GraSP23 || GravityShapePisa 2023

Gravitational Observables from Scattering Amplitudes

Based on [M. Bianchi, CG, F. Riccioni, JHEP 08 (2023) 188]

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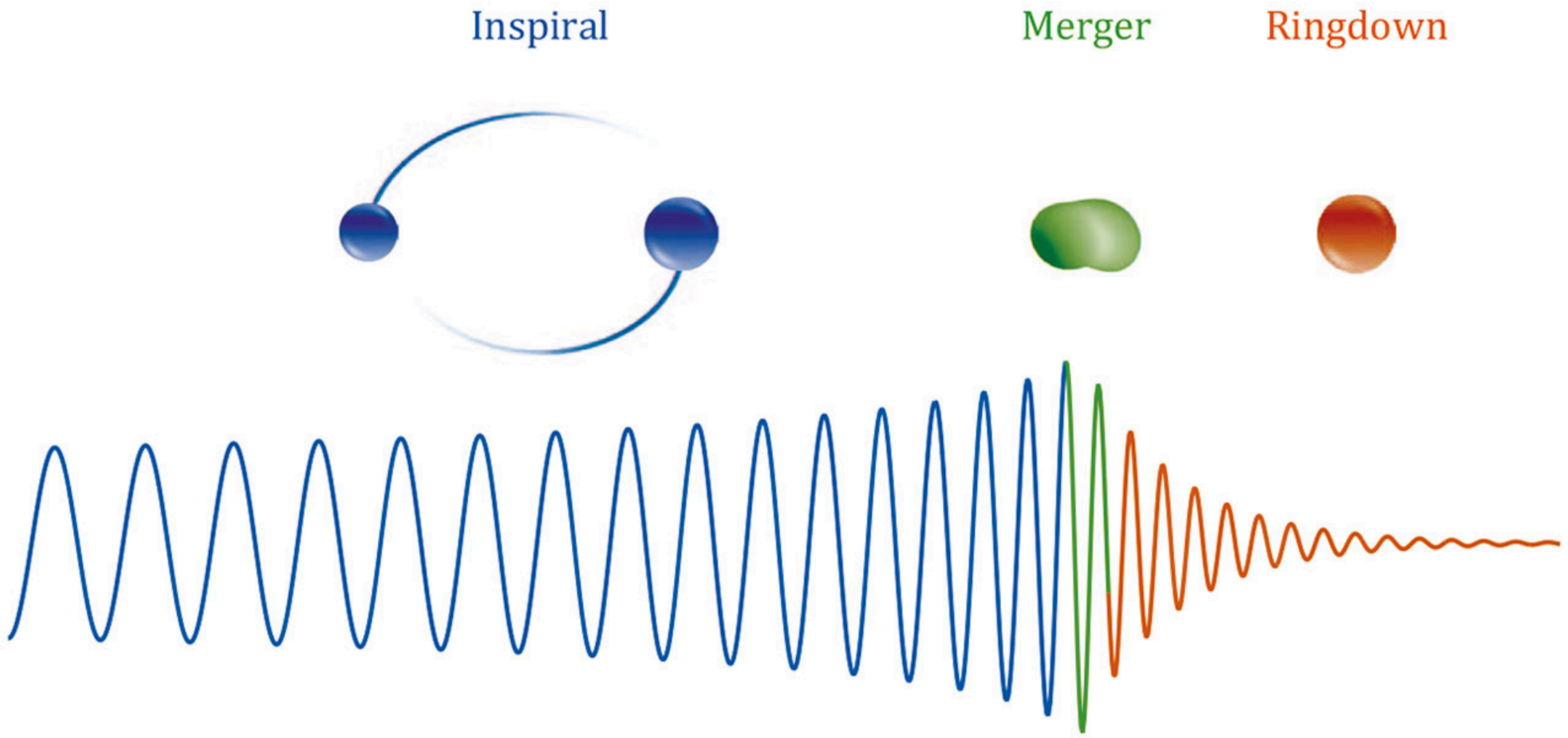
Outline

- Motivation: why gravity from scattering amplitudes?
- Classical physics from loop amplitudes
- State-of-the-art understanding of BH-particle correspondence
- Kerr-Newman metric in Kerr-Schild gauge
- Scalar probe scattering off KN background
- Scattering angle and comparison with the literature
- Conclusions

Motivation

The discovery of GWs started an incredible effort with the aim of describing (extreme) gravitational processes with analytic techniques.



Post-Minkowskian Expansion (Scattering Amplitudes)

Numerical Relativity

Quasi Normal Modes

Credit: *Eur.Phys.J.Plus* 132 (2017) 1, 10

The PM series is an expansion in G (weak field limit), which is the natural expansion parameter of gravitational scattering amplitudes.

↓

We can employ the very well-known machinery of scattering amplitudes to compute gravitational observables!

Why loop amplitudes can be classical?

"...we want to point out that there seems to exist an erroneous belief that only tree diagrams contribute to the classical process."

[Iwasaki, 1971]

$$L = P - V + 1 \;\longrightarrow\; \begin{matrix} V \sim \hbar^{-1} \\ P \sim \hbar \end{matrix} \;\longrightarrow\; Amplitude \sim \hbar^{P-V} = \hbar^{L-1} \to \frac{1}{\hbar}\left(1 + \hbar + \hbar^2 + \cdots \right)$$

