Non-Linear Black Hole Ringdowns

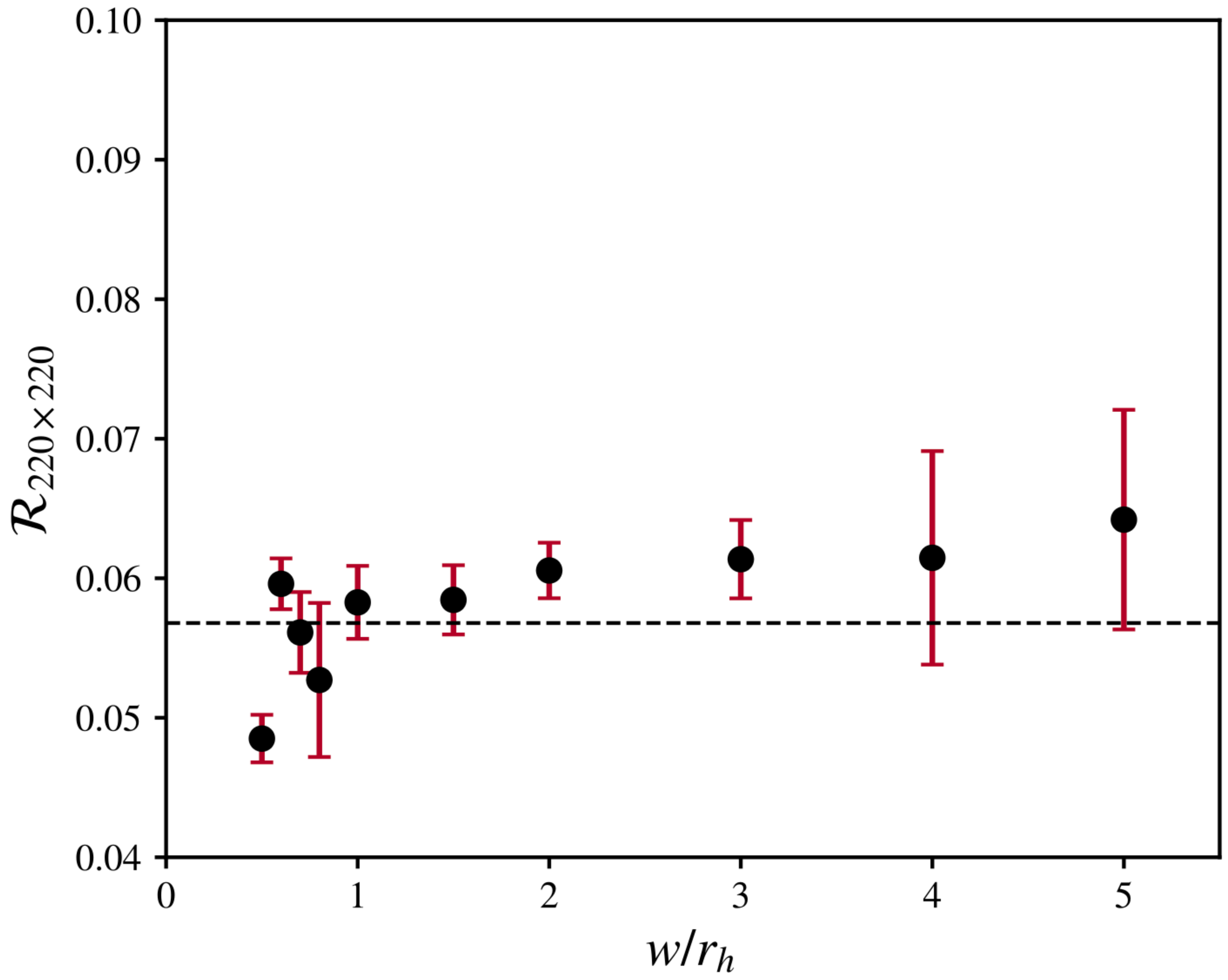
An Analytical Approach

Based on the work arXiv:2308.15886
in collaboration with T. Barreira, Prof. Kehagias and Prof. Riotto

