Quasi-resonant tidal perturbations of planetary orbits caused by the passage of a flying-by star

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Abstract

The aim of this research project is to investigate the properties of a planetary orbit which is perturbed by the passage of a flying-by star.
We shall see that this passage, under particular conditions, is able to produce strong variations in the kinetic energy of the perturbed body and, as a consequence, effective variations of the eccentricity of its orbit.
The study of tidal perturbations of planetary orbits is of great importance nowadays, because many studies have shown that this mechanism should play a crucial role in the determination of the final configuration of planetary systems.
This work is composed of different sections. The first part describes the problem from a general point of view. In the second part and in the third sections we shall investigate the difference between the analytical model we have employed in our simulations and one old model, which has proven to be of particular interest in the description of globular clusters. This difference is shown both from an analytical point of view and through simulations.
The fourth part describes the first results obtained with the simulation environment developed by the author, along with some analytical considerations on the problem under study.

1 Tidal interactions: a general introduction to the problem

The physical environment which we are aiming to describe is presented in the graphical representation below:

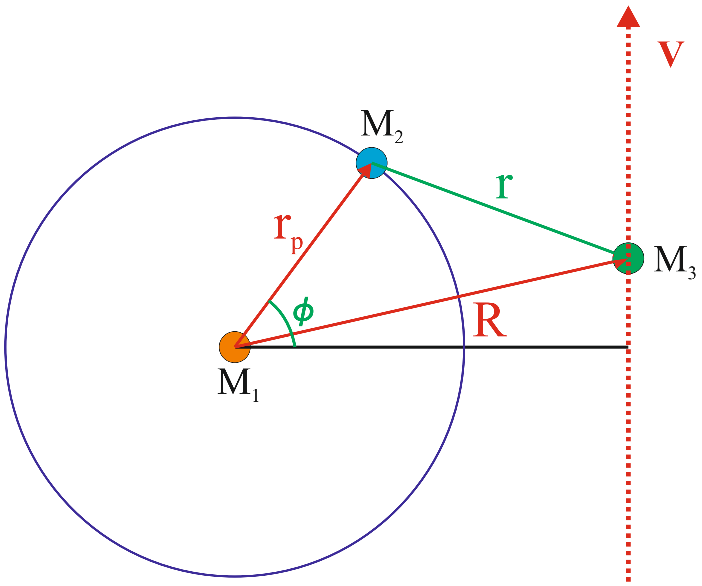

Fig. 1: Graphical representation of the physical environment described: an object (in blue), interacts with the center of mass of the proto-planetary system (in orange) and with an external perturber (in green).

An object, shown in blue, orbits around the center of mass of a gravitationally bound system. The orbit is supposed to be initially circular. This isolated system is perturbed by the passage of a flying-by star. The center of the system of reference is fixed on the center of mass of the gravitationally-bound system. Note that the system so described is scale-invariant: the analytical approach developed is equally able to describe the tidal perturbations active in galactic encounters and the perturbations of planetary orbits caused by passing-by stars. This is due to the fact that both of the cited systems are bound by the gravitational force, which is scale-invariant.
In the simulation of planetary systems, the following simplifying assumptions are used:

- The mass of the central star is equal to the mass of the perturbing star, while the mass of the perturbed body (a planet or a minor body) is several orders of magnitude smaller.
- The forces acting on the central star and on the perturbing star are subtracted off. In this way, the central star remains fixed on center of the reference system and the perturbing star, which possesses an initial velocity, follows a strictly straight line.
- The minor body is on a circular orbit around the center of gravity of its system.

Under these assumptions, the effective forces acting in the systems are:

1. The gravitational force exerted by the central star on the victim body, which allows it to stay on a stable circular orbit.
2. The tidal force, which is the difference between the gravitational force exerted by the perturbing star on the center of the planetary system and on the planet itself.

In particular, the tidal force deserves more explanation.
In the previous graphical representation, the expression for the tidal force is, taking $G = 1$

$$\mathbf{F_T} = -\left(\frac{M_2 M_3}{R^3}\mathbf{R} - \frac{M_2 M_3}{r^3}\mathbf{r}\right) \tag{1}$$

This expression is exact, but not quite useful in our approximation, where the radius of the planetary orbit is negligible with respect to the interstellar distance. In this approximation, the previous two quantities are very similar to each other and subject to large numerical errors.
For this reason, we expand the tidal force F_t in a series expansion for $R \approx r$, that is, calling r_p the radius of the planetary orbits, $r_p \approx 0$.
In this way, we obtain the following expression, valid for the x-axis (along the direction of minimum approach between the stars, toward the perturber):

$$a_x = -\frac{M_3 x_2}{R^3} - \frac{3M_3}{R^4}(x_3 - x_2)(r - R) \tag{2}$$

2 Comparison between two different analytical approaches

The aim of this section is to provide some possible solutions to the difference between the Bessel function arguments, present in the energy variation formula, of the following papers:

- Disruption of Galactic Clusters, Spitzer L., ApJ (1958)
- Quasi-Resonant Theory of Tidal Interactions, D'Onghia E. et al, ApJ (2010)

The first paper dates back to the 1958 and uses a differential equation approach to solve, by a series expansion, the equation of motion of a particle subject to a harmonic potential, with a time-dependent external force (the perturbing force). This potential is particularly useful when describing the potential well of globular cluster, while it fails to describe the Keplerian potential which acts in both the galactic and the planetary cases.
The second paper is more recent and it develops the analytical theory used in the present work.
In the first paper, the argument is:

$$A_1 = 2\omega t$$

while in the second paper the argument is:

$$A_2 = \omega t$$

where, in both cases, ω is the internal angular frequency of the particles inside the victim. Actually, there is a slight difference between the physical meaning of the two angular frequencies: while in the first paper ω actually represents the harmonic frequency, in the second paper it stands for the real circular frequency of the orbit. In any case, as far as the analysis is decomposed in the three cartesian coordinates, this difference is not significant at all.
The personal belief of the author is that the derivation in the paper by Spitzer is inaccurate in some of the derivations and in general not applicable to the planetary or galactic cases. The following calculations should allow to understand the reason.