1: Classical

$\hbar$: LO Quantum Correction

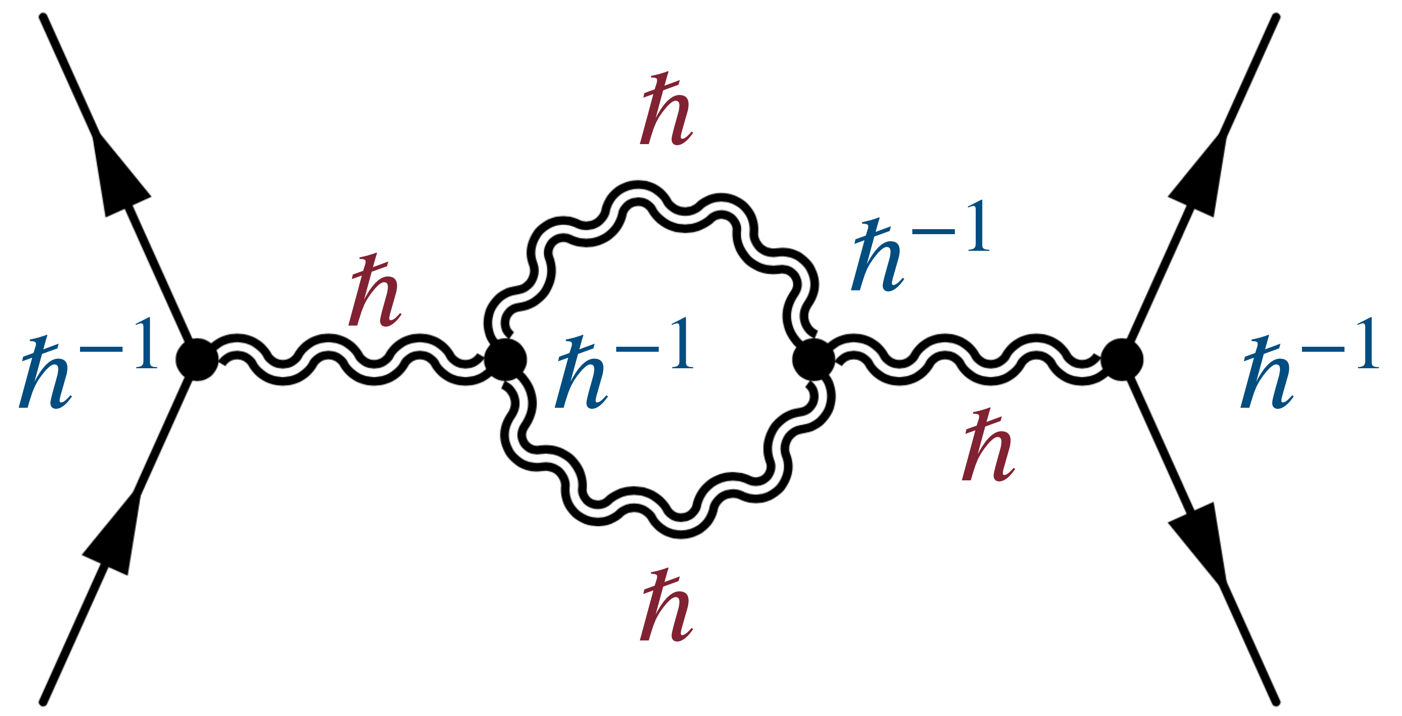
$\hbar^2$: NLO Quantum Correction

However, in the low-energy limit, internal massive lines and massless lines behave differently...

With $\hbar$'s restored, internal momenta are dimensionally wavelengths $\longrightarrow$

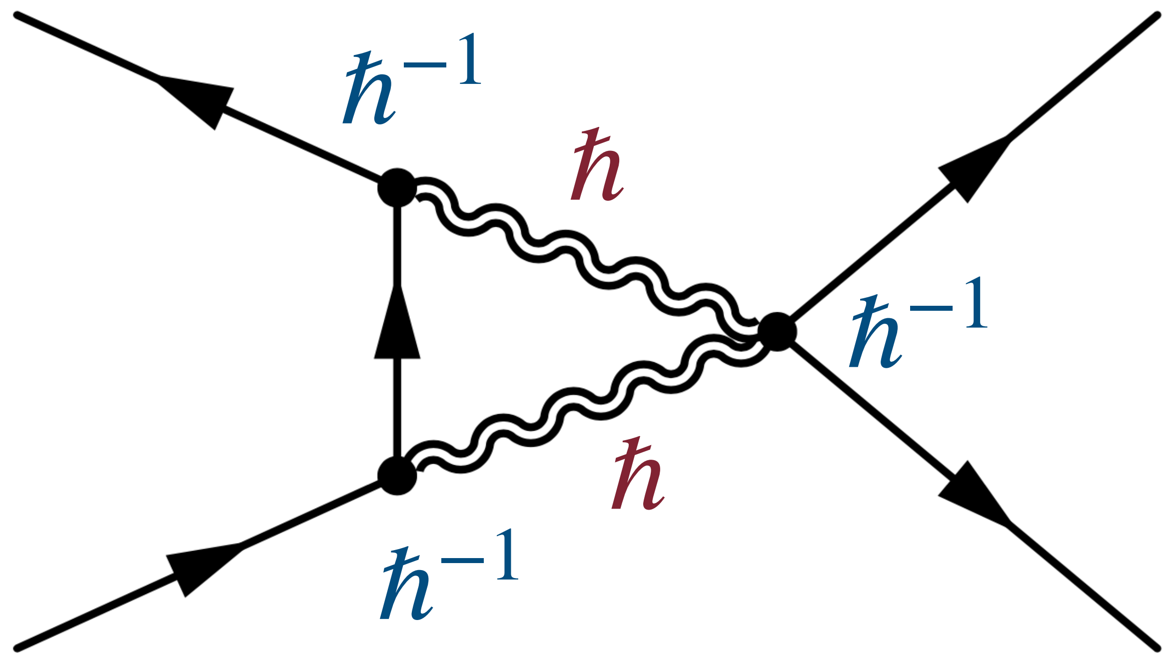
$$\frac{i\hbar}{(p - \hbar\, q)^2 - m^2 + i\epsilon} \sim \frac{i}{-2\; p \cdot q + i\epsilon}$$

[Donoghue, gr-qc/9405057]
[Bjerrum-Bohr, Damgaard, Festuccia, Planté, Vanhove, 1806.04920]



$\sim \frac{1}{\hbar}\left(\hbar\right)$

Quantum Amplitude



$\sim \frac{1}{\hbar}\left(1\right)$

Classical Amplitude

Loop scattering amplitudes with an internal tree structure are classical.

From this argument follows that every gravitational observables can be computed from graviton scattering amplitudes, organised in a PM expansion in which $PM = L + 1$.

