Introductions to accelerators I: Transverse Dynamics

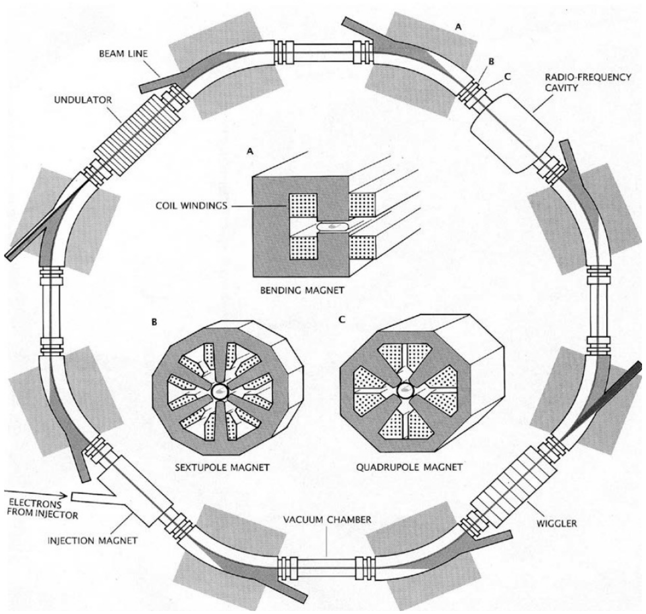
Dr Tatiana Pieloni (Tatiana.Pieloni@epfl.ch)
Laboratory for Particle Accelerator Physics, EPFL

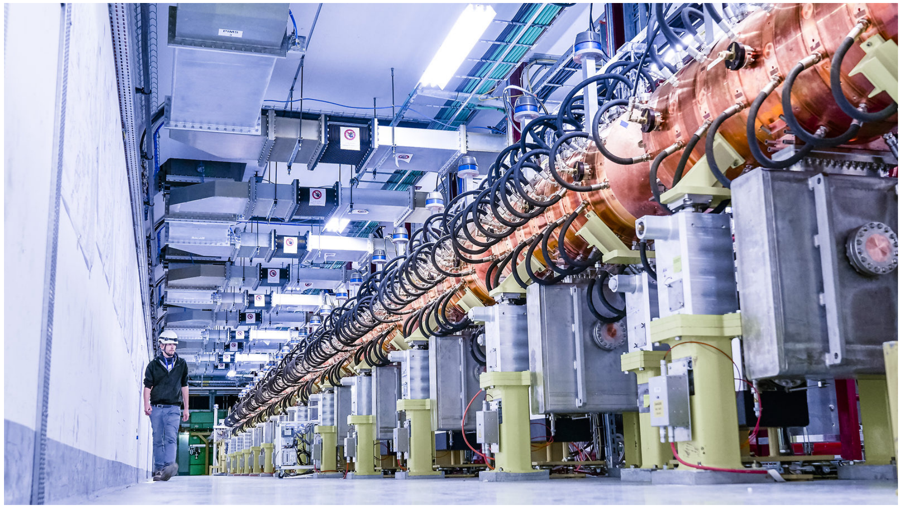
EPFL

Introduction to particle Dynamics

Accelerator = series of elements for **beam guiding** (bending, focusing) and **acceleration**

- often arranged in a **closed loop** (ring) or in a periodic **"straight" sequence** (linacs)
- guiding fields must ensure stability of circulating particles





Introduction to particle Dynamics

Accelerator = series of elements for **beam guiding** (bending, focusing) and **acceleration**

- often arranged in a **closed loop** (ring) or in a periodic **“straight” sequence** (linacs)
- guiding fields must ensure stability of circulating particles

questions to be answered:

- Can we describe the particle motion?
- How to ensure bound motion of a particle beam?
- What are conditions for stability?
- Statistical beam properties like beam width and angular spread?
- How to design magnet lattices (arrangements of dipoles and quads in a line)?
- What happens when non-linear effects occur?
- What is the impact of field errors in bending and focusing magnets?

References and accessible Reading Material

available on the internet:

P. Schmüser & J. Rossbach, Basic course on accelerator optics:

https://cds.cern.ch/record/247501/files/p17.pdf

F.Tecker, Longitudinal Dynamics:

https://arxiv.org/pdf/1601.04901.pdf

L.Rivkin, Electron dynamics in rings in the presence of radiation :

https://cds.cern.ch/record/375974/files/p45.pdf

Book, H.Wiedemann, Particle Accelerators, download pdf !:

https://link.springer.com/book/10.1007%2F978-3-319-18317-6

CERN Accelerator School (CAS) proceedings homepage (huge!)
http://cas.web.cern.ch/cas/CAS_Proceedings.html

CERN Accelerator School on Medical Applications:

https://cds.cern.ch/record/2271793/files/33-8-PB.pdf

books, papers:

S.Peggs, T.Satogata, *Introduction to Accelerator Dynamics*, Cambridge University Press, 2017

A. Wolski, *Beam Dynamics in high energy particle accelerators*, Imperial College Press, 2014

A. W. Chao, M. Tigner, *Handbook of Accelerator Physics and Engineering*, World Scientific 1999

E. D. Courant and H. S. Snyder, Annals of Physics: **3**, 1-48 (1958)

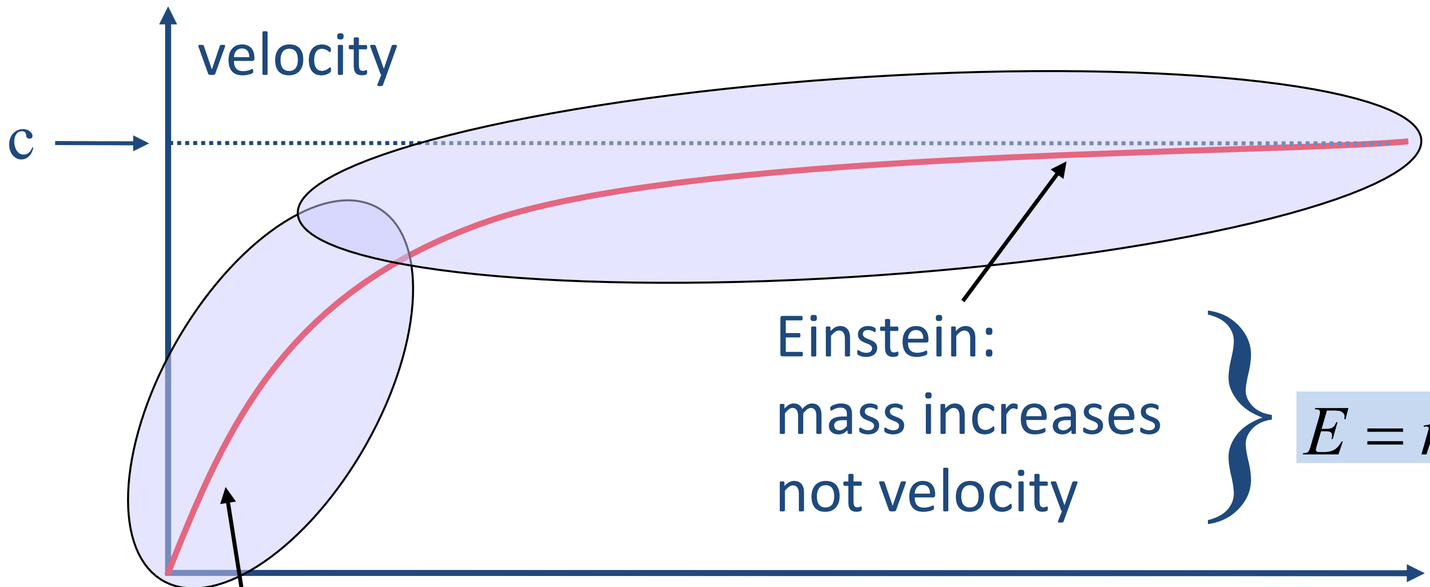
M. Sands, SLAC-121, 1969

Physics of Electron Storage Rings: An Introduction.

https://digital.library.unt.edu/ark:/67531/metadc865991/

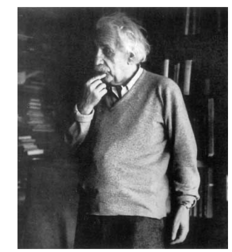
*** with focus on linear beam optics**

Accelerating particles → Towards Relativity



Einstein:
mass increases
not velocity

$E = mc^2$



Newton: $E = \frac{1}{2}mv^2$

Acceleration of particles: basic relativistic relations

relativistic energy-momentum relation:

$$E = \sqrt{m_0^2 c^4 + c^2 p^2}$$
$$= m_0 c^2 + E_k$$

$$v = c\sqrt{1 - m_0^2 c^4 / E^2}$$
$$= \beta c$$

$$p = \beta c \cdot \gamma m_0$$

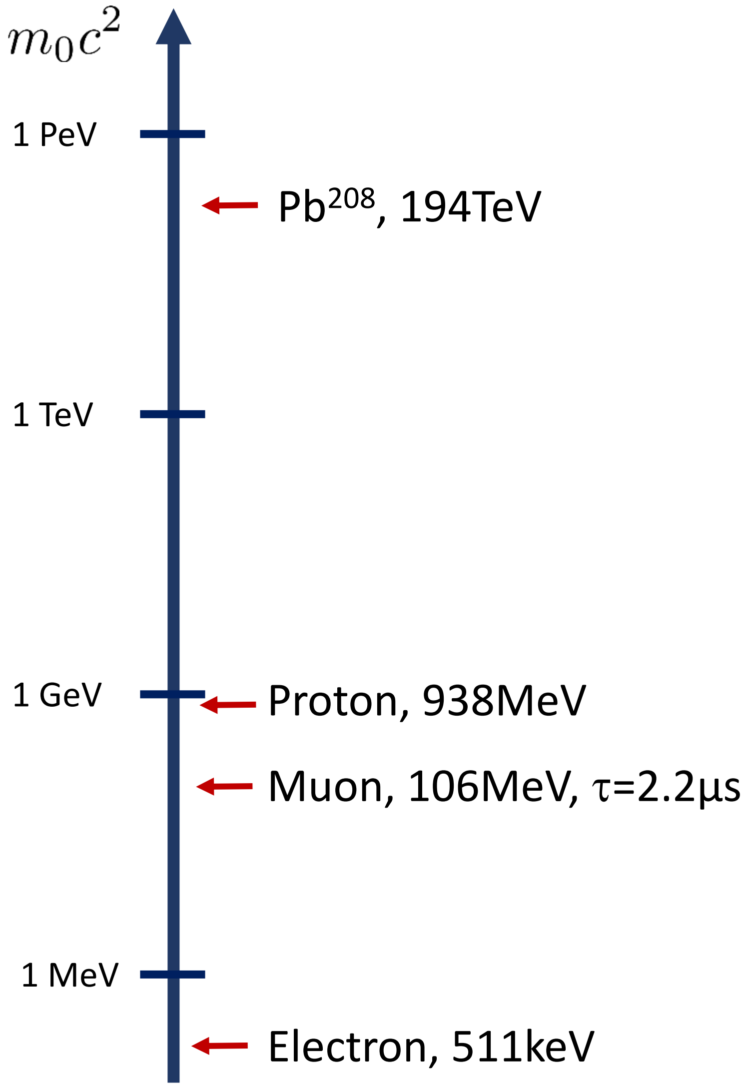
in the limit $E \rightarrow \infty$: $v \rightarrow c,\ m_{\text{eff}} \rightarrow \infty$

$$\gamma = \frac{E}{m_0 c^2} = 1 + \frac{E_k}{m_0 c^2}$$
$$\beta = \sqrt{1 - 1/\gamma^2}$$