Our starting point is the **equation 23** in the Spitzer's paper, which expresses the variation of the total energy along the z direction.

$$\Delta\left[\frac{1}{2}m\left(\frac{dz}{dt}\right)^2_{z=0}\right] = \frac{m\omega^2}{2}\left[\left(A_0 + \lambda A_1 + \lambda^2 A_2 + ...\right)^2 - A_0^2 + \left(B_0 + \lambda B_1 + \lambda^2 B_2 + ...\right)^2 - B_0^2\right] \tag{3}$$

where the meaning of the coefficients is explained in the paper, while λ is the expansion factor. All the coefficients are evaluated at t equal to $+\infty$.

Let us expand the previous relation by means of performing the squares. In doing this, we assume that we preserve only the terms up to the second order in λ, as Spitzer himself does.

$$\Delta U_z = \frac{m\omega^2}{2}\left[A_0^2 + \lambda^2 A_1^2 + 2\lambda A_0 A_1 + 2\lambda^2 A_0 A_2 - A_0^2 + B_0^2 + \lambda^2 B_1^2 + 2\lambda B_0 B_1 + 2\lambda^2 B_0 B_2 - B_0^2 + ... + O(\lambda^3)\right] \tag{4}$$

The zero-order terms in λ cancels. Let us group the terms of first and second order in λ.

$$\Delta U_z = \frac{m\omega^2}{2}\left[\lambda^2(A_1^2 + 2A_0A_2 + B_1^2 + 2B_0B_2) + \lambda(2A_0A_1 + 2B_0B_1) + ... + O(\lambda^3)\right] \tag{5}$$

It is necessary to report the meaning of the terms A_0, B_0, A_1, B_1, A_2, B_2
The terms A_0 and B_0 are the coefficients of the trigonometric functions of the zero-order solution:

$$z_0 = A_0 cos\omega t + B_0 sin\omega t \tag{6}$$

while the other terms are the coefficients of the higher order expansions:

$$z_1 = A_1(t)cos\omega t + B_1(t)sin\omega t \tag{7}$$

$$z_2 = A_2(t)cos\omega t + B_2(t)sin\omega t \tag{8}$$

These functions of t are all evaluated at $t \to \infty$ for energy calculations, as stated by Spitzer. Namely:

$$A_1 = -\frac{1}{\omega}\int_{-\infty}^{+\infty} f(\tau)sin\omega\tau(A_0cos\omega\tau + B_0sin\omega\tau)d\tau \tag{9}$$

$$B_1 = \frac{1}{\omega}\int_{-\infty}^{+\infty} f(\tau)cos\omega\tau(A_0cos\omega\tau + B_0sin\omega\tau)d\tau \tag{10}$$

$$\begin{aligned} A_2 \quad = \quad & \frac{1}{\omega^2}\int_{-\infty}^{+\infty} f(\tau)sin\omega\tau cos\omega\tau d\tau \int_{-\infty}^{\tau} (A_0cos\omega u + B_0sin\omega u)sin\omega u f(u)du \\ & -\frac{1}{\omega^2}\int_{-\infty}^{+\infty} f(\tau)sin^2\omega\tau d\tau \int_{-\infty}^{\tau} (A_0cos\omega u + B_0sin\omega u)cos\omega u f(u)du \end{aligned} \tag{11}$$

$$\begin{aligned} B_2 \quad = \quad & -\frac{1}{\omega^2}\int_{-\infty}^{+\infty} f(\tau)cos^2\omega\tau d\tau \int_{-\infty}^{\tau} (A_0cos\omega u + B_0sin\omega u)sin\omega u f(u)du \\ & +\frac{1}{\omega^2}\int_{-\infty}^{+\infty} f(\tau)sin\omega\tau cos\omega\tau d\tau \int_{-\infty}^{\tau} (A_0cos\omega u + B_0sin\omega u)cos\omega u f(u)du \end{aligned} \tag{12}$$

As stated by Spitzer, since the phases of the stars are at random, we may write, denoting by broken brackets an average over all stars:

$$< A_0B_0 >= 0 \tag{13}$$

$$< A_0^2 >=< B_0^2 >= z_c^2 \tag{14}$$

where z_c^2 is the mean-square value of z for the cluster stars.
Now, it is straightforward to see that first-order terms in λ in equation 5 cancels out. Indeed:

$$A_0 A_1 = -\frac{A_0}{\omega} \int_{-\infty}^{+\infty} f(\tau) sin\omega\tau (A_0 cos\omega\tau + B_0 sin\omega\tau) d\tau \tag{15}$$

since A_0 does not depend on time, we can put it inside the integral:

$$A_0 A_1 = -\frac{1}{\omega} \int_{-\infty}^{+\infty} f(\tau) sin\omega\tau (A_0^2 cos\omega\tau + A_0 B_0 sin\omega\tau) d\tau \tag{16}$$

taking the average of $A_0 A_1$ over the ensemble of stars:

$$< A_0 A_1 >= -\frac{1}{\omega} \int_{-\infty}^{+\infty} f(\tau) sin\omega\tau < (A_0^2 cos\omega\tau + A_0 B_0 sin\omega\tau) > d\tau \tag{17}$$

Note that the operator $< ... >$ acts only on the coefficients of the expansions. This is due to the fact that the average is performed on the ensemble of stars, not on time nor on any other quantity.
As a consequence of the linearity of the average operator and due to the relations 13 and 14, it is straightforward to calculate:

$$< A_0 A_1 >= -\frac{z_c^2}{\omega} \int_{-\infty}^{+\infty} f(\tau) sin\omega\tau cos\omega\tau d\tau \tag{18}$$

The same process is to performed on the product $B_0 B_1$:

$$< B_0 B_1 >= \frac{1}{\omega} \int_{-\infty}^{+\infty} f(\tau) cos\omega\tau < (A_0 B_0 cos\omega\tau + B_0^2 sin\omega\tau) > d\tau \tag{19}$$

the result being obviously:

$$< B_0 B_1 >= \frac{z_c^2}{\omega} \int_{-\infty}^{+\infty} f(\tau) sin\omega\tau cos\omega\tau d\tau \tag{20}$$