[Chung, Huang, Kim, Lee, 1812.08752]
[Arkani-Hamed, Huang, O'Connell, 1906.10100]

State of the art

BHs in the weak field regime can be described by minimally-coupled massive spinning particles in the limit $\hbar \to 0$.

- Tidal deformations are captured by higher-derivative operators.
- Generic compact objects are described by non-minimal terms.
- Spin-S fields reconstruct the $2S$ multipole moment of a rotating compact object: for a BH, the limit $S \to +\infty$ leads to Kerr.

Pick an observable → ***Compute it trough amplitude calculations*** → ***Analytically continue it to bound systems***

[Kalin, Porto, 1910.03008]

Kerr-Newman metric in Kerr-Schild gauge

$$K_\mu = \left(1, \frac{rx + ay}{r^2 + a^2}, \frac{ry - ax}{r^2 + a^2}, \frac{z}{r}\right)$$

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} = \eta_{\mu\nu} + \Phi K_\mu K_\nu$$

$$\Phi = -G\frac{2Mr - Q^2}{r^2 + a^2\cos^2\theta}$$

Oblate Spheroidal Coordinates

$$\begin{cases} x = \sqrt{r^2 + a^2}\sin(\theta)\cos(\varphi) \\ y = \sqrt{r^2 + a^2}\sin(\theta)\sin(\varphi) \\ z = r\cos(\theta) \end{cases}$$

$$g^{\mu\nu} = \eta^{\mu\nu} + h^{\mu\nu} = \eta^{\mu\nu} - \Phi K^\mu K^\nu \quad \longleftarrow \quad K_\mu K^\mu \equiv K_\mu K_\nu \eta^{\mu\nu} = 0$$

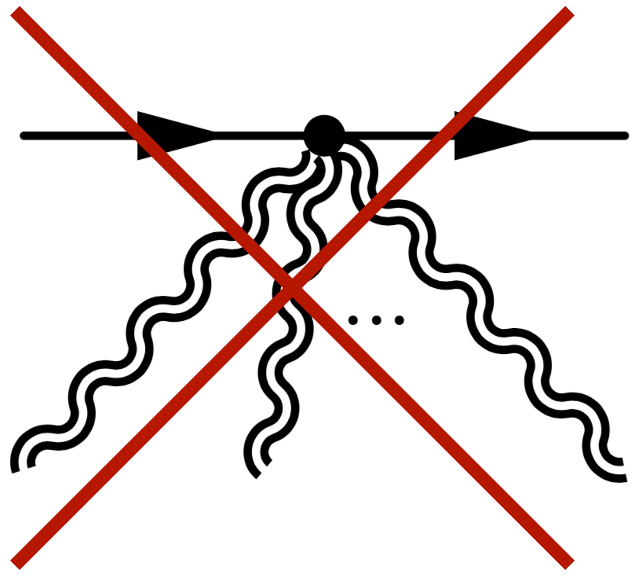
Most important feature of KS

Gauge potential associated to the charged BH: $A_\mu = V_A K_\mu$

$$V_A = Q\frac{r}{r^2 + a^2\cos^2\theta}$$

Scalar probe scattering off a KN background

Key points: $h^{\mu\nu} \sim G$ & $\sqrt{-g} = 1$.



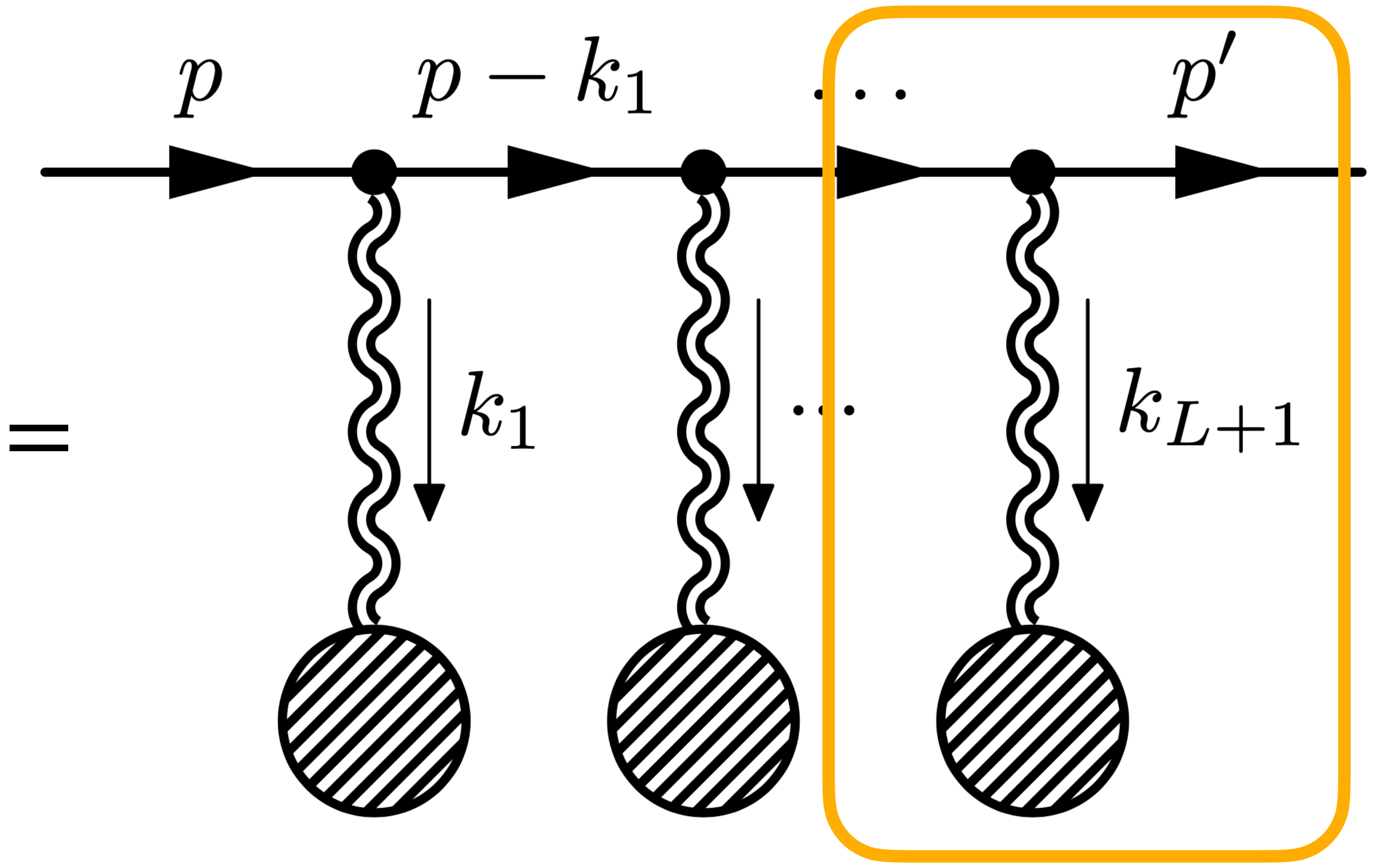
$$S = \int d^4x \sqrt{-g} \left(\frac{1}{2} \partial_\mu \phi \, \partial_\nu \phi \, g^{\mu\nu} - \frac{1}{2} m^2 \phi^2 \right) \Rightarrow \mathscr{L}_{int} = \frac{1}{2} h^{\mu\nu}(x) T^{\phi}_{\mu\nu}(x) \longrightarrow$$

