Sapienza
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Modeling and Control of a Free-Floating Planar Manipulator

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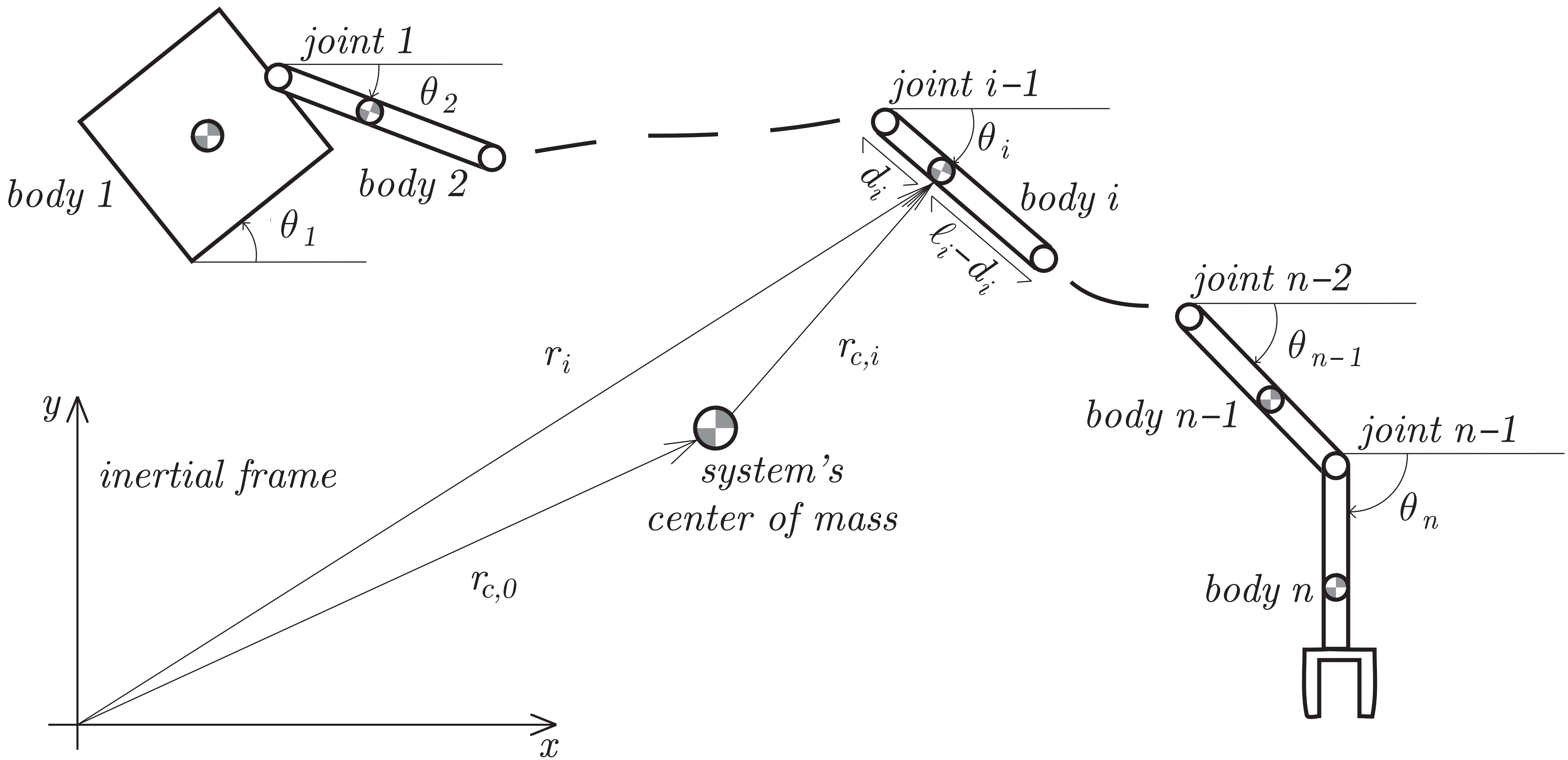
Modeling Assumptions

- n-body planar open kinematic chain
- inertial frame located at the system center of mass
- no external force is applied
- no gravity nor dissipative forces
- initial zero linear and angular momenta

n-Body Planar Manipulator



Conservation of linear momentum:

$$\sum_{i=1}^{n} m_i \dot{r}_i = 0 \rightarrow \int \rightarrow \sum_{i=1}^{n} m_i r_i = m_t r_{c,0} = c = 0$$

Conservation of angular momentum along the z axis (nonholonomic for $n>2$):

$$\sum_{i=1}^{n} p_i = \mathbf{1}_n^T B(\theta)\dot{\theta} = 0$$

where $\theta = (\theta_1, \ldots, \theta_n)^T$ is the generalized vector of coordinates, $B(q) \in \mathbb{R}^{n \times n}$ is the positive definite robot inertia matrix and $\mathbf{1}_n^T = (1, 1, \ldots, 1) \in \mathbb{R}^n$.

i-th Body Center of Mass

$$r_{c,i} = \begin{bmatrix} \sum_{j=1}^{n} k_{ij} \cos\theta_j \\ \sum_{j=1}^{n} k_{ij} \sin\theta_j \end{bmatrix} \qquad k_{ij} = \begin{cases} \frac{1}{m_t}[\ell_j \sum_{h=1}^{j-1} m_h + (\ell_j - d_j) m_j] & \text{for } j < i, \\ \frac{1}{m_t}[d_i \sum_{h=1}^{i-1} m_h - (\ell_i - d_i) \sum_{k=i+1}^{n} m_k] & \text{for } j = i, \\ -\frac{1}{m_t}[\ell_j \sum_{h=j+1}^{n} m_h + d_j m_j] & \text{for } j > i. \end{cases}$$

obtained from the conservation of linear momentum and the geometric relationships between consecutive links.

i-th Body Kinetic Energy

$$T_i = \frac{1}{2} m_i \dot{r}_{c,i}^T \dot{r}_{c,i} + \frac{1}{2} J_i \dot{\theta}_i^2 =$$

$$= \frac{1}{2} m_i \left[\sum_{h=1}^{n} \sum_{j=1}^{n} k_{ij} k_{ih} \cos(\theta_h - \theta_j) \dot{\theta}_h \dot{\theta}_j \right] + \frac{1}{2} J_i \dot{\theta}_i^2$$

where J_i is the inertia of the *i*-th body along the z axis.