GraSP, Pisa, 25 Oct 2023
Davide Perrone, University of Geneva

Recent work on Non-Linearity ratio

Estimate and independence on initial conditions



$$R_{220\times220} = \frac{|A^{440}_{220\times220}|}{|A_{220}|^2}$$

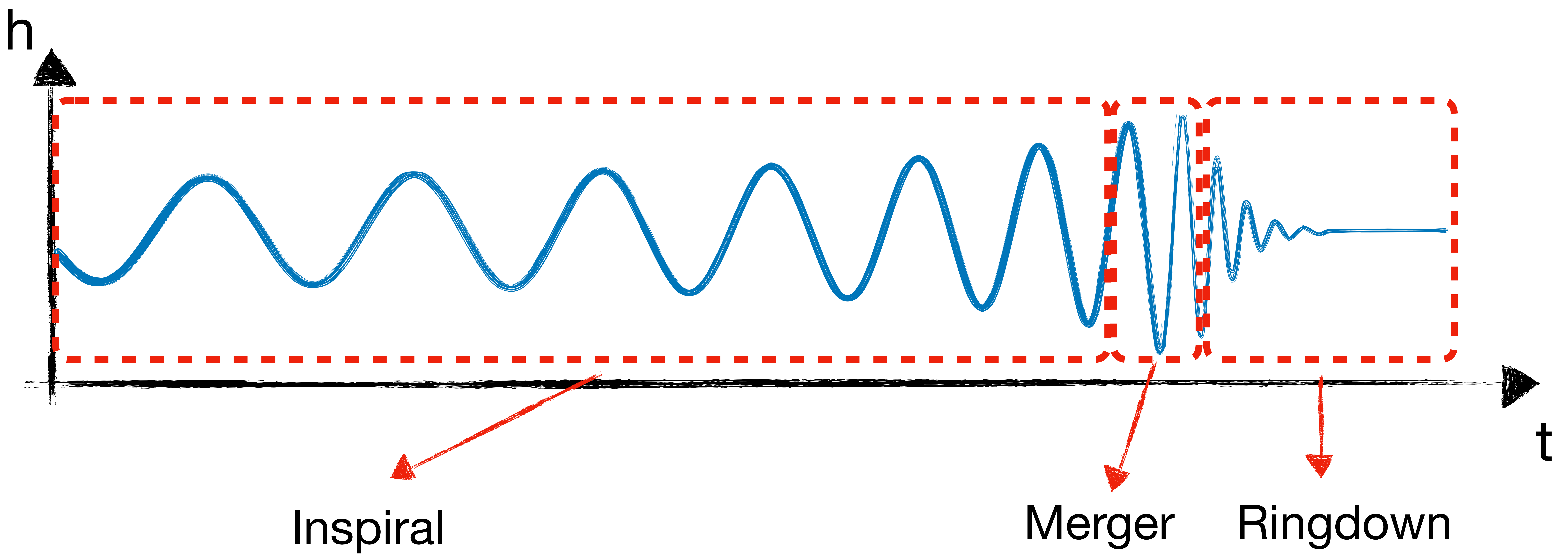
$$R_{220\times220}\Big|_{Schwarzschild} \simeq 0.06$$

Can we Explain this analytically?

J. Redondo-Yuste et al. (2023)

Gravitational Wave signal

What we see from a coalescence event



Ringdown, sum of Quasi-Normal Modes

And why are Quasi-Normal modes crucial

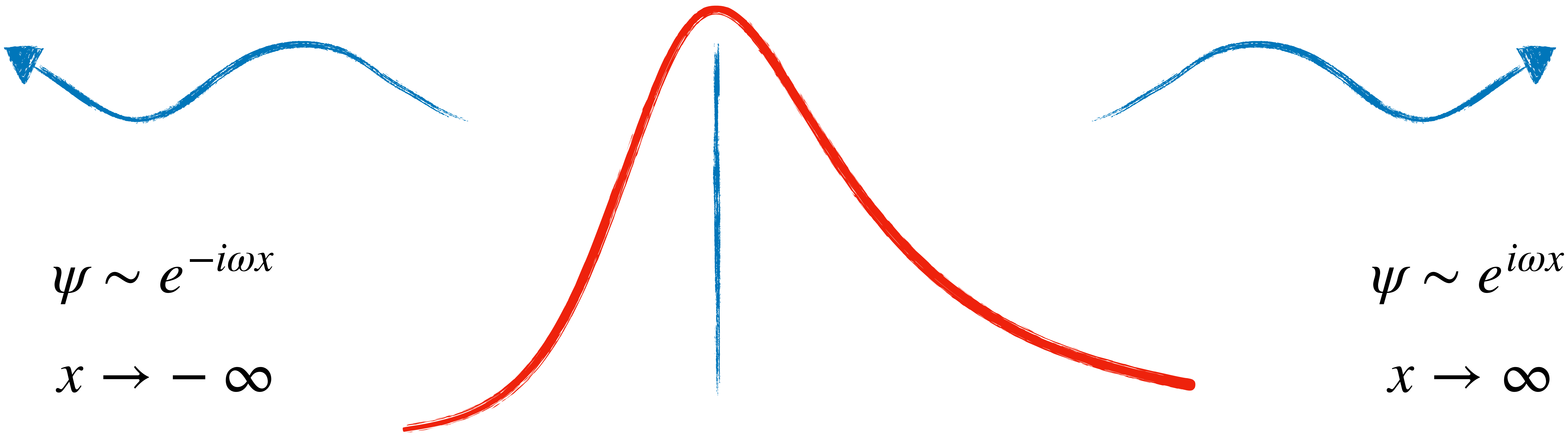
Proper response
to any kind of disturbance

