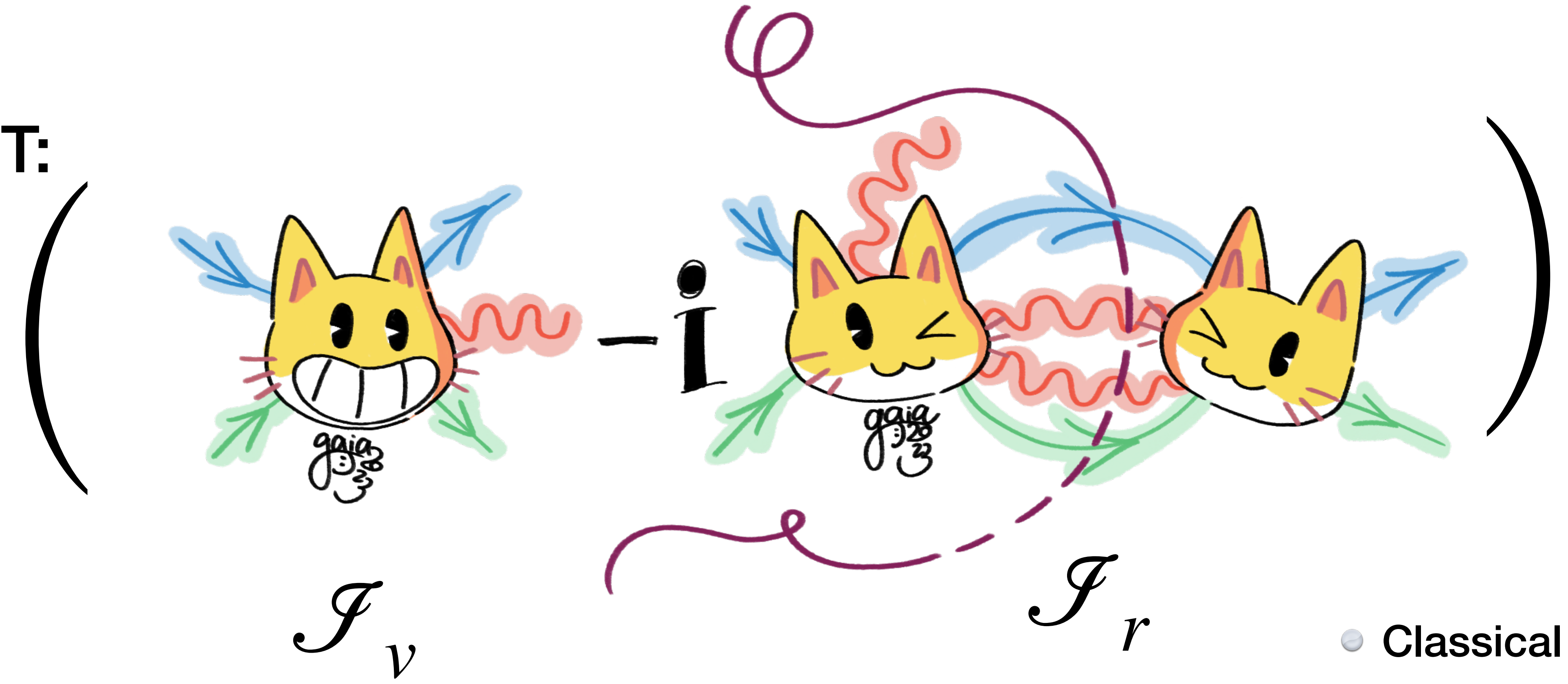
Iterations

$$d\mu^{(d)} = \frac{d^dq_1}{(2\pi)^{d-1}}\frac{d^dq_2}{(2\pi)^{d-1}}\delta(2\bar{p}_1\cdot q_1)\delta(2\bar{p}_2\cdot q_2)\delta^{(d)}(q_1+q_2-k)$$

