Introductions to accelerators II: Transverse Dynamics

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Linear Dynamics - continued

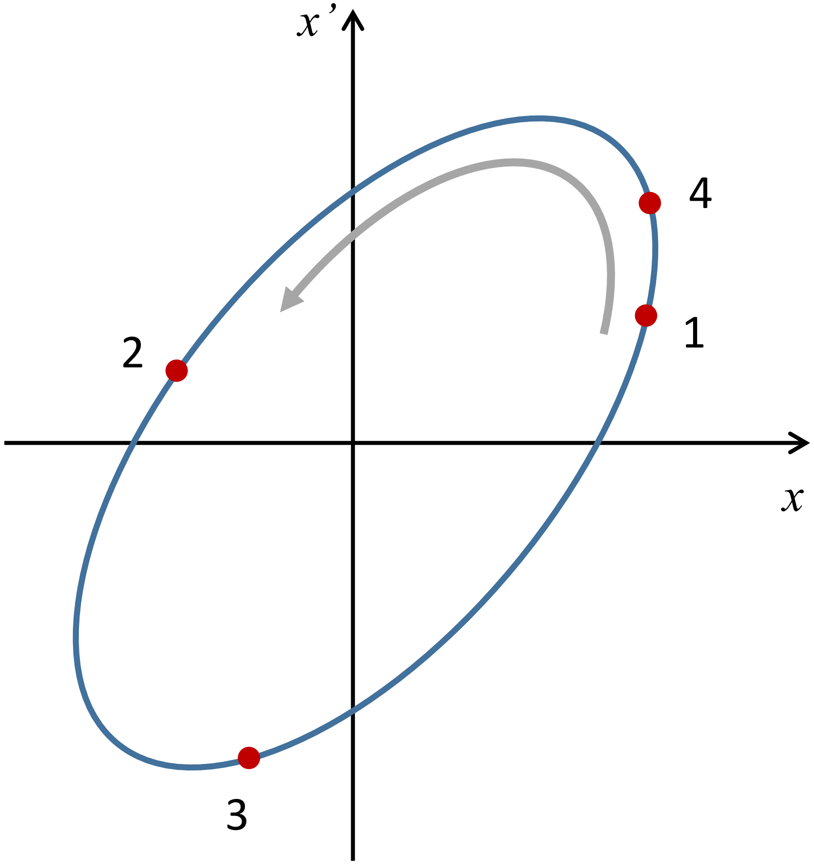
practical questions to be answered:

- ✓ How to ensure bound motion of a particle beam?
- ✓ What are conditions for stability?
- ✓ Amplitude and frequency of particle oscillations?
- Statistical beam properties like beam width and angular spread?
- How to design magnet lattices (arrangements of dipoles and quads in a line)?
- Chromaticity
- What is the impact of field errors in bending and focusing magnets?

Statistical Beam Properties

- Single particle Phase space ellipse
- Courant-Snyder Invariant of motion
- Particle Action versus beam emittance
- Emittance and distribution function
- Liouville theorem
- consequences of conservation of emittance

Phase Space Ellipse



$$x(s) = \sqrt{2J\beta}\cos(\varphi)$$

$$x'(s) = -\sqrt{\frac{2J}{\beta}}\left(\alpha\cos(\varphi) + \sin(\varphi)\right)$$

J = particle action (oscillation amplitude)

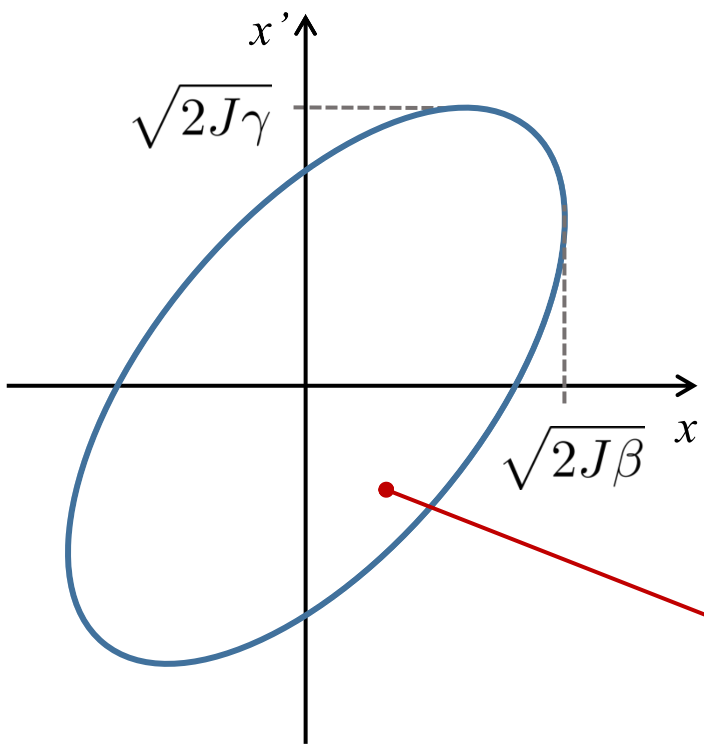
observing the coordinates of a particle at one location in a ring for consecutive turns

x, x' describe an ellipse in phase space when φ is varied

example $\mu_x \approx 0.37 \times 2\pi$ phase advance per turn

see also Schmüser/Rossbach sect. 4.3/4.4, Wiedemann chap. 8.1.2

Courant-Snyder Invariant



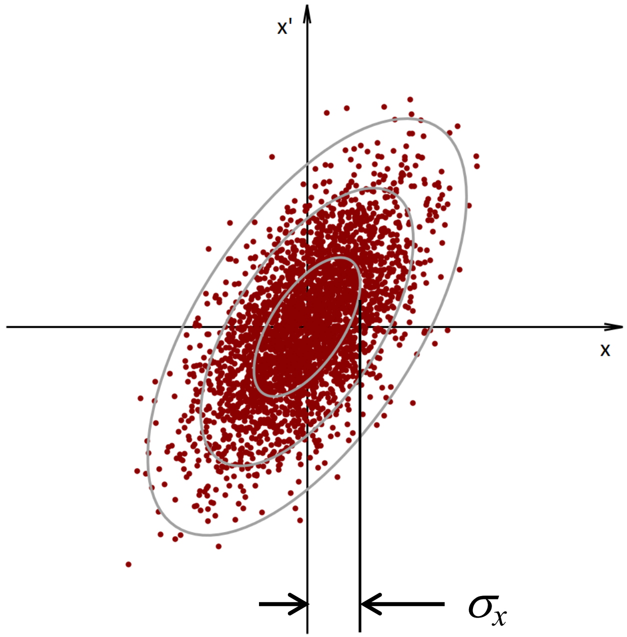
- at one location this phase space ellipse is sampled by a particle with action J
- when moving along the magnet lattice, the ellipse is changing shape, but it stays an ellipse
- the area of the ellipse is invariant (Courant-Snyder Invariant) and equals 2π × action.
- the unit is [m·rad], or often [mm·mrad]
- this constant of motion refers to a **single particle**, **emittance** is a statistical property

$$\text{area} = 2\pi J = \pi(\gamma x^2 + 2\alpha x x' + \beta x'^2)$$

$$\text{with}: \beta\gamma - \alpha^2 = 1$$

E.D. Courant and H.S. Snyder, Theory of the Alternating Synchrotron, Annals of Physics 3 (1958) 1

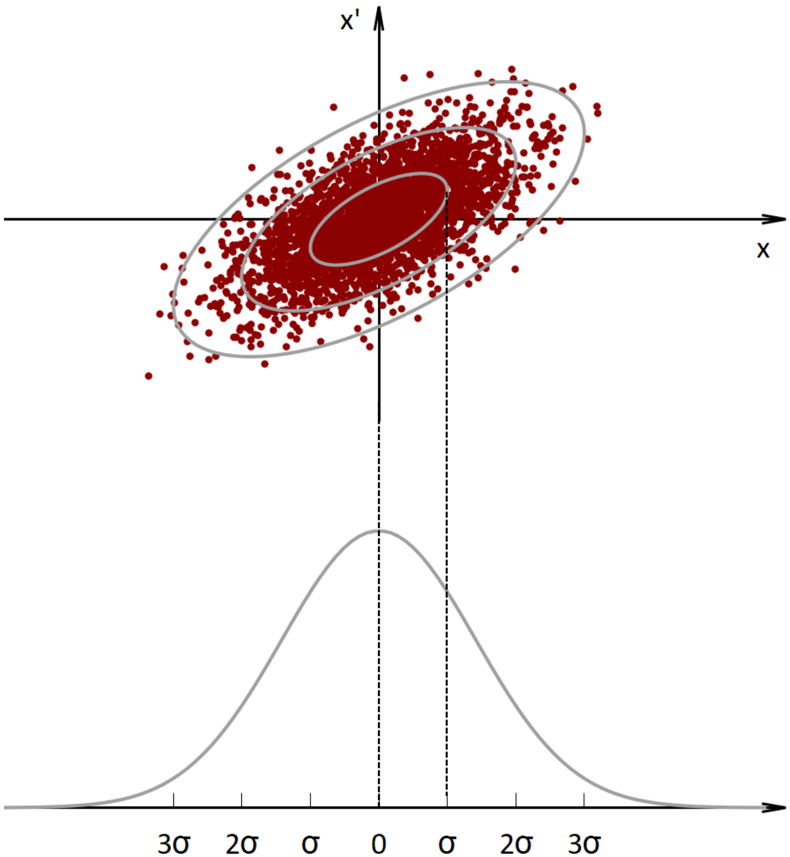
Action vs. Emittance



- single particles are associated with a particular ellipse
- when observing a beam with a wire scanner a (projected) rms (root mean square) width σ_x is measured; one ellipse corresponds to σ_x
- emittance ε is the average value of action *J*

$$\boxed{\varepsilon = \langle J \rangle}$$

Beam Emittance



beam emittance as statistical property:

$$\varepsilon_x = \sqrt{< x^2 >< x'^2 > - < xx' >^2}$$

two-dimensional Gaussian distribution:

$$f(x, x') = \frac{1}{2\pi\varepsilon_x} \exp\left(-\frac{\gamma x^2 + 2\alpha x x' + \beta x'^2}{2\varepsilon_x}\right)$$

projected Gaussian distribution:

$$f(x) = \frac{1}{\sqrt{2\pi}\sqrt{\beta_x \varepsilon_x}} \exp\left(-\frac{x^2}{2\beta_x \varepsilon_x}\right)$$

see Wiedemann sec. 8.1.3

Fractions of Beam in rms widths

From practical measurements (wire scan, beam screen) the projected rms width of the beam is determined.
What fraction of beam is contained in $n \times \sigma_{\mathrm{rms}}$?

compute the beam fraction inside an ellipse corresponding to n$\times\sigma_x$:

$$J(x = n\sigma_x) = \frac{n^2}{2}\varepsilon_x$$

$$r = \frac{1}{2\pi\varepsilon_x}\int_0^{\frac{n^2}{2}\varepsilon_x}\int_0^{2\pi}\exp\left(-\frac{J}{\varepsilon_x}\right)\, dJ\, d\varphi$$

note: This applies for a two dimensional Gaussian distribution.

$$\boxed{r = 1 - \exp(-n^2/2)}$$

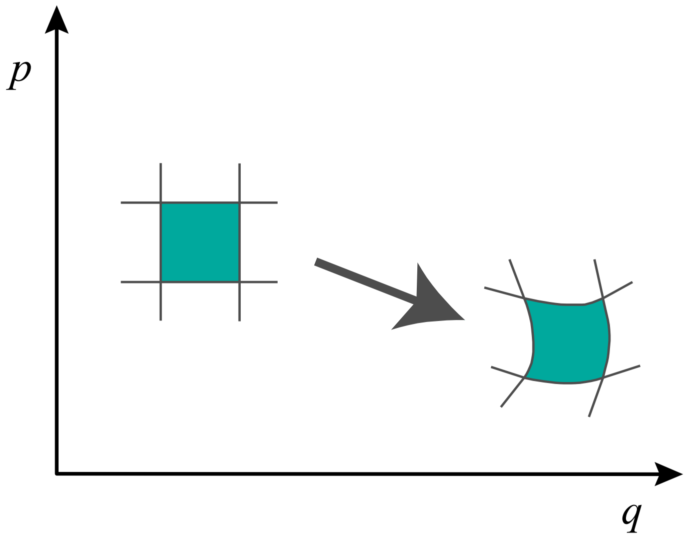
rms width n	beam fraction r
1	**39%**
2	**86%**
3	**99%**

Liouvilles Theorem

Beams subject to conservative forces as in our accelerator (without dissipative forces i.e. synchrotron radiation) → preserve the phase space density over time

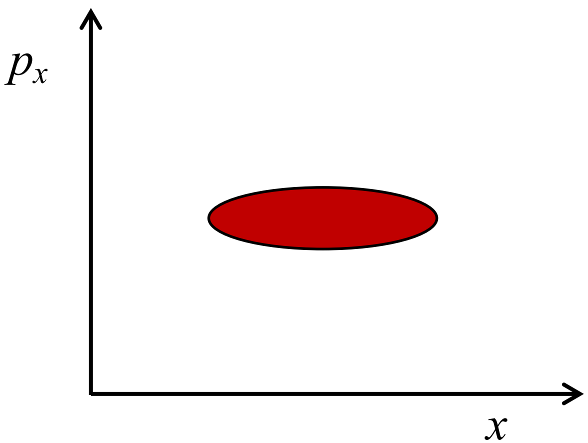
the phase space density is conserved $\boldsymbol{\frac{d\psi}{dt} = 0}$