No higher-order interaction terms!

$V_{\phi\phi h^n} = 0 \quad n \geq 2$

All the interaction information is contained in a tri-linear vertex.

$i\mathscr{M}^{(L+1)} =$



Comb-like diagram made of $3pt$ amplitude building blocks.

$$i\mathscr{M}_{KN}(p, p', \vec{q}) = i\, \tilde{h}^{\mu\nu}_{KN}(\vec{q})\, p_\mu p'_\nu$$

We need this FT.

Exact Fourier transform

Setting up $\tilde{h}^{KN}_{\mu\nu}(\vec{q}\,) = \int d^3\vec{x} e^{-i\vec{q}\cdot\vec{x}} \Phi_G(\vec{x}\,) K_\mu(\vec{x}\,) K_\nu(\vec{x}\,)$ **Miraculous cancellation** $\longrightarrow$ $d^3x\, \Phi(\vec{x}) = dr d\Omega\; G(-2Mr + Q^2)$

It is possible to compute the FT exactly with a variable change

$K_\mu(\vec{x}) \Rightarrow \hat{K}_\mu(i\partial_{\vec{q}})$ $\quad \vec{u} = \left(q_x\sqrt{r^2+a^2}, q_y\sqrt{r^2+a^2}, q_z r\right)$ $\quad \vec{q}\cdot\vec{x} = \vec{u}\cdot\vec{n} = u\cos(\theta)$ $\quad |\vec{u}| = u = \sqrt{r^2q^2 + a^2q_\perp^2}$

$$\tilde{h}^{KN}_{\mu\nu}(\vec{q}) = -8\pi G M \tilde{h}_{\mu\nu}(\vec{q}) + 4\pi G Q^2 \Delta\tilde{h}_{\mu\nu}(\vec{q})$$

$\tilde{h}_{\mu\nu}(\vec{q}) = \int_{q_\perp}^{+\infty} \frac{udu}{|\vec{q}|^2} \hat{K}_\mu \hat{K}_\nu\, j_0(u)$ $\quad \Delta\tilde{h}_{\mu\nu}(\vec{q}) = \int_{q_\perp}^{+\infty} \frac{udu}{|\vec{q}|^2} \frac{1}{r(u)} \hat{K}_\mu \hat{K}_\nu\, j_0(u)$ $\quad \tilde{A}_\mu(\vec{q}) = 4\pi G \int_{q_\perp}^{+\infty} \frac{udu}{|\vec{q}|^2} \hat{K}_\mu\, j_0(u)$

$$i\mathcal{M}(p,p',\vec{q}) = i\frac{8\pi GM}{|\vec{q}|^2}\Bigg\{E^2\cos|\vec{a}\times\vec{q}| + iE\Bigg(-\frac{\vec{q}\cdot(\vec{p}'+\vec{p})}{|\vec{q}|}\frac{\pi}{2}J_0(|\vec{a}\times\vec{q}|)$$

$$+j_0(|\vec{a}\times\vec{q}|)(\vec{a}\times\vec{q})\cdot(\vec{p}'+\vec{p})\Bigg) + j_0(|\vec{a}\times\vec{q}|)\left(\vec{p}\cdot\vec{p}' - 2\frac{\vec{q}\cdot\vec{p}\;\vec{q}\cdot\vec{p}'}{|\vec{q}|^2}\right)$$

$$-\frac{j_1(|\vec{a}\times\vec{q}|)}{|\vec{a}\times\vec{q}|}(\vec{a}\times\vec{q})\cdot\vec{p}\;(\vec{a}\times\vec{q})\cdot\vec{p}'$$

$$+\frac{1}{|\vec{q}|}\frac{\pi}{2}\frac{J_1(|\vec{a}\times\vec{q}|)}{|\vec{a}\times\vec{q}|}\Big(\vec{q}\cdot\vec{p}\,(\vec{a}\times\vec{q})\cdot\vec{p}' + \vec{q}\cdot\vec{p}'\;(\vec{a}\times\vec{q})\cdot\vec{p}\Big)\Bigg\}$$

Results:

- p and p' are off-shell momenta.
- When put on-shell:
  - In $\mathcal{M}$ the J_n's disappear.
  - In $\Delta\mathcal{M}$ the j_n's disappear.
- $\Delta\mathcal{M}$ is subleading.