E_k [MeV]	γ	β	p [MeV/c]
590	1.63	0.79	1207

numerical example for protons

Particles to Accelerate



Wide range of rest masses from electron to heavy ions

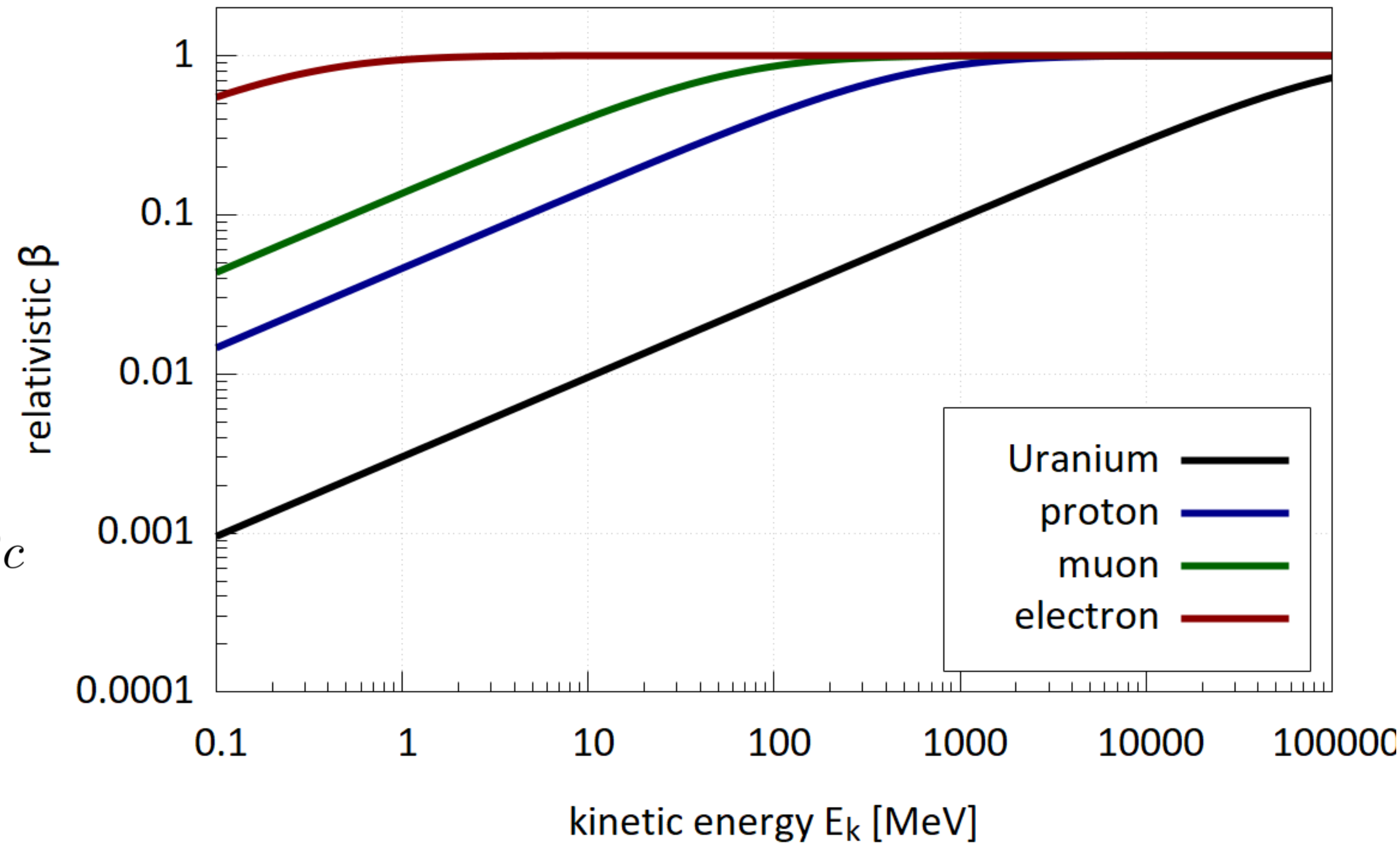
The accelerators differ vastly, e.g.

- particle speed in cavities
- synchrotron radiation power
- activation by losses
- requirements for vacuum

The muon as a heavy lepton is interesting for particle physics

$1\text{eV} = 1.602\times10^{-19}$ Joule
$1\text{eV} \cong 1.783\times10^{-36}$ kg

Speed of different particles vs energy

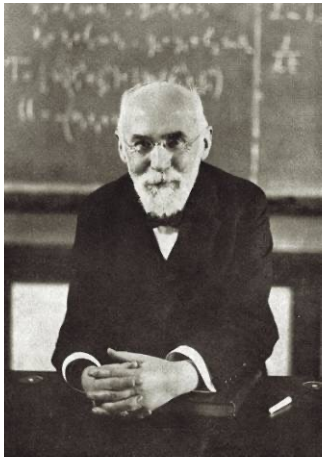


$$v = \beta c$$

Relativistic electrons at ~ MeV
Relativistic protons ~ GeV

Guiding charged particles: Lorentz Force

$$\vec{F} = e\vec{E} + e\vec{v} \times \vec{B}$$ (charge = e)



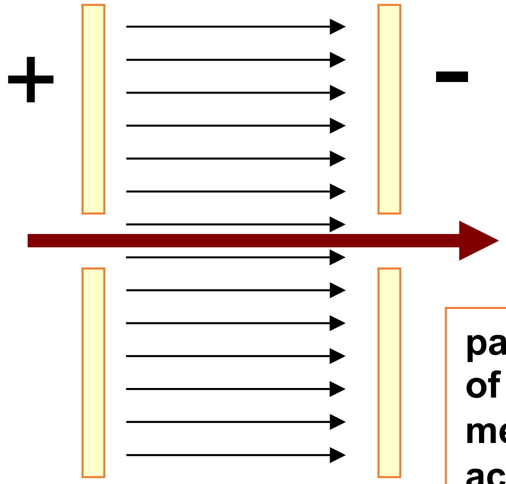
H.A.Lorentz
1853-1928

electric field

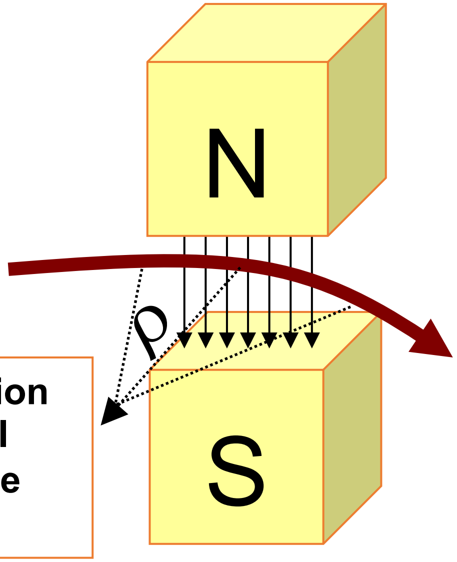
energy gain: $\Delta E_k = eU$

magnetic field

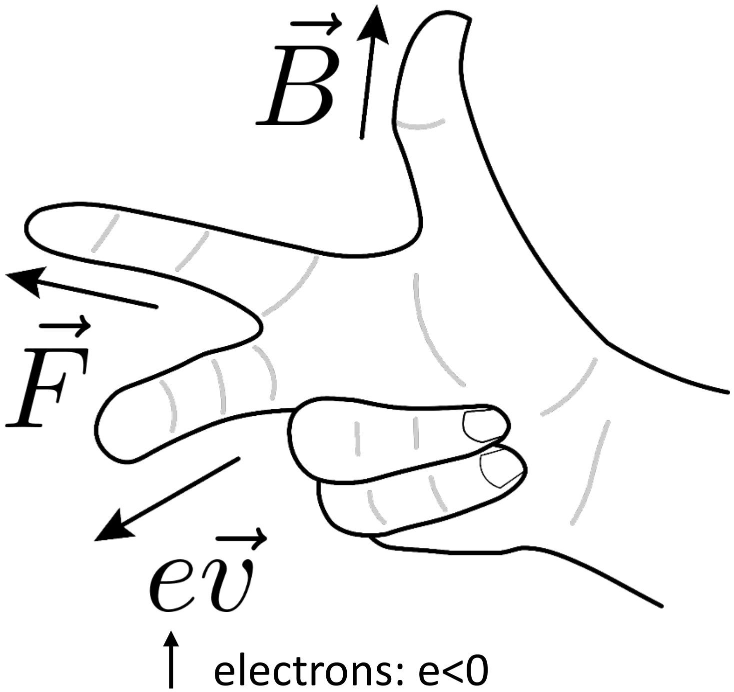
bending: $B\rho = p/e, \ \Delta E_k = 0$



path bending and acceleration of particles are fundamental mechanisms of each particle accelerator

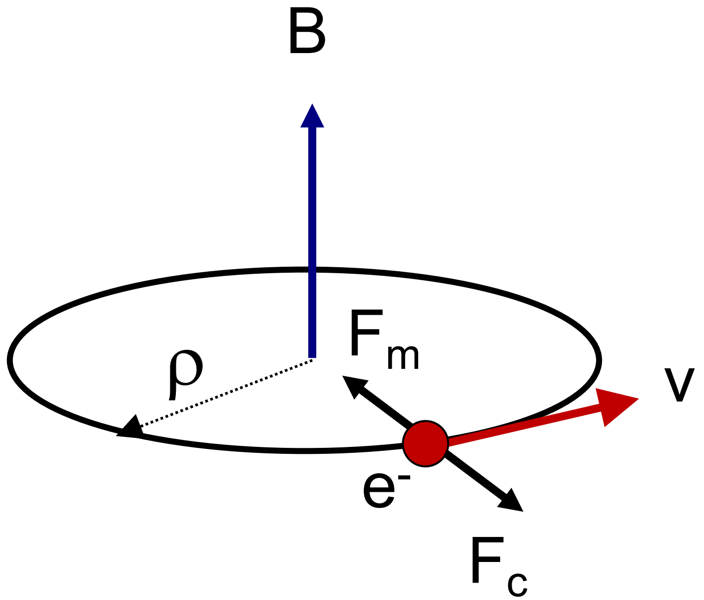


Lorentz Force – getting it right



$$B\rho[\mathrm{Tm}] \quad = \quad 3.3356 \cdot \beta E_{\mathrm{tot}}[\mathrm{GeV}]$$

[see appendix for derivation]



Comparison E and B field

Bending radius for protons in B and E:

example: electric and magnetic force on protons

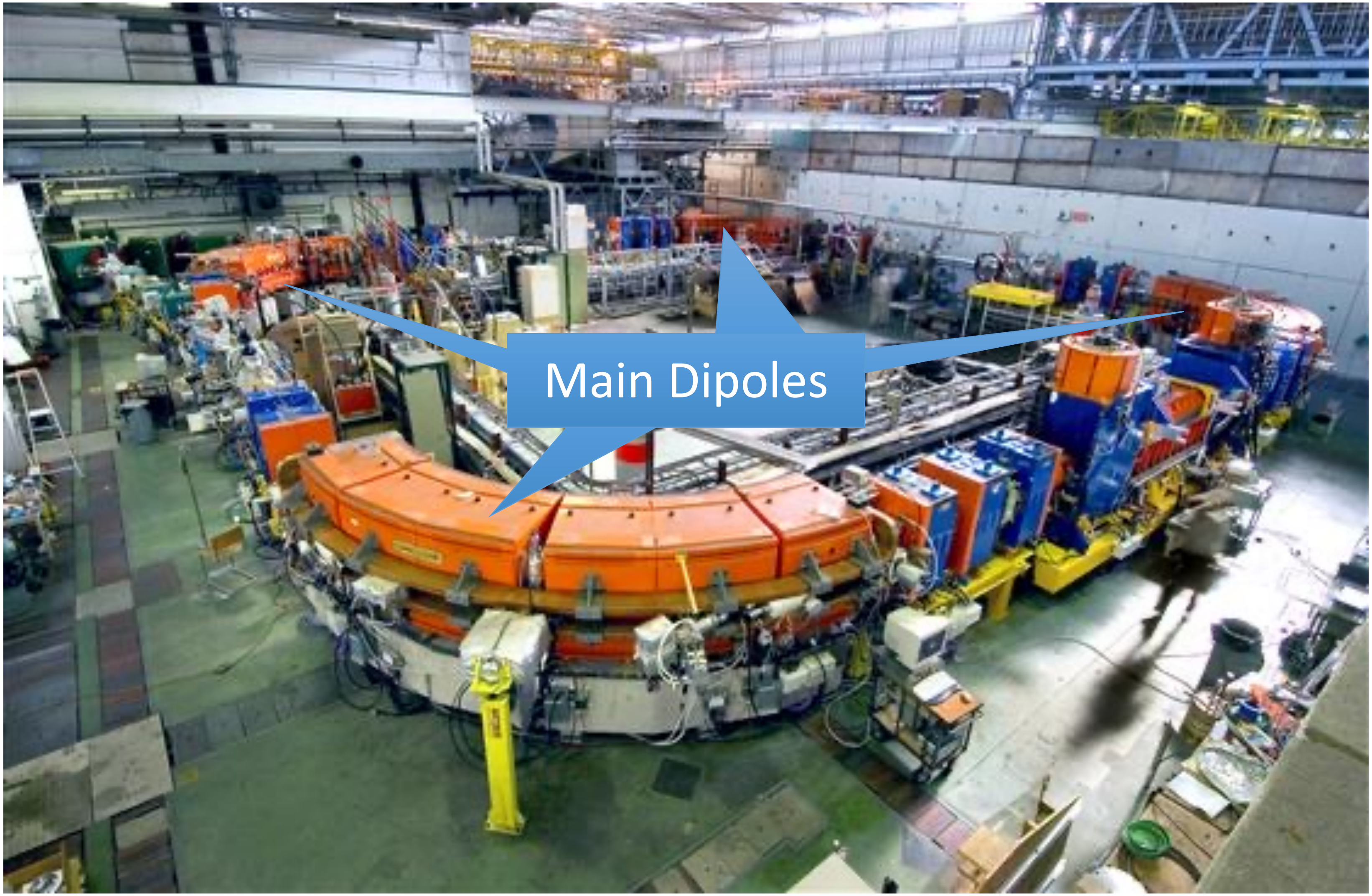
$$\vec{F_E} = e \cdot \vec{E}, \quad \vec{F_B} = e \cdot \vec{v} \times \vec{B}$$