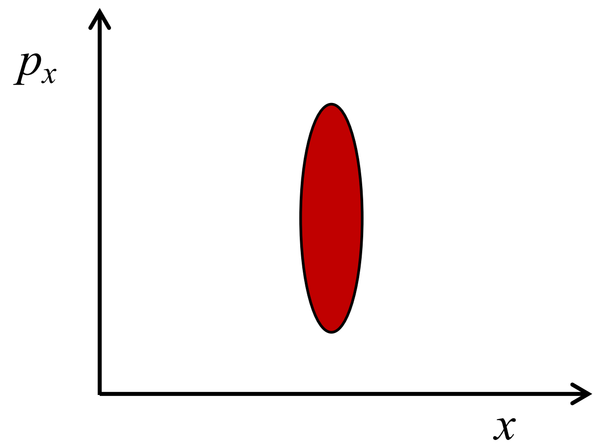
The phase space density behaves like an incompressible liquid.



see also Wiedemann sec. 8.1.1

Conservation of Emittance

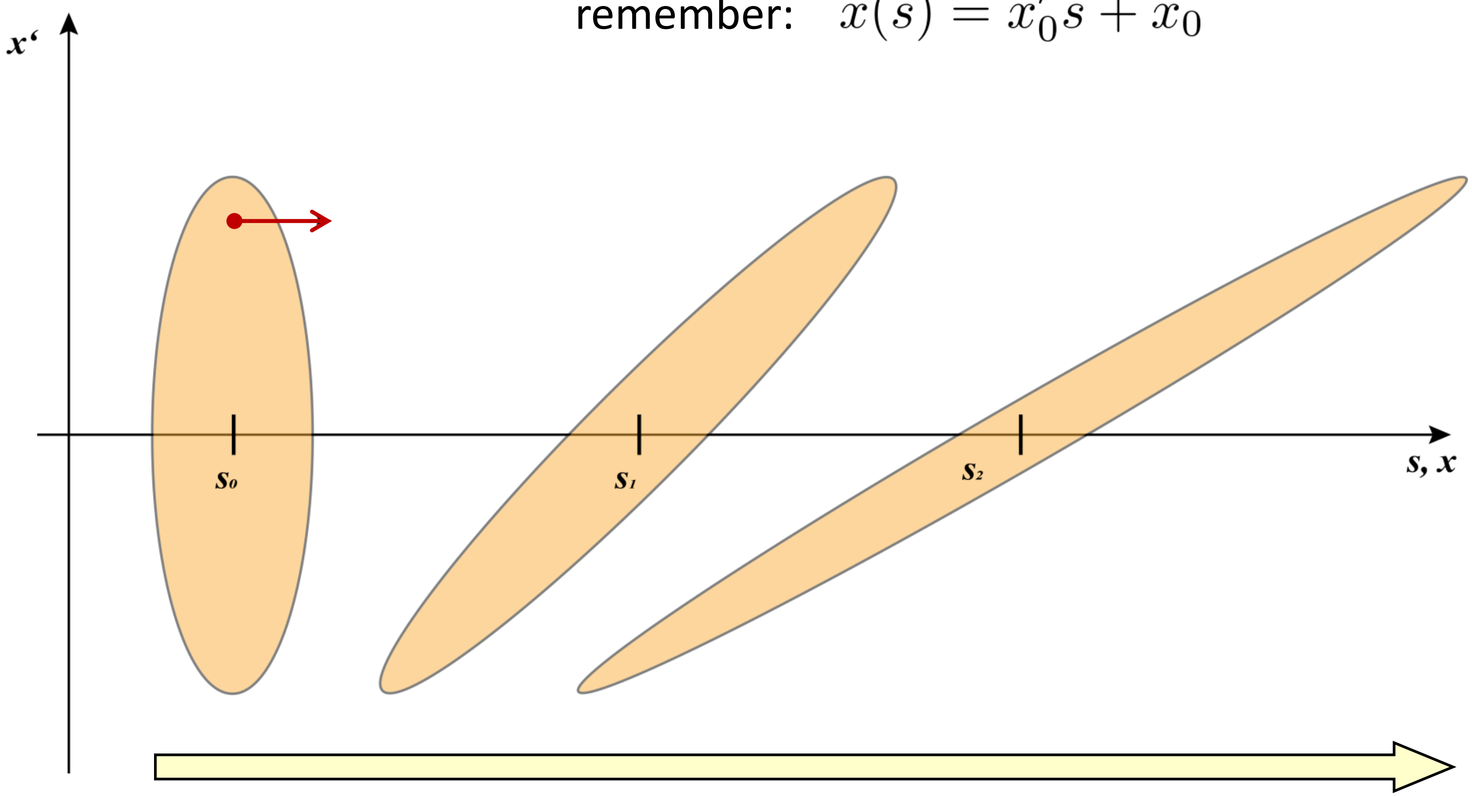




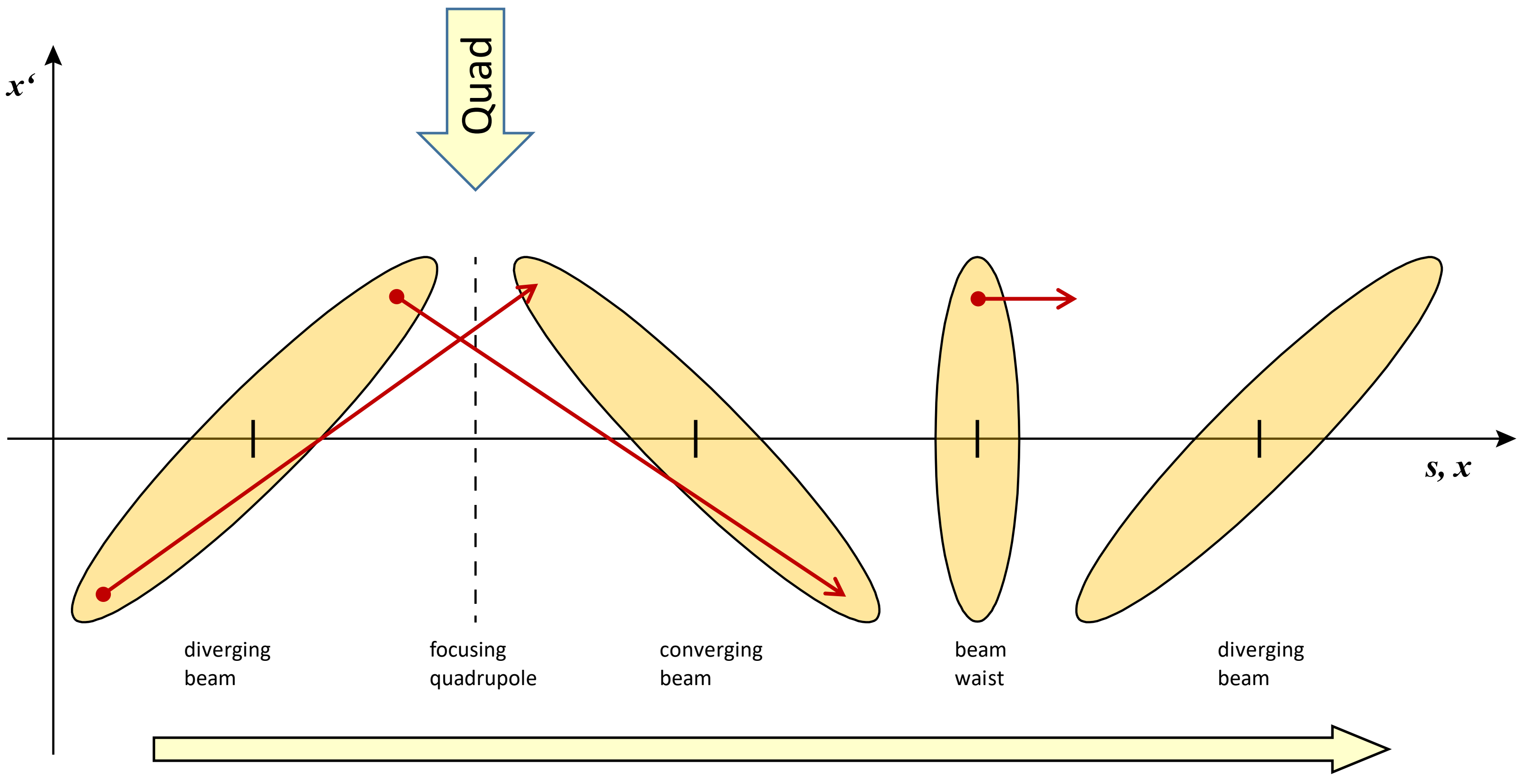
with a given emittance a beam can be made small with large angular spread, or can have small angular spread with a large size

Phase Space Ellipse in Drift Space

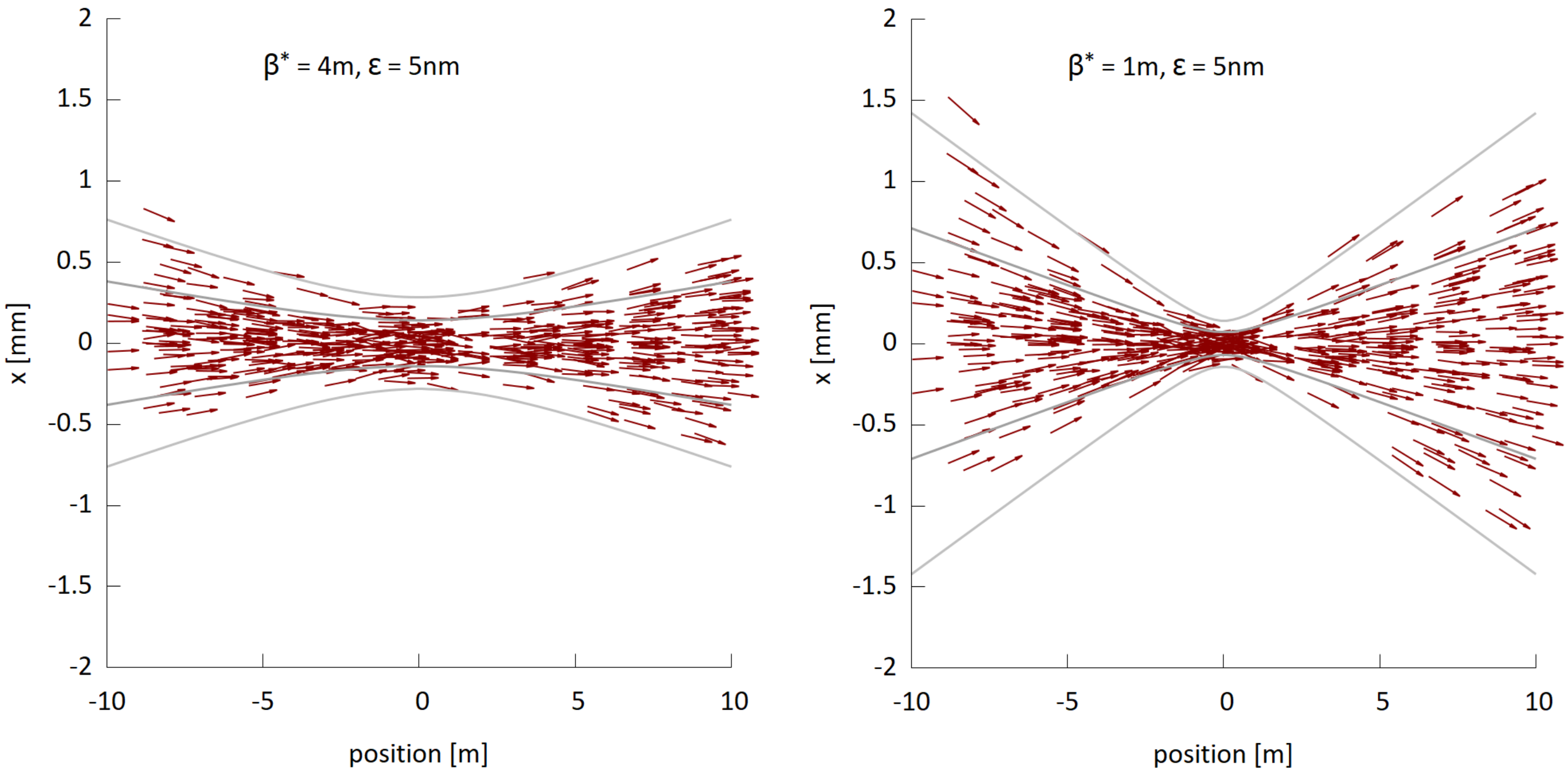
remember: $x(s) = x'_0 s + x_0$



Phase Space Ellipse after focusing



Beam Waist (e.g. interaction point collider)

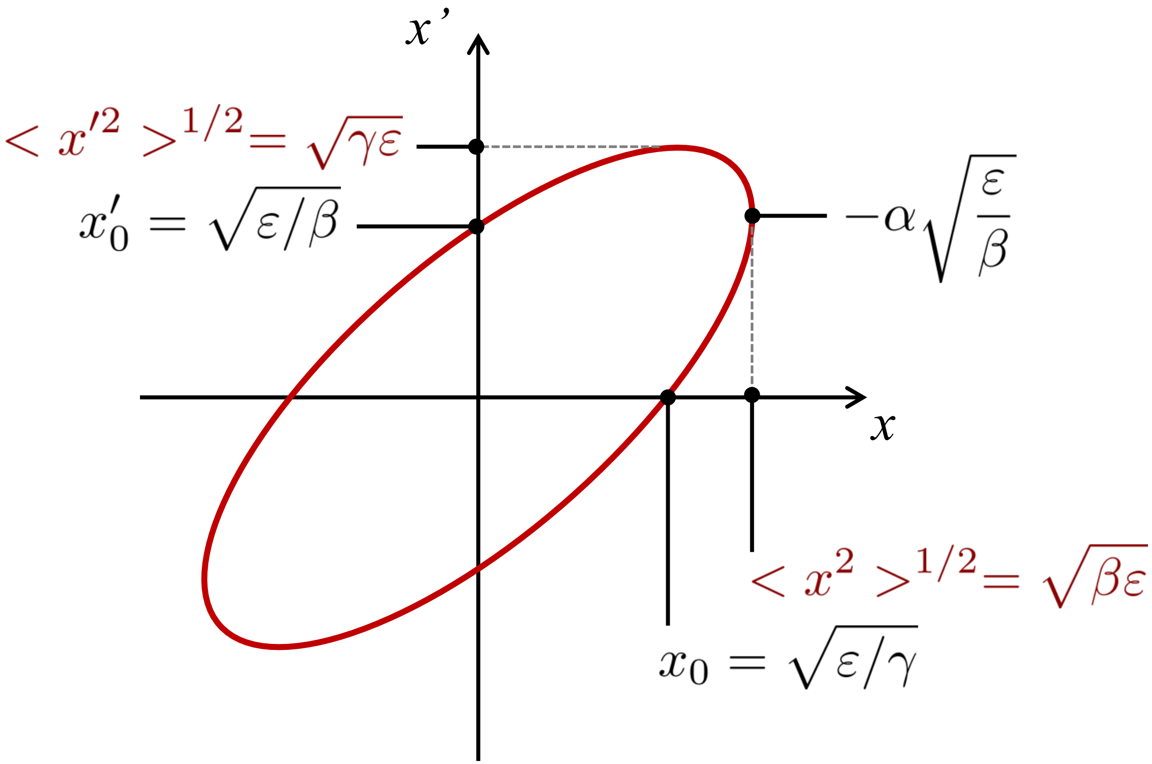


$$\beta(s) = \beta^* + \frac{s^2}{\beta^*}$$

$$\sigma_{\mathrm{rms}} = \sqrt{\varepsilon\beta^*},\ \sigma'_{\mathrm{rms}} = \sqrt{\frac{\varepsilon}{\beta^*}}$$