$$i\Delta\mathcal{M}(p,p',\vec{q}) = \left(-i\frac{4\pi GQ^2}{|\vec{q}|}\right)\Bigg\{E^2\frac{\pi}{2}J_0(|\vec{a}\times\vec{q}|) + iE\Bigg(-\frac{\vec{q}\cdot(\vec{p}+\vec{p}')}{|\vec{q}|}j_0(|\vec{a}\times\vec{q}|)$$

$$+\frac{\pi}{2}(\vec{a}\times\vec{q})\cdot(\vec{p}+\vec{p}')\frac{J_1(|\vec{a}\times\vec{q}|)}{|\vec{a}\times\vec{q}|}\Bigg) + \frac{\pi}{2}\frac{J_1(|\vec{a}\times\vec{q}|)}{|\vec{a}\times\vec{q}|}\left(\vec{p}\cdot\vec{p}' - \frac{\vec{q}\cdot\vec{p}\;\vec{q}\cdot\vec{p}'}{|\vec{q}|^2}\right)$$

$$+\frac{1}{|\vec{q}|}\frac{j_1(|\vec{a}\times\vec{q}|)}{|\vec{a}\times\vec{q}|}\Big(\vec{q}\cdot\vec{p}\;(\vec{a}\times\vec{q})\cdot\vec{p}' + \vec{q}\cdot\vec{p}'\;(\vec{a}\times\vec{q})\cdot\vec{p}\Big)$$

$$-\frac{\pi}{2}\frac{J_2(|\vec{a}\times\vec{q}|)}{|\vec{a}\times\vec{q}|^2}(\vec{a}\times\vec{q})\cdot\vec{p}\;(\vec{a}\times\vec{q})\cdot\vec{p}'\Bigg\}$$

$$\mathcal{M}_{KN} = \mathcal{M} + \Delta\mathcal{M}$$

Up to calculations, the problem is completely worked out, and we can extrapolate gravitational observables exact in the spin at any PM order!

The eikonal expansion

[M. Levy, J. Sucher, 69′]
[D. Amati, M. Ciafaloni , G. Veneziano, 87′-90′]

An example of a systematic approach to derive gravitational observables out from scattering amplitudes is the **eikonal exponentiation**.

Impact Parameter Space: $$\widetilde{\mathcal{M}}^{(n)}(p,\vec{b}\,) = \frac{1}{2|\vec{p}\,|}\int \frac{d^2q}{(2\pi)^2} e^{i\vec{q}\cdot\vec{b}}\mathcal{M}^{(n)}$$

Eikonal Phase

$$\widetilde{S}(p,\vec{b}\,) = 1 + i\widetilde{T}(p,\vec{b}\,) = e^{2i\delta(p,\vec{b}\,)} \Rightarrow i\widetilde{T}(p,\vec{b}\,) = i\sum_{n=1}^{+\infty}\widetilde{\mathcal{M}}^{(n)}(p,\vec{b}\,) = \sum_{m=1}^{+\infty}\frac{1}{m!}\left(2i\sum_{n=1}^{+\infty}\delta^{(n)}(p,\vec{b}\,)\right)^m$$

$$\widetilde{\mathcal{M}}^{(1)}(p,\vec{b}\,) = 2\delta^{(1)}(p,\vec{b}\,)$$

Leading order classical contribution

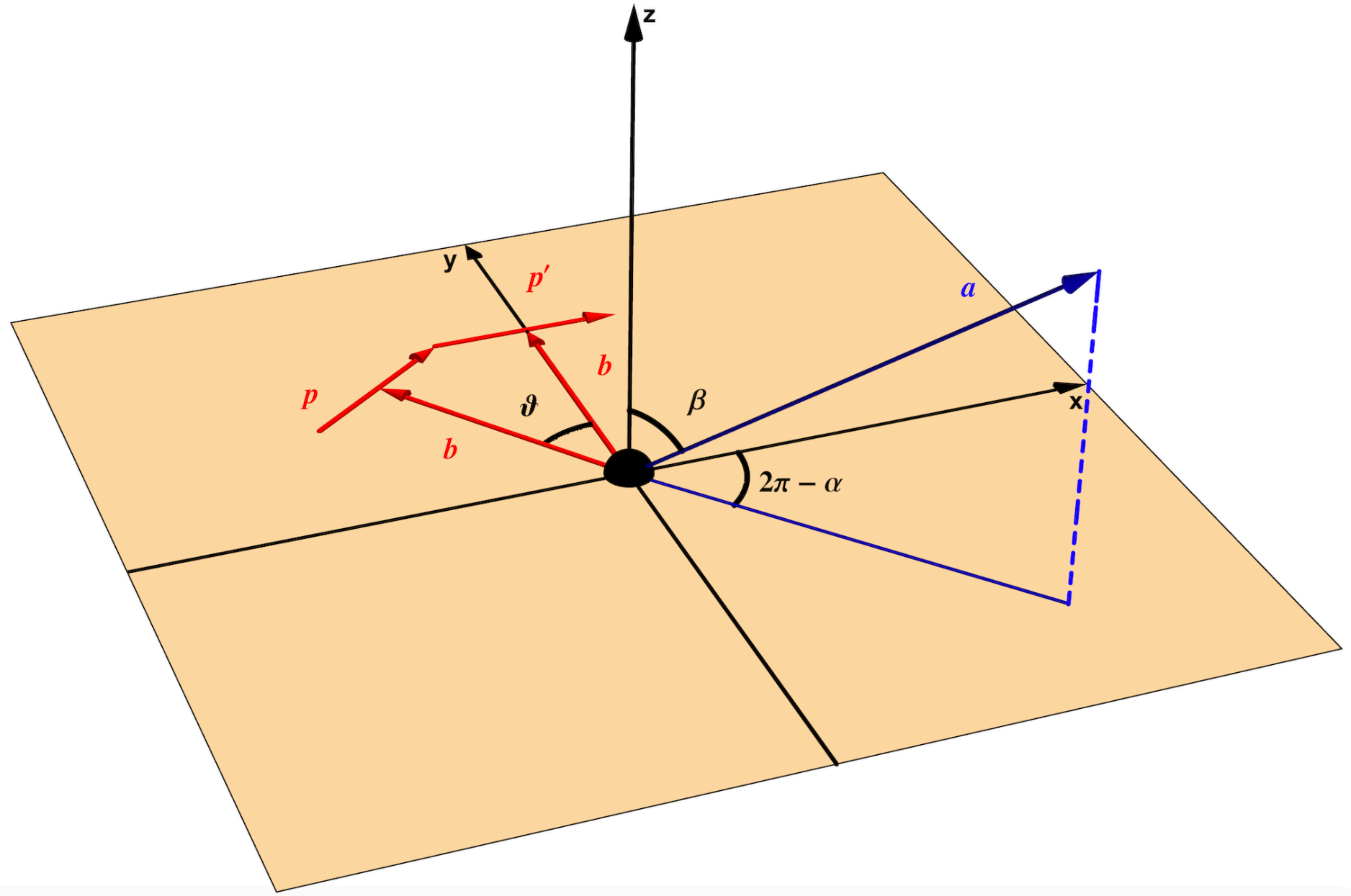
$$\widetilde{\mathcal{M}}^{(2)}(p,\vec{b}\,) = 2\delta^{(2)}(p,\vec{b}\,) - \frac{i}{2}\left(2i\delta^{(1)}(p,\vec{b}\,)\right)^2$$