Now, it has been proved that:

$$< A_0 A_1 + B_0 B_1 >= 0 \tag{21}$$

so the terms linear in λ cancels out, as stated in the paper by Spitzer.
The remaining terms in the expansion of the variation of energy are (forgetting about the $O(\lambda^3)$ terms, as in Spitzer):

$$\Delta U_z = \frac{m\omega^2}{2} \lambda^2 (A_1^2 + 2A_0 A_2 + B_1^2 + 2B_0 B_2) \tag{22}$$

There are several ways to group these terms. Obviously, finding the way used by Spitzer himself is not trivial, considering that there are no clues in the paper.
What we will do next is to group these terms in the most logical way possible, and to find out if the expression found by Spitzer and our expression are compatible or not.
Firstly, we focus on **equation 26** in the paper by Spitzer.

$$
\begin{aligned}
\int_{-\infty}^{+\infty} f(\tau)F(\omega\tau)d\tau \int_{-\infty}^{\tau} f(u)G(\omega u)du &= \\
|\int_{-\infty}^{t} f(\tau)F(\omega\tau)d\tau \times \int_{-\infty}^{t} f(u)\, G(\omega u)du|_{-\infty}^{+\infty} &+ \\
-\int_{-\infty}^{+\infty} f(t)G(\omega t)dt \int_{-\infty}^{t} f(u)F(\omega u)du &
\end{aligned}
\tag{23}
$$

The $\times$ operator is simply the usual product between real numbers. This relation is definitely correct and it derives simply by using the usual formula of integration by parts:

$$
\int_{-\infty}^{+\infty} \frac{df(t)}{dt} g(t)dt = f(t)g(t)|_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} f(t)\frac{dg(t)}{dt}dt \tag{24}
$$

where:

$$
\frac{df(t)}{dt} = f(t)F(\omega t) \tag{25}
$$

and $g(t)$ is an integral function:

$$
g(t) = \int_{-\infty}^{t} f(u)G(\omega u)du \tag{26}
$$

The product of functions in 23 is evaluated at t equal to $-\infty$ and $+\infty$. As a consequence of the fact that the lower limit of integration in those integrals is $-\infty$, the evaluation performed in this limit vanishes. So, we have:

$$
\begin{aligned}
\int_{-\infty}^{+\infty} f(\tau)F(\omega\tau)d\tau \int_{-\infty}^{\tau} f(u)G(\omega u)du &= \\
\left[\int_{-\infty}^{+\infty} f(\tau)F(\omega\tau)d\tau\right] \times \left[\int_{-\infty}^{+\infty} f(u)G(\omega u)du\right] &+ \\
-\int_{-\infty}^{+\infty} f(t)G(\omega t)dt \int_{-\infty}^{t} f(u)F(\omega u)du &
\end{aligned}
\tag{27}
$$

This relation is employed to derive the **equation 27** in the Spitzer's paper. Actually, this relation, as it is written in the paper, is wrong. The correct relation is the following one:

$$
\begin{aligned}
&\int_{-\infty}^{+\infty} f(\tau)F(\omega\tau)d\tau \int_{-\infty}^{\tau} f(u)G(\omega u)du + \\
&+\int_{-\infty}^{+\infty} f(\tau)G(\omega\tau)d\tau \int_{-\infty}^{\tau} f(u)F(\omega u)du = \\
&\left[\int_{-\infty}^{+\infty} f(t)F(\omega t)dt\right] \times \left[\int_{-\infty}^{+\infty} f(t)G(\omega t)dt\right]
\end{aligned}
\tag{28}
$$

This is a very straightforward application of the integration by parts relation:

$$
\begin{aligned}
\int_{-\infty}^{+\infty} f(\tau)F(\omega\tau)d\tau \int_{-\infty}^{\tau} f(u)G(\omega u)du + \int_{-\infty}^{+\infty} f(\tau)G(\omega\tau)d\tau \int_{-\infty}^{\tau} f(u)F(\omega u)du &= \\
\left[\int_{-\infty}^{+\infty} f(\tau)F(\omega\tau)d\tau\right] \times \left[\int_{-\infty}^{+\infty} f(u)G(\omega u)du\right] - \int_{-\infty}^{+\infty} f(t)G(\omega t)dt \int_{-\infty}^{t} f(u)F(\omega u)du &+ \\
+\int_{-\infty}^{+\infty} f(\tau)G(\omega\tau)d\tau \int_{-\infty}^{\tau} f(u)F(\omega u)du &= \\
\left[\int_{-\infty}^{+\infty} f(t)F(\omega t)dt\right] \times \left[\int_{-\infty}^{+\infty} f(t)G(\omega t)dt\right]
\end{aligned}
\tag{29}
$$

The aim of Spitzer was clearly to prove this last relation, not the one reported in his paper, which is likely to be a typo. In fact, the expression

$$
\begin{aligned}
\int_{-\infty}^{+\infty} f(\tau)F(\omega\tau)d\tau \int_{-\infty}^{\tau} f(u)G(\omega u)du + \int_{-\infty}^{+\infty} f(\tau)G(\omega\tau)d\tau \int_{-\infty}^{\tau} f(u)G(\omega u)du &= \\
\left[\int_{-\infty}^{+\infty} f(\tau)F(\omega\tau)d\tau\right] \times \left[\int_{-\infty}^{+\infty} f(u)G(\omega u)du\right] - \int_{-\infty}^{+\infty} f(t)G(\omega t)dt \int_{-\infty}^{t} f(u)F(\omega u)du + \\
+\frac{1}{2}\left[\int_{-\infty}^{+\infty} f(\tau)G(\omega\tau)d\tau\right]^2
\end{aligned}
\tag{30}
$$

does not allow a transformation in a single multiplication. By the way, the last squared term is derived from the very same expression of integration by parts:

$$\int_{-\infty}^{+\infty} f(\tau)F(\omega\tau)d\tau \int_{-\infty}^{\tau} f(u)F(\omega u)du \quad = \tag{31}$$

$$\left[\int_{-\infty}^{+\infty} f(\tau)F(\omega\tau)d\tau\right] \times \left[\int_{-\infty}^{+\infty} f(u)G(\omega u)du\right] \quad + \tag{32}$$

$$-\int_{-\infty}^{+\infty} f(t)F(\omega t)dt \int_{-\infty}^{t} f(u)F(\omega u)du \tag{33}$$