table: bending radius, varying E_k

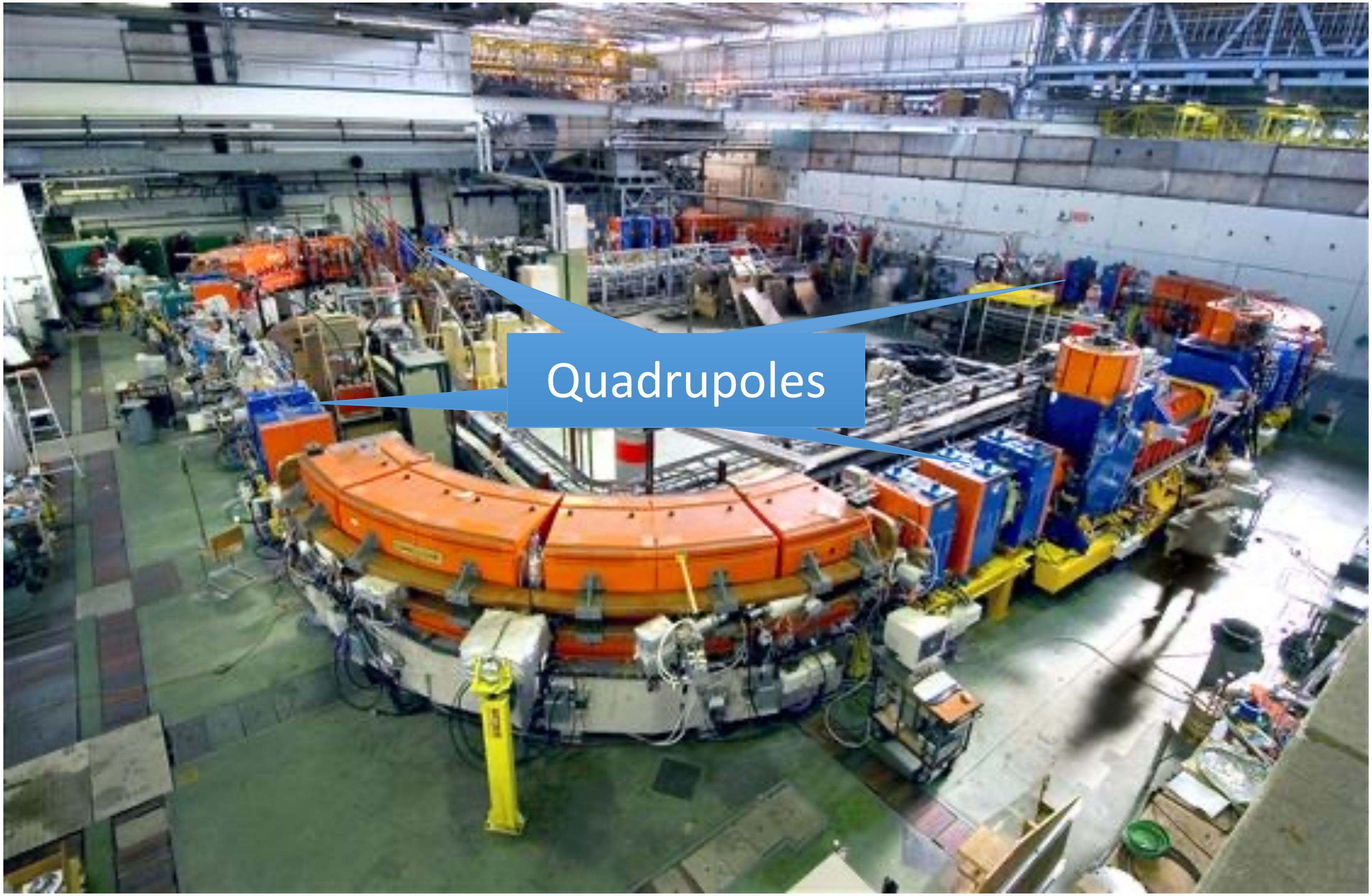
E_k	B = 1T	E = 10MV/m
60 keV	35 mm	12 mm
1 MeV	140 mm	200 mm
1 GeV	5.6 m	150 m

Magnetic fields are used exclusively to bend and focus ultra-relativistic particles

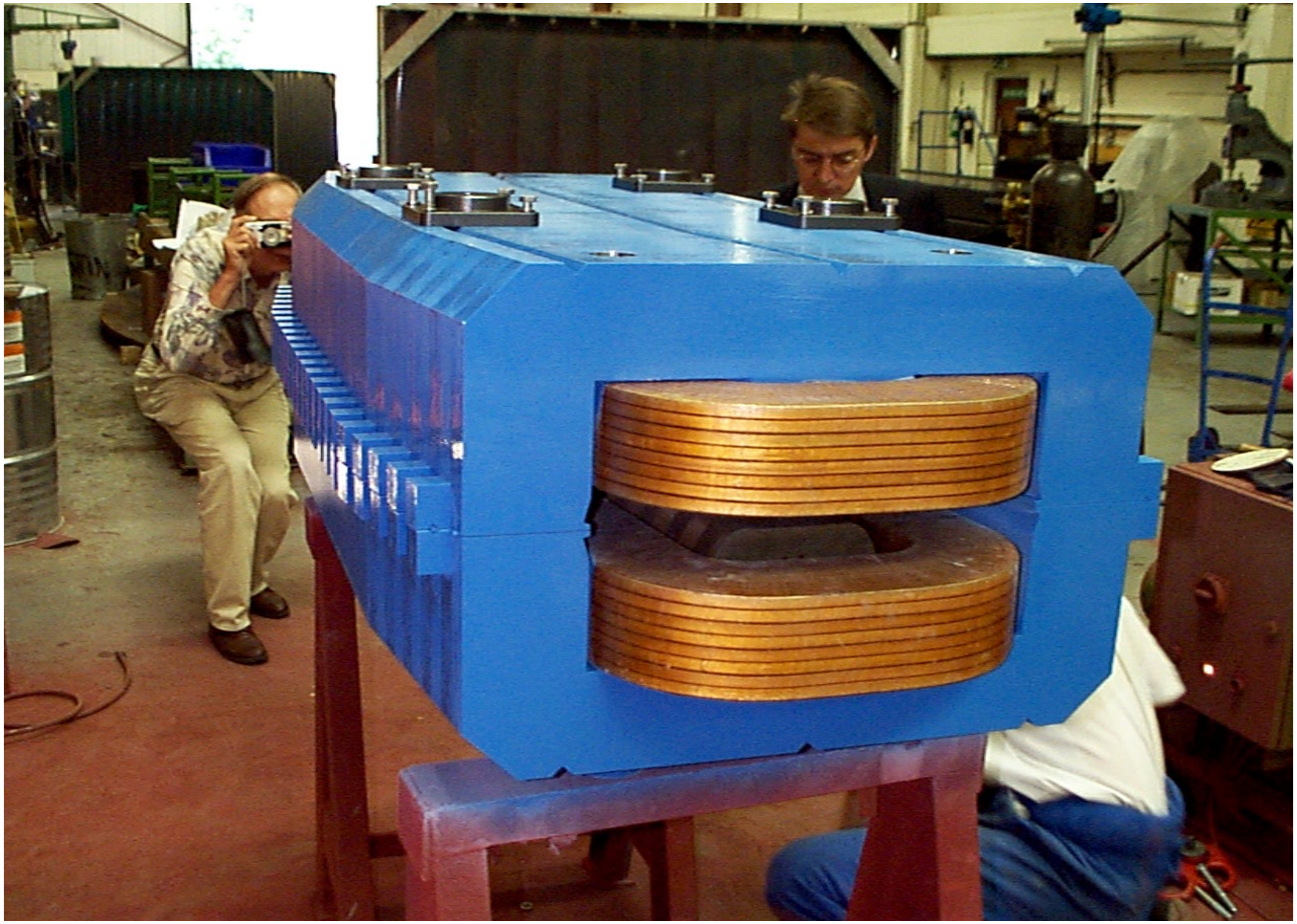
Make Particles Circulate



Focusing the Particles



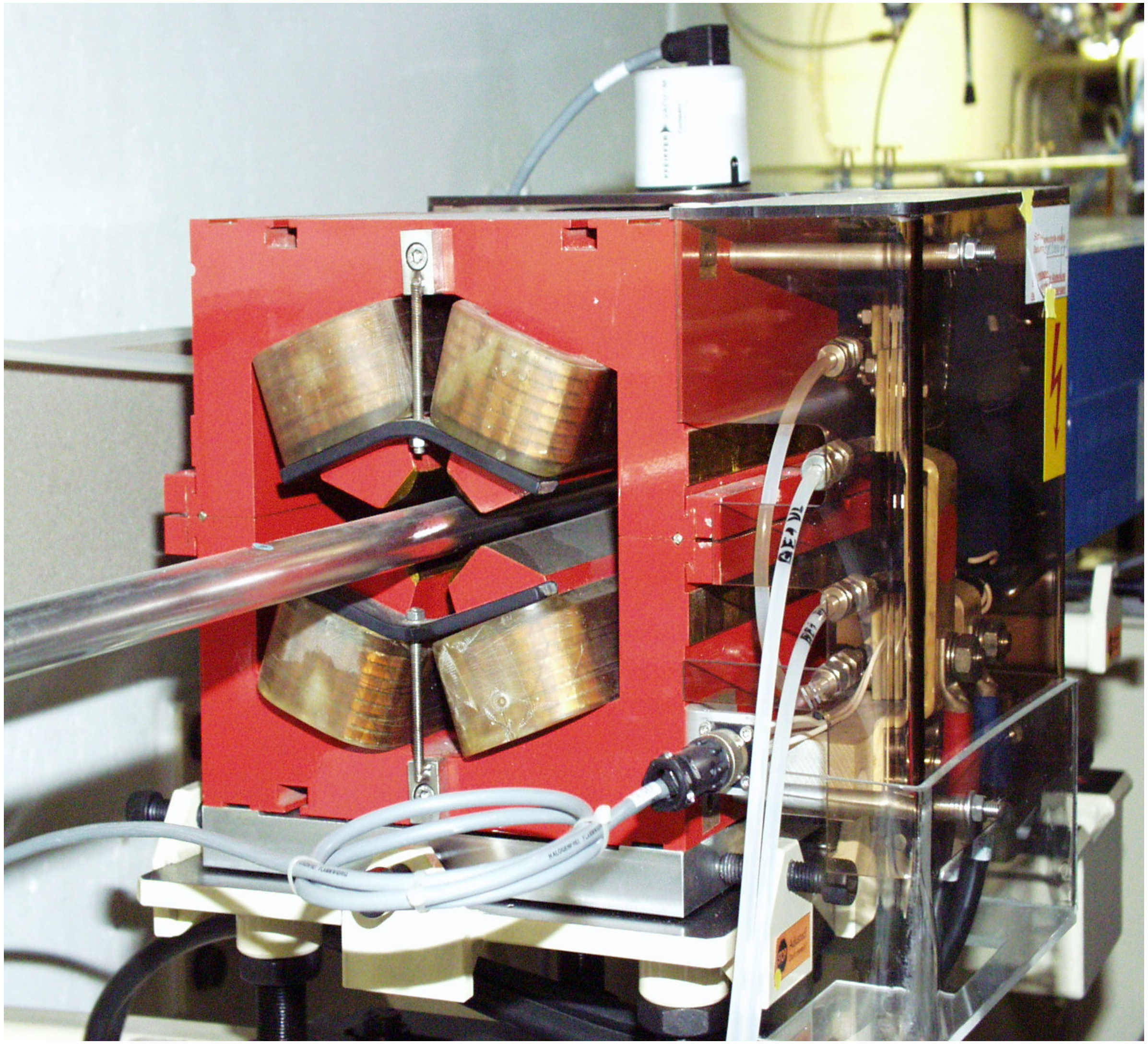
Bending Magnet - SLS dipole



magnetic rigidity:

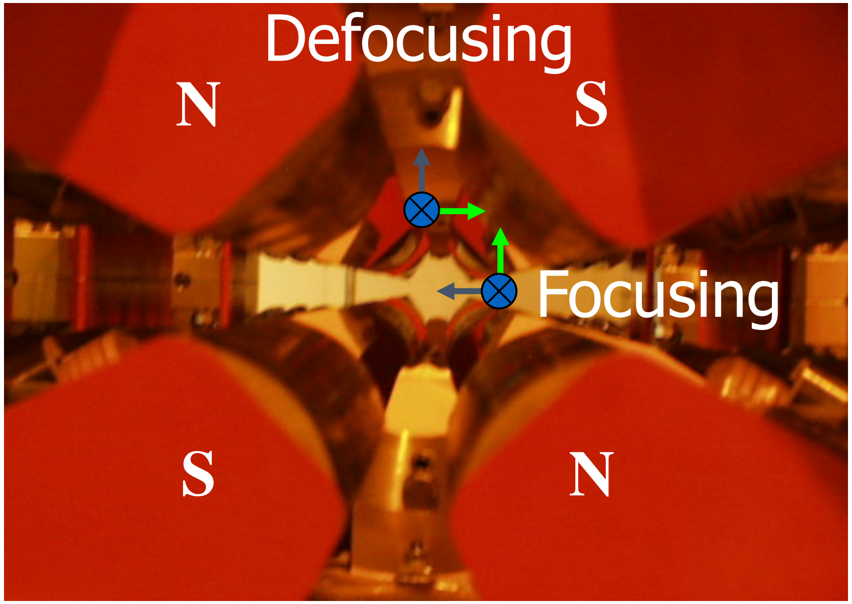
$$B\rho = \frac{p}{e}$$

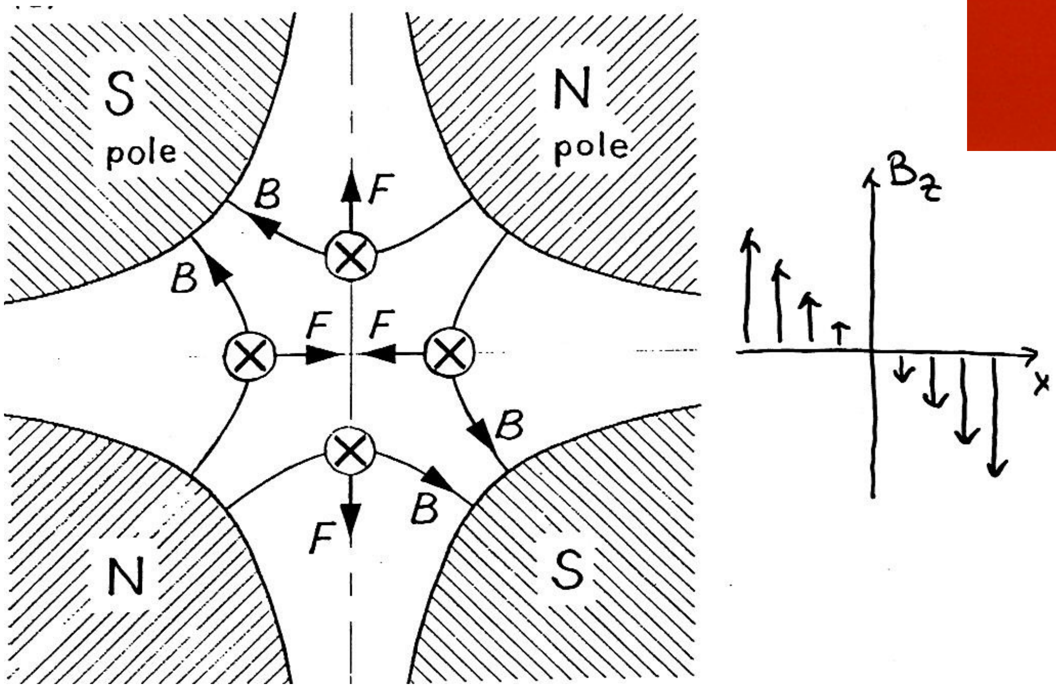
Quadrupole Magnet - Focusing Element



Quadrupole magnets

- Focusing in one plane
- Defocusing in the other plane





$$\nabla \times \boldsymbol{B} = 0 \rightarrow \frac{\partial B_y}{\partial x} = \frac{\partial B_x}{\partial y}$$

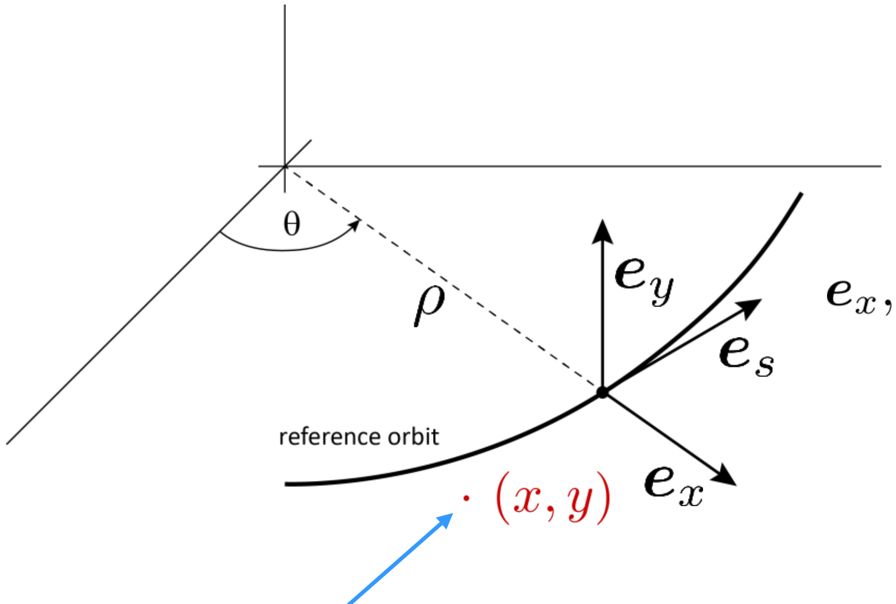
Gradients g

Next: Equation of Motion

- suited coordinate system
- linearizing forces and deriving differential equation

Curvelinear Coordinate System

aim: derive a set of equations that describe the motion of a single particle wrt. a curved coordinate system around the reference orbit of a beam, (x, y)



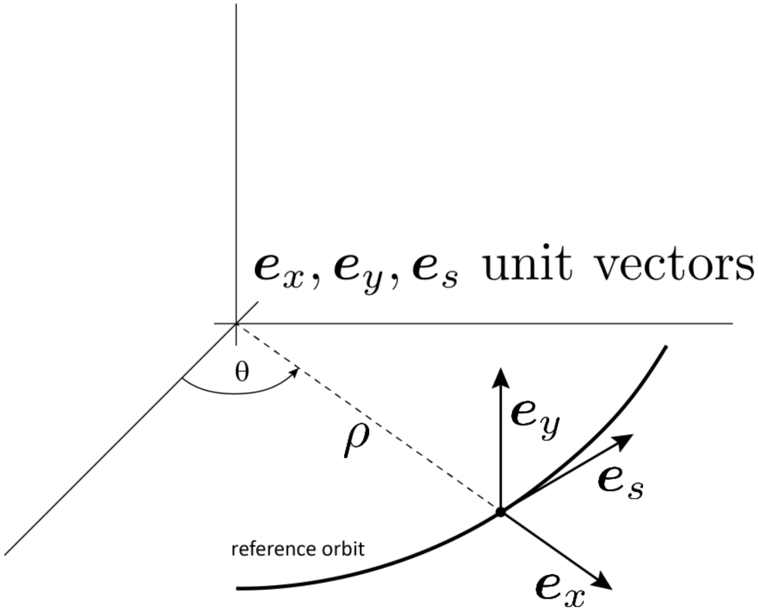
$\boldsymbol{e}_x, \boldsymbol{e}_y, \boldsymbol{e}_s$ unit vectors

particle coordinate: $\vec{R} = r\boldsymbol{e}_x + y\boldsymbol{e}_y, \;\; r \equiv \rho + x$

$$x, y \ll \rho$$

see also: Frenet-Serret coordinates, e.g. Wiedemann chap 4.3

Deriving the Equation of Motion in x-plane (see Appendix)



the effect of the curved coordinate system, i.e. the moving unit vectors e_x, e_s must be *included in the calculation*

starting with general equation of motion: $$\frac{d\vec{p}}{dt} = \gamma m_0 \ddot{\vec{R}} = \vec{F}$$

$$B_y = B_0 + gx, \quad B_x = gy$$ dipole and quadrupole field

$$\frac{1}{\rho} = \frac{eB_0}{\gamma m_0 v}$$ orbit curvature

$$g \equiv \frac{\partial B_y}{\partial x} = \frac{\partial B_x}{\partial y}$$ Quadrupole field gradient sign convention!

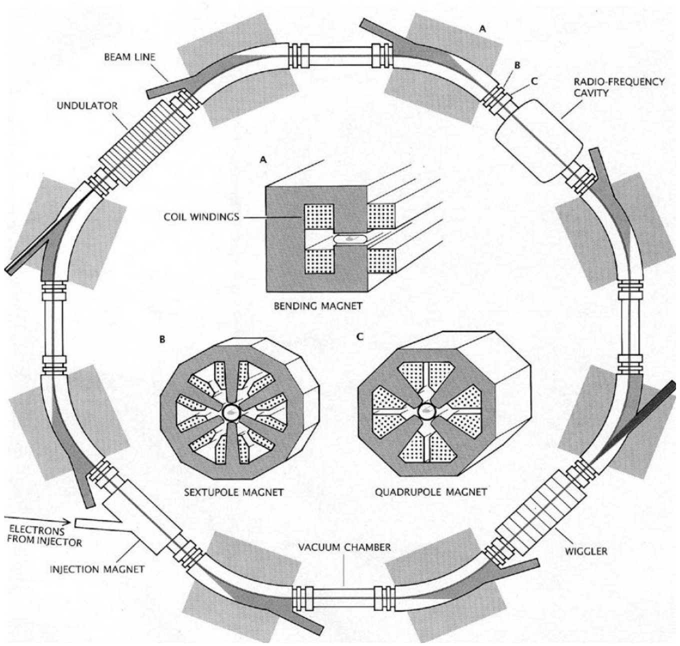
$$k = \frac{eg}{\gamma m_0 v}$$ k - value

$$x'' + \left(\frac{1}{\rho^2} + k\right) x = \frac{1}{\rho}\frac{\Delta p}{p_0}$$

- derivative w.r.t. path-length *s*, not time *t*
- curved coordinates
- k - value
- off momentum term

Equation of Motion in x and y planes:

$$
\begin{aligned}
x'' + \left(\frac{1}{\rho^2} + k\right) x &= \frac{1}{\rho}\frac{\Delta p}{p_0} \\
y'' - ky &= 0
\end{aligned}
$$

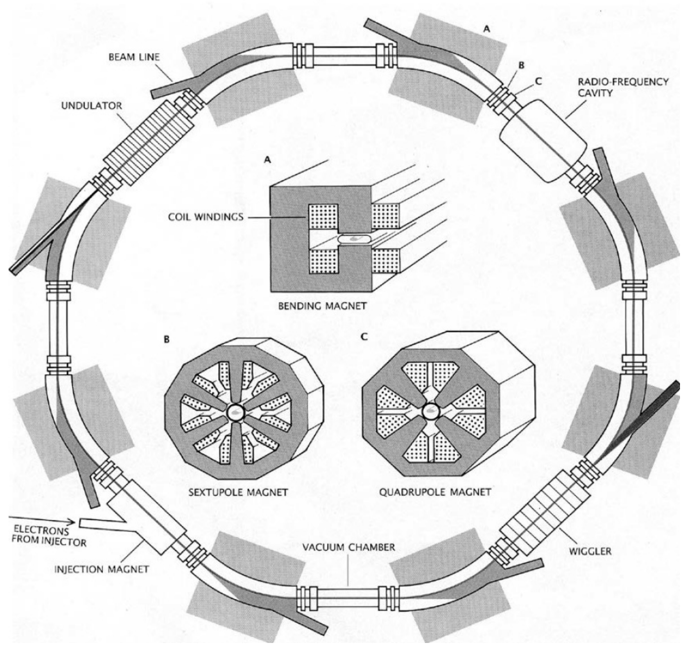


Equation of Motion in x and y planes:

$$x'' + \left(\frac{1}{\rho^2} + k\right) x = \frac{1}{\rho}\frac{\Delta p}{p_0}$$
$$y'' - ky = 0$$

generalised form:

$$x'' + K(s)x = \frac{1}{\rho(s)}\frac{\Delta p}{p_0}$$



*see also Wiedemann sec. 1.5.8

Equation of Motion in x and y planes:

$$x'' + \left(\frac{1}{\rho^2} + k\right)x = \frac{1}{\rho}\frac{\Delta p}{p_0}$$
$$y'' - ky = 0$$

generalised form:

$$x'' + K(s)x = \frac{1}{\rho(s)}\frac{\Delta p}{p_0}$$

Differential Equation valid for:

- drift spaces
- Quadrupoles ($k \neq 0$)
- combined function magnets ($k \neq 0$, $1/\rho \neq 0$)
- off-momentum particles ($\Delta p \neq 0$, first order)

we discuss solutions of different cases of this equations in single accelerator magnets, depending on K(s), ρ(s), Δp

*see also Wiedemann sec. 1.5.8

Summary on Approximations used

- small displacements $x \ll \rho, y \ll \rho, \ddot{s} \approx 0$ **(paraxial optics)**
- only dipole and quadrupole magnets **(linear field changes)**
- design orbit lies in a plane **(flat accelerator)**
- no coupling between motion in hor. and vert. plane **(upright magnets)**
- small momentum deviations **(quasi monochromatic beam)**
- **in general: no quadratic or higher order terms (linear beam optics)**

geometric meaning of coefficients

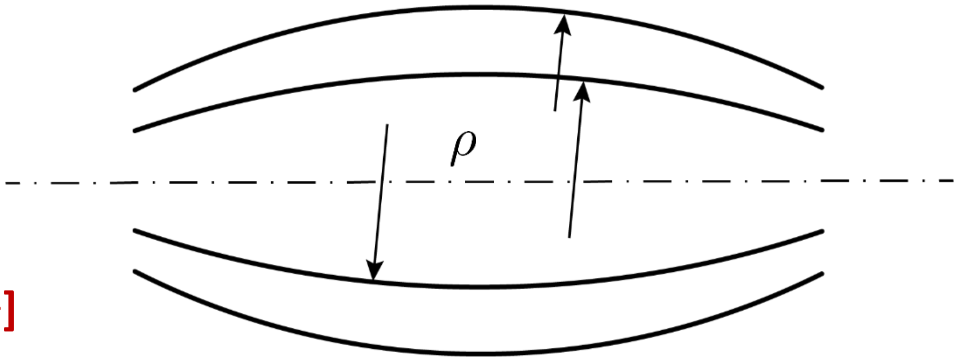
$$x'' + K(s)x = \frac{1}{\rho(s)}\frac{\Delta p}{p_0}$$

$$\frac{d\theta}{ds} = \frac{1}{\rho}(s)$$

= curvature 1/ρ [m^{-1}]

$$K(s) \cdot x = \frac{1}{\rho}(x, s)$$

K = amplitude dependent curvature K[m^{-2}]