Next-to-leading order classical + hyper-classical

Scattering angle in a generic orientation

From the eikonal phase we can derive many gravitational observables, one of which is the scattering angle.

$$\vartheta(p, \vec{b}) = -\frac{2}{|\vec{p}\,|}\frac{\partial \delta(p, \vec{b})}{\partial b}$$



We managed to derive the scattering angle of a scalar probe scattering off a Kerr-Newman BH at $1PM$, exactly in a, and for a generic spin orientation.

Leading eikonal phase

[Y. F. Bautista, A. Guevara, C. Kavanagh, J. Vines, 2107.10179]

$$\delta^{(1)}_{KN} = \frac{1}{4|\vec{p}\,|}\int \frac{d^2q}{(2\pi)^2} e^{i\vec{q}\cdot\vec{b}} \mathcal{M}^{(1)}_{KN}$$

Equipped with →

$$\begin{cases} \mathcal{F}(d,\nu) = \int \frac{d^dq}{(2\pi)^d} e^{i\vec{q}\cdot\vec{x}} |\vec{q}\,|^{2\nu} = \frac{2^{2\nu}}{\pi^{d/2}} \frac{\Gamma(\nu + d/2)}{\Gamma(-\nu)} \frac{1}{|\vec{x}\,|^{2\nu+d}} \\ \mathcal{F}(d,-d/2) = \int \frac{d^dq}{(2\pi)^d} e^{i\vec{q}\cdot\vec{x}} |\vec{q}\,|^{-d} = -\frac{2^{1-d}}{\pi^{d/2}\Gamma(d/2)} \log \mu |\vec{x}\,| \end{cases}$$

Restricting to Kerr: $|\vec{a}\times\vec{q}\,| \to \pm \vec{q}\cdot\hat{\ell}\times\vec{a}$

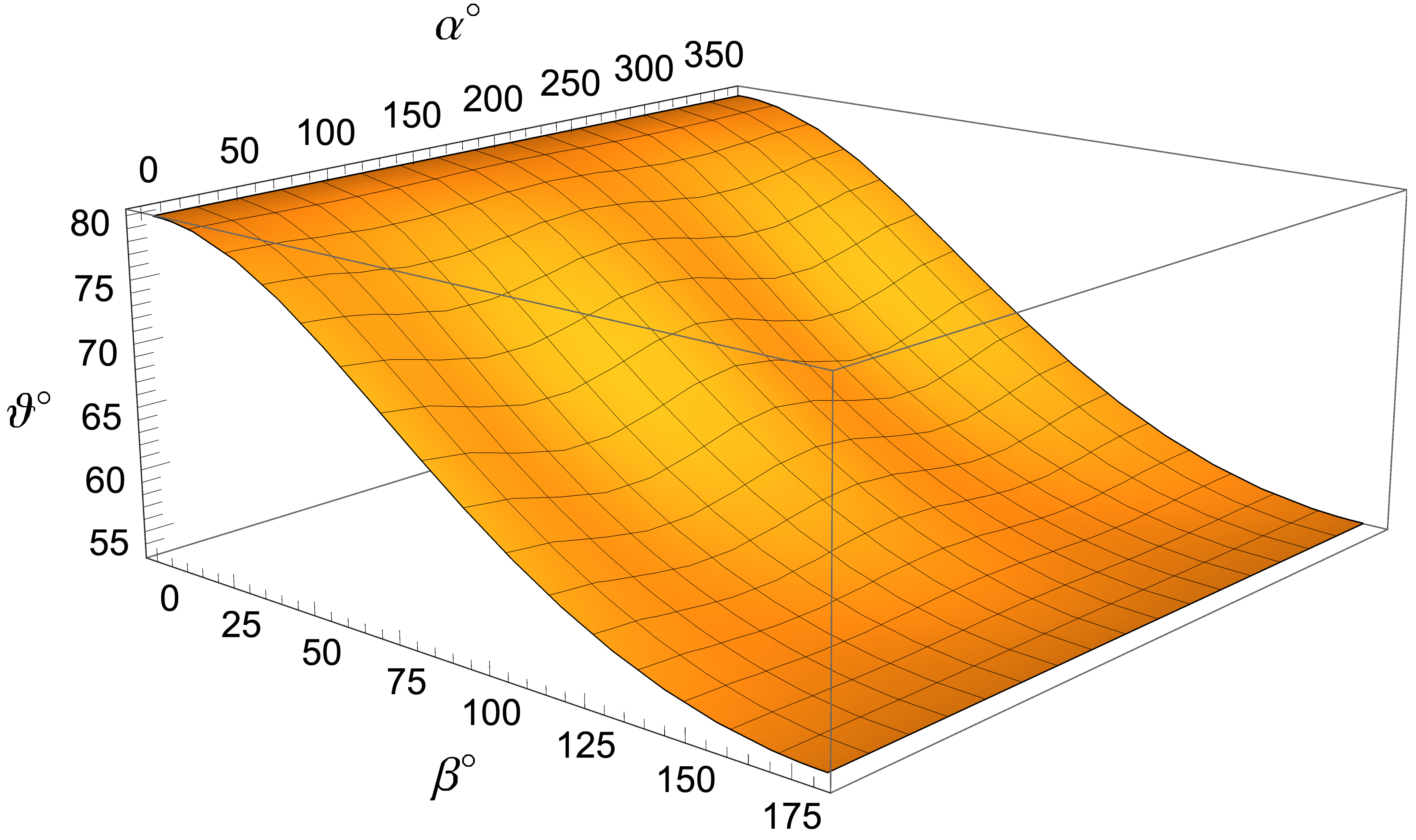
We neglect local terms, *i.e.* powers of q^{2n}!