β* = Beta function at waist

Phase Space Ellipse - Parameters



reminder:

$$\gamma = \frac{1+\alpha^2}{\beta}$$

$$\alpha = -\frac{1}{2}\beta'$$

for upright ellipse:

$$< xx' > = -\alpha = 0$$

Remarks on Beam Distributions

Electrons

in a ring electrons radiate photons which continuously mixes particles in phase space and generates an equilibrium Gaussian distribution

i.e. a large injected beam will shrink to equilibrium while a small beam will grow

Protons, Ions

“protons never forget” G.Voss

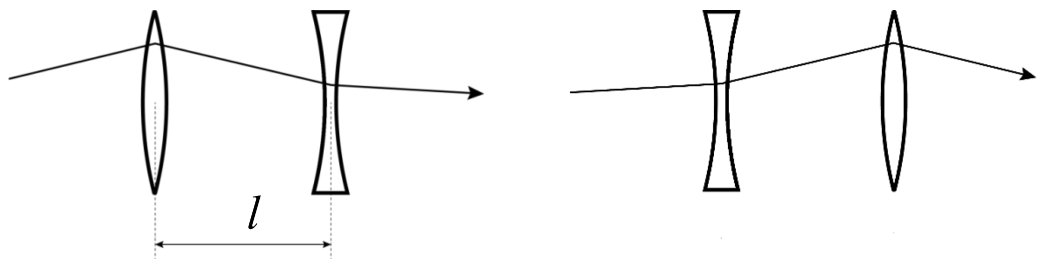
can have “strange” distributions since those depend on the history of beam generation and acceleration; i.e. no damping mechanism

however: in practice often close to Gaussian distribution

Next: FODO Lattices

- FODO parameter space
- FODO with bending magnets

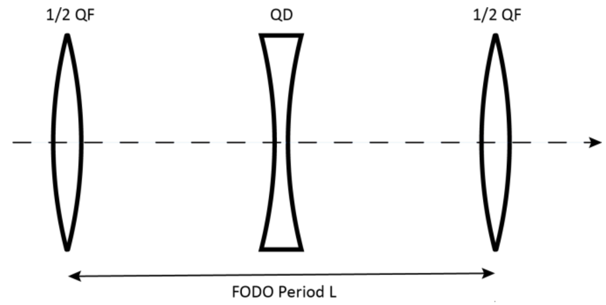
Reminder: Quadrupole Doublet



$$\boldsymbol{M}_{\mathrm{doublet}} = \begin{pmatrix} 1+\frac{l}{f} & l \\ -\frac{1}{f^*} & 1-\frac{l}{f} \end{pmatrix}$$

$$f^* = \frac{f^2}{l} > 0$$ $\rightarrow \mathrm{M}_{\mathrm{doublet}}$ is always focusing

FODO Cell



$$\boldsymbol{M}_{\mathrm{FODO}} = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix} = \begin{pmatrix} 1 - \frac{L^2}{8f^2} & L\left(1 + \frac{L}{4f}\right) \\ -\frac{1}{f^*} & 1 - \frac{L^2}{8f^2} \end{pmatrix}, \quad \frac{1}{f^*} = \frac{L}{4f^2}\left(1 - \frac{L}{4f}\right)$$

Unit sequence of magnets used to build an accelerator
Alternating gradients → net focusing!

FODO Cell II

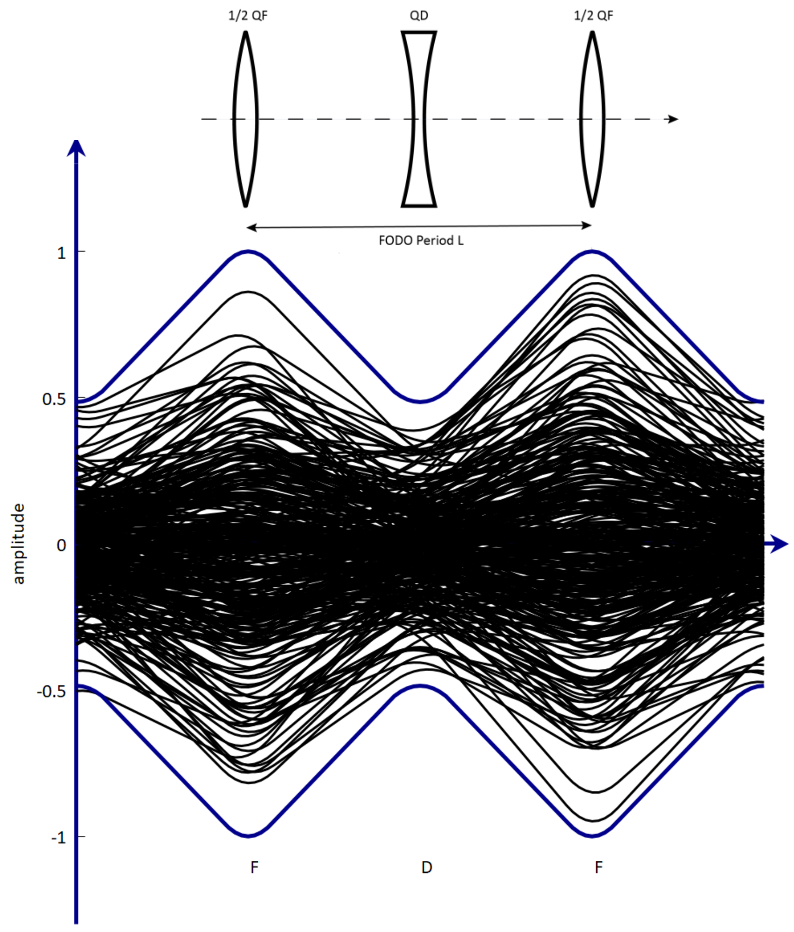


illustration:
particle trajectories of varying phase and amplitude in a FODO cell

Gaussian (projected) profile

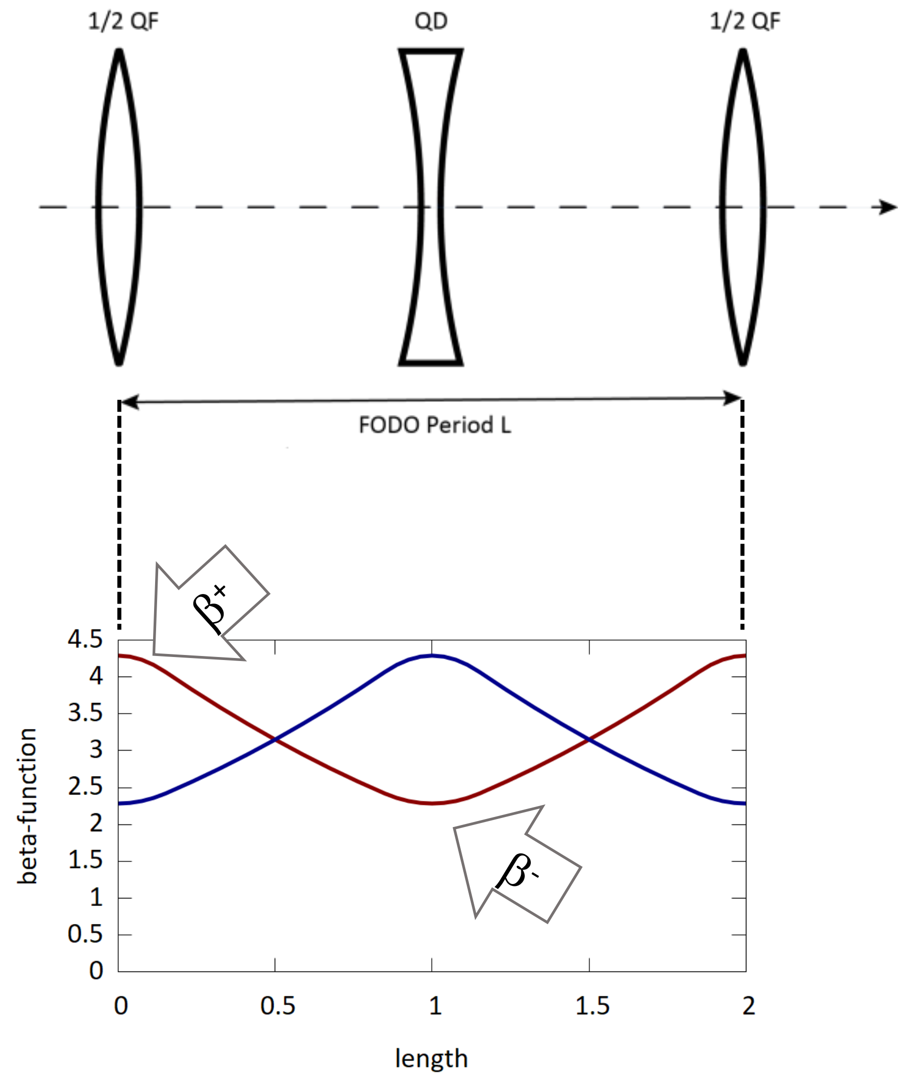
FODO Cell Parameters

we obtain for β+ in the focusing quad and β- in the defocusing:

$$\beta^{\pm} = \frac{L}{2\sin\mu/2}\sqrt{\frac{1\pm\sin\mu/2}{1\mp\sin\mu/2}}$$

phase advance per cell:

$$\sin(\mu/2) = \frac{L}{4f}$$



see Wiedemann sec. 10.1

FODO Cell: choice of phase advance

β^+ reaches minimum at μ_{opt}=76.3deg
at this point the vacuum chamber
needs a minimal size

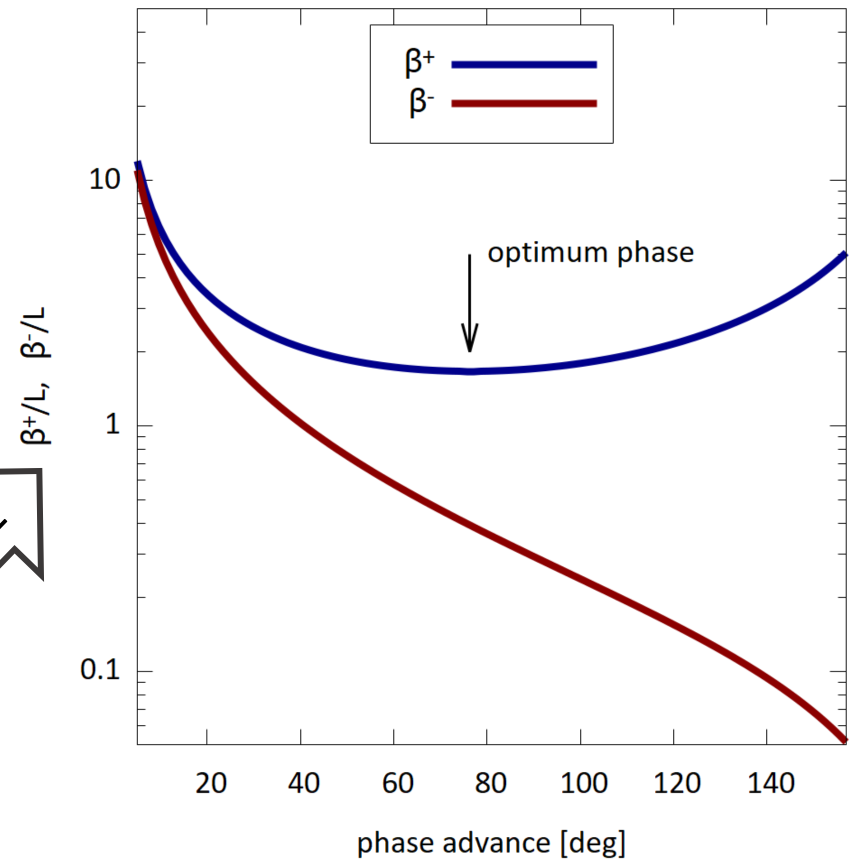
$$\sin(\mu_{\rm opt}) = \frac{2}{1+\sqrt{5}}$$

Example LEP(CERN)
operating modes:

μ_x/μ_y [degrees]:
60/60, 90/60, 90/90, 102/90

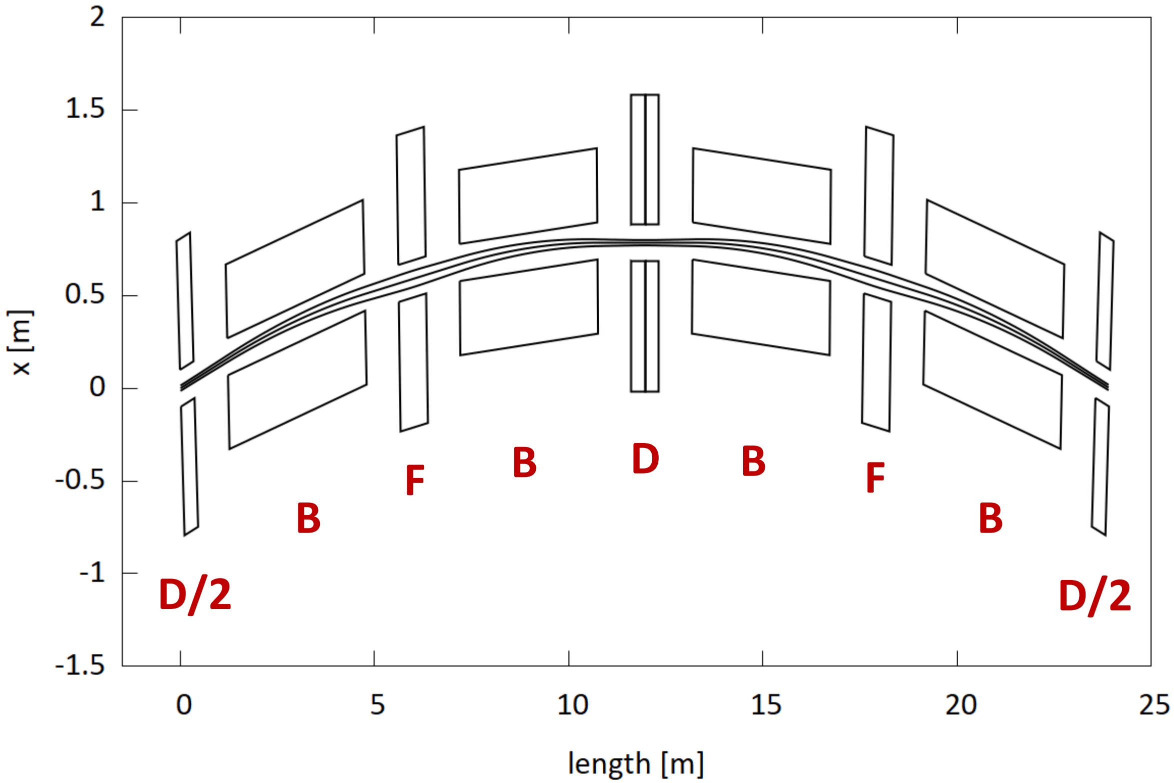
(stronger foc. = smaller emittance)

note scaling per L



FODO Cell with Bending Magnets

FODO structure with bending magnets to form a ring - the standard scheme for synchrotrons



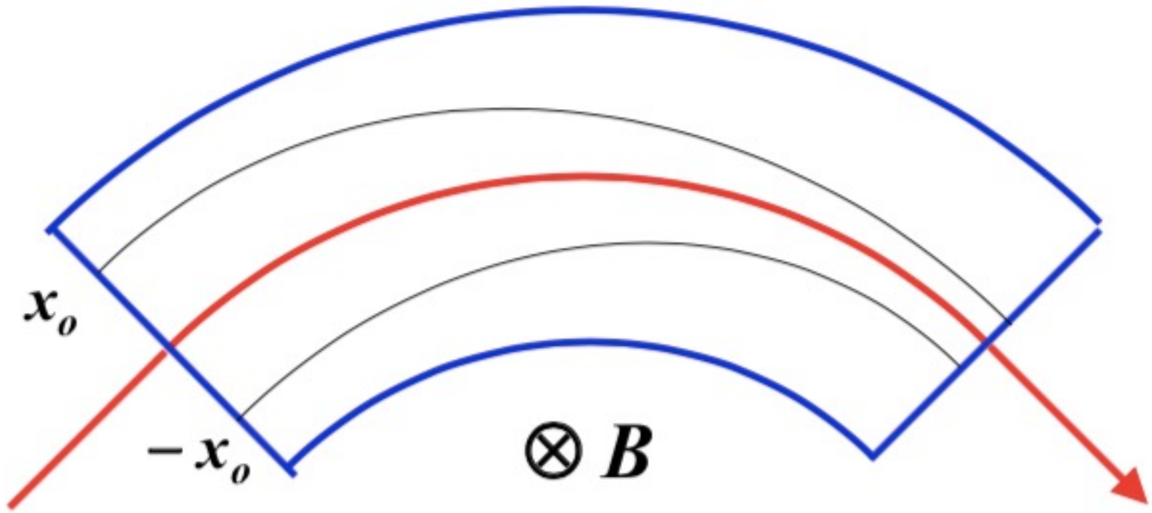
example:

- **2 FODO cells**
- **ρ = 55m**
- **separated function**

$$
\begin{aligned}
x'' + \left(\frac{1}{\rho^2} + k\right) x &= \frac{1}{\rho}\frac{\Delta p}{p_0} \\
y'' - ky &= 0
\end{aligned}
$$

SECTOR BENDING MAGNET

$$x'' + \left(\frac{1}{\rho^2} + k\right) x = \frac{1}{\rho}\frac{\Delta p}{p_0}$$
$$y'' - ky = 0$$



Positive displacement x_o of the initial coordinate from the center axis leads to a longer path inside the magnet, i.e. more deflection

Negative displacement $-x_o$ of the initial coordinate from the center axis leads to a shorter path inside the magnet, i.e. less deflection

In both cases the trajectory comes closer to the central orbit → **FOCUSING**

Weak focusing can be neglected for machines with large bending radius

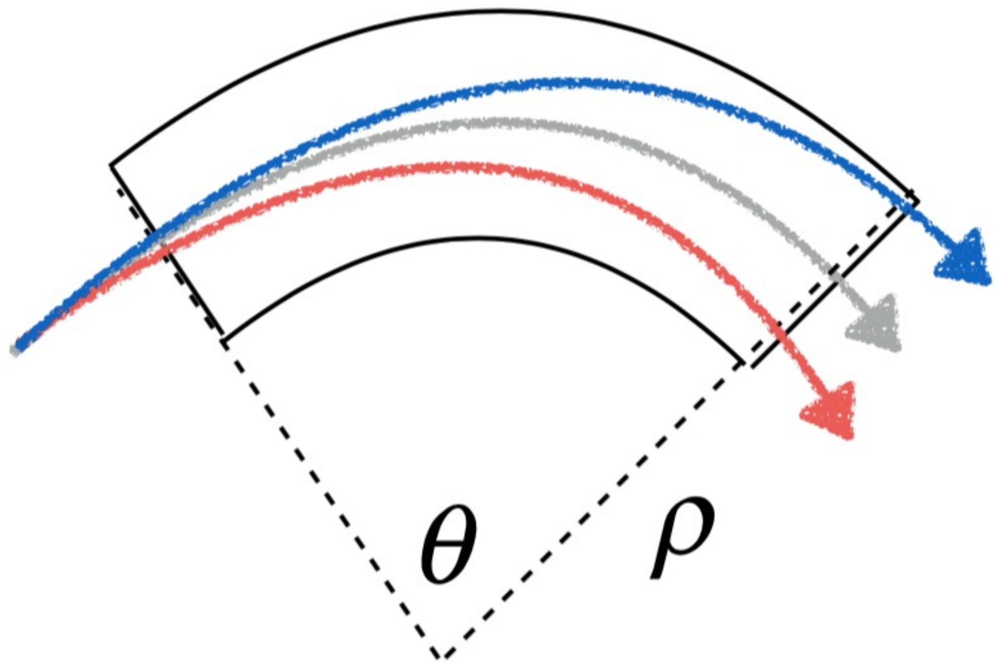
Dispersion

....for particles with δ energy equations of motion change!

$$\delta \neq 0$$

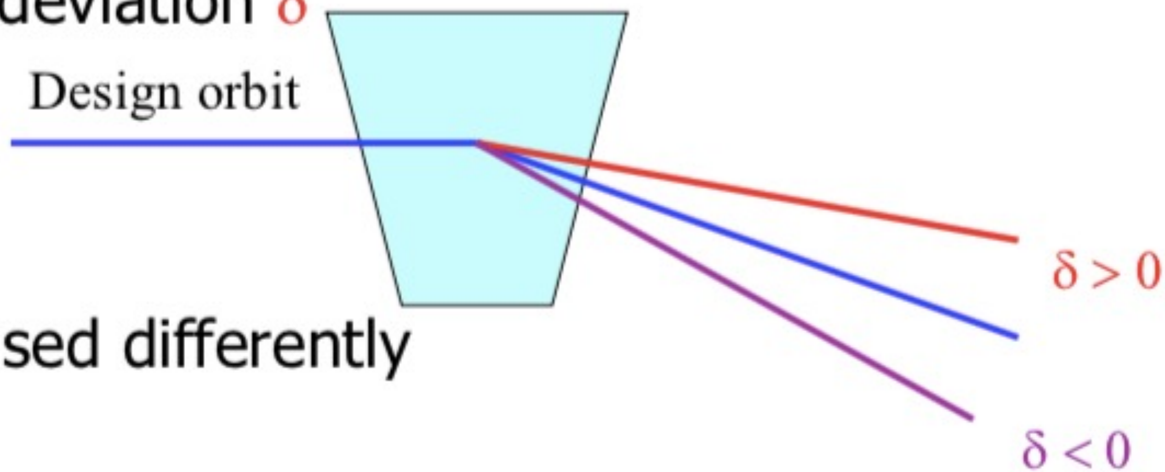
$$\boxed{\begin{aligned} x'' + \left(\frac{1}{\rho^2} + k\right) x &= \frac{1}{\rho}\frac{\Delta p}{p_0} \\ y'' - ky &= 0 \end{aligned}}$$

Bending in a dipole changes with the particle energy...



Off-momentum particles

Particle with an energy deviation δ



- Will be bent and focused differently
- The equation of motion: non-homogeneous Hill equation

$$\boxed{x'' + k(s)\,x = \frac{\delta}{\rho}}$$

- The motion is a sum of the
  solution of homogeneous equation + a particular solution

Dispersion function

Particle deviation from ideal orbit

$$x = x_\beta + x_\varepsilon = x_\beta + D(s) \cdot \delta$$

D(s) - dispersion function

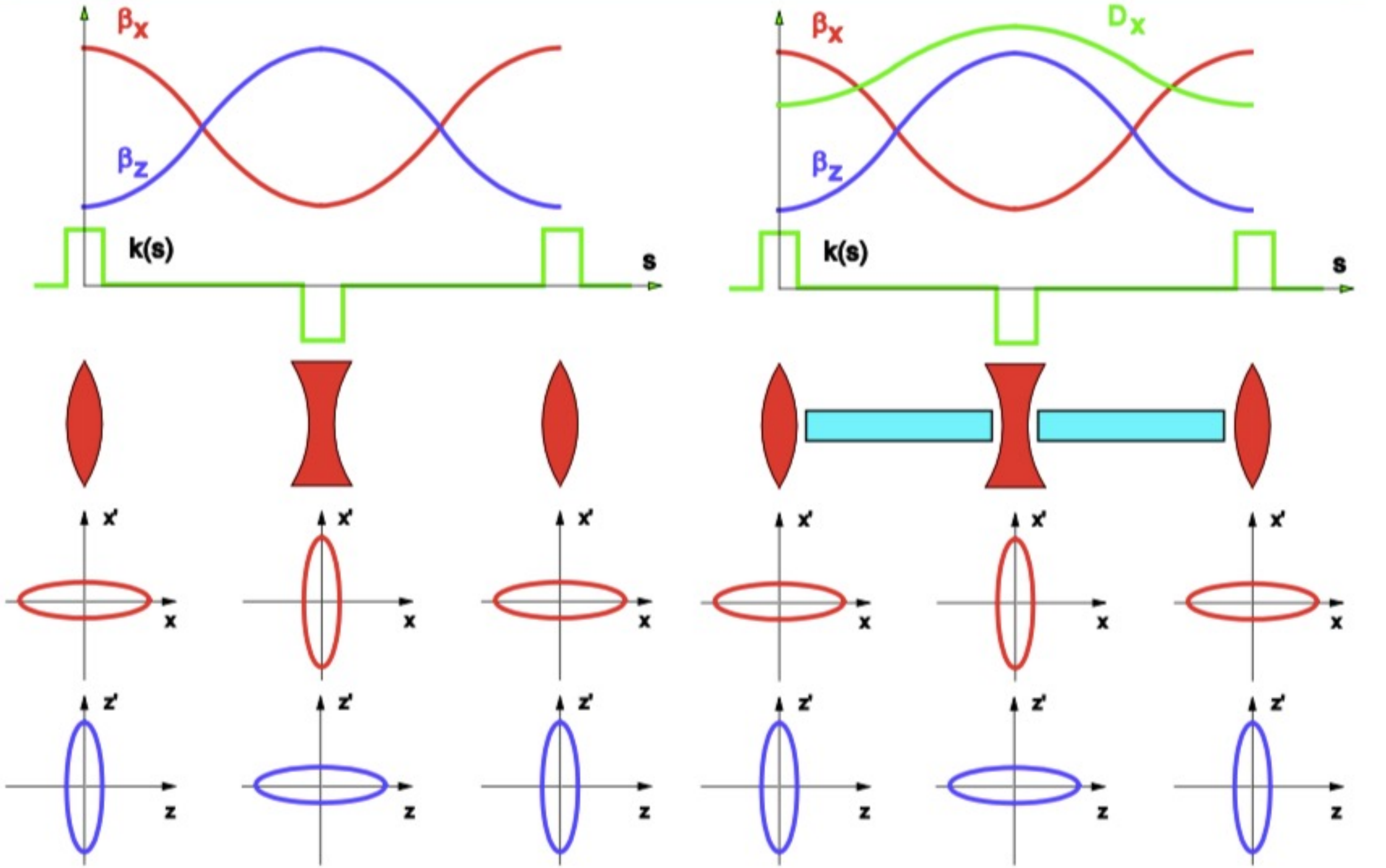
- Periodic solution of the inhomogeneous Hill equation

$$D'' + k(s)D = \frac{1}{\rho(s)} \quad \left\{ \begin{aligned} &= 0 \text{ in straights} \\ &= \frac{1}{\rho} \text{ in bends} \end{aligned} \right\}$$