Robot Inertia Matrix

The kinetic energy of the system is

$$T = \sum_{i=1}^{n} T_i = \frac{1}{2}\dot{\theta}^T B(\theta)\dot{\theta}$$

and from this we can obtain

$$B(\theta) \in \mathbf{R}^{n \times n} \quad b_{ij}(\theta_i, \theta_j) = \begin{cases} \sum_{h=1}^{n} m_h k_{hi} k_{hj} \cos(\theta_i - \theta_j) & \text{for } j \neq i, \\ J_i + \sum_{h=1}^{n} m_h k_{hh}^2 & \text{for } j = i. \end{cases}$$

New Generalized Coordinates

$$q = (\theta_1, \phi_1, \ldots, \phi_{n-1})^T \qquad \phi_i = \theta_{i+1} - \theta_i \qquad i = 1, \ldots, n-1$$

with the following inverse mapping

$$\theta = \begin{bmatrix} 1 & 0 & 0 & \ldots & \ldots \\ 1 & 1 & 0 & \ldots & \ldots \\ 1 & 1 & 1 & 0 & \ldots \\ & & \ldots & & \\ 1 & \ldots & 1 & 1 & 1 \end{bmatrix} q = \begin{bmatrix} \mathbf{1}_n & S \end{bmatrix} \begin{bmatrix} \theta_1 \\ \phi \end{bmatrix}$$

Kinematic Model

From the conservation of angular momentum

$$\mathbf{1}_n^T B(\theta)\dot{\theta} = \mathbf{1}_n^T B(q) \begin{bmatrix} \mathbf{1}_n & S \end{bmatrix} \dot{q} = \mathbf{1}_n^T B(\phi) \left(\mathbf{1}_n \dot{\theta}_1 + S\dot{\phi} \right) = 0$$

we obtain

$$\dot{\theta}_1 = -\frac{\mathbf{1}_n^T B(\phi) S}{\mathbf{1}_n^T B(\phi) \mathbf{1}_n} u$$

where $u = \dot{\phi}$ are the commanded joint velocities.

Holonomy Angle

Given $\gamma = \theta_{1,d} - \theta_1$ and t_ℓ determine the value of Δ that identify Γ such that

$$\oint_\Gamma d\theta_1(u) = \gamma$$

with

$$u_i(t) = \begin{cases} \Delta / t_\ell & t \in [t_\ell(i-1), t_\ell i), \\ -\Delta / t_\ell & t \in [t_\ell(n+i-2), t_\ell(n+i-1)), \\ 0 & \text{otherwise.} \end{cases}$$

Cyclic Control Strategy

1. First drive ϕ to the desired value ϕ_d using

$$u = K(\phi_d - \phi)$$

2. Determine Δ and apply the piecewise-constant input so to achieve the desired base reorientation

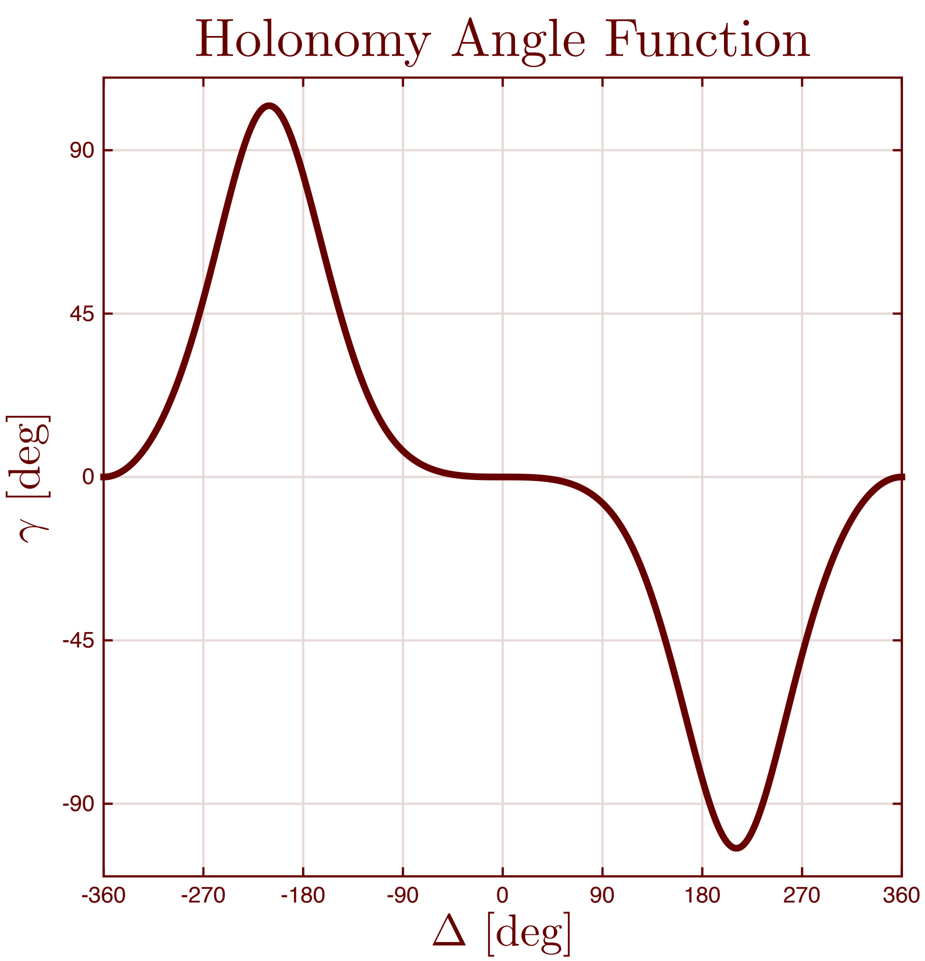
$$u_i(t) = \begin{cases} \Delta / t_\ell & t \in [t_\ell(i-1), t_\ell i), \\ -\Delta / t_\ell & t \in [t_\ell(n+i-2), t_\ell(n+i-1)), \\ 0 & \text{otherwise.} \end{cases}$$