$$\text{Ringdown} \sim \sum_n \text{QNM}_n$$

They dominate the
Ringdown phase

Quasi-Normal modes
Boundary conditions

Ingoing at horizon and outgoing at infinity



$$\psi \sim e^{-i\omega x}$$

$$x \to -\infty$$

Black Hole Horizon

Light ring

$$\psi \sim e^{i\omega x}$$

$$x \to \infty$$

Free Space

Linear Quasi-Normal modes

Master Equation, Schwarzschild Black Hole

$$\left(\partial_x^2 + \omega^2 - V(x)\right)\psi(x) = 0$$

→ Schrödinger-like Equation

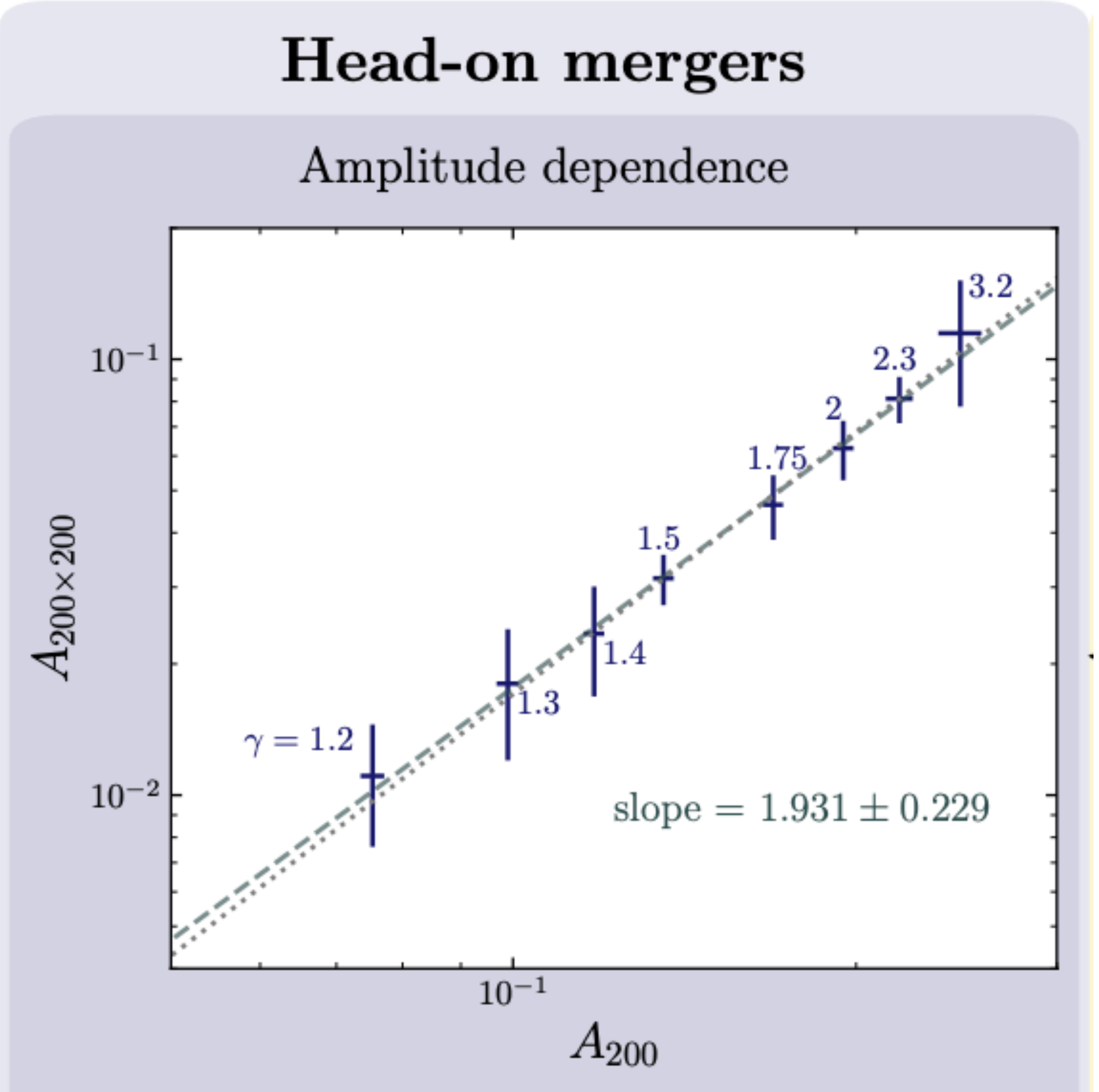
$$h_{l,m}(t-r) = \frac{1}{r}\sum_n A_{l,m,n}\, e^{-i\omega_{l,m,n}(t-r)}$$