- New equilibrium orbit of a particle with energy deviation δ
- Betatron oscillations are executed around this new equilibrium

FODO cell lattice

D(s) is a periodic function in FODO cells



Dispersion Function in a Ring

the dispersion function at position s is calculated by integrating over contributions from bending magnets ($1/\rho \neq 0$) around the ring:

$$D(s) = \frac{\sqrt{\beta(s)}}{2\sin(\pi Q)} \oint dt\ \frac{\sqrt{\beta(t)}}{\rho(t)} \cos\left(\varphi(t) - \varphi(s) - \pi Q\right)$$

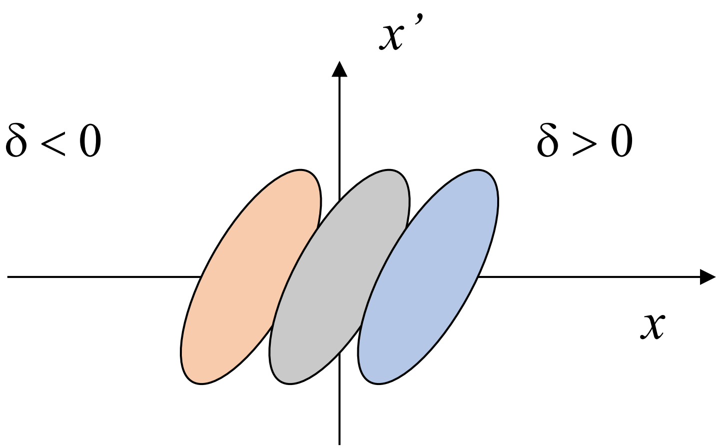
$$D'(s) = \frac{1}{2\sqrt{\beta(s)}\sin(\pi Q)} \oint dt\ \frac{\sqrt{\beta(t)}}{\rho(t)} \left(\alpha(s)\cos(\dots) + \sin(\dots)\right)$$

D, D' are periodic functions:

$$D(s+C) = D(s),\ \ D'(s+C) = D'(s)$$

Beam size with finite momentum spread

besides emittance also momentum spread may contribute to beam size and angular spread, via dispersion function; when the beam momentum spread is δ:



quadratic addition of transverse and longitudinal contributions:

$$\sigma_{\text{tot}}^2 = \sigma_\varepsilon^2 + \sigma_\delta^2 = \varepsilon\beta + D^2 \frac{\delta p^2}{p_0^2}$$

$${\sigma'_{\text{tot}}}^2 = {\sigma'_\varepsilon}^2 + {\sigma'_\delta}^2 = \varepsilon \frac{1+\alpha^2}{\beta} + D'^2 \frac{\delta p^2}{p_0^2}$$

at some locations the momentum contribution should be suppressed by designing for D=0, D'=0

examples:

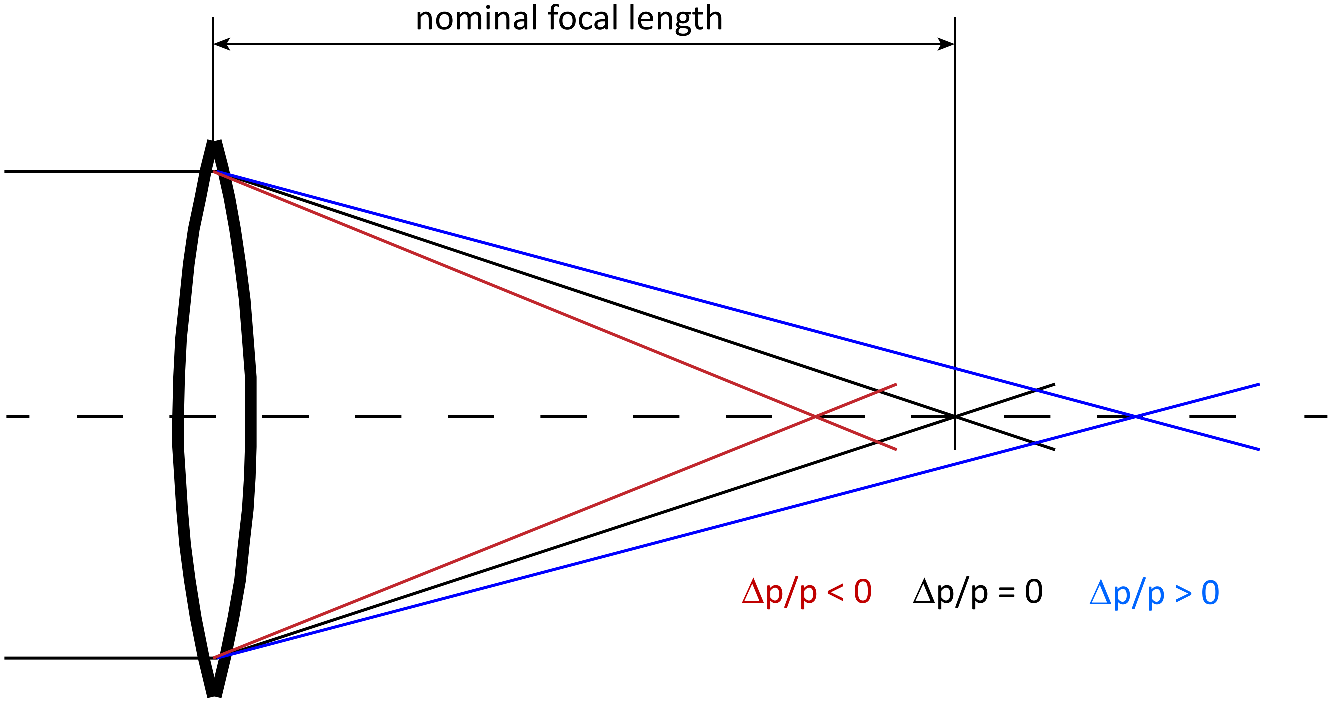
- interaction point in a collider where beams should be as small as possible
- undulators/source magnets, where divergence of emitted radiation should be small

Next: Chromatic Focusing Error

- Focusing Error - What happens?
- Chromaticity
- Correction using Sextupole Magnets

Chromatic Errors

a spread of momentum leads to chromatic aberrations, similarly to aberrations of optical lenses:



Chromaticity

particles with momentum deviation are focused differently, leading to a shift of the betatron frequency

$$K = \frac{eg}{p} \quad dK = -\frac{eg\,dp}{p^2} = -K_0\frac{dp}{p}$$

Chromaticity ξ = change of tune per relative change of momentum:

$$\Delta Q = \xi\,\frac{\Delta p}{p_0}$$

integration over gradients around ring, betafunction as “sensitivity factor”:

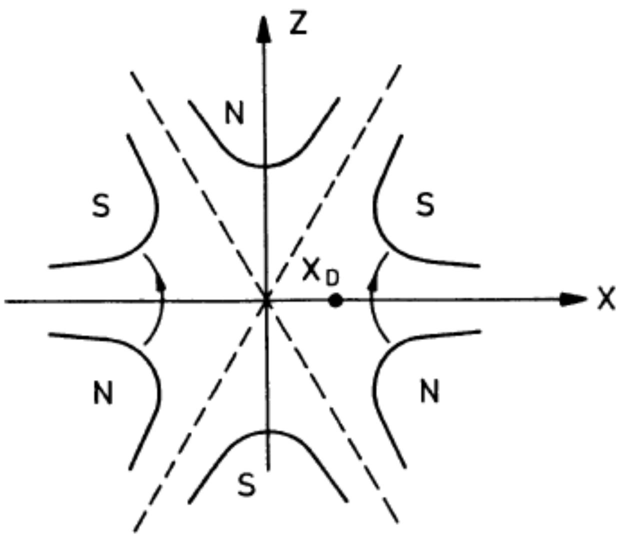
$$\xi_x = -\frac{1}{4\pi}\oint K(s)\beta_x(s)ds$$

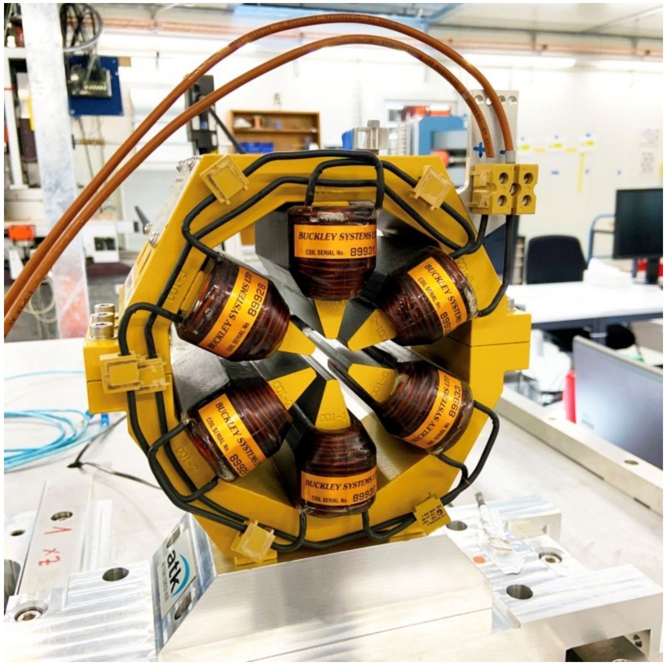
see Wiedemann sec. 15.4.1

Sextupol Magnet

Sextupoles are placed in a region of finite dispersion:
sort particles according to their energy deviation

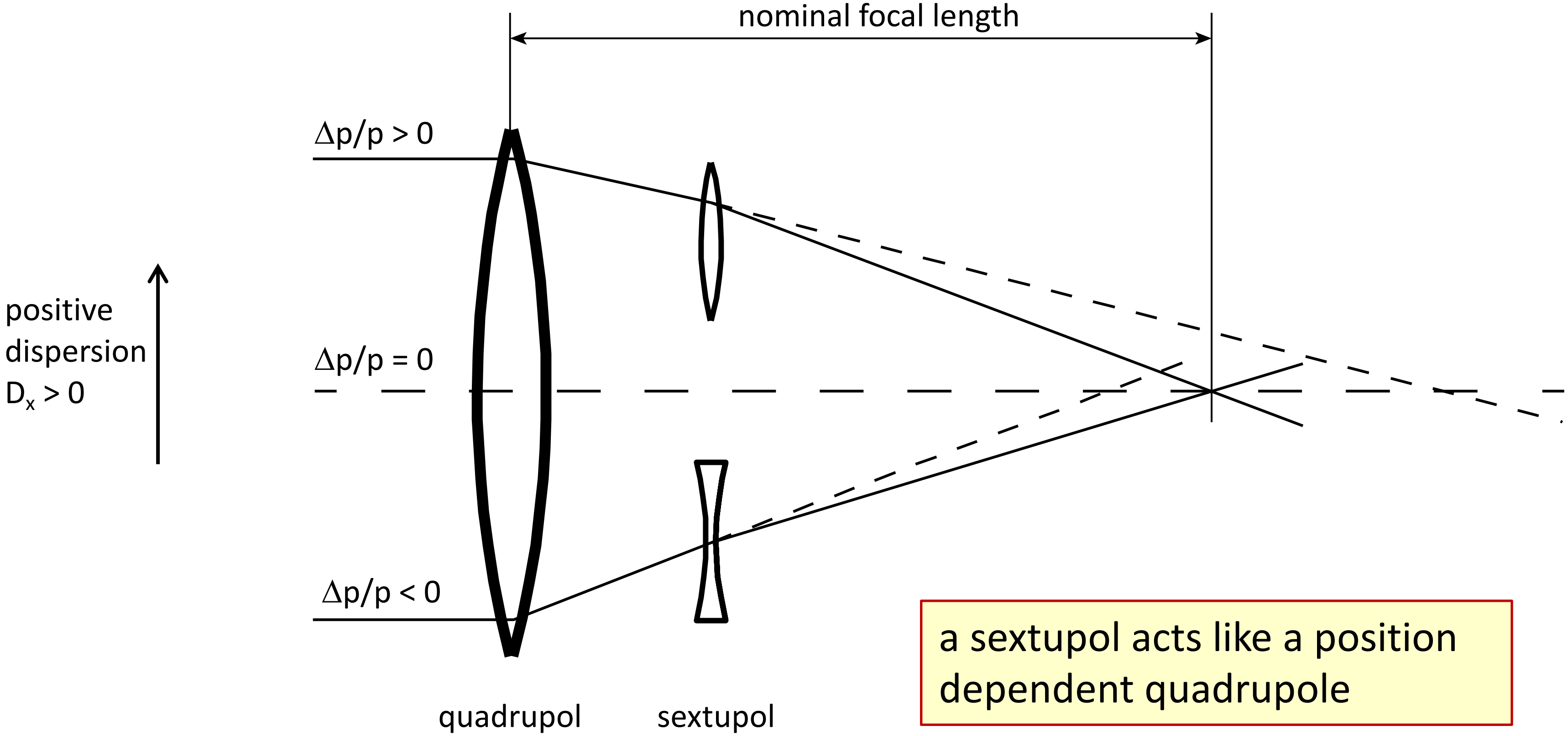
$$x_d = D(s) \cdot \frac{\Delta p}{p}$$





[PSI / SLS Sextupol]