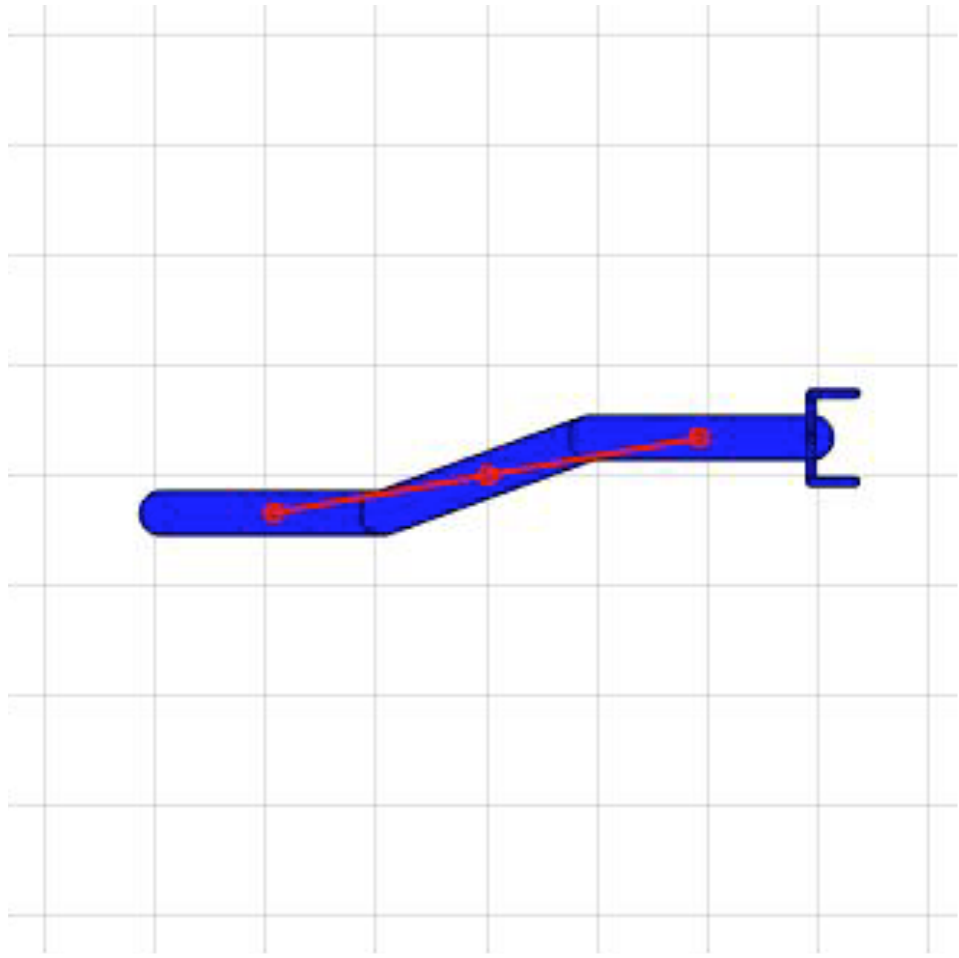
Simulation
Three-Body Free-Floating Manipulator

$m_i = 10\ \text{kg} \quad \ell_i = 1\ \text{m} \quad d_i = 0.5\ \text{m} \quad J_i = m_i \ell_i^2\ /\ 12 \quad n = 3$

$q_0{}^T = [0°, -20°, 20°] \quad q_d{}^T = [90°, 30°, -30°] \quad K_\phi = 0.5 \cdot I_2$