Ringdown, linear

How large are Non-linearities from interactions?

Numerical Nonlinearities

For head-on merger into Schwarzschild Black Hole



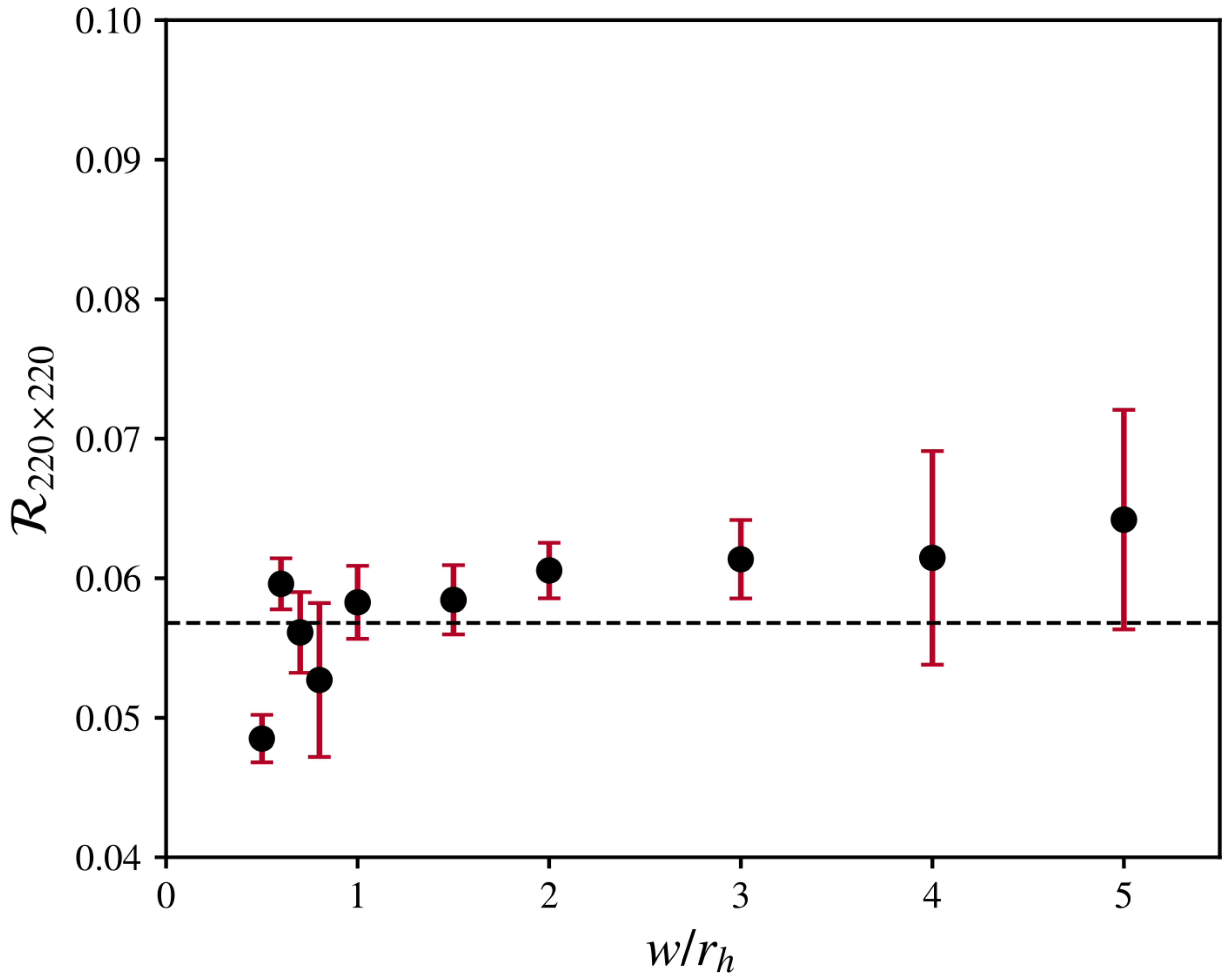
$$R_{200\times200} = \frac{|A^{400}_{200\times200}|}{|A_{200}|^2}$$

$$R_{200\times200} = 1.7 \pm 0.1$$

M. H. Cheung et al. (2022)

Our target, again

For Schwarzschild Black Hole, but not Head-on



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J. Redondo-Yuste et al. (2023)