$$\delta^{(1)}(p,\vec{b}) = -\frac{GME^2}{2|\vec{p}\,|}\sum_{\pm}(1\pm v)^2 \log \mu |\vec{b} \pm \hat{p}\times\vec{a}\,|$$

$$\vartheta^{(1)} = \frac{GM}{v^2}\sum_{\pm}\frac{(1\pm v)^2(b \mp a\cos\beta)}{a^2\sin^2\alpha\sin^2\beta + (b\mp a\cos\beta)^2}$$

Deflection angle vs BH angular momentum orientation

$a = 0.8\ [GM] \quad v = 0.9 \quad b = 4\ [GM]$



Gauge potential contribution

$$i\mathcal{M}^{A}_{on-shell}(p,\vec{q}\,) = -i\frac{4\pi QQ_\phi}{|\vec{q}\,|^2}\left(2E\cos|\vec{a}\times\vec{q}\,| + 2i\sin|\vec{a}\times\vec{q}\,|\frac{\vec{a}\times\vec{q}\cdot\vec{p}}{|\vec{a}\times\vec{q}\,|}\right)$$

$$\delta_A(p,\vec{b}\,) = \frac{QQ_\phi}{2v}\sum_{\pm}(1\pm v)\log|\vec{b}\pm\hat{p}\times\vec{a}\,|$$

$$\vartheta_A(p,\vec{b}\,) = -\frac{QQ_\phi}{v^2E}\sum_{\pm}\frac{(1\pm v)(b\mp a\cos\beta)}{a^2\sin^2\alpha\sin^2\beta + (b\mp a\cos\beta)^2}$$

The gauge potential contribution is dominant for reasonable energies.

In order to be $\frac{QQ_\phi}{Eb}\sim\frac{GM}{b}$ $\longrightarrow$ $E\sim M_p$ for $Q_\phi\sim 1$

In the KN case the integration is more subtle, and a more suitable alternative approach is proposed.

Why? $\Delta\mathscr{M} \sim q^{2n+1}$! $\longrightarrow$

$$\mathscr{M} \sim \mathscr{F}(2,n) \sim \frac{1}{\Gamma(-n)} = 0$$

$$\Delta\mathscr{M} \sim \mathscr{F}(2,n+1/2) \sim \frac{1}{\Gamma(-n-1/2)} \neq 0$$

$$\delta^{(1)}_{KN}(p,\vec{b}\,) = \frac{1}{2}\int \frac{d^3q}{(2\pi)^3} e^{i\vec{q}\cdot\vec{b}}\, 2\pi\delta(\vec{q}\cdot\vec{\ell}\,)\mathscr{M}^{KN}_{on-shell}(p,\vec{q}\,) = \frac{1}{2}\int_{-\infty}^{+\infty} d\xi\ \mathrm{FT}\Big[\mathscr{M}^{KN}_{on-shell}\Big](p,\vec{b}+\xi\vec{\ell}\,)$$

We know exactly the FT

$$\mathrm{FT}\Big[\mathscr{M}^{KN}_{on-shell}\Big] = -\,h_{\mu\nu}(\vec{x})p^\mu p^\nu \longrightarrow$$

$$\delta^{(1)}_{KN}(p,\vec{b}\,) = \frac{1}{2}\int_{-\infty}^{+\infty} d\xi \frac{2GMr^3 - GQ^2r^2}{r^4 + z^2a^2}\Big(K_\mu p^\mu\Big)^2\Big|_{\vec{x}=\vec{b}+\xi\vec{\ell}}$$

From $\delta^{(1)}_{KN} = \delta^{(1)} + \Delta\delta^{(1)}$ we can extrapolate $\Delta\delta^{(1)}$ at arbitrary high-order in $O(a^n)$.

Analytic formulae can be given for special configurations, *e.g.*

$$\Delta\delta^{(1)}(p,\vec{b})\Big|_{\vec{a}=(0,0,a)} = \frac{G\pi Q^2}{4a^2|\vec{p}\,|^2}\left(-\frac{(aE+b|\vec{p}\,|)^2}{\sqrt{b^2-a^2}}+2aE|\vec{p}\,|+|\vec{p}\,|^2 b\right)$$

We conjecture that through analytic continuation an analytic formula at all orders in the spin can be recovered!