Next: Solving the Equation of Motion using Matrices

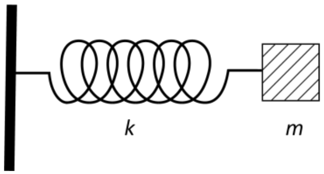
- drift space, focusing and defocusing quadrupole
- TWISS matrix and stability criterion

$$x'' + K(s)x = \frac{1}{\rho(s)} \frac{\Delta p}{p_0}$$

EPFL

Piecewise Solution of Equation

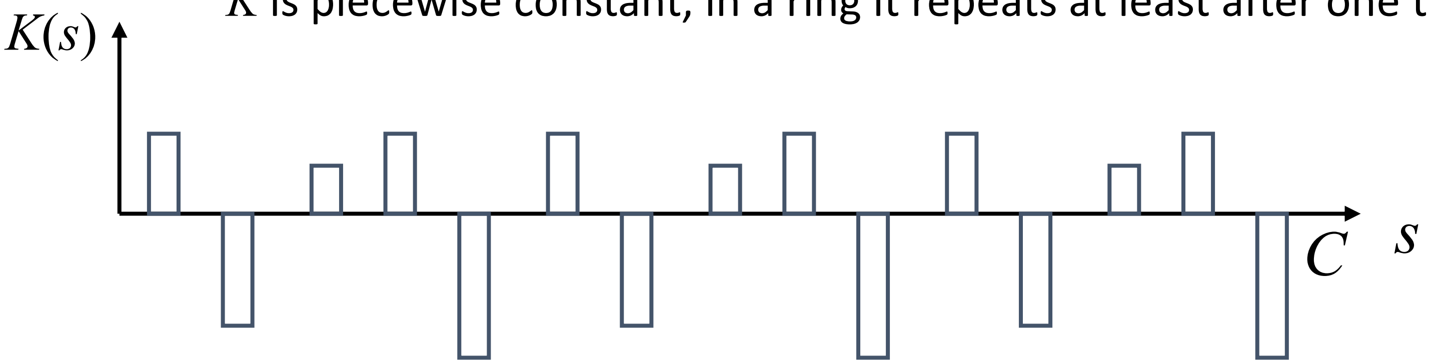
$$x'' + K(s)x \quad = \quad 0$$



general form of equation similar to harmonic oscillator with three cases: *K*=0, *K*<0, *K*>0

$$m\ddot{x} + kx = 0, \;\; \omega = \sqrt{\frac{k}{m}}$$

K is piecewise constant, in a ring it repeats at least after one turn ($s=C$)



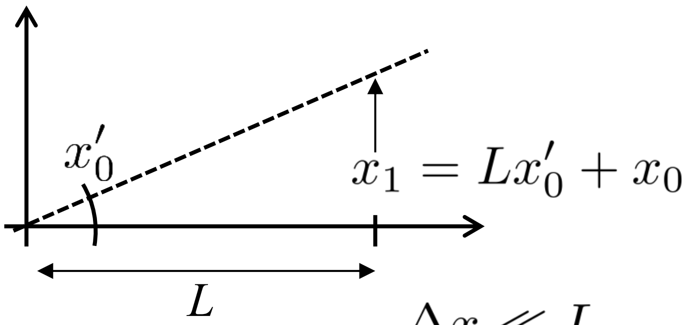
Drift Space

On momentum particles ($\Delta p = 0$) moves straight

$$x'' + K(s)x = 0$$

1) K=0 → Drift Space

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{\text{out}} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}_{\text{in}}$$



$\Delta x \ll L$

Focusing Quadrupole

On momentum particles ($\Delta p = 0$)

$$x'' + K(s)x = 0$$

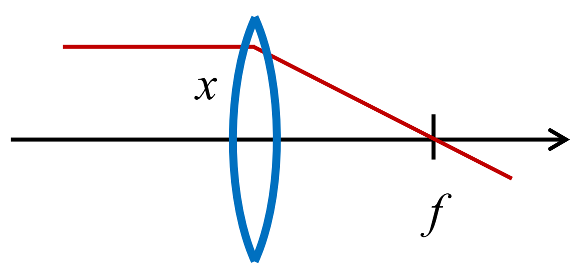
2) K>0: Focusing Quadrupole

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{\text{out}} = \begin{pmatrix} \cos(\sqrt{K}L) & \sin(\sqrt{K}L)/\sqrt{K} \\ -\sin(\sqrt{K}L)\sqrt{K} & \cos(\sqrt{K}L) \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}_{\text{in}}$$

thin lens approximation:

$$K = \frac{1}{Lf}, \quad \lim_{L \to 0} \left(\sin\left(\sqrt{L/f}\right) \frac{1}{\sqrt{Lf}} \right) = \frac{1}{f}$$

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{\text{out}} = \begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}_{\text{in}}$$



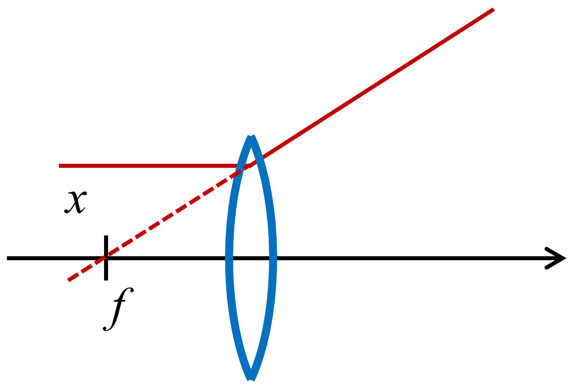
Defocusing Quadrupole

3) K<0: Defocusing Quadrupole

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{\text{out}} = \begin{pmatrix} \cosh(\sqrt{|K|}L) & \sinh(\sqrt{|K|}L)/\sqrt{|K|} \\ \sinh(\sqrt{|K|}L)\sqrt{|K|} & \cosh(\sqrt{|K|}L) \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}_{\text{in}}$$

thin lens approximation for defocusing quad:

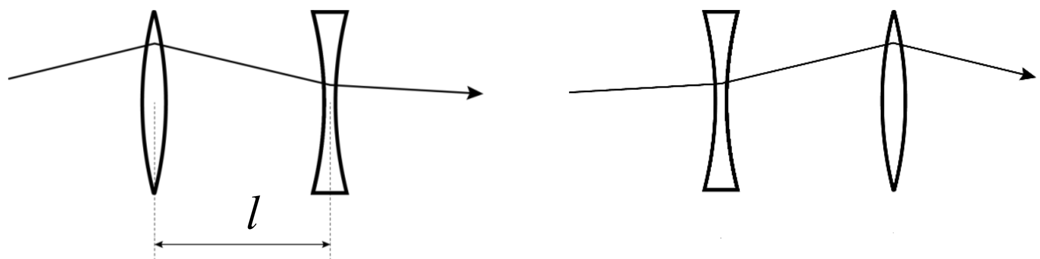
$$\begin{pmatrix} x \\ x' \end{pmatrix}_{\text{out}} = \begin{pmatrix} 1 & 0 \\ 1/f & 1 \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}_{\text{in}}$$



Quadrupole Doublet

concatenation of particle transport through a series of elements:

$$\boldsymbol{M} = \boldsymbol{M}_n \ldots \boldsymbol{M}_2 \cdot \boldsymbol{M}_1$$ (***M*** = transport matrix 2x2)



$$\boldsymbol{M}_{\text{doublet}} = \begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 1/f & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 - \frac{l}{f} & l \\ -\frac{1}{f^*} & 1 - \frac{l}{f} \end{pmatrix}$$

$$f^* = \frac{f^2}{l} > 0$$ $\rightarrow \mathrm{M}_{\text{doublet}}$ is always focusing

Courant - Snyder parameters: transfer matrices

Consider a transfer matrix M for a full turn starting at some point

- We know that **det M = 1**
- Any such matrix with unit determinant can be parameterized:

$$M_{period} = \begin{pmatrix} \cos\mu + \alpha \sin\mu & \beta \sin\mu \\ -\gamma \sin\mu & \cos\mu - \alpha \sin\mu \end{pmatrix}$$

where in order to satisfy the condition of unit determinant

$$\beta\gamma - \alpha^2 = 1$$

we can regard this now as just a formal parameterization

Twiss Parameters:

$$\beta(s)$$

$$\alpha(s) = -\frac{1}{2}\beta'(s)$$

$$\gamma(s) = \frac{1 + \alpha^2}{\beta}$$

β - famous **Betafunction** of accelerators

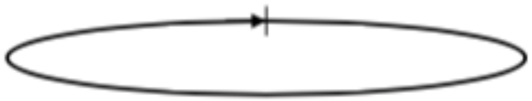
these variables **are not related to relativistic factors**

(sorry for the historic nomenclature)

Stability of Motion

The transfer matrix of a beamline that consists of elements with individual matrices M_1 , M_2 , ... M_n $\mathbf{M_{tot} = M_n \cdot \;...\; \cdot M_2 \cdot M_1}$
(N.B. the order in which matrices are multiplied!)

- Full turn matrix M



$$\begin{pmatrix} x \\ x' \end{pmatrix}_n = M^n \begin{pmatrix} x \\ x' \end{pmatrix}_0$$

- After n turns must remain finite for arbitrarily large n

Stability condition

Let v_1 and v_2 be eigenvectors and λ_1 and λ_2 eigenvalues of M

$$\begin{pmatrix} x \\ x' \end{pmatrix}_0 = A\mathrm{v}_1 + B\mathrm{v}_2 \qquad\qquad M^n \begin{pmatrix} x \\ x' \end{pmatrix}_0 = A\lambda_1^n \mathrm{v}_1 + B\lambda_2^n \mathrm{v}_2$$

- For stability λ_1^n, λ_2^n must not grow with n
- since the product of eigenvalues is unity:

$$\det M = 1 \quad \Rightarrow \quad \lambda_1 \cdot \lambda_2 = 1$$

we can write in general $\lambda_1 = e^{i\mu}, \lambda_2 = e^{-i\mu}$

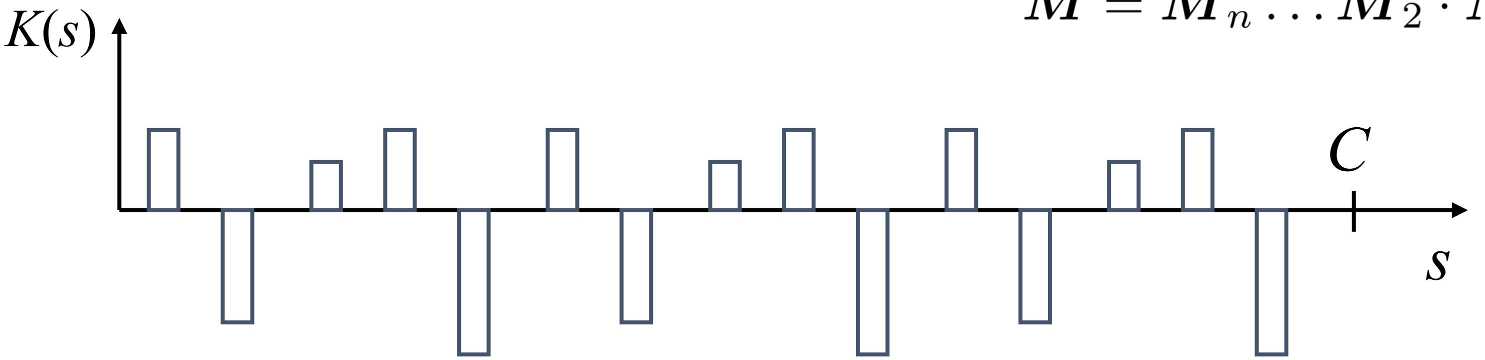
- For stability μ should be real! $\mathrm{Tr}\, M = \lambda_1 + \lambda_2 = 2\cos\mu$

$$-1 \le \frac{1}{2}\mathrm{Tr}\, M \le 1$$

Summary Matrix Treatment

- equation of motion is piecewise solved for constant K(s)
- coordinates x, x' are transported by multiplication with a 2x2 matrix
- matrixes can be concatenated
- defocusing and focusing quadrupoles are combined in overall focusing doublets
- $\det \mathbf{M} = 1$
- Courant-Snyder Parametrization of **M**
- linear motion in a ring is stable if $|\mathrm{Tr}\,\mathbf{M}|<2$

$$\boldsymbol{M} = \boldsymbol{M}_n \ldots \boldsymbol{M}_2 \cdot \boldsymbol{M}_1$$