$\gamma = 65.9° \quad \Delta = -165°$





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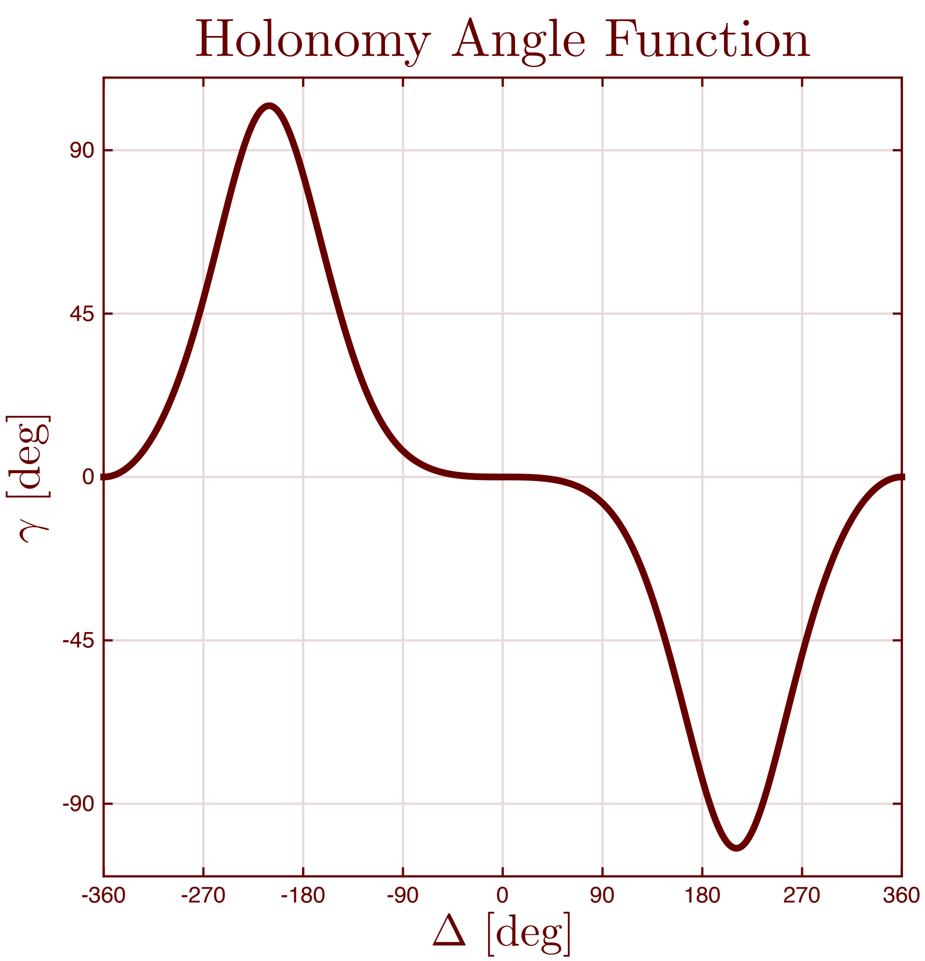
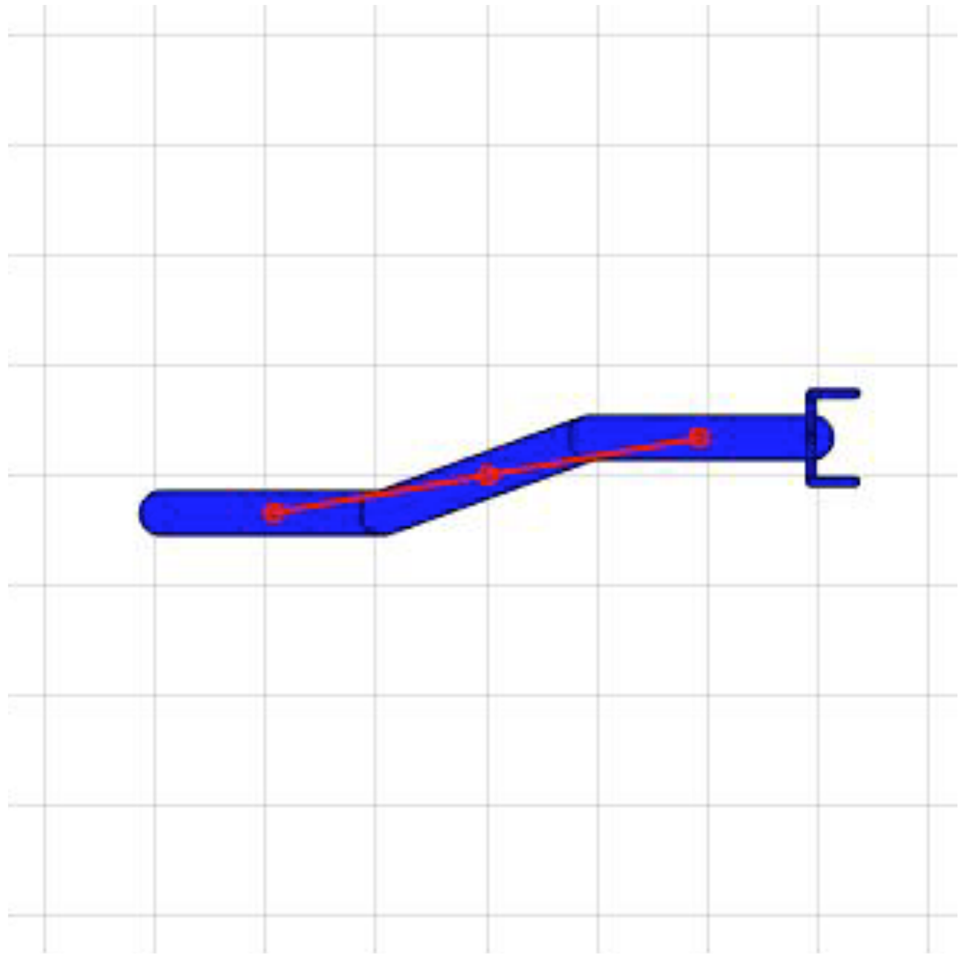
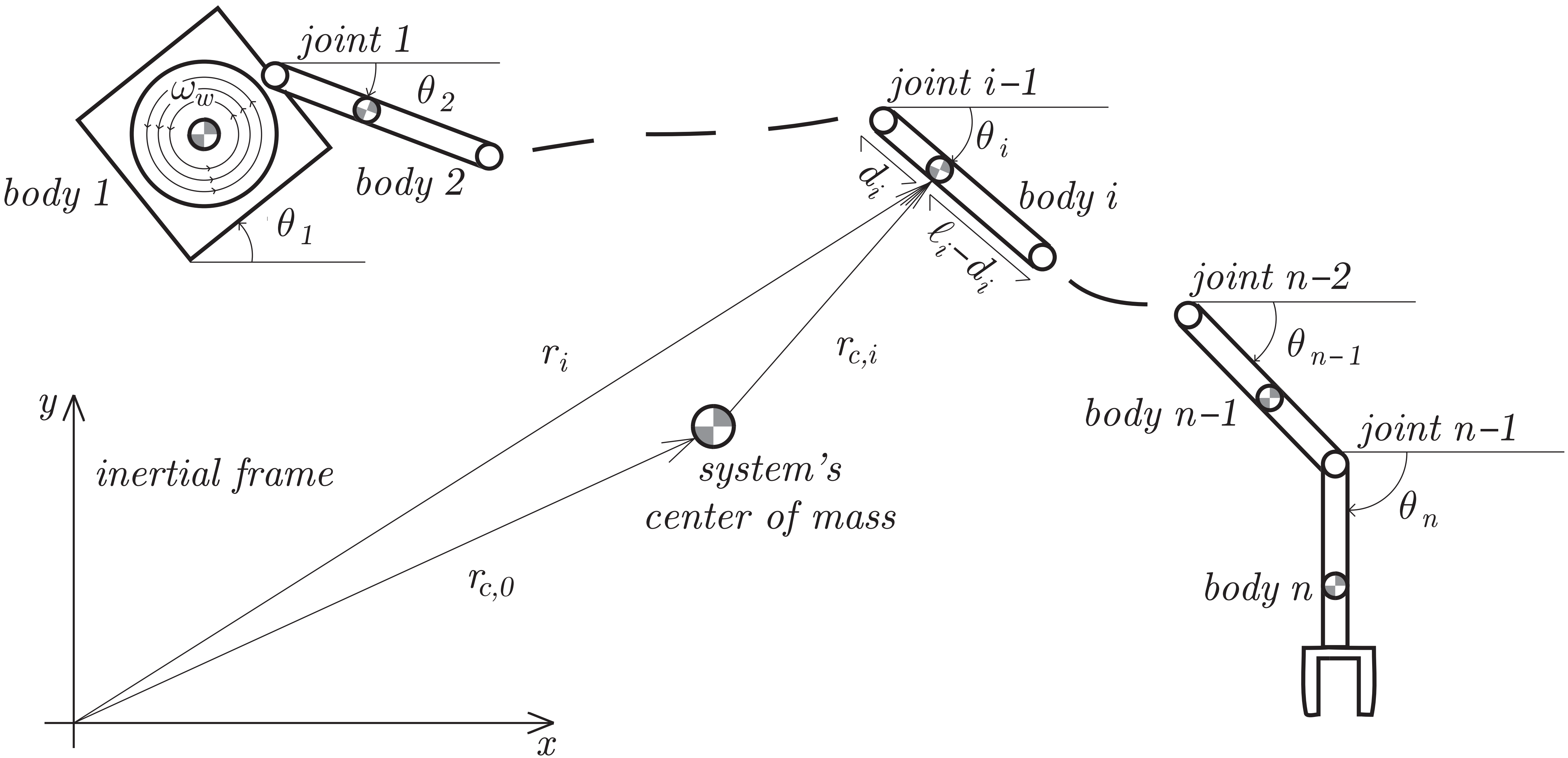