Green’s function approach

Laplace transform of the master equation

$$\left(\partial_x^2 - s^2 - V(x)\right)\psi_1(x,s) = j_1(x,s)$$

$$\left(\partial_x^2 - s^2 - V(x)\right)\psi_2(x,s) = S(x,s) + j_2(x,s)$$

$\psi_\pm(x,s)$

Homogeneous solutions

Source $S\left(\psi_1(x,s),\, \psi_1(x,s)\right)$

Initial Conditions $j_{1,2}(x,s)$

Linear and Non-Linear solutions

Using analytic structure in s

Green's Function $$G(x, x', s) = \frac{\psi_+ \left(\max(x, x'), s\right) \cdot \psi_- \left(\min(x, x'), s\right)}{W(s)}$$

$$\psi_1(x, t) = \int_s \int_{x'} e^{ts} \, G(x, x', s) \; j_1(x', s)$$

$$\psi_2(x, t) = \int_s \int_{x'} e^{ts} \, G(x, x', s) \left[j_2(x', s) + S(x', s) \right]$$

Linear, $\omega_{\mathrm{l,m,n}}$

$\mathrm{Non-Linear}, \ 2\omega_{\mathrm{l,m,n}}$

Non-linearity ratio estimate

The source is localised at the light ring

$$R_{220\times220} = \frac{|\psi_2(x \to \infty, s = 2s_0 = 2\omega_{220})|}{|\psi_1(x \to \infty, s = s_0 = \omega_{220})|^2}$$