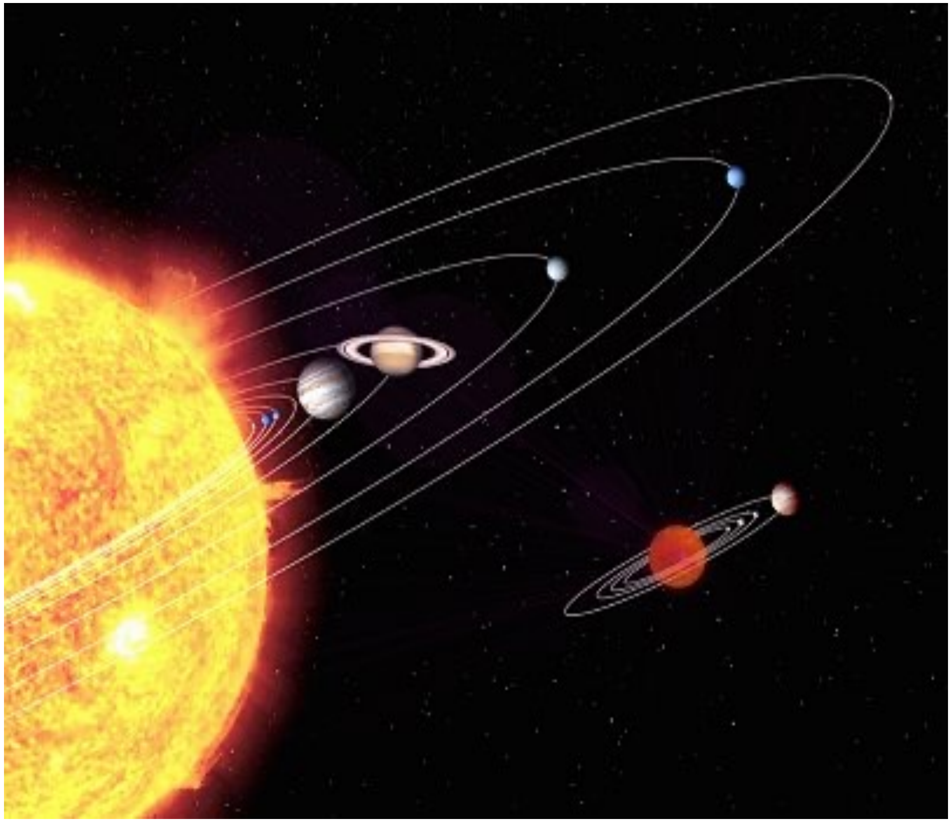
Next: Analytical Solution

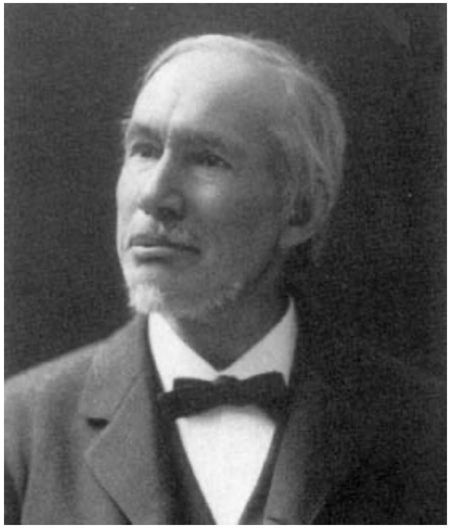
- Hills equation
- Beta function
- phase space ellipse
- include momentum offset
- tune Q_x, Q_y

Hill equation

- First used by an astronomer G. Hill in his studies of the motion of the moon, a motion under the influence of periodically changing forces

$$x'' + K(s) \cdot x = 0$$





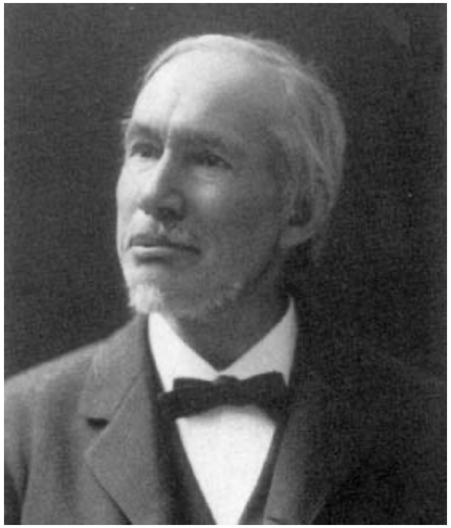
1838 -- 1914

$$K(s) = K(s + C)$$

Periodic over one full revolution
C = 29 days

Hill equation

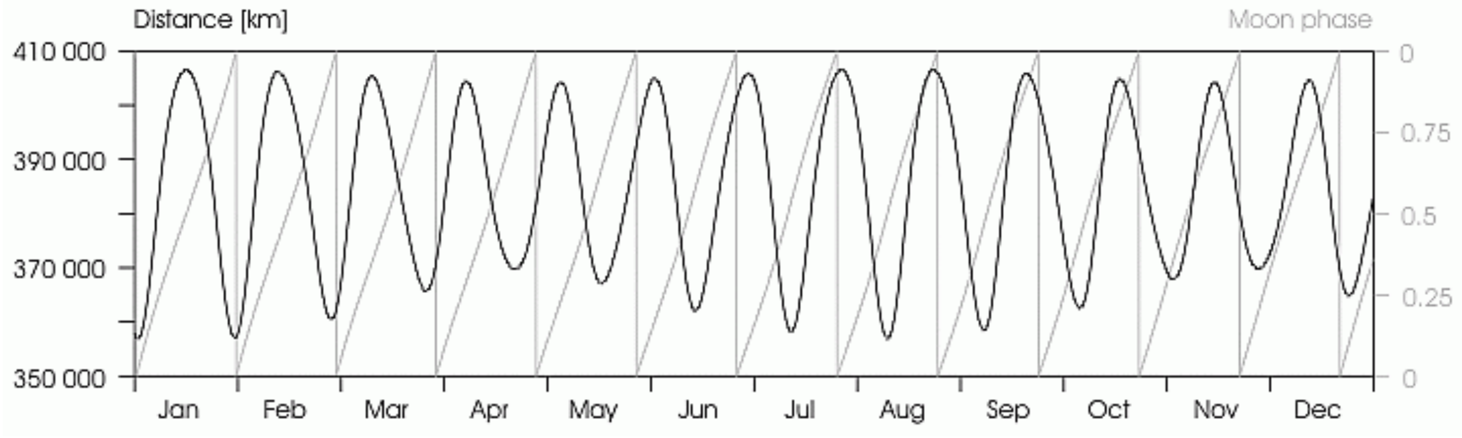
- First used by an astronomer G. Hill in his studies of the motion of the moon, a motion under the influence of periodically changing forces



Solution is of the type:

$$u(s) = A\sqrt{\beta(s)}\cos\left[\phi(s)\right]$$

Pseudo-harmonic oscillator



$$x'' + K(s) \cdot x = 0$$

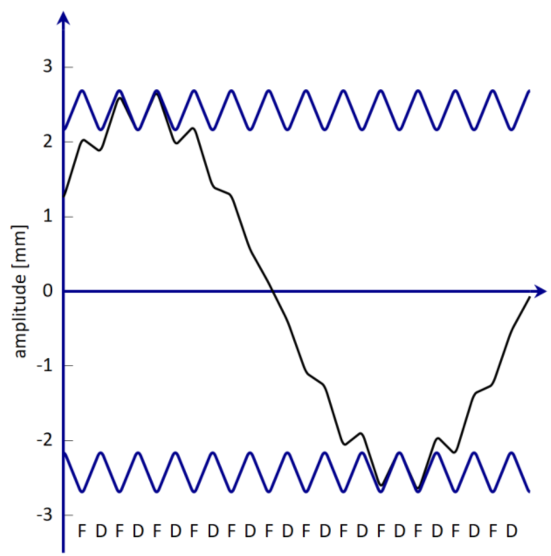
$$K(s) = K(s + C)$$

Hill: Solution for periodic *K*

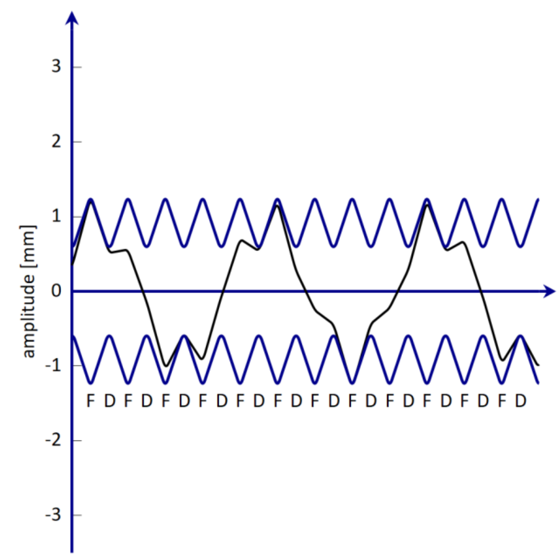
$$x(s) = A\sqrt{\beta(s)}\cos(\varphi(s) - \varphi_0), \;\; \varphi(s) = \int_{t=s_0}^{s} \frac{dt}{\beta(t)}$$

$\rightarrow$ the **beta function is a scaling factor** for the amplitude of orbit oscillations and their **local wavelength**

A, φ_0 are constants of motion



weak quads



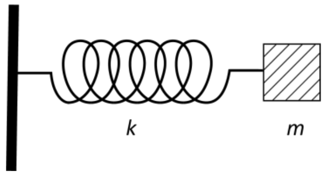
strong quads

$$A\sqrt{\beta(s)}$$

Comparison to Classical Harmonic Oscillator

$$\ddot{u} + \omega^2 u = 0$$

$$u(t) = A\cos\omega t,\ \ \omega = \sqrt{\frac{k}{m}}$$



amplitude is fixed: $A = \text{const}$

phase grows linear with time: $\sqrt{\frac{k}{m}}t$

conserved (energy): $\frac{k}{2}u^2 + \frac{m}{2}\dot{u}^2 = \frac{k}{2}A^2$

Hill Equation (pseudo harmonic equation)

$$x(s) = \sqrt{2J\beta}\cos(\varphi)$$

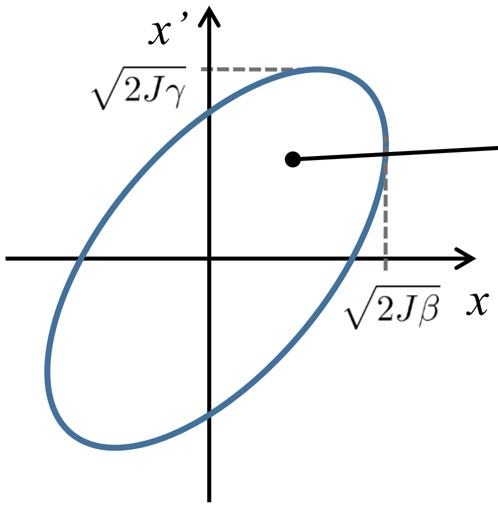
$$x'(s) = -\sqrt{\frac{2J}{\beta}}\left(\alpha\cos(\varphi) + \sin(\varphi)\right)$$

amplitude varies: $x(s) \propto \sqrt{\beta(s)}$

phase increases monotonically
but growth rate varies as 1/β : $d\varphi = \frac{ds}{\beta(s)}$

conserved (action): $\gamma x^2 + 2\alpha x x' + \beta x'^2 = 2J = \text{const}$

Conserved action : invariant on motion



$$\text{area} = 2\pi J = \pi(\gamma x^2 + 2\alpha x x' + \beta x'^2)$$

phase space area is preserved

tune = number of oscillations per turn:

$$Q_x = \frac{1}{2\pi} \oint \frac{ds}{\beta_x(s)}$$

Particle motion in phase space

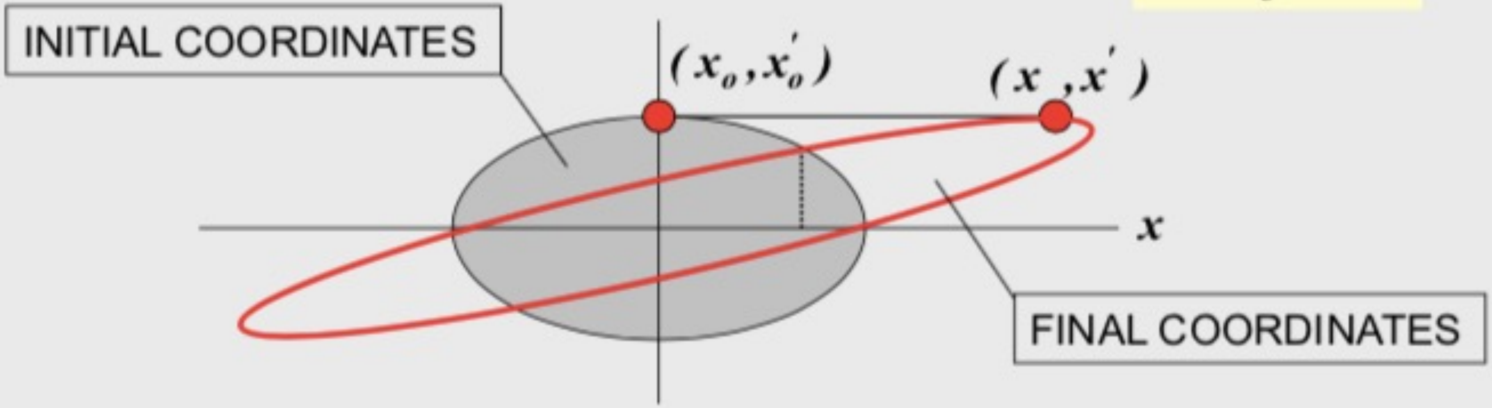
CHANGE OF PARTICLE DISTRIBUTION IN PHASE SPACE