Problems

- The holonomy angle function requires a significant computational effort
- Open-loop control
- Desired reorientation may not be achievable in a single maneuver
- Unnecessary movements

n-Body Planar Manipulator with Momentum Wheel



Modifications

Include the mass of the momentum wheel to the mass of the first body: $m'_1 = m_1 + m_w$

A new state vector: $\zeta = (\theta_1 \quad \phi_1 \quad \ldots \quad \phi_{n-1} \quad \theta_w)^T$

A new inertia matrix: $B'(\zeta) = B'(\phi) = \begin{bmatrix} B(\phi) & \mathbf{0}_n \\ \mathbf{0}_n^T & J_w \end{bmatrix}$

Conservation of angular momentum becomes

$$\mathbf{1}_{n+1}^T B'(\zeta)\dot{\zeta} = \mathbf{1}_n^T B(\phi)\left(\mathbf{1}_n \dot{\theta}_1 + S\dot{\phi}\right) + J_w \dot{\theta}_w = 0$$

Kinematic Model

$$\dot{\zeta} = \begin{bmatrix} \dot{\theta}_1 \\ \dot{\phi} \\ \dot{\theta}_w \end{bmatrix} = \begin{bmatrix} -\mathbf{1}_n^T B(\phi) S \big/ \mathbf{1}_n^T B(\phi) \mathbf{1}_n & -J_w \big/ \mathbf{1}_n^T B(\phi) \mathbf{1}_n \\ I_{n-1} & \mathbf{0}_{n-1} \\ \mathbf{0}_{n_1}^T & 1 \end{bmatrix} \begin{bmatrix} u \\ \omega_w \end{bmatrix}$$

where $u = \dot{\phi}$ and $\omega_w = \dot{\theta}_w$ are, respectively, the joints and the momentum wheel commanded angular velocities.

Feedback Linearization

Using the following control law

$$\omega_w = -\frac{\mathbf{1}_n^T B(\phi) S}{J_w} u - \frac{\mathbf{1}_n^T B(\phi) \mathbf{1}_n}{J_w} u_w$$

the evolution of the base orientation becomes

$$\dot{\theta}_1 = -\frac{\mathbf{1}_n^T B(\phi) S}{\mathbf{1}_n^T B(\phi) \mathbf{1}_n} u - \frac{J_w}{\mathbf{1}_n^T B(\phi) \mathbf{1}_n} \omega_w = u_w$$

After Feedback Linearization

$$\dot{\zeta} = \begin{bmatrix} \dot{\theta}_1 \\ \dot{\phi} \\ \dot{\theta}_w \end{bmatrix} = \begin{bmatrix} \mathbf{0}_{n-1}^T & 1 \\ I_{n-1} & \mathbf{0}_{n-1} \\ -\mathbf{1}_n^T B(\phi) S \big/ J_w & -\mathbf{1}_n^T B(\phi) \mathbf{1}_n \big/ J_w \end{bmatrix} \begin{bmatrix} u \\ u_w \end{bmatrix}$$

The dynamics of θ_1 and ϕ can now be modeled as simple integrators and proportional control actions will be appropriate, i.e.