Shifting the second term of the right hand side to the left hand side, we have two equal expressions, so that:

$$\int_{-\infty}^{+\infty} f(\tau)F(\omega\tau)d\tau \int_{-\infty}^{\tau} f(u)F(\omega u)du = \frac{1}{2}\left[\int_{-\infty}^{+\infty} f(\tau)F(\omega\tau)d\tau\right] \times \left[\int_{-\infty}^{+\infty} f(u)G(\omega u)du\right] \tag{34}$$

Now, with all these tools we can go straight to the derivation of the final expression for the energy variation.
Let us now explicit the terms in the equation 22:

$$\begin{aligned} 2A_0A_2 \quad = \quad & \frac{2A_0}{\omega^2}\int_{-\infty}^{+\infty} f(\tau)sin\omega\tau cos\omega\tau d\tau \int_{-\infty}^{\tau} (A_0cos\omega u + B_0sin\omega u)sin\omega u f(u)du + \\ & -\frac{2A_0}{\omega^2}\int_{-\infty}^{+\infty} f(\tau)sin^2\omega\tau d\tau \int_{-\infty}^{\tau} (A_0cos\omega u + B_0sin\omega u)cos\omega u f(u)du \end{aligned} \tag{35}$$

$$\begin{aligned} 2B_0B_2 \quad = - \quad & \frac{2B_0}{\omega^2}\int_{-\infty}^{+\infty} f(\tau)cos^2\omega\tau d\tau \int_{-\infty}^{\tau} (A_0cos\omega u + B_0sin\omega u)sin\omega u f(u)du + \\ & +\frac{2B_0}{\omega^2}\int_{-\infty}^{+\infty} f(\tau)sin\omega\tau cos\omega\tau d\tau \int_{-\infty}^{\tau} (A_0cos\omega u + B_0sin\omega u)cos\omega u f(u)du \end{aligned} \tag{36}$$

$$A_1^2 = \frac{1}{\omega^2}\left[\int_{-\infty}^{+\infty} f(\tau)sin\omega\tau(A_0cos\omega\tau + B_0sin\omega\tau)d\tau\right]^2 \tag{37}$$

$$B_1^2 = \frac{1}{\omega^2}\left[\int_{-\infty}^{+\infty} f(\tau)cos\omega\tau(A_0cos\omega\tau + B_0sin\omega\tau)d\tau\right]^2 \tag{38}$$

Now we need to focus only on the first two terms, $2A_0A_2$ and $2B_0B_2$. We put the constant factors A_0 and B_0 inside the integrals, and take the average over the ensemble of stars, like performed before.

$$\begin{aligned} 2A_0A_2 \quad = \quad & \frac{2}{\omega^2}\int_{-\infty}^{+\infty} f(\tau)sin\omega\tau cos\omega\tau d\tau \int_{-\infty}^{\tau} (A_0^2cos\omega u + A_0B_0sin\omega u)sin\omega u f(u)du + \\ & -\frac{2}{\omega^2}\int_{-\infty}^{+\infty} f(\tau)sin^2\omega\tau d\tau \int_{-\infty}^{\tau} (A_0^2cos\omega u + A_0B_0sin\omega u)cos\omega u f(u)du \end{aligned} \tag{39}$$

$$2B_0B_2 = -\frac{2}{\omega^2}\int_{-\infty}^{+\infty} f(\tau)cos^2\omega\tau d\tau \int_{-\infty}^{\tau} (A_0B_0cos\omega u + B_0^2 sin\omega u)sin\omega u f(u)du + \tag{40}$$
$$+\frac{2}{\omega^2}\int_{-\infty}^{+\infty} f(\tau)sin\omega\tau cos\omega\tau d\tau \int_{-\infty}^{\tau} (A_0B_0cos\omega u + B_0^2 sin\omega u)cos\omega u f(u)du$$

and hence:

$$< 2A_0A_2 > = \frac{2z_c^2}{\omega^2}\int_{-\infty}^{+\infty} f(\tau)sin\omega\tau cos\omega\tau d\tau \int_{-\infty}^{\tau} cos\omega u sin\omega u f(u)du + \tag{41}$$
$$-\frac{2z_c^2}{\omega^2}\int_{-\infty}^{+\infty} f(\tau)sin^2\omega\tau d\tau \int_{-\infty}^{\tau} cos^2\omega u f(u)du$$

and

$$< 2B_0B_2 > = -\frac{2z_c^2}{\omega^2}\int_{-\infty}^{+\infty} f(\tau)cos^2\omega\tau d\tau \int_{-\infty}^{\tau} sin^2\omega u f(u)du + \tag{42}$$
$$+\frac{2z_c^2}{\omega^2}\int_{-\infty}^{+\infty} f(\tau)sin\omega\tau cos\omega\tau d\tau \int_{-\infty}^{\tau} sin\omega u cos\omega u f(u)du$$

Let us sum together the first term in equation 41 and the second term in equation 42.

$$\frac{2z_c^2}{\omega^2}\int_{-\infty}^{+\infty} f(\tau)sin\omega\tau cos\omega\tau d\tau \int_{-\infty}^{\tau} cos\omega u sin\omega u f(u)du + \frac{2z_c^2}{\omega^2}\int_{-\infty}^{+\infty} f(\tau)sin\omega\tau cos\omega\tau d\tau \int_{-\infty}^{\tau} sin\omega u cos\omega u f(u)du = \tag{43}$$
$$\frac{4z_c^2}{\omega^2}\int_{-\infty}^{+\infty} f(\tau)sin\omega\tau cos\omega\tau d\tau \int_{-\infty}^{\tau} sin\omega u cos\omega u f(u)du = \frac{z_c^2}{\omega^2}\int_{-\infty}^{+\infty} f(\tau)2sin\omega\tau cos\omega\tau d\tau \int_{-\infty}^{\tau} 2sin\omega u cos\omega u f(u)du \tag{44}$$