Waveform from HEFT

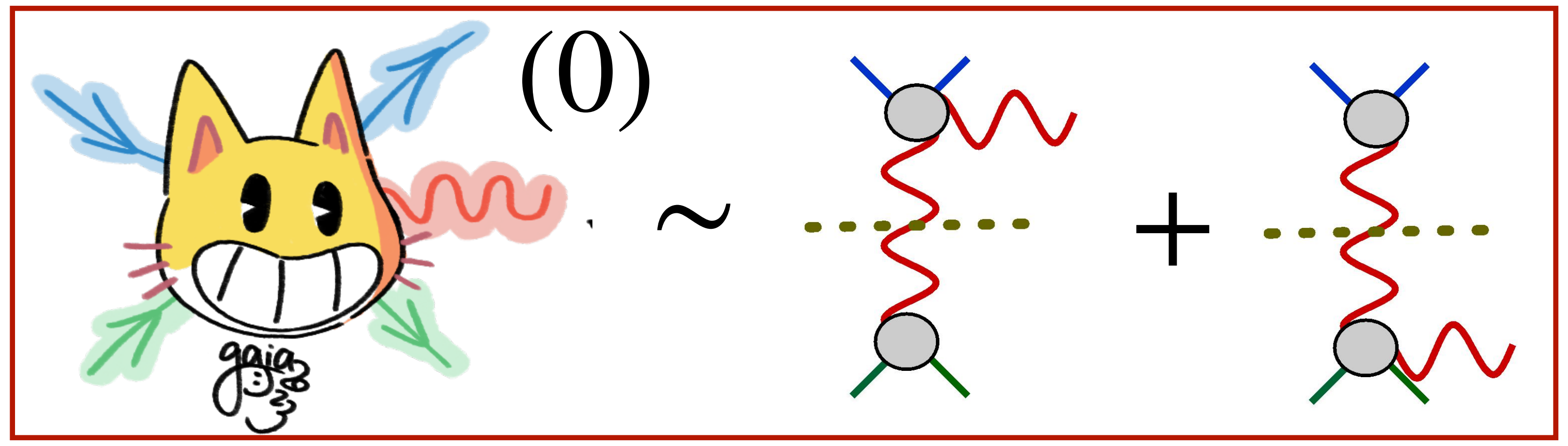
- The spectral waveform can be evaluated from HEFT:

$$\tilde{W} = \int d\mu^{(d)} e^{i(q_1\cdot b_1+q_2\cdot b_2)}$$

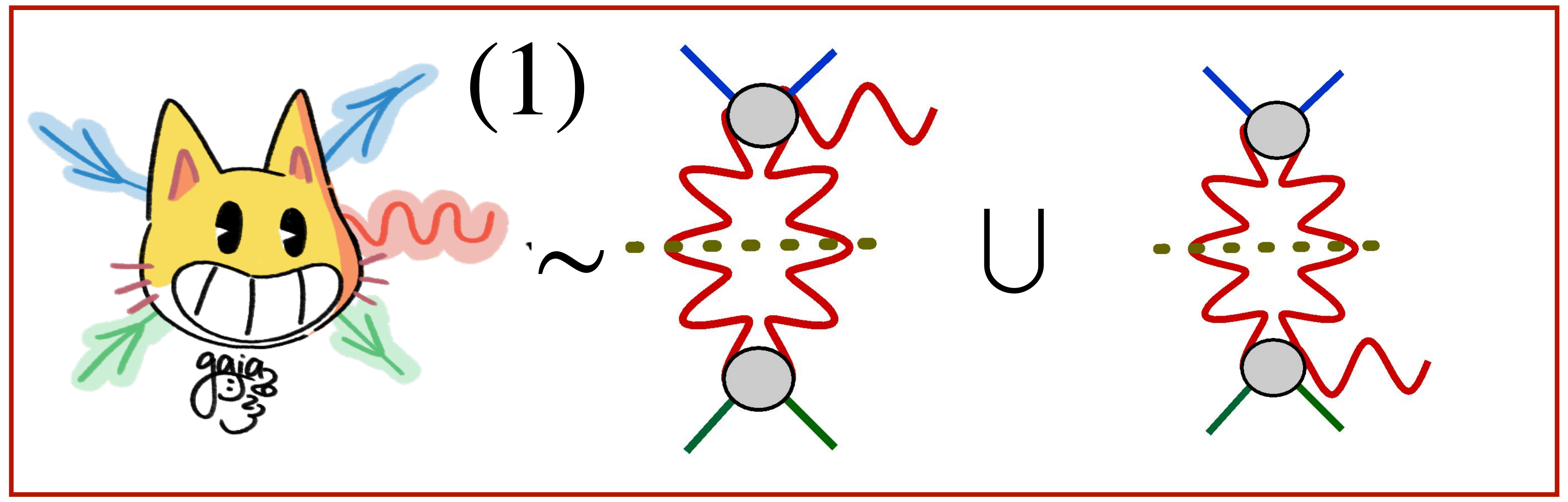


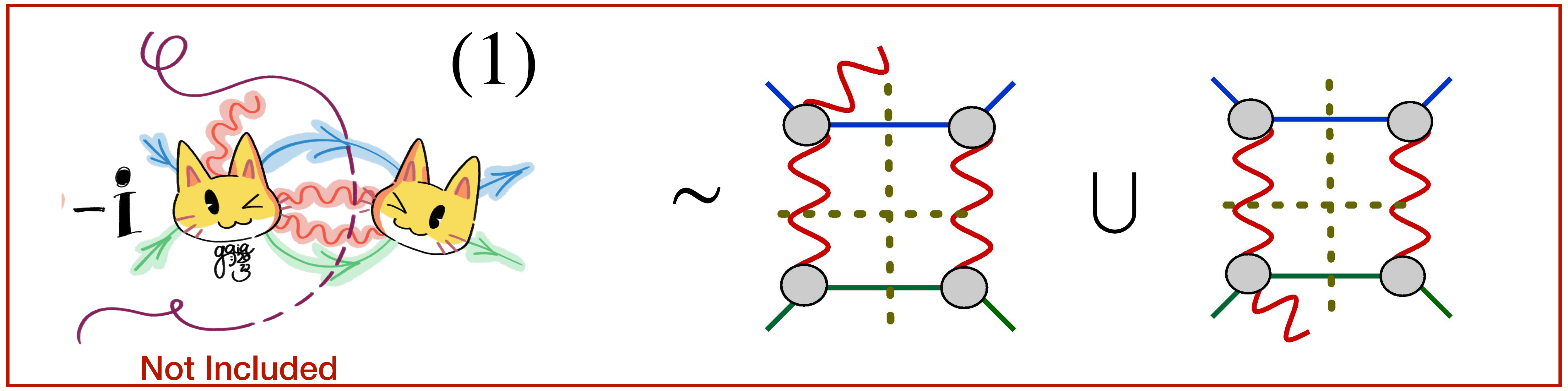
- Classical computation: [Eath (1978)][Kovacs, Thorne (1977)]
- Worldline: [Jacobsen, Mogull, Plefka, Steinhoff (2021)] [Mougiakakos, Riva, Vernizzi (2021)]
- Eikonal: [Di Vecchia, Heisenberg, Russo, Veneziano (2023)]

- At leading order only virtual contribution appears:



- NLO has been recently computed: [Brandhuber, Brown, Chen, De Angelis, Gowdy, Travaglini (2023)] [Herderschee, Roiban, Teng (2023)] [Georgoudis, Heissenberg, Varquez-Holm (2023)]