$$u = K_\phi(\phi_d - \phi) \qquad u_w = k_\theta(\theta_{1,d} - \theta_1)$$

Simulation Parameters

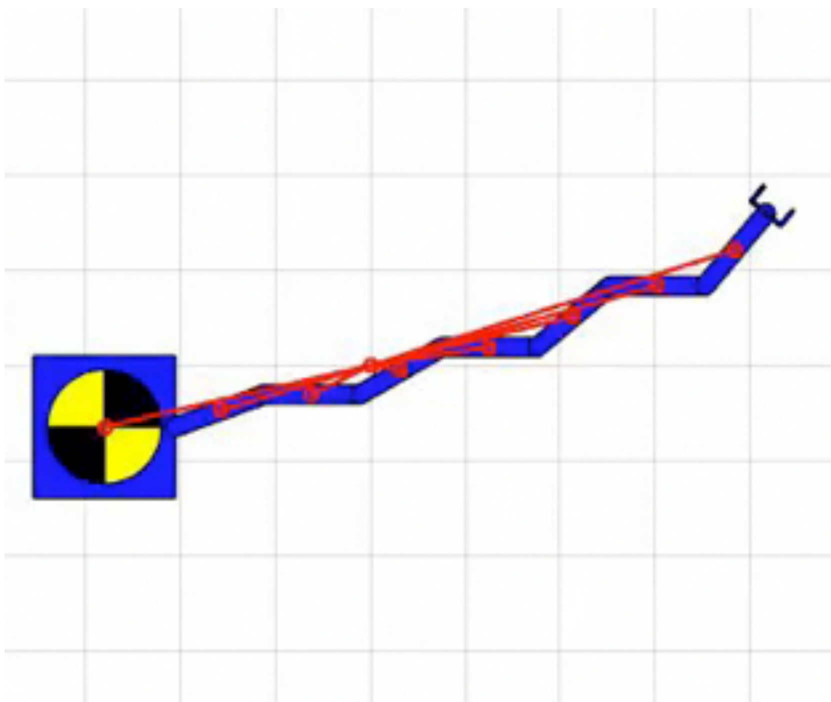
$$m_i = \begin{cases} 12\text{ kg} & i = 1 \\ 8\text{ kg} & i = 2,\ldots,8 \end{cases} \qquad \ell_i = \begin{cases} 1.5\text{ m} & i = 1 \\ 1\text{ m} & i = 2,\ldots,8 \end{cases} \qquad d_i = \frac{1}{2}\ell_i$$

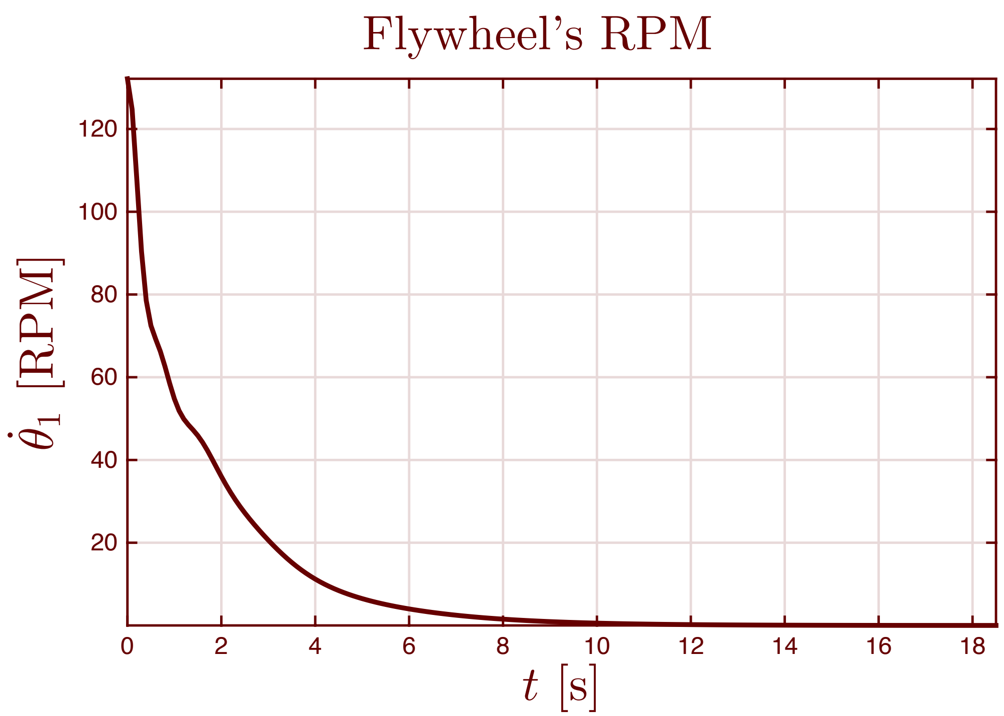
$$J_w = \frac{m_w \ell_w^2}{2} = 2.16\text{ kg}\cdot\text{m}^2 \qquad J_i = \begin{cases} \dfrac{m_i \ell_i^2}{6} = 4.5\text{ kg}\cdot\text{m}^2 & i = 1, \\ \dfrac{m_i \ell_i^2}{12} = 0.\overline{6}\text{ kg}\cdot\text{m}^2 & i = 2,\ldots,8 \end{cases}$$

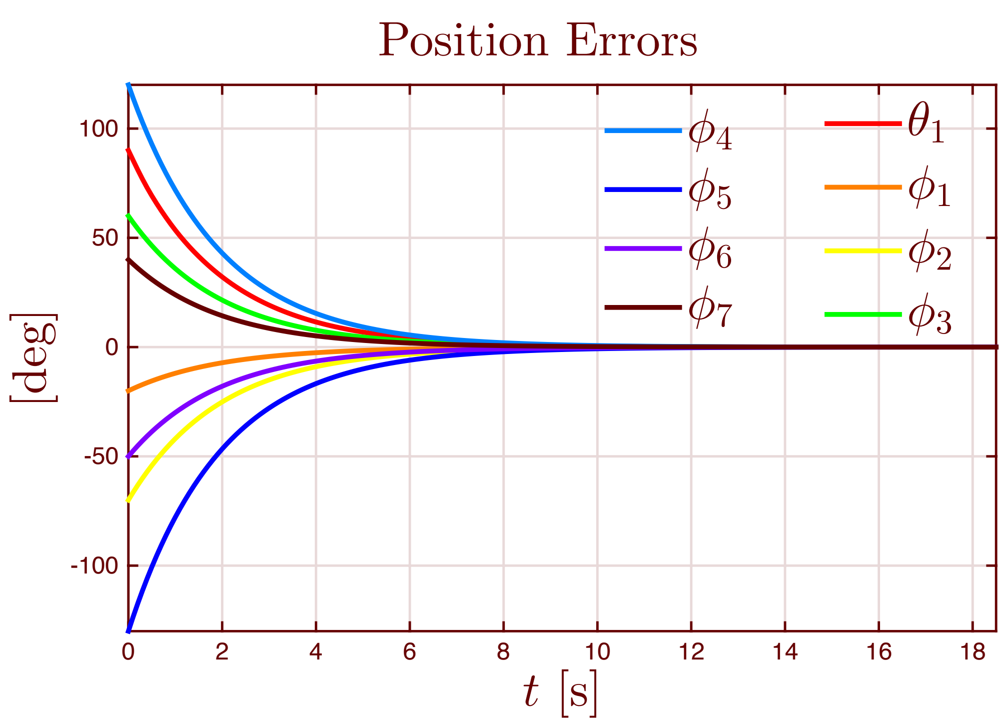
$$\zeta_0 = [0°, 20°, -20°, 30°, -30°, 40°, -40°, 50° \mid 0°]^T \qquad K_\phi = 0.5 \cdot I_7$$