Chromaticity – Correction using Sextupoles



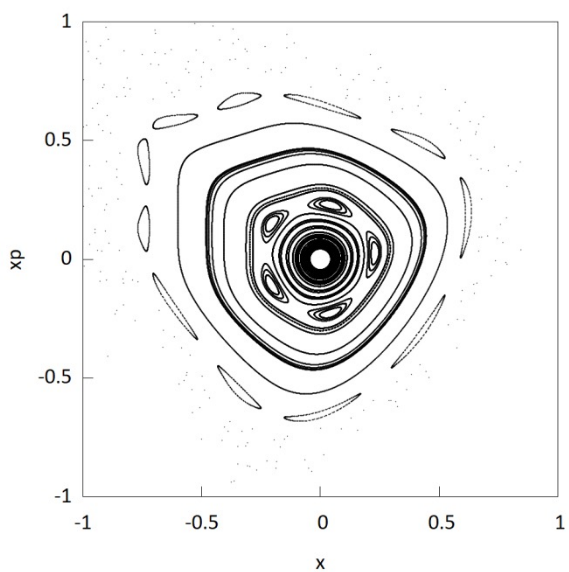
a sextupol acts like a position dependent quadrupole

total chromaticity in a ring: $$\xi_{\mathrm{tot}} = \frac{1}{4\pi}\oint (m(s)D(s) - K(s))\beta_x(s)ds$$

see Wiedemann sec. 15.4.2

Caution with Sextupoles

- while sextupoles can correct chromatic focusing errors, they are **nonlinear elements**
- nonlinear elements **drive resonances and reduce the dynamic aperture** of a ring, which must be carefully optimized when designing a ring



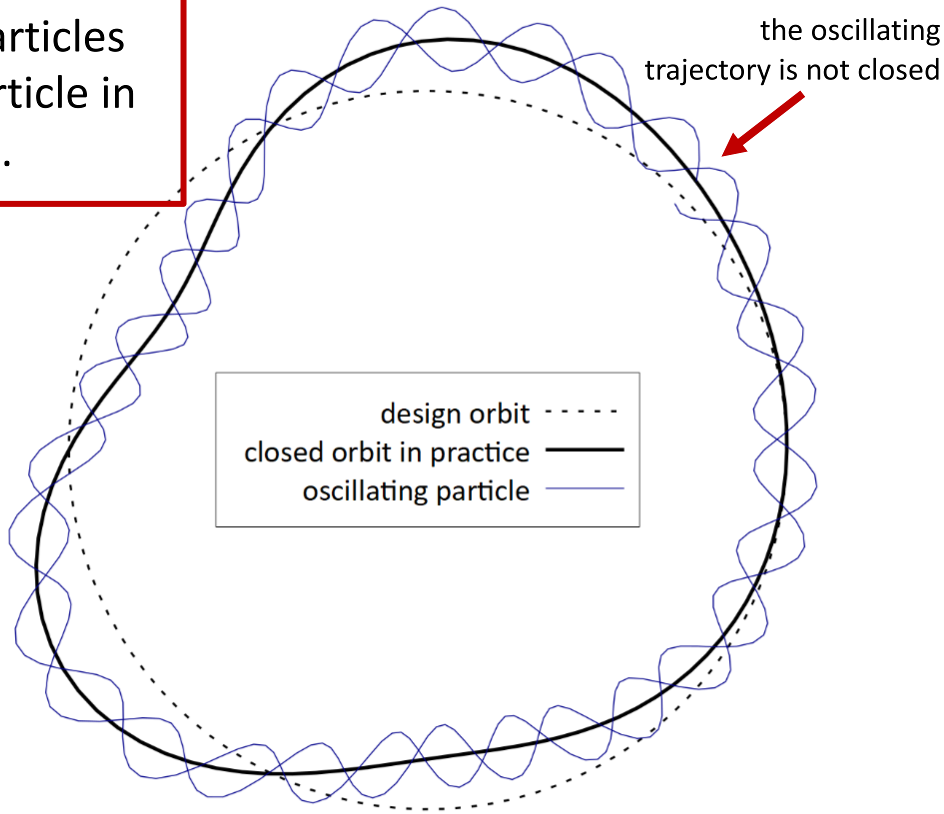
phase space portrait with sextupole kick

Next: Lattice Imperfections

- closed orbit distortion
- gradient errors

Closed Orbit

The stable mean value around which the particles oscillate is called the **closed orbit**. Every particle in the beam oscillates around the closed orbit.



[closed orbit – conceptual sketch]

- in practice the **closed orbit** does not exactly follow the **design orbit**, but deviates due to small errors
- the closed orbit represents the beam center, **particles with nonzero actions oscillate around it**
- to assess practical implications and tolerances the effect of orbit distortions must be estimated

see Wiedemann sec. 15.2.3

The closed orbit

The general expression of the **closed orbit $X_{co}(s)$** in the presence of a **deflection θ** is:

amplitude modulated by the envelope β

oscillating term

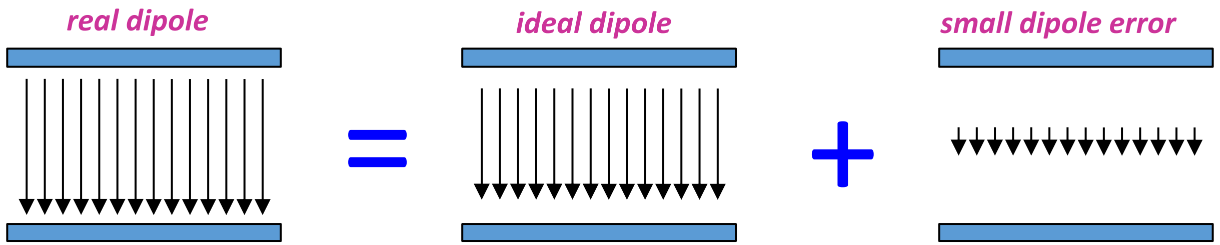
kink at the location of the deflection

$$X_{co}(s) = \frac{\sqrt{\beta(s)\beta_\theta}\cos(|\mu(s)-\mu_\theta|-\pi Q)}{2\sin(\pi Q)}\theta$$

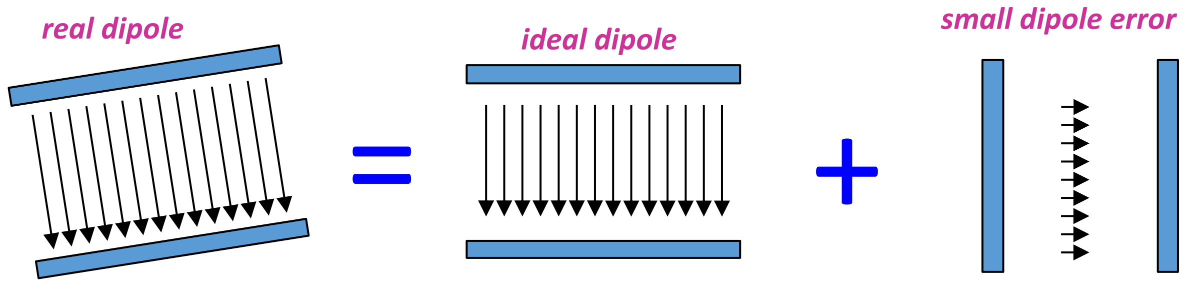
divergence for Q = n

Unintended deflection

- The first source is a field error (deflection error) of a dipole magnet.
- This can be due to an **error** in the **magnet current** or in the **calibration table** (measurement accuracy etc).
  - The imperfect dipole can be expressed as a perfect one + a small error.

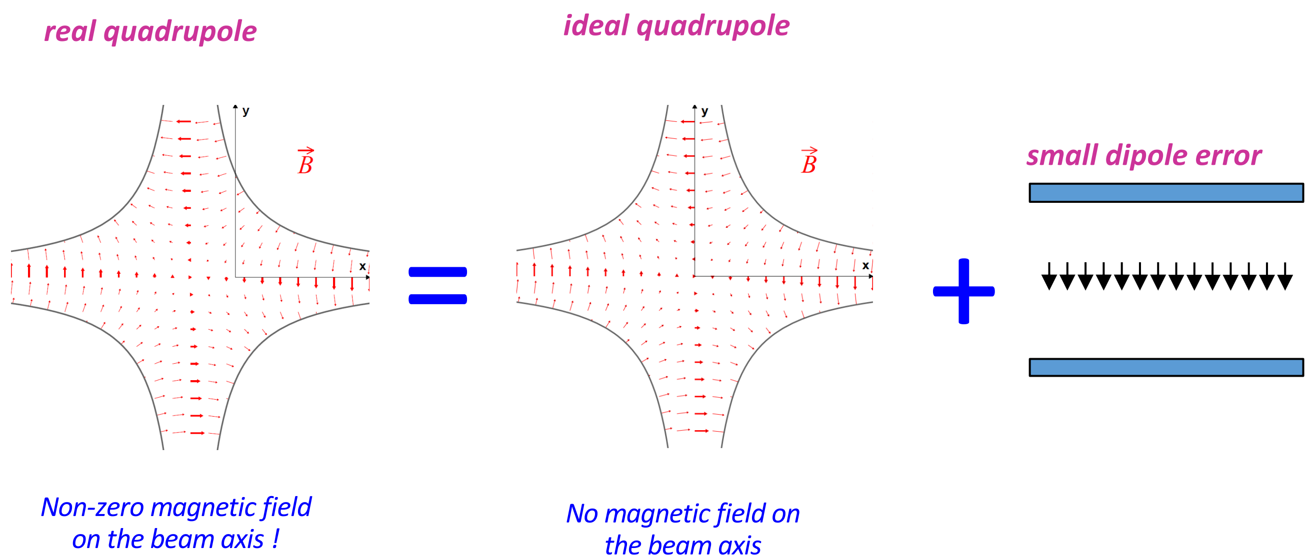


- A small rotation (misalignment) of a dipole magnet has the same effect, but in the other plane.

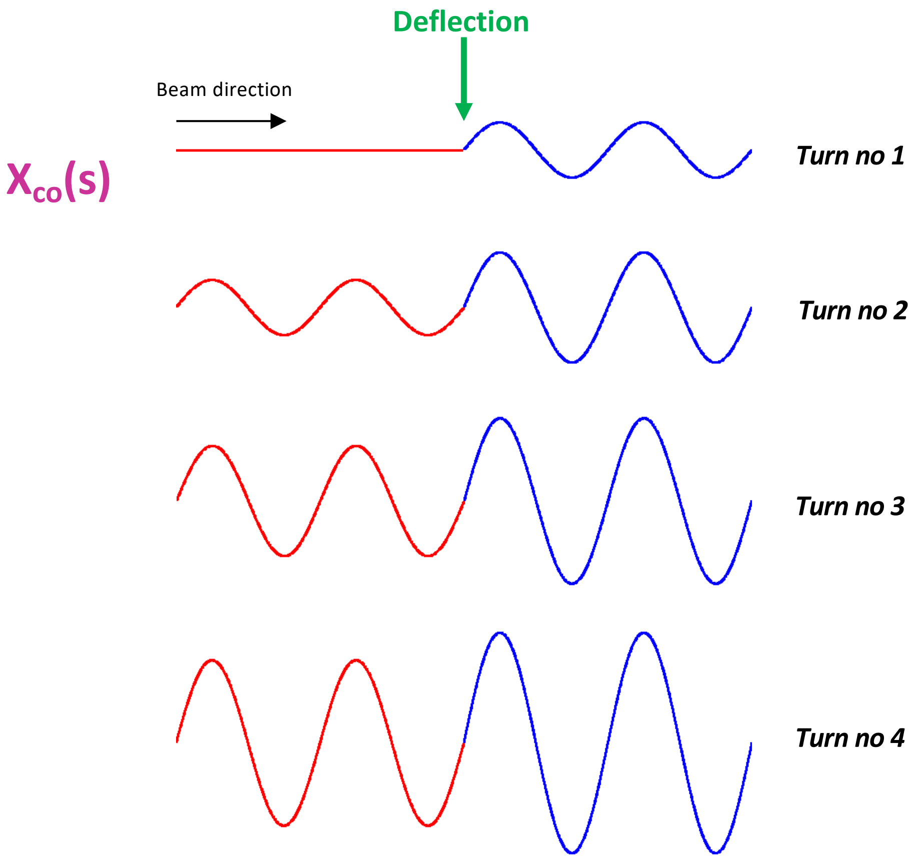


Unintended deflection

- The second source is a misalignment of a <u>quadrupole</u> magnet.
  - The misaligned quadrupole can be represented as a perfectly aligned quadrupole plus a small deflection.



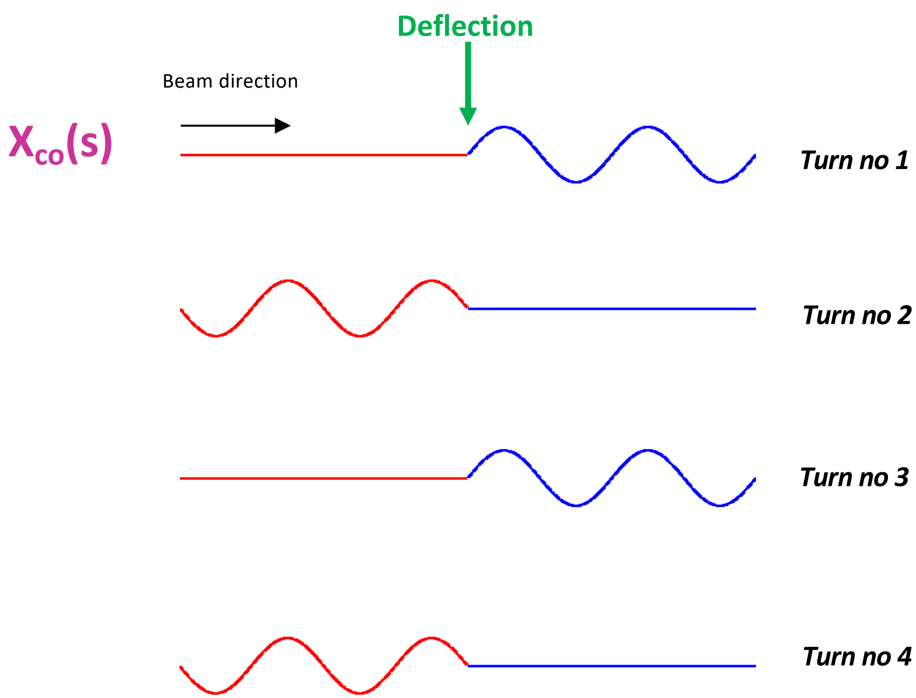
Effect of a deflection



- We set the machine tune to an integer value:
  - $Q = n \in N$

- When the tune is an integer number, **the deflections add up on every turn** !
  - The amplitudes diverge, the particles do not stay within the accelerator vacuum chamber.