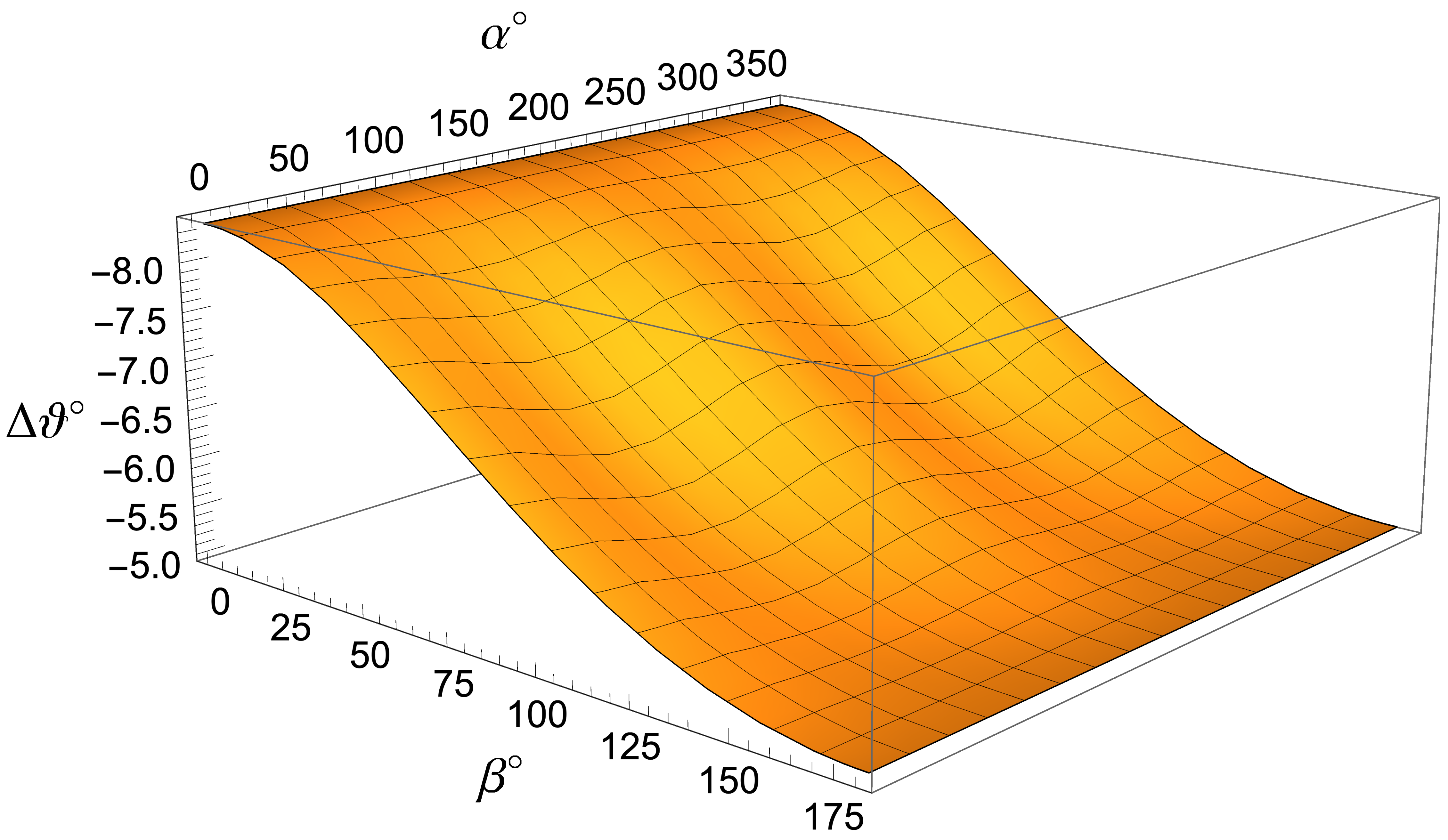
$$\Delta\delta^{(1)}\Big|_{1/b} = -\frac{G\pi Q^2 E}{8vb}(2+v^2)$$

$$\Delta\delta^{(1)}\Big|_{1/b^2} = -\frac{G\pi Q^2 aE\cos\beta}{4b^2}$$

$$\Delta\delta^{(1)}\Big|_{1/b^3} = -\frac{G\pi Q^2 a^2 E}{128vb^3}\Big(8+5v^2+(8+7v^2)\cos 2\beta +4(2+v^2)\cos 2\alpha\sin^2\beta\Big)$$

Deflection angle vs BH angular momentum orientation

$a = 0.8\ [GM]$ $Q = 0.8\ [G^{1/2}M]$ $v = 0.9$ $b = 4\ [GM]$



Comparison with the literature

$$\Delta\vartheta^{(1)}\Big|_{1/b^2} = -\frac{G\pi Q^2}{4v^2b^2}(2+v^2)$$

$$\Delta\vartheta^{(1)}\Big|_{1/b^3} = -\frac{G\pi Q^2 a\cos\beta}{vb^3}$$

$$\Delta\vartheta^{(1)}\Big|_{1/b^4} = -\frac{3G\pi Q^2a^2}{64v^2b^4}\Big(8+5v^2+(8+7v^2)\cos 2\beta + 4(2+v^2)\cos 2\alpha\sin^2\beta\Big)$$

We obtain the scattering angle at $O(G)$, but exact in the spin of the BH and in a **generic orientation**!

There exist results in literature at arbitrary high order in G, exact in a, but **only for equatorial scattering**!

These are obtained by geodesic calculations.

(n,k)	$\chi_n^{(k)}/\frac{G^nQ^{2k}M^{n-k}}{v^{2n}(b^2-a^2)^{(3n+k-1)/2}}$
(1,0)	$2\left(-2av+bv^2+b\right)$
(1,1)	$\pi/(2a^2)\left(2a^3v-a^2\left(-v^2\sqrt{b^2-a^2}+2bv^2+b\right)+b^2v^2\left(b-\sqrt{b^2-a^2}\right)\right)$
(2,0)	$\pi/(2a^2)\big[-4a^5v-4a^3b^2v\left(3v^2+2\right)+2a^2b^2v^2\left(b\left(2v^2+3\right)-v^2\sqrt{b^2-a^2}\right)$ $+b^4v^4\left(\sqrt{b^2-a^2}-b\right)+a^4\left(v^4\sqrt{b^2-a^2}+3b\left(4v^2+1\right)\right)\big]$
(2,1)	$(8a^3+24ab^2)(1+v^2)v+(-6a^2b-2b^3)(1+6v^2+v^4)$
(2,2)	$-(3\pi)/(16a^4)\big[4a^7\left(2v^3+v\right)+4a^5b^2v\left(3v^2+4\right)+2b^6v^4\left(b-\sqrt{b^2-a^2}\right)+a^2b^4v^4\left(6\sqrt{b^2-a^2}-7b\right)$ $-a^6\left(b\left(8\left(v^2+3\right)v^2+3\right)-2v^4\sqrt{b^2-a^2}\right)+2a^4b^2\left(b\left(4v^4-3v^2-1\right)-3v^4\sqrt{b^2-a^2}\right)\big]$

[Hoogeveen, 2303.00317]

Conclusions

- We have seen why and how loop scattering amplitudes contain classical physics
- We introduced the Kerr-Schild gauge and computed analytically the FT of the Kerr-Newman metric
- This result is then used to obtain the scattering angle of a scalar probe scattering off a Kerr-Newman BH

Future directions

- Results for the $2PM$ eikonal phase
- Extend the analysis to probe with spin, in particular the $s = 2$ case is phenomenologically relevant for the scattering of GWs
- Understand if and how it is possible to generalize such approach to the 2-to-2 scattering of KN BHs

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Thank you!