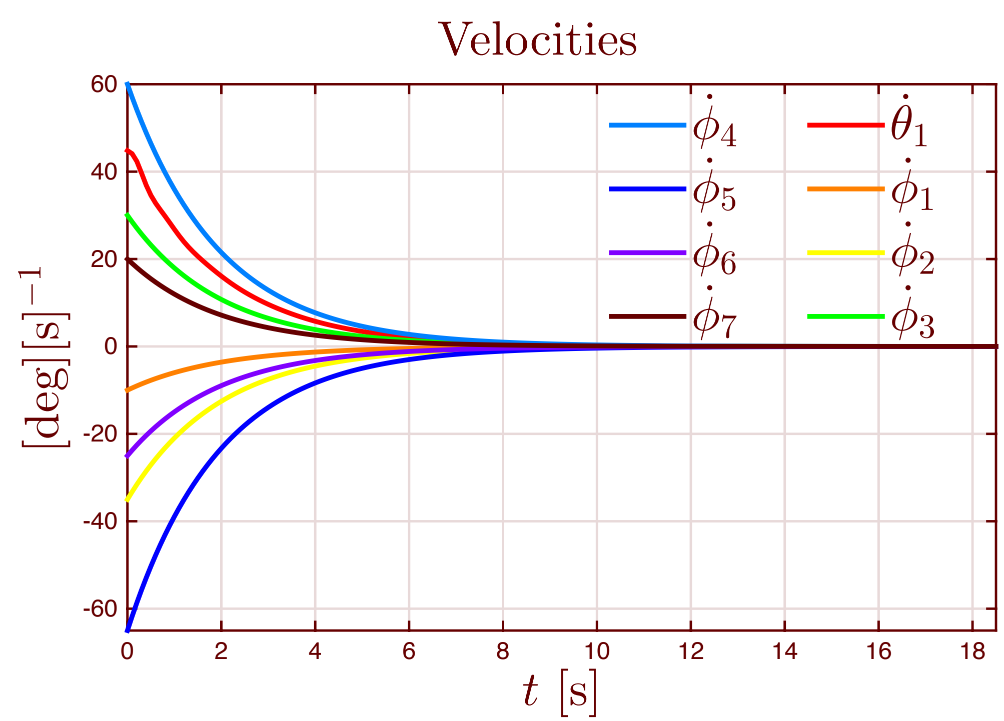
$$\zeta_d = [90°, 0, -90°, 90°, -90°, -90°, -90°, 90° \mid \star]^T \qquad k_{\theta_1} = 0.5$$