Effect of a deflection



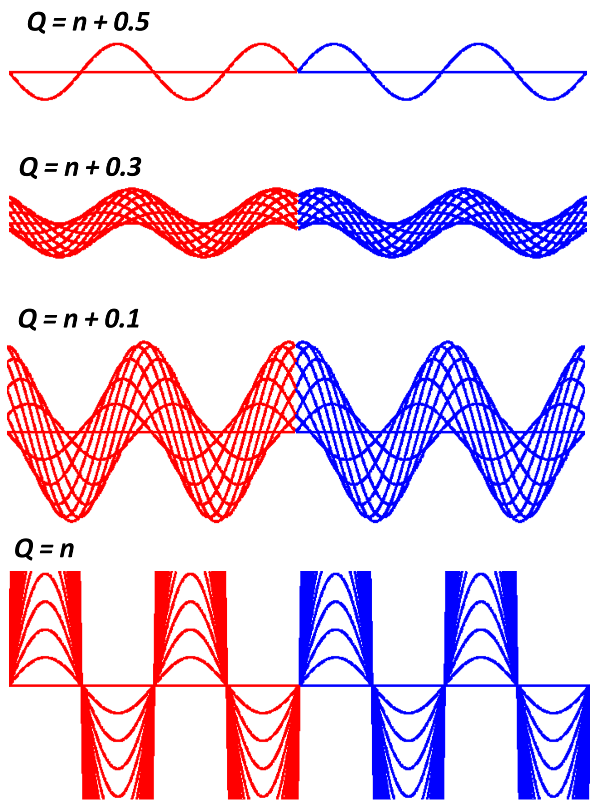
- We set the machine tune to a half integer value:
  - $Q = n+0.5,\ n \in N$

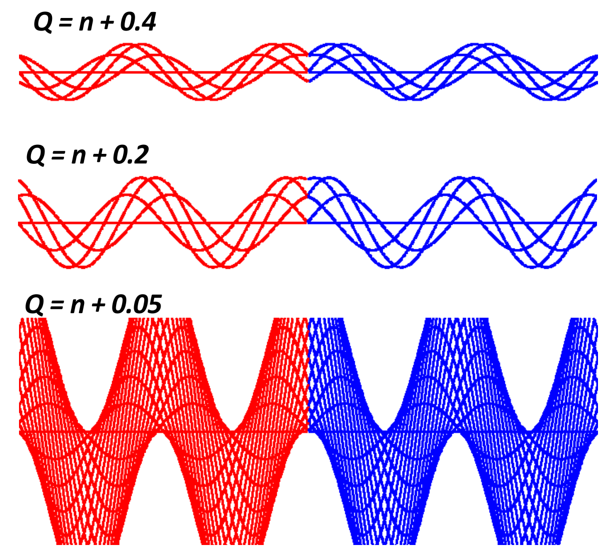
- For half integer tune values, **the deflections compensate on every other turn** !
  - The amplitudes are stable.

- This looks like a much better working point for Q!

Many turns reveal something

- Let's plot the 50 first turns on top of each other and change Q.
  - All plots are on the same scale



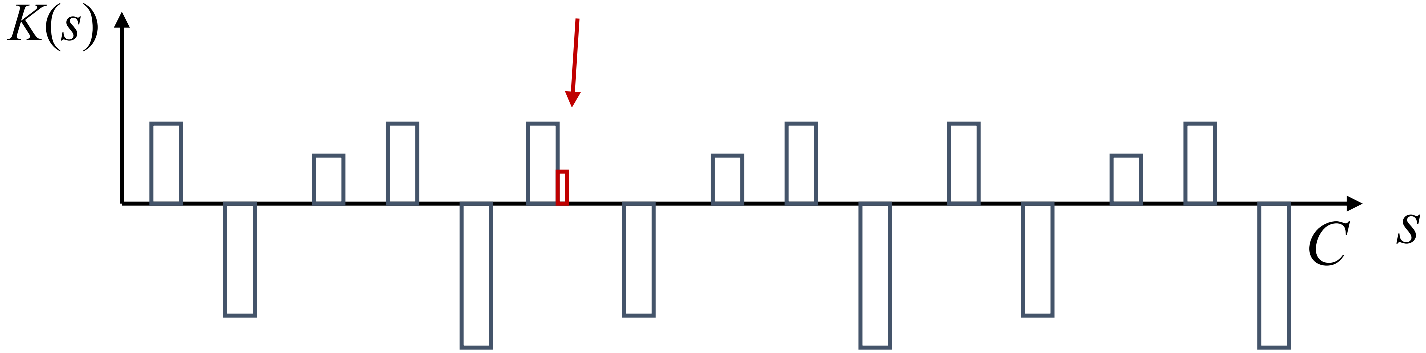


- ❑ The particles **oscillate around a stable mean value** ($Q \neq n$)!
- ❑ The amplitude diverges as we approach Q = n → integer resonance
- ❑ We just encountered our first **resonance** – the <u>integer resonance</u> that occurs when $Q = n \in N$

Gradient Error

$$x'' + (K_0(s) + \Delta K(s))\,x \quad = \quad 0$$

most simple case: the distortion of the gradient is short and can be treated as a thin lens



we want to know:

1. the **tune shift** caused by the error
2. the modification of the beam width (via computing **Δβ(s)**)

Gradient Error - Calculation

method: modify 1-turn transport matrix by multiplying thin lens error matrix

resulting tune shift for distributed gradient errors: $$\Delta Q = \frac{1}{4\pi} \oint \beta(s) \Delta K(s) ds$$

see Wiedemann sec. 15.3.1

$$\Delta\beta(s) = \frac{\beta(s)}{2\sin(2\pi Q)} \oint dt\ \beta(t) \Delta K(t) \cos\left(2(\varphi(s) - \varphi(t) - \pi Q)\right)$$

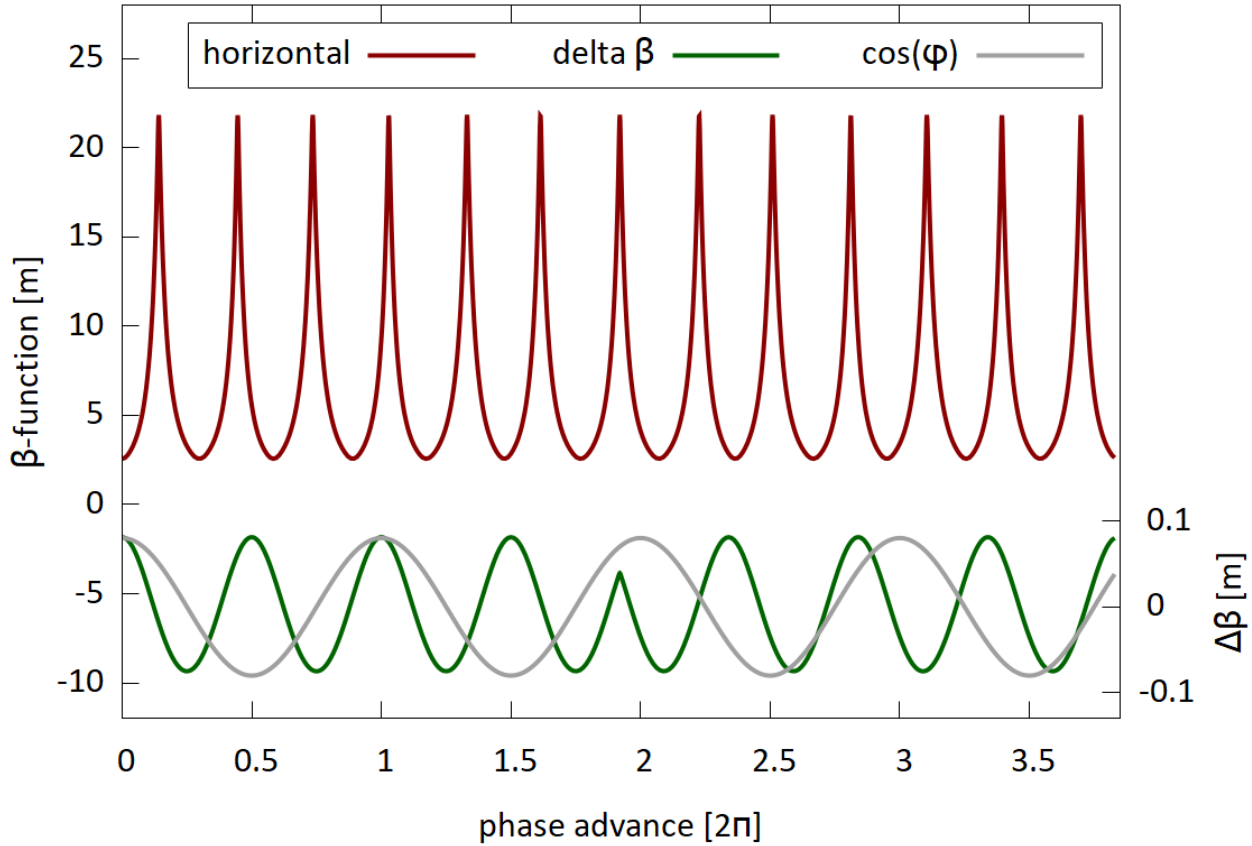
solution explodes for Q $\rightarrow$ Integer × 0.5

note: double frequency

- this error modulates the beam width around the ring
- the effect is called **„Beta-Beat“**
- the Beta-Beat propagates at the double frequency of an orbit distortion

Gradient Error Example continued

accelerator lattice with 13 regular FODO cells, one quad in center has an error of +2%
when $\Delta\beta/\beta$ is plotted against phase advance we see the "error kick" and the double beat frequency