- Real contributions were not included [Caron-Huot, Giroux, Hannesdottir, Mizera (2023)]
- Partial agreement with MPM formalism [Bini, Damour, Geralico, 2023]
- Fourier transform not performed
- New terms appearing coming from the cut contribution [G.B., De Angelis, Kosower (in progress)]
- Recently computed with spin effects at LO [De Angelis, Novichkov, Gonzo (2023)] [Brandhuber, Brown, Gowdy, Travaglini [Aoude, Haddad, Heissenberg, Helset (2023)]

Simplifying Waveform via Analiticity and Unitarity

[G.B., De Angelis, Kosower (in progress)]

- Do we really need the full amplitude to get the physical observable?

$$\tilde{W} = \int d\mu^{(d)} e^{i(q_1\cdot b_1 + q_2\cdot b_2)}\left(\mathcal{I}_v + \mathcal{I}_r\right)$$

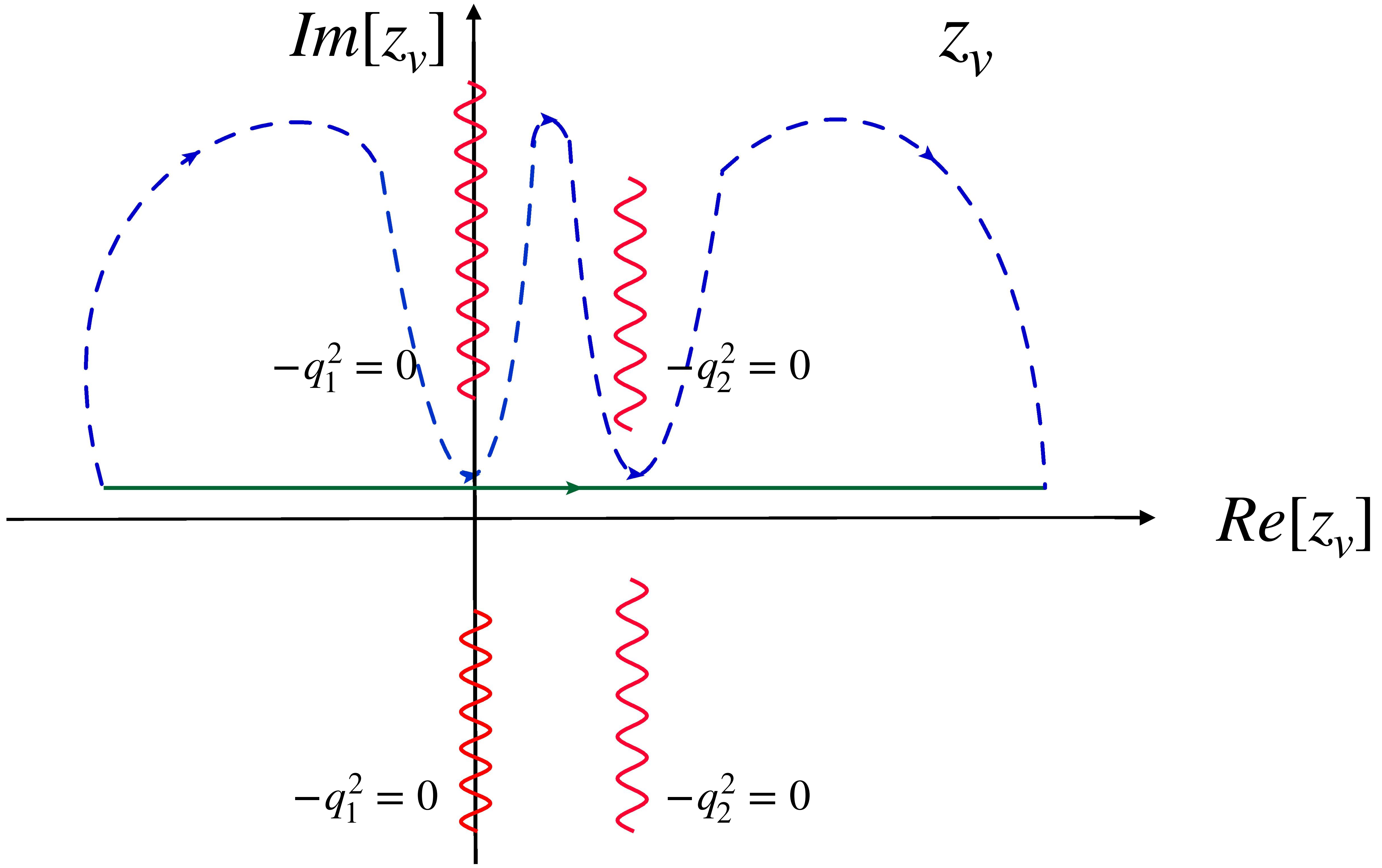
$$d\mu^{(d)} = \frac{d^d q_1}{(2\pi)^{d-1}}\frac{d^d q_2}{(2\pi)^{d-1}}\delta(2\bar{p}_1\cdot q_1)\delta(2\bar{p}_2\cdot q_2)\delta^{(d)}(q_1+q_2-k)$$