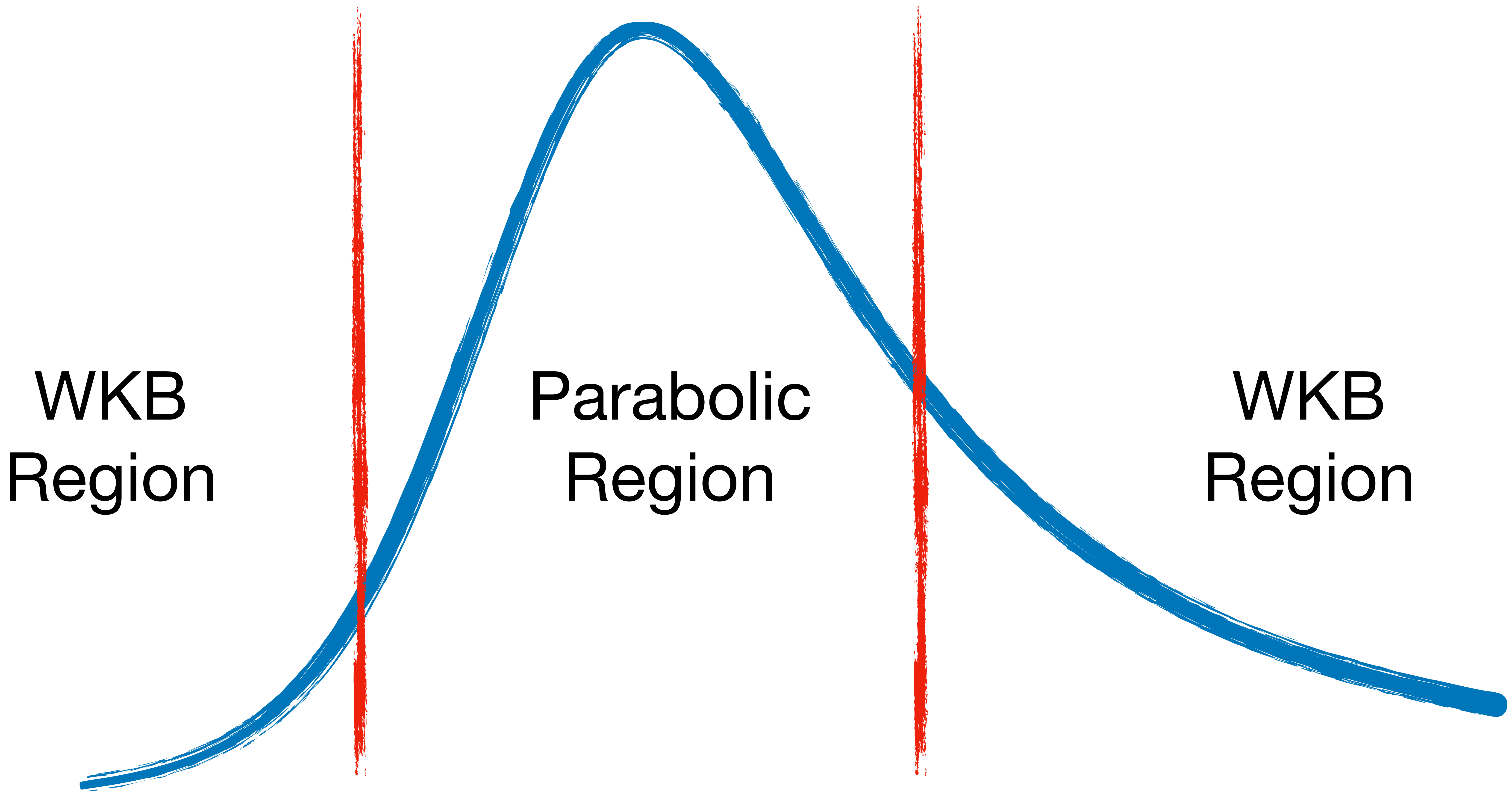
↓ Localisation hypotesis

$$R_{220\times220} \propto \frac{\left| \frac{A_{220}^2}{W(2s_0)} \cdot \int \psi_-(x', 2s_0)\tilde{S}\, dx' \right|}{\left| A_{220} + \frac{A_{220}^2}{s_0\, W'(s_0)} \cdot \int \psi_-(x', s_0)\tilde{S}\, dx' \right|^2}$$

A. Kehagias, D.P., A. Riotto (2023)

WKB approach for the estimate

Parabolic potential between WBK regions



B. F. Schutz et al. (1985)

Matching between regions

To find both solutions and Quasi-Normal mode frequencies

Matching

Quasi-Normal modes approximate frequencies $\omega_{n,l,m}$

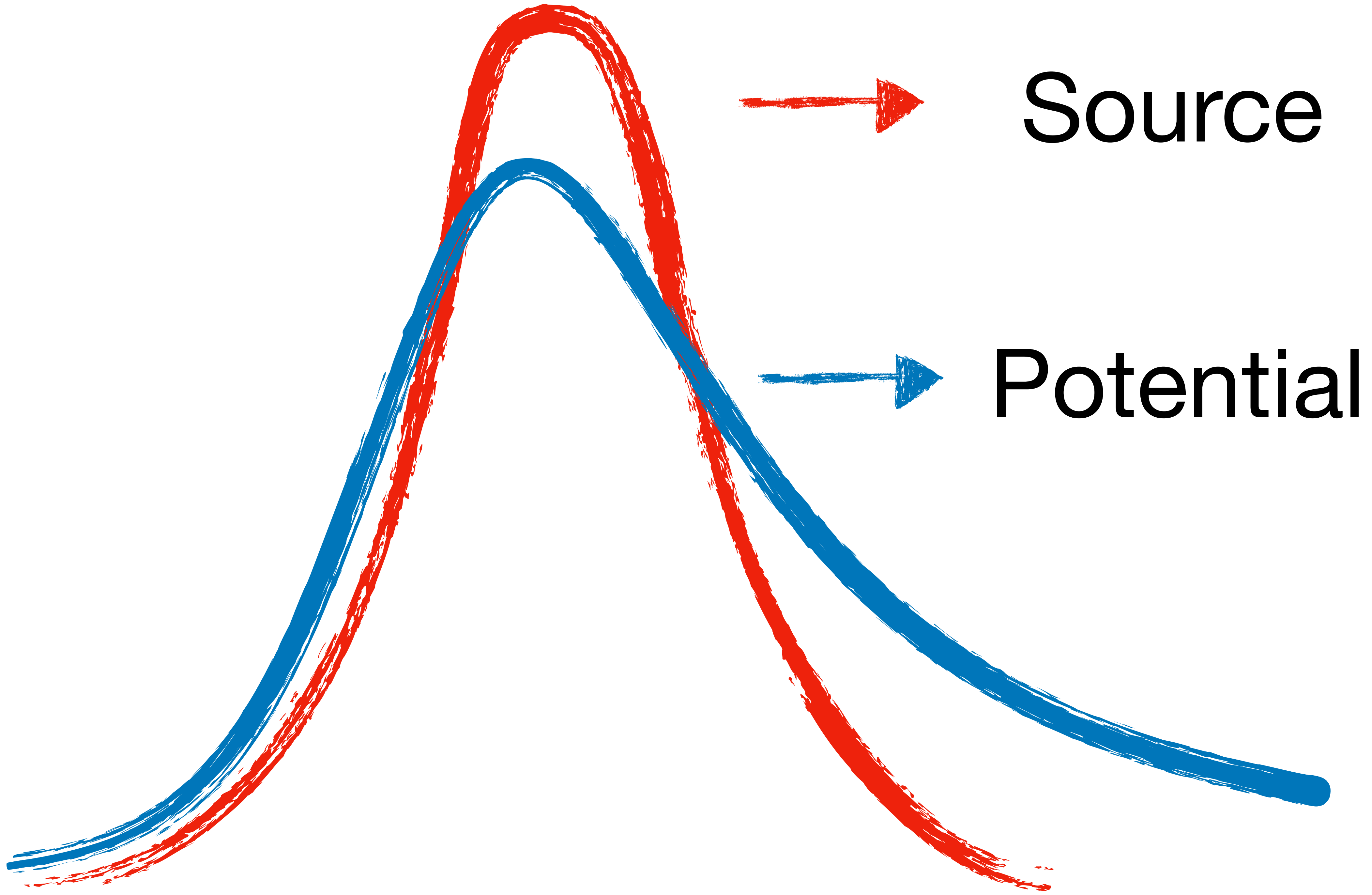
Approximate Solution for each frequency $\psi(x,t)$