Now, recalling the trigonometric relation:

$$sin2\theta = 2sin\theta cos\theta \tag{45}$$

and using again the expression 34, the previous expression equals:

$$2A_0A_2 = \frac{z_c^2}{2\omega^2}\left[\int_{-\infty}^{+\infty} f(\tau)sin2\omega\tau d\tau\right]^2 \tag{46}$$

Now, we focus our attention on the second term of the right hand side of 41 and on the first term on the right hand side of 42.
Summing them:

$$-\frac{2z_c^2}{\omega^2}\int_{-\infty}^{+\infty} f(\tau)sin^2\omega\tau d\tau\int_{-\infty}^{\tau} cos^2\omega u f(u)du - \frac{2z_c^2}{\omega^2}\int_{-\infty}^{+\infty} f(\tau)cos^2\omega\tau d\tau\int_{-\infty}^{\tau} sin^2\omega u f(u)du \quad = \tag{47}$$

$$-\frac{2z_c^2}{\omega^2}\left(\int_{-\infty}^{+\infty} f(\tau)sin^2\omega\tau d\tau\int_{-\infty}^{\tau} cos^2\omega u f(u)du + \int_{-\infty}^{+\infty} f(\tau)cos^2\omega\tau d\tau\int_{-\infty}^{\tau} sin^2\omega u f(u)du\right) \tag{48}$$

Now, we use the equation 28 in its correct form, not the form in the Spitzer's paper. Actually, it is very likely that the expression was derived to handle exactly this sum of integrals.
A direct application of the formula leads to:

$$-\frac{2z_c^2}{\omega^2}\left[\int_{-\infty}^{+\infty} f(\tau)sin^2\omega\tau d\tau\right]\times\left[\int_{-\infty}^{+\infty} f(\tau)cos^2\omega\tau d\tau\right] \tag{49}$$

Recalling the trigonometric formulas:

$$cos2\theta = cos^2\theta - sin^2\theta = 2cos^2\theta - 1 = 1 - 2sin^2\theta \tag{50}$$

we may write:

$$sin^2\omega\tau = \frac{1 - cos2\omega\tau}{2} \tag{51}$$

$$cos^2\omega\tau = \frac{1 + cos2\omega\tau}{2} \tag{52}$$

Coming back to the expression 49 and substituting:

$$-\frac{z_c^2}{2\omega^2}\left[\int_{-\infty}^{+\infty} f(\tau)(1 - cos2\omega\tau)d\tau\right]\times\left[\int_{-\infty}^{+\infty} f(\tau)(1 + cos2\omega\tau)d\tau\right] \quad = \tag{53}$$

$$-\frac{z_c^2}{2\omega^2}\left\{\left[\int_{-\infty}^{+\infty} f(\tau)d\tau - \int_{-\infty}^{+\infty} f(\tau)cos2\omega\tau d\tau\right]\times\left[\int_{-\infty}^{+\infty} f(\tau)d\tau + \int_{-\infty}^{+\infty} f(\tau)cos2\omega\tau d\tau\right]\right\} \quad =$$

$$-\frac{z_c^2}{2\omega^2}\left\{\left[\int_{-\infty}^{+\infty} f(\tau)d\tau\right]^2 - \left[\int_{-\infty}^{+\infty} f(\tau)cos2\omega\tau d\tau\right]^2\right\} \quad =$$

$$\frac{z_c^2}{2\omega^2}\left\{\left[\int_{-\infty}^{+\infty} f(\tau)cos2\omega\tau d\tau\right]^2 - \left[\int_{-\infty}^{+\infty} f(\tau)d\tau\right]^2\right\}$$

because the two mixed terms cancel out.
Then:

$$2B_0B_2 = \frac{z_c^2}{2\omega^2}\left\{\left[\int_{-\infty}^{+\infty} f(\tau)cos2\omega\tau d\tau\right]^2 - \left[\int_{-\infty}^{+\infty} f(\tau)d\tau\right]^2\right\} \tag{54}$$

To summarize, at this stage of the work we have the following expression for the variation of the total energy:

$$\Delta U_z = \frac{m\omega^2}{2}\lambda^2 \left(\frac{z_c^2}{2\omega^2} \left\{ \left[\int\limits_{-\infty}^{+\infty} f(\tau) sin2\omega\tau d\tau \right]^2 + \left[\int\limits_{-\infty}^{+\infty} f(\tau) cos2\omega\tau d\tau \right]^2 - \left[\int\limits_{-\infty}^{+\infty} f(\tau) d\tau \right]^2 \right\} + A_1^2 + B_1^2 \right) \tag{55}$$

There are still the expressions for A_1^2 and B_1^2 that needs to be worked out. Actually, this is the most tricky calculation, because the absence of any constant A_0 or B_0 leads to the impossibility of using the average over the ensemble in a direct way. Nonetheless, it is necessary to add the averaging procedure in some ways, because the constants A_0 and B_0 needs to disappear.
We will perform the entire calculation only for A_1^2, because for B_1^2 it is exactly the same, with obvious modifications.

$$A_1^2 = \frac{1}{\omega^2} \left[\int\limits_{-\infty}^{+\infty} f(\tau) sin\omega\tau (A_0 cos\omega\tau + B_0 sin\omega\tau) d\tau \right]^2 \tag{56}$$

we multiply and divide two times for B_0 (the coefficient of $sin\omega\tau$, the reason will be evident soon):

$$A_1^2 = \frac{1}{\omega^2} \left[\int\limits_{-\infty}^{+\infty} f(\tau) \frac{B_0}{B_0}\frac{B_0}{B_0} sin\omega\tau (A_0 cos\omega\tau + B_0 sin\omega\tau) d\tau \right]^2 \tag{57}$$

On average, the following expression holds, due to the properties 13 and 14:

$$< A_1^2 > = < \frac{1}{\omega^2} \left[\int\limits_{-\infty}^{+\infty} f(\tau) \frac{B_0}{B_0^2} (A_0 sin\omega\tau + B_0 sin\omega\tau)(A_0 cos\omega\tau + B_0 sin\omega\tau) d\tau \right]^2 > \tag{58}$$

hence, squaring the parentheses and simplifying B_0:

$$< A_1^2 > = < \frac{1}{\omega^2} \left[\int\limits_{-\infty}^{+\infty} f(\tau) \frac{1}{B_0} (A_0 sin\omega\tau + B_0 sin\omega\tau)^2 d\tau \right]^2 > \tag{59}$$

performing the square of the parentheses:

$$< A_1^2 > = < \frac{1}{\omega^2} \left[\int\limits_{-\infty}^{+\infty} f(\tau) \frac{1}{B_0} \left(A_0^2 cos^2\omega\tau + B_0^2 sin^2\omega\tau + 2A_0B_0 sin\omega\tau cos\omega\tau \right) d\tau \right]^2 > \tag{60}$$