Simulation
Eight-Body with Momentum Wheel









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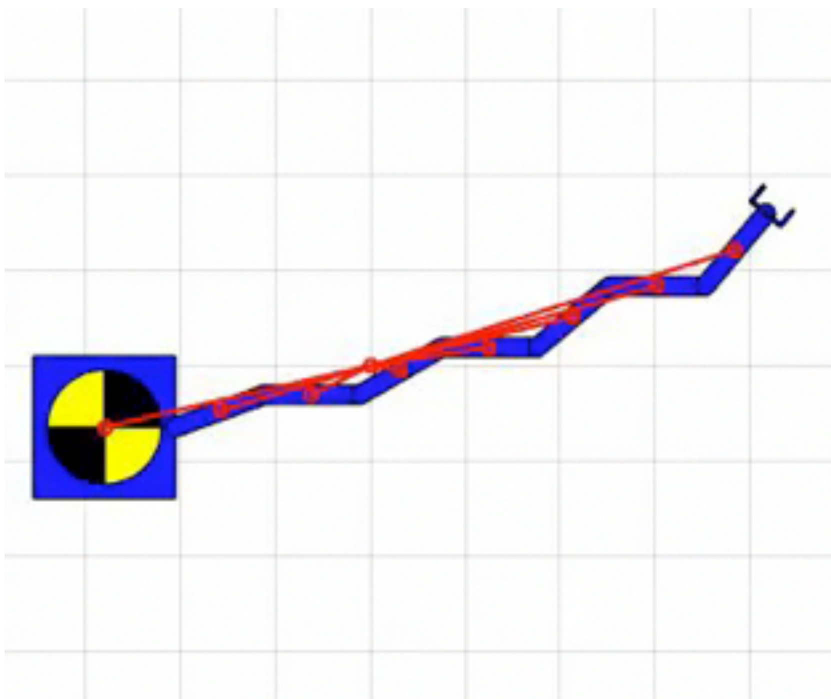
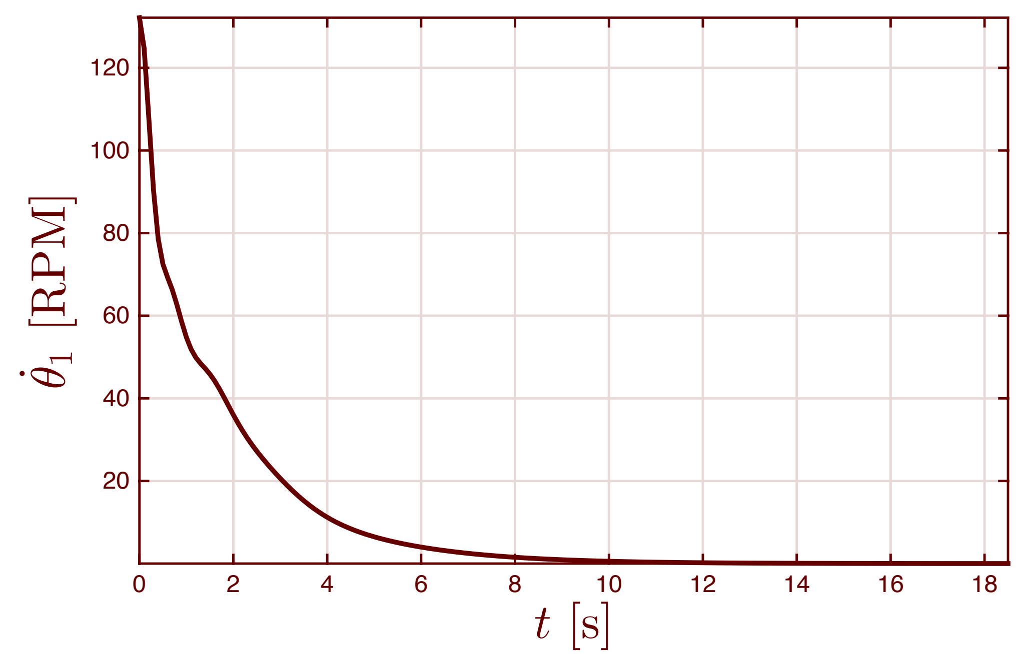
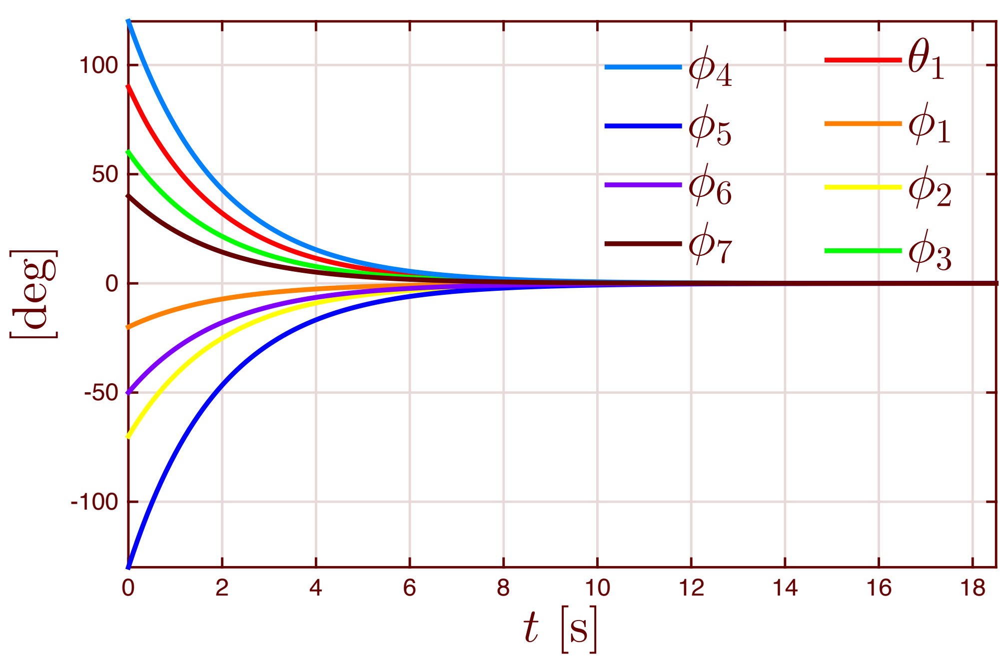


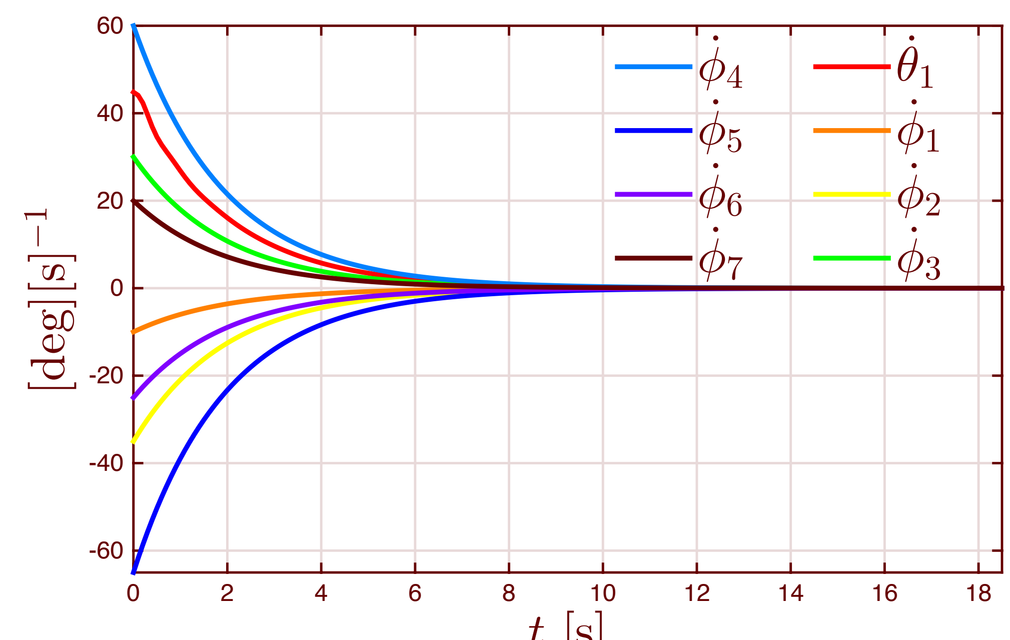
Flywheel's RPM



Position Errors



Velocities



Conclusions

- It is possible to reach any configuration in finite time
- No redundant maneuvers
- System dynamics reduced to a set of integrators
- Useful in case of path planning