Wronskian estimated in the Parabolic region

$$W(s) = \psi_{-,parabolic}(x,s)\,\psi'_{+,parabolic}(x,s) - \psi_{+,parabolic}(x,s)\,\psi'_{-,parabolic}(x,s)$$

Estimate source peak and nonlinearities

With a real sources this time!



$$\int \psi_{-}(x', 2s_0)\tilde{S}(x') \propto$$

$$\int \psi_{-,parabolic}(x', 2s_0)\ \psi^2_{-,parabolic}(x', s_0)\ R(x')$$

Steepest Descent

$$\sim R(x_0) \int \psi_{-,parabolic}(x', 2s_0)\psi^2_{-,parabolic}(x', s_0)$$

A. Kehagias, D.P., A. Riotto (2023)

Close estimate with numerical results

And independence on initial conditions

$$R_{220\times220} \propto \left| \frac{1}{W(2s_0)} \cdot \int \psi_{-}(x', 2s_0)\tilde{S}(x')\, dx' \right|$$

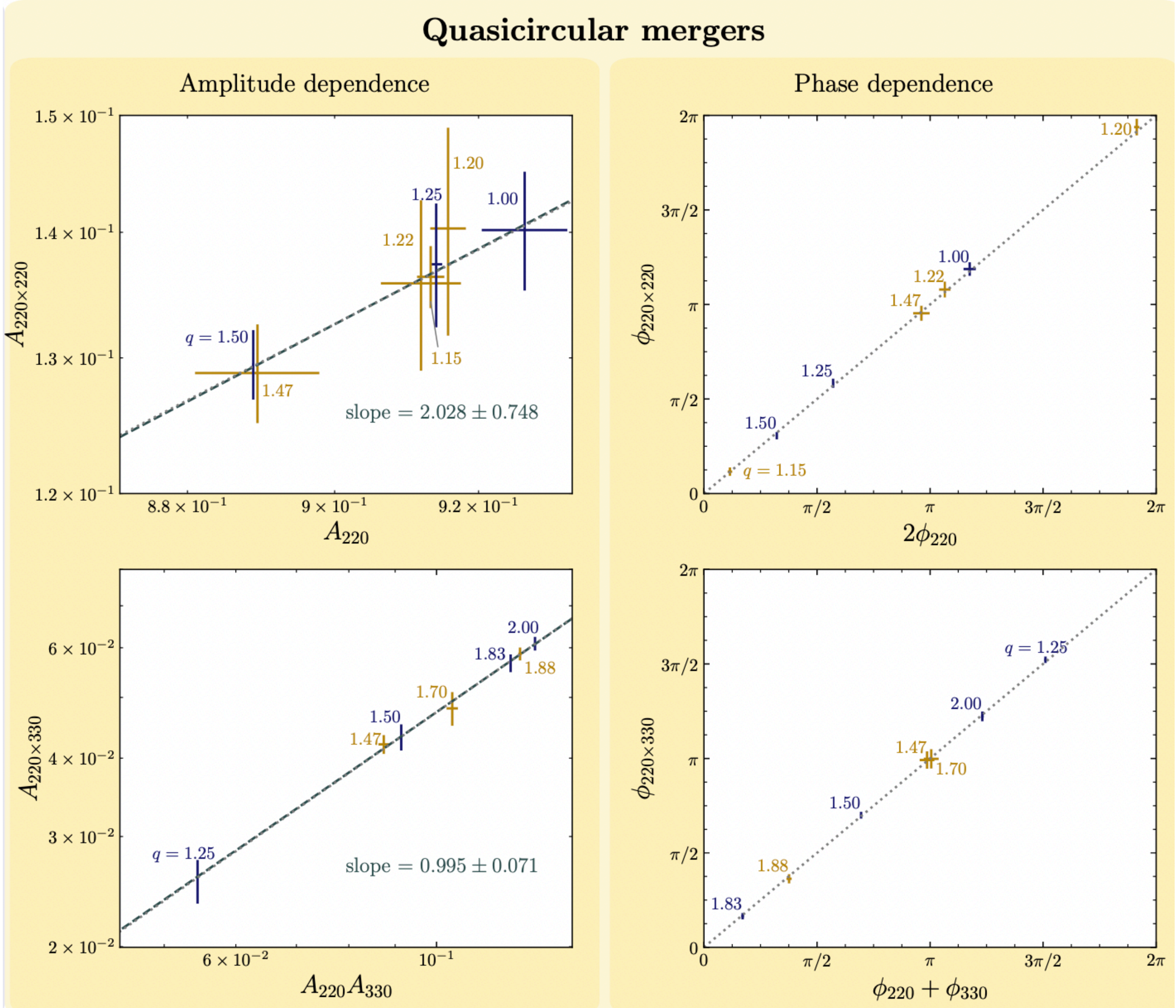
$$R_{220\times220}^{(estimate)} \simeq 0.069 \qquad \text{Vs} \qquad R_{220\times220}^{(numerical)}\Big|_{Schwarzschild} \simeq 0.06$$