Putting B_0 inside the internal parentheses, we finally obtain:

$$< A_1^2 > = < \frac{1}{\omega^2} \left[\int\limits_{-\infty}^{+\infty} f(\tau) \left(\frac{A_0^2}{B_0} cos^2\omega\tau + B_0 sin^2\omega\tau + 2A_0 sin\omega\tau cos\omega\tau \right) d\tau \right]^2 > \tag{61}$$

Now, a very important point: we must compute the average over the entire ensemble of objects for the quantities A_0^2/B_0, B_0, A_0.
As also Spitzer pointed out, phases of the objects are at random, so the A_0 and B_0 are random variables. In addition, as the property 13 states, they can be positive or negative.
So, let us suppose that both A_0 and B_0 are random variables with a uniform distribution between $[-k, k]$, with $k > 0$.
The expected value of the variable X is known to be:

$$< X >= \int_{-k}^{k} X P(X) dX \tag{62}$$

being $P(X)$ the uniform distribution in the selected range. Then, for example:

$$< X^2 >=< B_0^2 >= \frac{1}{2k} \int_{-k}^{k} X^2 dX = \frac{k^2}{3} \equiv z_c^2 \tag{63}$$

The last equality is of course the definition given by Spitzer.
When it comes to the expected value of the product of two variables, we have to take into account that A_0 and B_0 are uncorrelated, as a consequence of the following argument. If A_0 and B_0 were two dependent variables, taken from the uniform distribution stated before, we would write:

$$< A_0 B_0 >=< A_0 >< B_0 > + Cov[A_0, B_0] \tag{64}$$

but

$$< A_0 >=< B_0 >= 0 \tag{65}$$

and

$$Cov[A_0, B_0] = k^2/3 \neq 0$$

so the property 13 would not be true anymore. So A_0 and B_0 are uncorrelated variables.
In order to study the expected value of the variable $\frac{1}{B_0} = \frac{1}{X}$, we define

$$Y = 1/X \tag{66}$$

From the relation between the probability densities:

$$P(X)dX = P(Y)dY \tag{67}$$

and calculating the absolute value of the derivative

$$|\frac{dX}{dY}| = X^2 \tag{68}$$

we obtain:

$$P(Y) = \frac{X^2}{2k} \tag{69}$$

The expected value of $1/B_0 = 1/X$ is:

$$< Y >=< X^{-1} >= \int_{-1/k}^{1/k} Y P(Y) dY = 0 \tag{70}$$

The expected values of A_0 and B_0 are obviously also null, so that the expected value of A_1^2 is zero.
The very same argument is applicable to B_1^2, this time evidently multiplying and dividing by A_0, the coefficient of $cos\omega\tau$.
Then, **on average**, $< A_1^2 >=< B_1^2 >= 0$.

Finally, recalling equation 55, the variation in the total energy of the system along the z axis is:

$$\Delta U_z = \frac{m}{4}\lambda^2 z_c^2 \left(\left[\int_{-\infty}^{+\infty} f(\tau) sin 2\omega\tau d\tau \right]^2 + \left[\int_{-\infty}^{+\infty} f(\tau) cos 2\omega\tau d\tau \right]^2 - \left[\int_{-\infty}^{+\infty} f(\tau) d\tau \right]^2 \right) \tag{71}$$

In order for this relation to be equal to the relation found by Spitzer, the following equation should hold:

$$\left[\int_{-\infty}^{+\infty} f(\tau) d\tau \right]^2 = - \left\{ \left[\int_{-\infty}^{+\infty} f(\tau) sin 2\omega\tau d\tau \right]^2 + \left[\int_{-\infty}^{+\infty} f(\tau) cos 2\omega\tau d\tau \right]^2 \right\} \tag{72}$$

which has no solution, for every possible function $f(\tau)$. In particular, with the expression of $f(\tau)$, the previous result does not hold, giving:

$$4\frac{p^2}{v^2} = \frac{16\omega^2 K_1\left(\frac{2\omega p}{v}\right)^2}{v^4} \tag{73}$$

The previous, very long, derivation shows that there are at least some doubts about the accuracy of the derivation in the original paper.
The following step has been to develop a simple simulation environment, devised especially to simulate the case of gravitational encounters between stellar systems. This program is described in the following section.

3 Simulation of the kinetic energy transfer: a comparison between the two predictions

As anticipated before, the two papers yield different expressions for the arguments in the kinetic energy change of the perturbed body, as a consequence of the difference in the arguments of the Bessel functions. In particular, the coefficient in the paper by Spitzer is 4α, while in the paper by D'Onghia et al. is 2α, where the meaning of α is described below.
In order to determine, from a simulation point of view, the analytical expression which describes more accurately the physical environment under study, a specific piece of software was devised.
The system of units chosen is based on the following assumptions:

- The gravitational constant is unitary: $G = 1$
- The unit of mass is the mass of the Earth
- The unit of distance is the astronomical unit (AU), the average distance between the Earth and the Sun

Having fixed these units, the time unit follows directly, as the unit of velocity.
The main parameters of the simulations executed are listed in the following table:

Distance between victim planet and central star	10^1
Interstellar distance	10^2
Velocity of the perturbing star	10^3
Mass ratio between the star and the planet	10^5
Orbital velocity of the planet	$V_c = \sqrt{G\frac{(M_1+M_2)}{r_p}}$