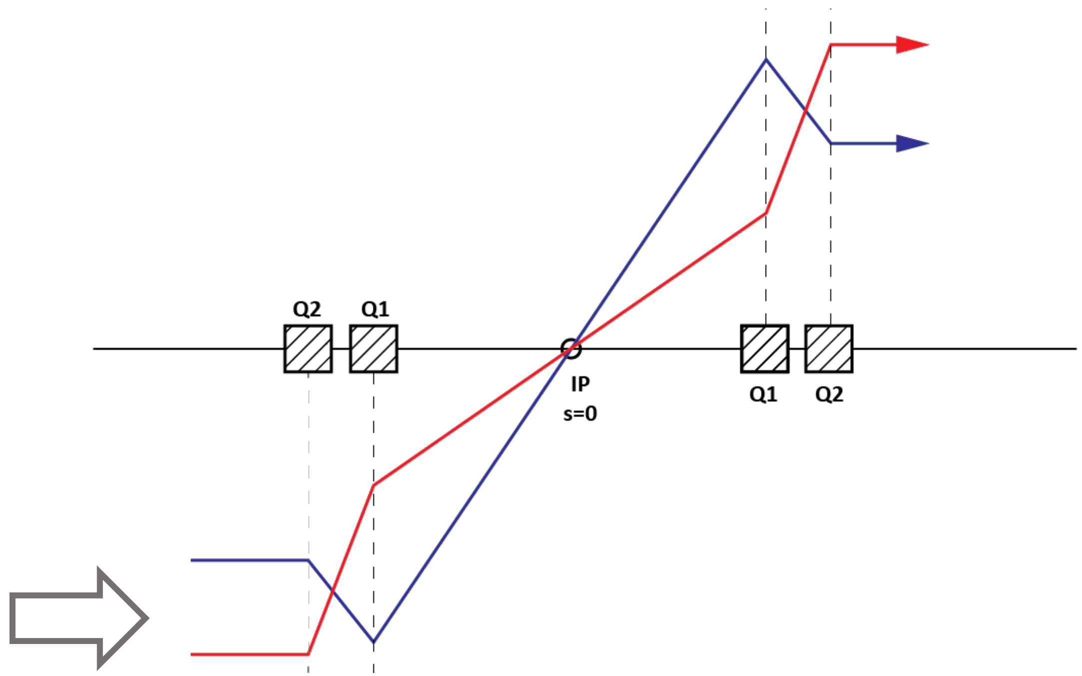


Next: Lattice Insertions

- low Beta insertion

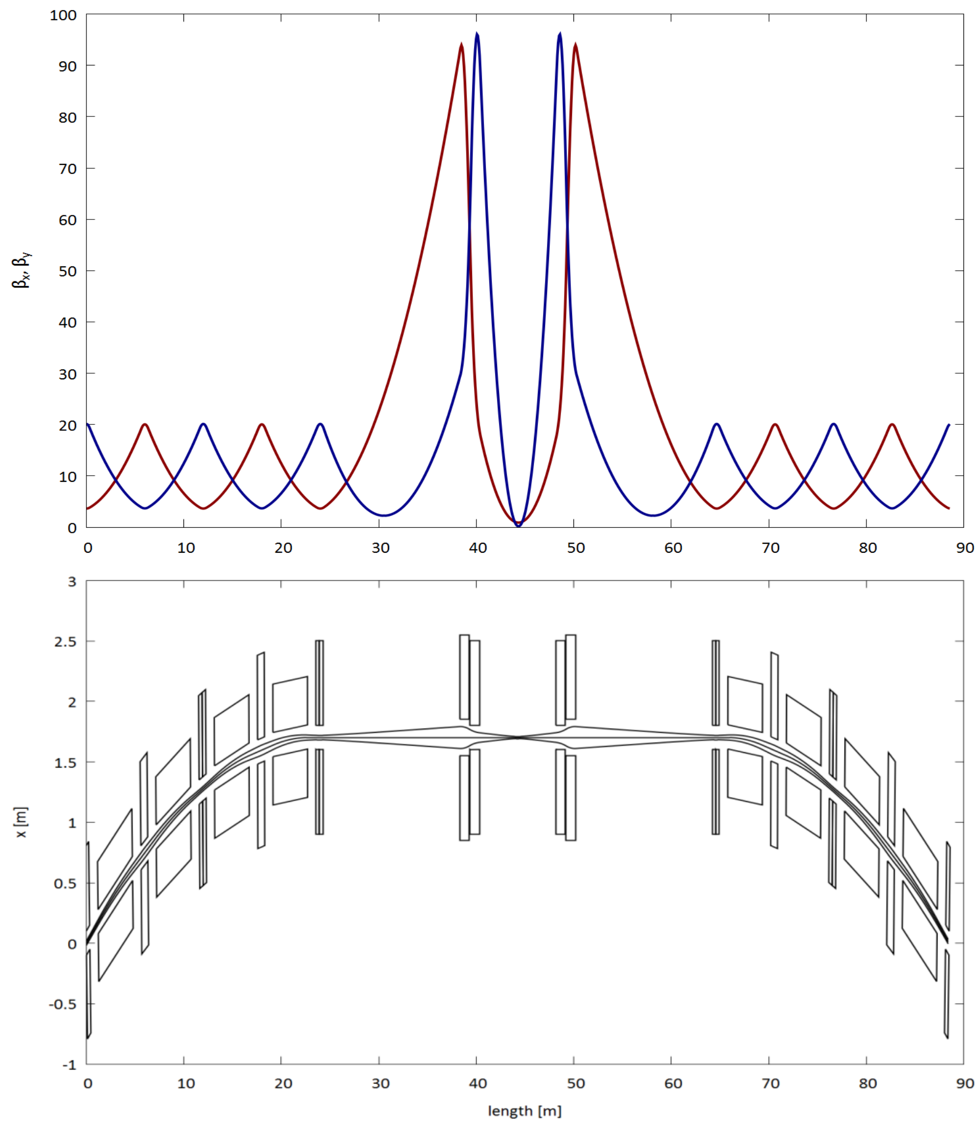
Low Beta Insertion

concept sketch: using a quadrupole doublet it is possible to focus particles in the horizontal and vertical planes simultaneously through the interaction point



incoming trajectories, parallel to reference orbit, in x,y

Low Beta Insertion



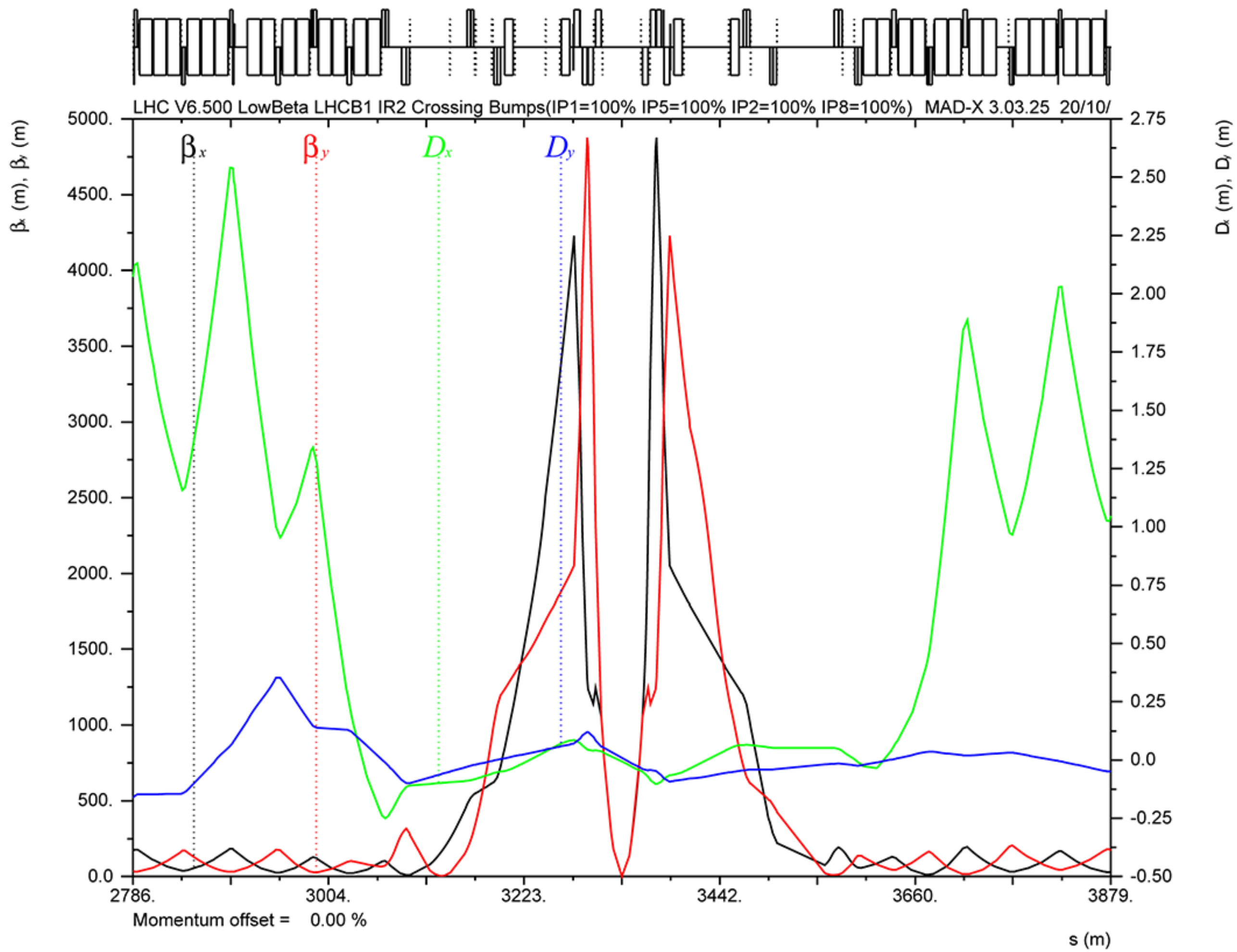
the most simple IR configuration

- doublet focusing
- large beta function in doublet
  $\rightarrow$ aperture limitation for ring

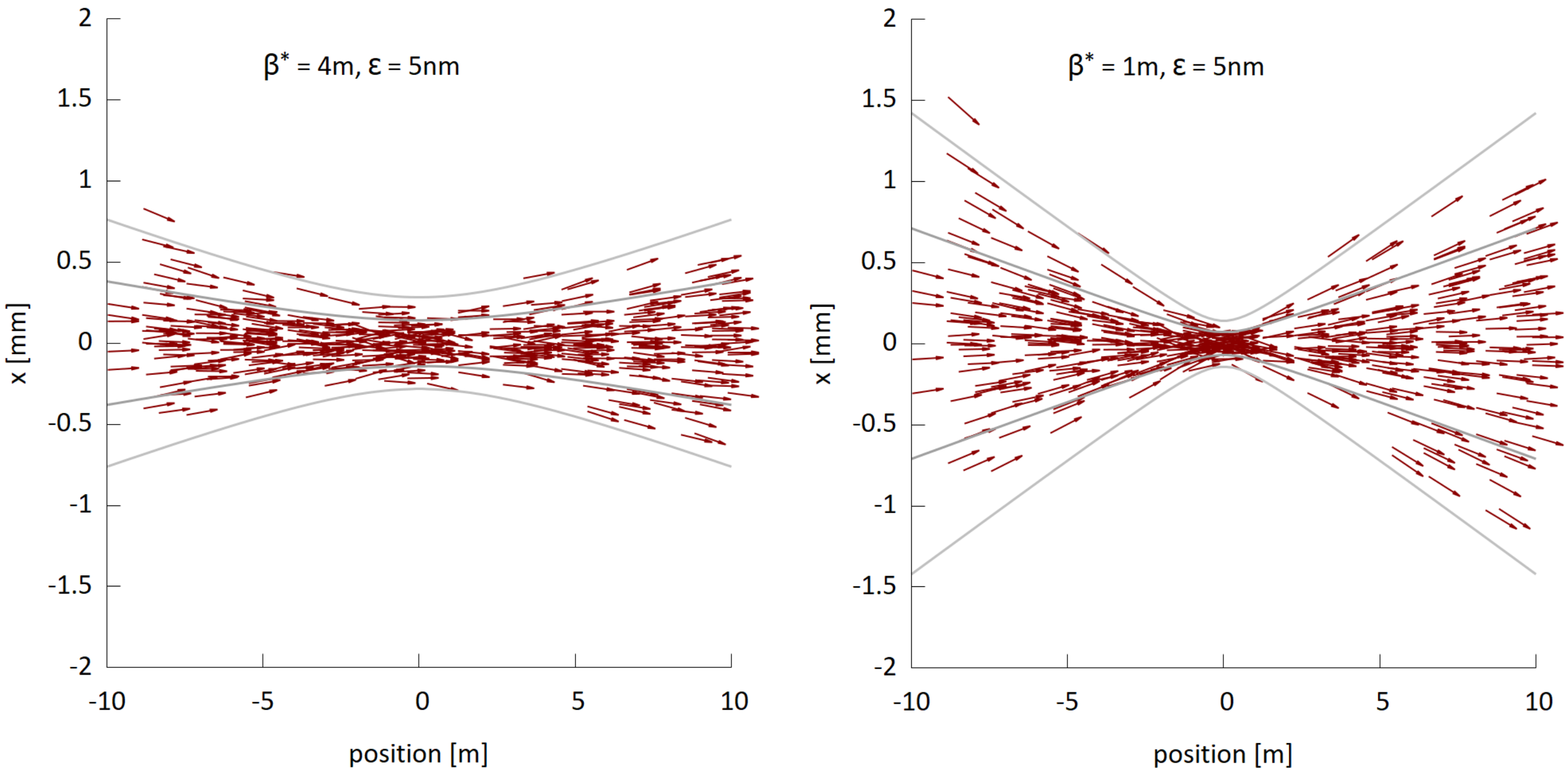
see also Wiedemann
sec. 10.2.4

Low Beta Insertion – Example of LHC

LHC interaction region with Low-Beta + D.S.



Beam Waist (e.g. interaction point collider)



$$\beta(s) = \beta^* + \frac{s^2}{\beta^*}$$

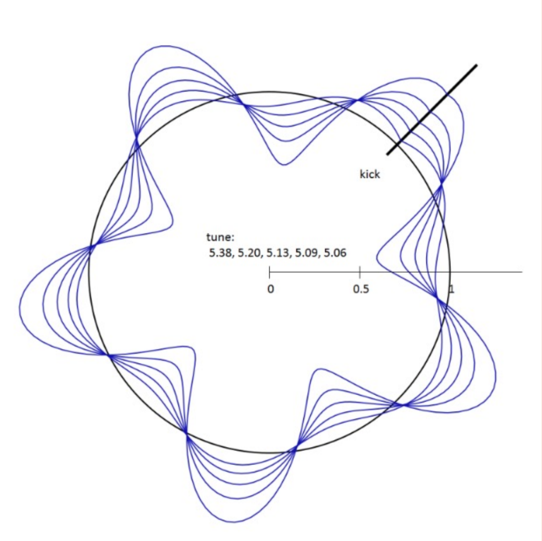
$$\sigma_{\text{rms}} = \sqrt{\varepsilon\beta^*},\ \sigma'_{\text{rms}} = \sqrt{\frac{\varepsilon}{\beta^*}}$$

β* = Beta function at waist

Orbit Correction:
Reminder - Closed Orbit Distortion by Kick θ

$$x(s) = \frac{\theta\sqrt{\beta(s)\beta_0}}{2\sin(\pi Q)} \ \cos(\varphi(s) - \pi Q)$$

kick θ is caused by an unwanted magnetic field, or an off-set quadrupole (errors) however, kick can be applied also on purpose to correct the orbit



with several kicks θ_j the contributions are added:

$$x_k = \sum_j \frac{\sqrt{\beta_k\beta_j}\ \cos(|\varphi_k - \varphi_j| - \pi Q)}{2\sin(\pi Q)} \ \theta_j$$

Orbit Correction

given is a set of beam positions representing an orbit x_k

calculate a set of (wanted) corrector strengths θ_j to minimize the orbit amplitide

this can be formulated as a problem of linear algebra (**R$_{\mathbf{kj}}$** coefficients last slide):

$$\vec{x}_{\mathrm{pos}} + \boldsymbol{R}\vec{\theta}_{\mathrm{cor}} = 0$$

this is solved exactly for $N_{pos} = N_{cor}$, however in practice we need flexible solutions

Singular Value Decomposition (SVD) is one approach:

$$\boldsymbol{R} = \boldsymbol{U} \cdot \boldsymbol{W} \cdot \boldsymbol{V}^{\mathrm{T}}$$

$$\boldsymbol{R}_{\mathrm{inv}} = \boldsymbol{V} \cdot \boldsymbol{W}^{-1} \cdot \boldsymbol{U}^{\mathrm{T}}$$

W = diagonal matrix with singular values, inversion simple

solution:

$$\boxed{\vec{\theta}_{\mathrm{cor}} = -\boldsymbol{V}\boldsymbol{W}^{-1}\boldsymbol{U}^{\mathrm{T}} \cdot \vec{x}_{\mathrm{pos}}}$$

- $N_{pos} = N_{cor}$: exact solution
- $N_{pos} < N_{cor}$: minimizes $|\vec{\theta}|$ (magnet currents)
- $N_{pos} > N_{cor}$: minimizes $|\vec{x}|$ (rms orbit deviation)

→ in practice this is done using computer codes of the accelerator control system

see Wiedemann sec. 15.4.2