A. Kehagias, D.P., A. Riotto (2023)

Non-Linearities are important!

- Crucial for future of Gravitational Wave analysis
- A lot of techniques are being developed to evaluate them
- Could hint to deviation from Einstein Gravity

Kerr Black Hole Non-Linearity



$$\frac{\left|A^{(2,2,0)\times(2,2,0)}_{(4,4)}\right|}{\left|A_{(2,2,0)}\right|^2} = 0.1637 \pm 0.0018$$

$$\frac{\left|A^{(2,2,0)\times(3,3,0)}_{(5,5)}\right|}{\left|A_{(2,2,0)}\right|\left|A_{(3,3,0)}\right|} = 0.4735 \pm 0.0062$$

M. H. Cheung et al. (2022)

Kerr Black Hole Non-Linearity

And its estimate in Extremal limit

$$\frac{\langle h_{(\ell_1,m_1)} h_{(\ell_2,m_2)} h_{(\ell_1+\ell_2,m_1+m_2)}\rangle}{\langle h^2_{(\ell_1,m_1)}\rangle\langle h^2_{(\ell_2,m_2)}\rangle} = \frac{6\sqrt{2}}{2\pi}\,{}_{-2}C^{m_1,m_2,m_1+m_2}_{\ell_1,\ell_2,\ell_1+\ell_2}\,\frac{G^{3,3}_{3,3}\left(\begin{matrix} -im_1, & 0, & im_2 \\ 1-im_1, & 1, & 1+im_2 \end{matrix}\,\middle|\, e^{i\pi}\right)}{\left|\Gamma\left(2-i(m_1+m_2)\right)\right|^2}$$

$$\frac{\langle h_{(2,2)} h_{(2,2)} h_{(4,4)}\rangle}{\langle h^2_{(2,2)}\rangle^2} \simeq 0.62\cdot\frac{5}{24}\sqrt{\frac{7}{\pi}} \simeq 0.17 \qquad \text{vs} \qquad \frac{\left|A^{(2,2,0)\times(2,2,0)}_{(4,4)}\right|}{\left|A_{(2,2,0)}\right|^2} = 0.1637 \pm 0.0018$$

$$\frac{\langle h_{(2,2)} h_{(3,3)} h_{(5,5)}\rangle}{\langle h^2_{(2,2)}\rangle\langle h^2_{(3,3)}\rangle} \simeq 1.57\cdot\frac{2}{3}\sqrt{\frac{7}{11\pi}} \simeq 0.47 \qquad \text{vs} \qquad \frac{\left|A^{(2,2,0)\times(3,3,0)}_{(5,5)}\right|}{\left|A_{(2,2,0)}\right|\left|A_{(3,3,0)}\right|} = 0.4735 \pm 0.0062$$

A. Kehagias, D.P., F. RIva, A. Riotto (2023)