The expression for α is the following one:

$$\alpha = \Omega\frac{R}{V_{sl}} \tag{74}$$

where $\Omega = \frac{V_c}{r_p}$ is the angular velocity of the planetary motion around the central star, R is the interstellar distance and V_{sl} is the velocity of the perturbing star, moving on a straight line trajectory. The complete expression for the α coefficient is then found to be the following one:

$$\alpha = \frac{\sqrt{M_1 + M_2}}{r_p^{3/2}} \frac{R}{V_{sl}} \tag{75}$$

With the previous parameters of choice, α is equal almost exactly to one. As α is linear with R, a linear increase in the interstellar distance leads to a linear increase in the tidal parameter.
We will restrict our analysis to the case in which α is quite large and the encounter is prograde, that is, the angular momentum of the victim planet (orbiting around the central star) and the angular momentum of the perturber (on a straight-line trajectory) have the same sign. In this case, the expression of the energy variation as a function of the tidal parameter α has a very simple expression. In addition, the magnitude of the effect is much larger when compared to the retrograde case.
From the paper by D'Onghia et al. it is possible to see that the functional dependence of the energy variation on α is the following one:

$$\Delta E_{pro} \sim \alpha^3 e^{-2\alpha} \tag{76}$$

The simulation environment has been used to estimate the variation of the kinetic energy of the planet as a function of the tidal parameters, with α ranging from 3 to 6. The results are reported in the following graphs:

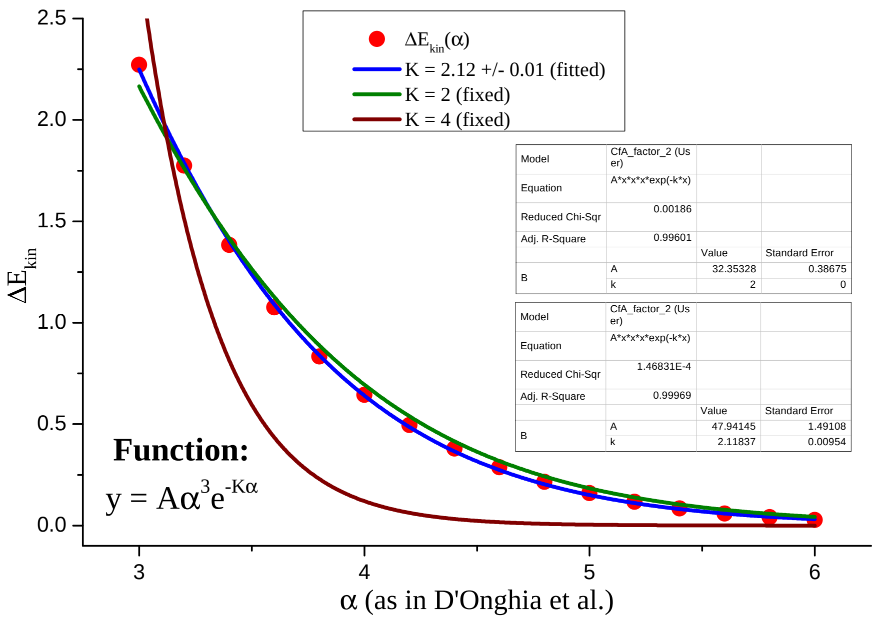

Fig. 2: Data points represent the kinetic energy variation of the victim planet as a function of the tidal parameter α. The fitting functions predicted by the two papers under analysis are reported. It seems clear that the expression predicted in the paper by D'Onghia et al. offers a much better fit to the data points. Remarkably, a direct fitting of the data (without requiring the K coefficient to be fixed to 2 or 4) yields $K = 2.12$.

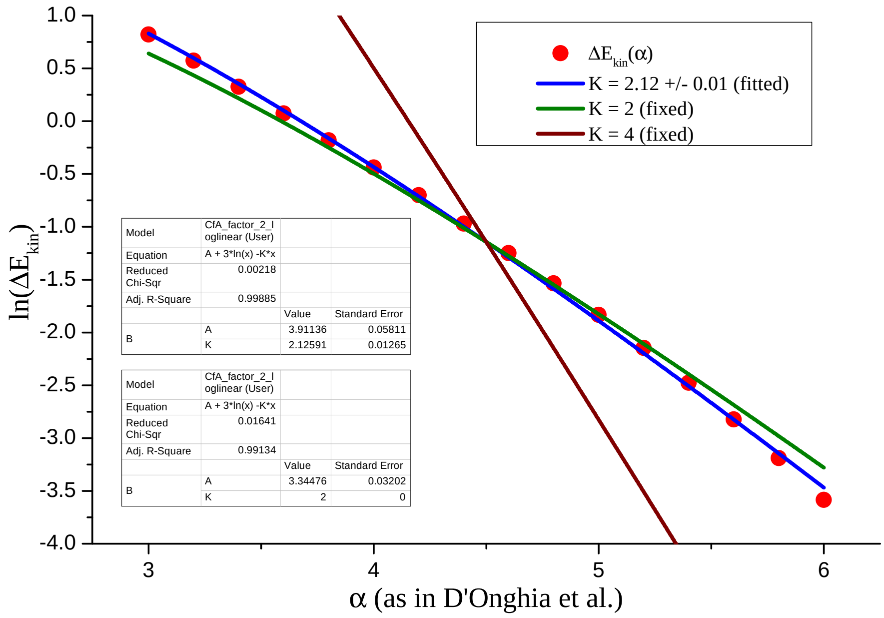

Fig. 3: Same data points and fitting functions as the previous one, but in log-linear scale. Here, the almost perfect agreement with the theoretical prediction of the paper by D'Onghia et al. is even more evident.