The initial coordinates of a particle ensemble in the transverse phase plane are contained in the ellipse:

$$x = x_o + s.x_o'$$

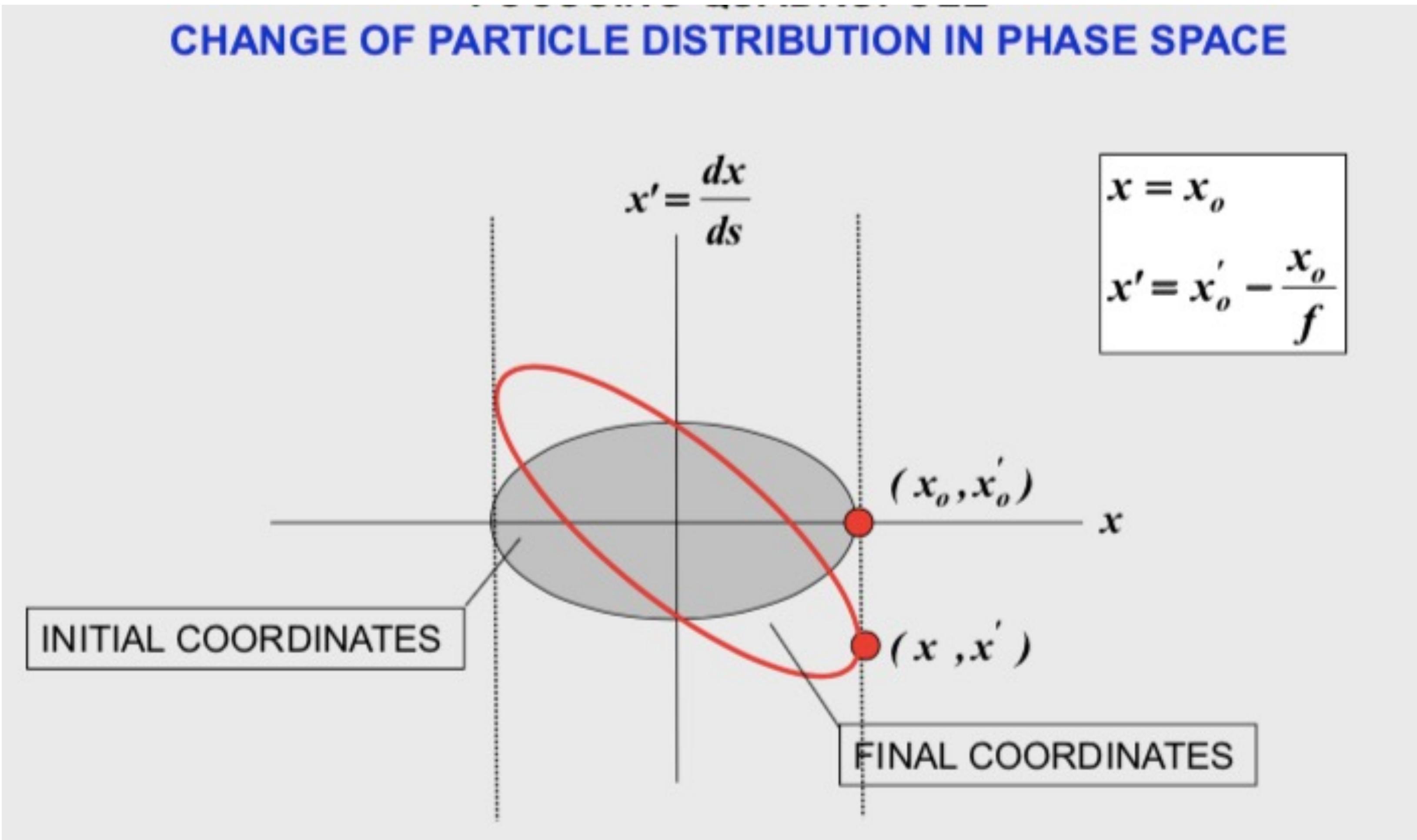
$$x' = x_o'$$

$$x' = \frac{dx}{ds}$$

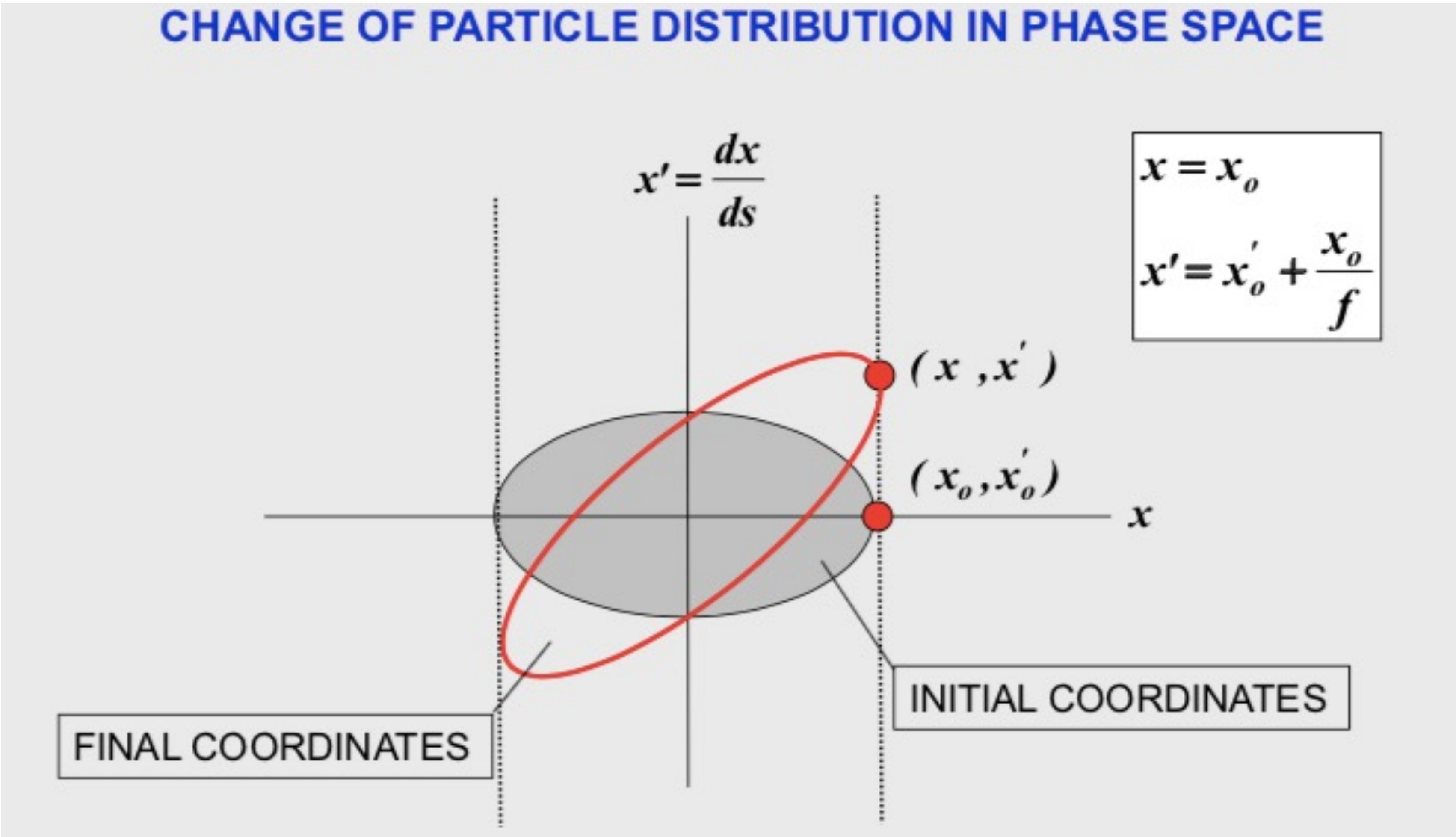


Focusing is needed to avoid beam blow up !

Focusing Quadrupole



De-Focusing Quadrupole



Computing the transport matrix from Twiss Parameters

transport matrix $s=s_0 \ldots s_1$ (arbitrary section):

$$\begin{pmatrix} C & S \\ C' & S' \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{\beta}{\beta_0}}(\cos\Delta\varphi + \alpha_0 \sin\Delta\varphi) & \sqrt{\beta\beta_0}\sin\Delta\varphi) \\ -\frac{1}{\sqrt{\beta\beta_0}}((\alpha-\alpha_0)\cos\Delta\varphi + (1+\alpha\alpha_0)\sin\Delta\varphi) & \sqrt{\frac{\beta_0}{\beta}}(\cos\Delta\varphi - \alpha\sin\Delta\varphi) \end{pmatrix}$$

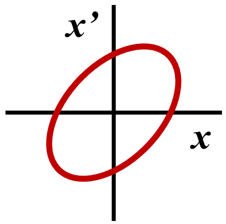
β_0, α_0 at s_0
β, α at s_1
$\Delta\varphi$ = phase advance between s_0, s_1

The one turn matrix $\boldsymbol{M}_{\text{rev}}$

transport matrix for one revolution, “One Turn Matrix”

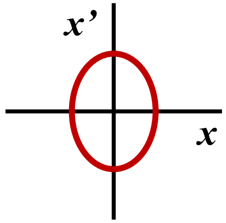
- same location: $\beta = \beta_0$
- $\Delta\varphi = 2\pi Q$ phase advance for complete turn, Q = “Tune” of accelerator

$$\boldsymbol{M}_{\text{rev}} = \begin{pmatrix} \cos 2\pi Q + \alpha \sin 2\pi Q & \beta \sin 2\pi Q \\ -\frac{1+\alpha^2}{\beta} \sin 2\pi Q & \cos 2\pi Q - \alpha \sin 2\pi Q \end{pmatrix}$$



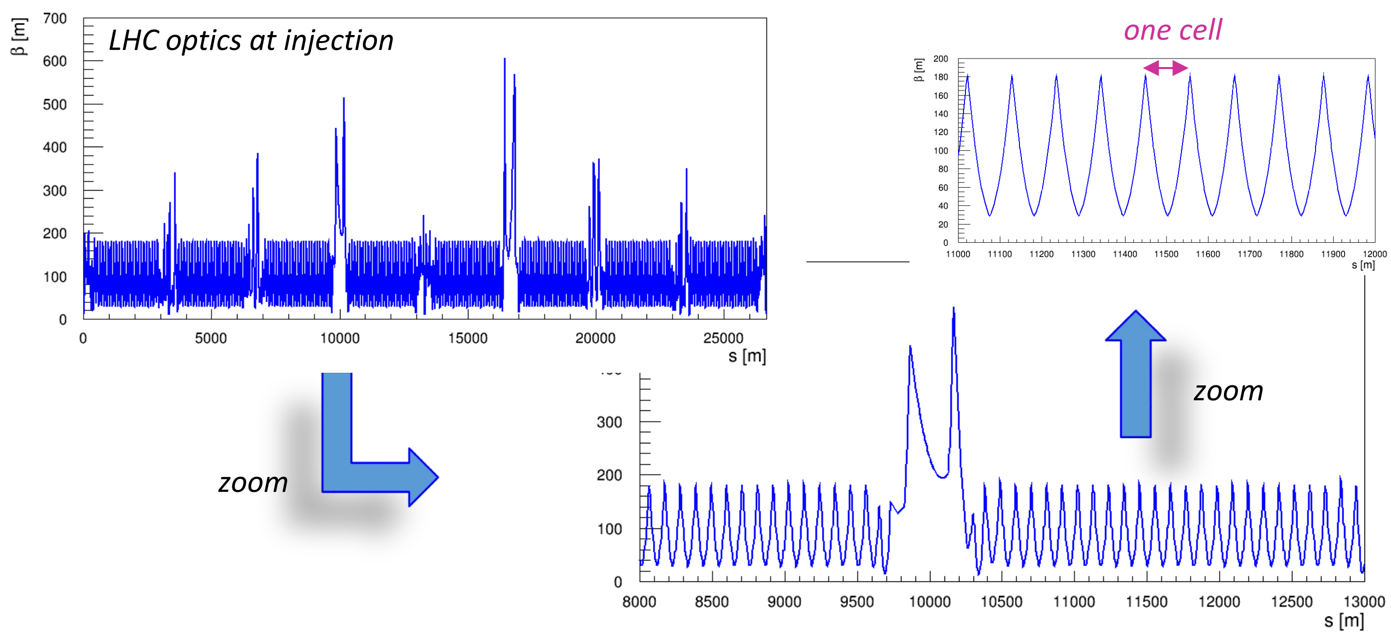
- special case: choose symmetry point $\alpha = 0$

$$\boldsymbol{M}_{\text{rev}} = \begin{pmatrix} \cos 2\pi Q & \beta \sin 2\pi Q \\ -\frac{1}{\beta} \sin 2\pi Q & \cos 2\pi Q \end{pmatrix}$$