- Smart change of coordinates: $q_1 = z_1 v_1 + z_2 v_2 + z_b \tilde{b} + z_v v$

$$v_1^\mu = \frac{p_1^\mu}{m_1}, \quad v_2^\mu = \frac{p_2^\mu}{m_2}, \quad \tilde{b}^\mu = \frac{b^\mu}{\sqrt{-b^2}}, \quad v^2 = -1.$$

$$\tilde{W} = \frac{1}{(2\pi)^{D-2}(4\bar{m}_1\bar{m}_2)\sqrt{\gamma^2-1}}\int d^{D-4}v\, dz_v dz_b\, z_v^{D-4} e^{-iz_b\sqrt{-b^2}}\left(\mathcal{I}_v + \mathcal{I}_r\right)\Big|_{z_1=\frac{\gamma}{\gamma^2-1}w_2,\, z_2=-\frac{1}{\gamma^2-1}w_2}$$

- Deform the integration contour in the z_v plane:
  - The integral localise along the branch cuts: $-q_1^2 = 0$. $-q_2^2 = 0$,
  - Only that part contributes to the integration kernel,
  - This simplifies the components of the amplitude needed for the computation,
  - Easier to perform the Fourier integrals



Outlooks

- Gravitational Waves physics is an exciting field where high precision predictions are required

- Scattering Amplitudes provide a systematic framework to compute physical observables in Post-Newtonian and Post-Minkowskian approximation
  - 5PN sector on-going [G.B, M.K. Mandal, P. Mastrolia, R. Patil (in progress)]
  - 4PM sector completed

- Many techniques have been developed to simplify the computation:
  - On-shell methods
  - Heft Expansion
  - Efficient IBP decomposition
  - Differential Equations for Master Integrals

- New techniques for Integral decomposition:
  - ✴Intersection Theory for Feynman Integrals [G.B, V. Chestnov, G.E. Crisanti, H. Frellesvig, F. Gasparotto, M.K. Mandal, P. Mastrolia, R. Patil (in progress)]
  - ✴Intersection Theory for Fourier Integrals [G.B, V. Chestnov, G.E. Crisanti, M. Giroux, P. Mastrolia, S. Smith (in progress)]

- Using an observable-based approach it is possible to compute the GWs waveform from Amplitudes

- The puzzle to get the NLO waveform has still to be solved: [G.B., De Angelis, Kosower (in progress)]
  - ✓Evaluation of the 1-loop 5-pts amplitudes
  - ✴Inclusion of Cut contribution
  - ✴Fourier transform to time domain

- Scattering amplitudes can be applied also in other fields, like for cosmological correlation functions.

[P. Benincasa, G.B., M.K. Mandal, P. Mastrolia F, Vazao (in progress)]

Thanks for the attention!