Beta function

- The **betatron function** **(β)** that defines the beam envelope,
  - **Beam size / envelope is proportional to √β**



The Betatron Frequency Q (tune of accelerator)

$$Q_x = \frac{1}{2\pi} \oint \frac{ds}{\beta_x(s)}$$

around ring

Tune = Number of Betatron Oscillations per Turn
the choice of tune is important to avoid resonant behaviour
Both integer and fractional part are important for machine design

Tunes and Orbit in LHC

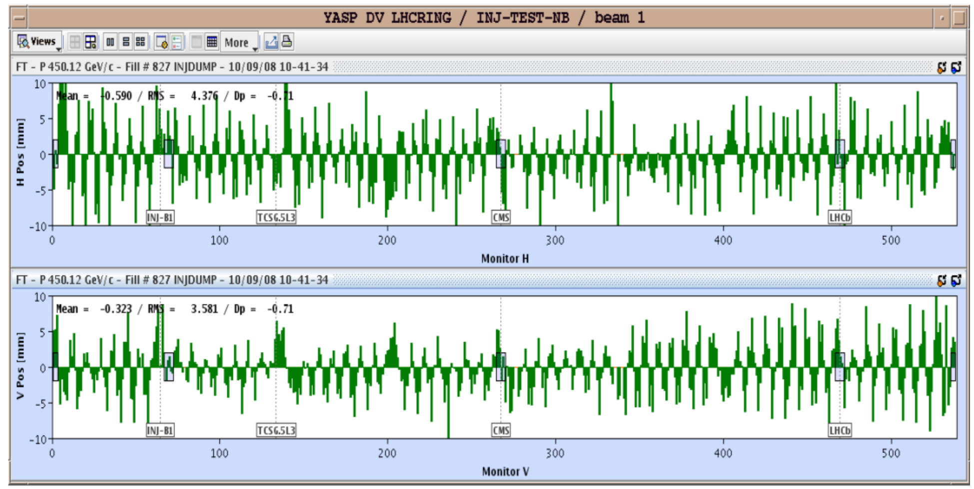
Example Orbit Oscillations:

LHC Tunes:

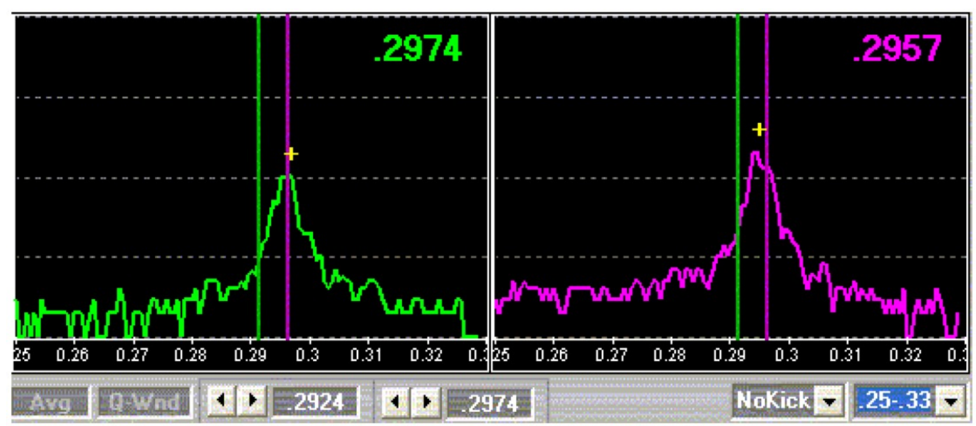
$Q_x = 64.31$

$Q_y = 59.32$

relevant for stability: non-integer part



Example Measured Beam Spectrum:



LHC Revolution Frequency: 11.3kHz

peak position: 3.5kHz = 0.31×11.3kHz

Appendix: Magnetic Rigidity (proton)

Lorentz force $\vec{F}_B = e \cdot \vec{v} \times \vec{B}$

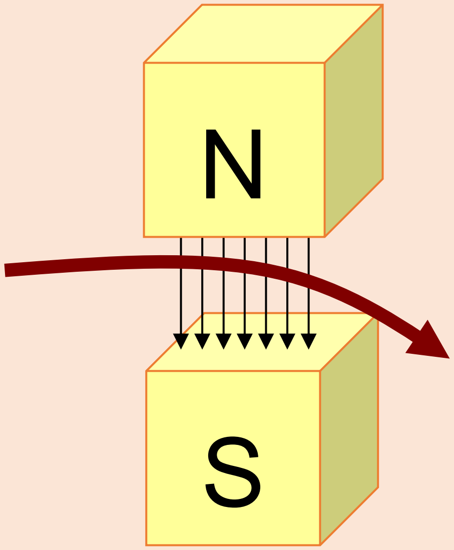
B, v perpendicular $F_B = evB$

centrifugal force $F_c = -m\frac{v^2}{\rho}$

$$F_B + F_c = 0 \longrightarrow evB = m\frac{v^2}{\rho}$$

$$B\rho = \frac{mv}{e}$$

Magnetic rigidity $\boxed{B\rho = \frac{p}{e}}$



B = magnetic field
ρ = local bending radius
p = momentum
e = elementary charge

Appendix: Magnetic Rigidity in Practical Units

$$B\rho = \frac{p}{e} = \frac{mv}{e} = \beta\gamma\frac{m_0 c}{e}$$

$$= \beta\gamma \, \frac{m_0 c^2}{ce}$$

$$= \beta \, \frac{E_{\mathrm{tot}}}{ce}$$

$$= \beta \, \frac{10^9}{c} E_{\mathrm{tot}}[\mathrm{GeV}]$$

$$\downarrow$$

$$B\rho[\mathrm{Tm}] \approx 3.3356 \cdot E_k[\mathrm{GeV/c}]$$

$$B\rho[\mathrm{Tm}] = 3.3356 \cdot p[\mathrm{GeV/c}]$$

B = magnetic field
ρ = local bending radius
p = momentum
e = elementary charge
E_k = kinetic energy

total energy:

$$E_{\mathrm{tot}} = E_k + m_0 c^2$$

approximations:

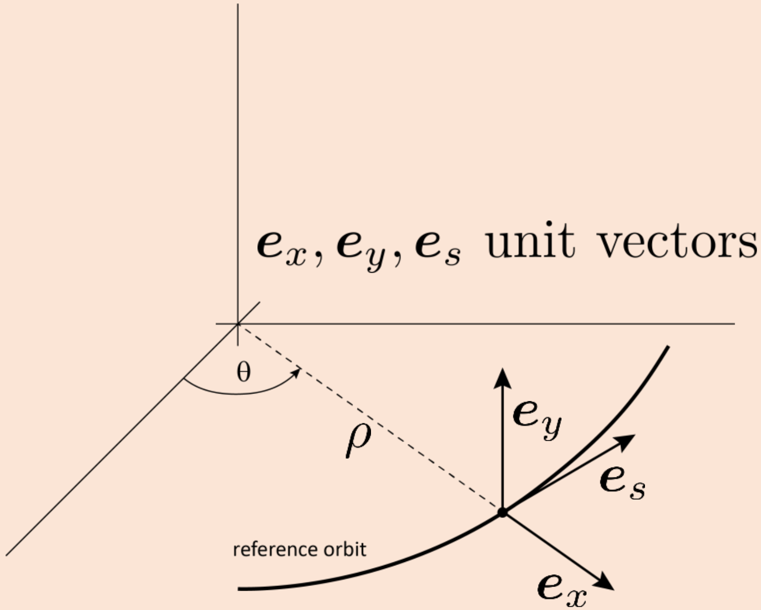
$$\beta \approx 1, cp \approx E_k$$

$$\text{for } E_k \gg m_0 c^2$$

see also Wiedemann, p.101, eq.5.6

Appendix, Derivation: Equation of Motion I

starting with general equation of motion: $\frac{d\vec{p}}{dt} = \gamma m_0 \ddot{\vec{R}} = \vec{F}$



$$
\begin{aligned}
\vec{R} &= r\boldsymbol{e}_x + y\boldsymbol{e}_y, \;\; r \equiv \rho + x \\
\dot{\vec{R}} &= \dot{r}\boldsymbol{e}_x + r\dot{\boldsymbol{e}}_x + \dot{y}\boldsymbol{e}_y \\
\dot{\vec{R}} &= \dot{r}\boldsymbol{e}_x + r\dot{\theta}\boldsymbol{e}_s + \dot{y}\boldsymbol{e}_y \\
\ddot{\vec{R}} &= \ddot{r}\boldsymbol{e}_x + (2\dot{r}\dot{\theta} + r\ddot{\theta})\boldsymbol{e}_s + r\dot{\theta}\dot{\boldsymbol{e}}_s + \ddot{y}\boldsymbol{e}_y \\
\ddot{\vec{R}} &= (\ddot{r} - r\dot{\theta}^2)\boldsymbol{e}_x + (2\dot{r}\dot{\theta} + r\ddot{\theta})\boldsymbol{e}_s + \ddot{y}\boldsymbol{e}_y
\end{aligned}
$$

used here: $\dot{\boldsymbol{e}}_x = \dot{\theta}\boldsymbol{e}_s, \;\; \dot{\boldsymbol{e}}_s = -\dot{\theta}\boldsymbol{e}_x$

comment: the main purpose here is to correctly treat the effect of the curved coordinate system, i.e. the moving unit vectors e_x, e_s

Derivation: Equation of Motion II

right side of equation, the force:

$$\vec{F} = e\vec{v} \times \vec{B}$$

$$\vec{v} \times \vec{B} = \begin{vmatrix} \boldsymbol{e}_x & \boldsymbol{e}_y & \boldsymbol{e}_s \\ v_x & v_y & v_s \\ B_x & B_y & 0 \end{vmatrix}$$

$$= -v_s B_y \boldsymbol{e}_x + v_s B_x \boldsymbol{e}_y + (v_x B_y - v_y B_x)\boldsymbol{e}_s$$

assumptions:

- no B_s
- $B_x(y=0) = 0$

use:

$$B_y = B_0 + gx$$

$$B_x = gy$$

$$g \equiv \frac{\partial B_y}{\partial x} = \frac{\partial B_x}{\partial y}$$

result: two equations hor/vert from x,y components:

$$\gamma m_0(\ddot{r} - r\dot{\theta}^2) = -ev_s(B_0 + gx)$$

$$\gamma m_0 \ddot{y} = ev_s gy$$

in literature g has varying sign conventions
Wiedemann, Table 6.2: $g = +dB_y/dx$
Schmüser/Hillert: $g = -dB_y/dx$

Derivation: Equation of Motion III

introduce path length s as independent variable:

$$\begin{aligned} \gamma m_0(\ddot{r} - r\dot{\theta}^2) &= -ev_s(B_0 + gx) \\ \gamma m_0 \ddot{y} &= ev_s g y \end{aligned}$$

$$\begin{aligned} x'' &= \frac{1}{r} - \frac{e}{\gamma m_0 v}(B_0 + gx) \\ y'' &= \frac{e}{\gamma m_0 v} g y \end{aligned}$$

use:

$$v_s = r\dot{\theta} \approx v$$

$$\ddot{r} = \ddot{x}$$

$$\ddot{x} = v^2 x'', \;\; x'' \equiv \frac{\partial^2 x}{\partial s^2}$$

$$\ddot{y} = v^2 y'', \;\; y'' \equiv \frac{\partial^2 y}{\partial s^2}$$

Derivation: Equation of Motion IV

$$
\begin{aligned}
x'' &= \frac{1}{r} - \frac{e}{\gamma m_0 v}(B_0 + gx) \\
y'' &= \frac{e}{\gamma m_0 v} g y
\end{aligned}
$$

$$
\begin{aligned}
x'' &= \frac{1}{\rho}\left(1 - \frac{x}{\rho}\right) - kx - \frac{1}{\rho\left(1 + \frac{\Delta p}{p_0}\right)} \\
&= -\left(\frac{1}{\rho^2} + k\right) x + \frac{1}{\rho}\frac{\Delta p}{p_0} \\
y'' &= ky
\end{aligned}
$$

use:

$$\frac{1}{r} = \frac{1}{\rho + x} \approx \frac{1}{\rho}\left(1 - \frac{x}{\rho}\right)$$

$$\frac{eB_0}{\gamma m_0 v} = \frac{eB_0}{p} = \frac{1}{\rho}$$

$$p = p_0\left(1 + \frac{\Delta p}{p_0}\right)$$

$$k = \frac{eg}{\gamma m_0 v}$$