Improved agreements can be achieved by expanding the Bessel function to higher orders in α. In fact, the original paper consider only the zero-th order Taylor expansion. Extending the fitting function up to the first order in α, the new expression for the energy variation, in the prograde case, is found to be the following:

$$\Delta E_{pro} \sim \alpha^3 e^{-2\alpha} \left(1 + \frac{1}{4\alpha} + \frac{1}{64\alpha^2}\right) \tag{77}$$

As the following graph clearly shows, the agreement between this very last expression and the data points is excellent:

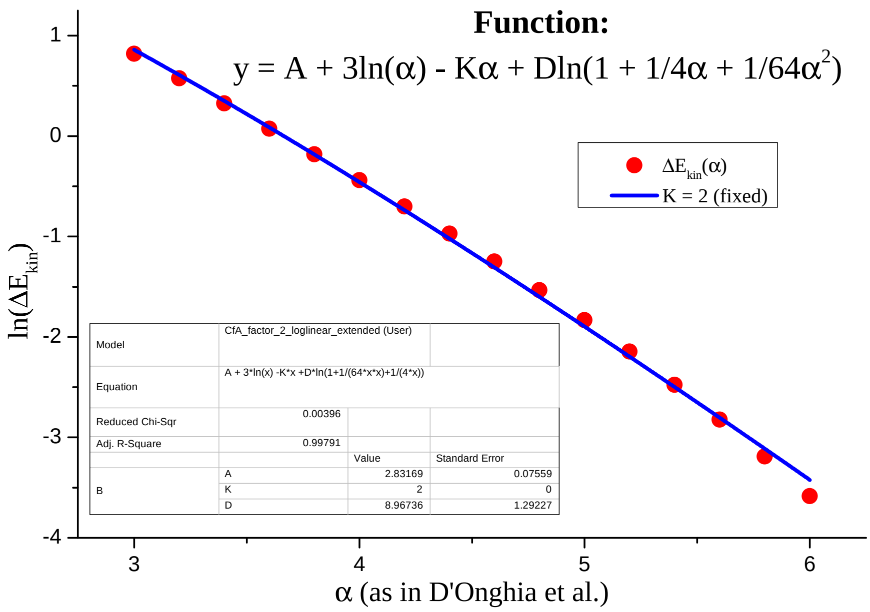

Fig. 4: Same data points as before, but fitted with the new expression for the energy variation, extended up to the first order in α in the Bessel function expansion. Now the agreement is excellent.

Slight deviations from the expected trend of the energy variation are evident for large values of α, where the kinetic energy variation of the victim is much smaller and becomes comparable to the intrinsic error in the determination of the energy shift.
From the previous simulations it could be possible to conclude that the expression for the kinetic energy variation devised in the paper by D'Onghia et al. is able to accurately describe the gravitational encounter in keplerian potential.

4 Planetary simulations

To conclude, we started to execute some actual planetary simulations, in order to see the effect on bodies with masses and distances comparable to the real ones in a planetary system.
The exact procedure we used is summarized in the following points:

1. We set the masses of the bodies involved. In particular, we used two stars of solar mass and a victim planet of cometary mass.
2. We set the distance between the victim body and the central star. In the example reported in this work, a distance of 50 000 AU has been used, which corresponds to a comet at the far edge of the Oort's Cloud.
3. We tried to determine the interstellar distance range where the prograde encounter is most effective. For most effective we mean that it is able to change the kinetic energy of the victim and the eccentricity of its orbit in the most efficient way.

The relation between the total energy (potential + kinetic) and the orbital eccentricity is the following:

$$e = \sqrt{1 + 2\frac{EL^2}{\mu\chi^2}} \tag{78}$$

where L is the angular momentum, μ is the reduced mass of the system, $\chi = M_s M_p$.

In order to determine the range of R where the encounter is most effective, we used two different parameters:

1. The tidal parameter, α, which describes the strength of the tidal coupling. A good range for α is between 0.5 and 2.0. Outside this range, the energy variation associated with the tidal encounter becomes very small.

2. The coupling coefficient, CC, which is defined as the ratio between the asymptotic behavior of the tidal force, M_{pert}/R^2, and α. The higher this parameter is, the better is the gravitational coupling. Even with good values of alpha, if the CC is too small, the gravitational encounter will not be effective because the magnitude of the tidal force is too small.

A very good example of this procedure is reported in the following graph, where the case of a cometary mass placed at the distance of 50 000 AU from the center of the stellar system is discussed.

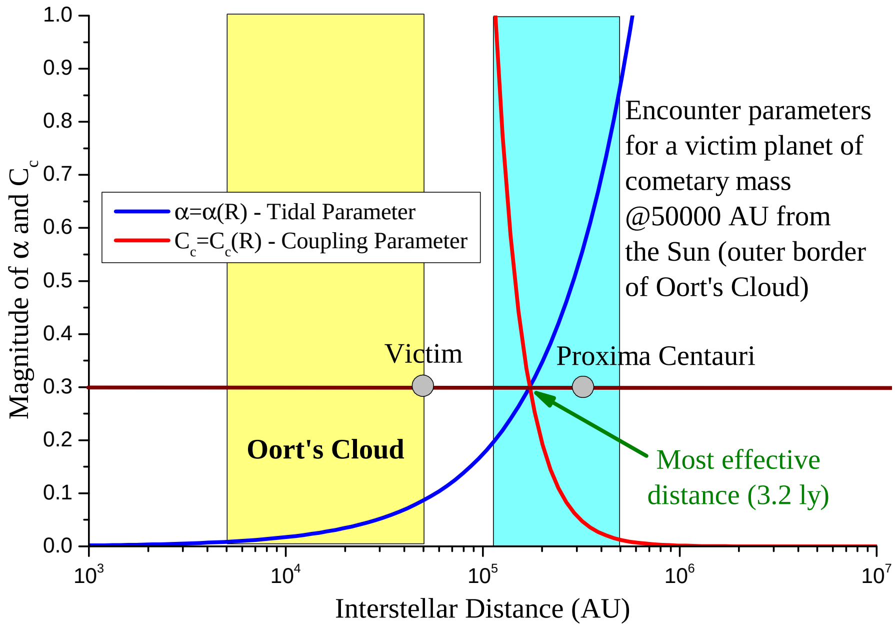

Fig. 5: Behaviors of the tidal parameter α and the coupling coefficient CC as a function of the interstellar distance, for a cometary mass placed at 50 000 AU from the center of the stellar system. In the solar system, this mass should be at the far edge of the Oort's cloud. The position of the victim and the position of Proxima Centauri, the nearest star (4.26 light years), are shown. The region where the tidal coupling is most effective is shaded in blue. From this graph it is possible to say that the bodies at the far edge of the Oort's cloud are well inside the region of tidal influence of Proxima Centauri.

The following graph, instead, represents the variations in kinetic energy and eccentricities for some interstellar distances inside the blue-shaded region of the previous graph.

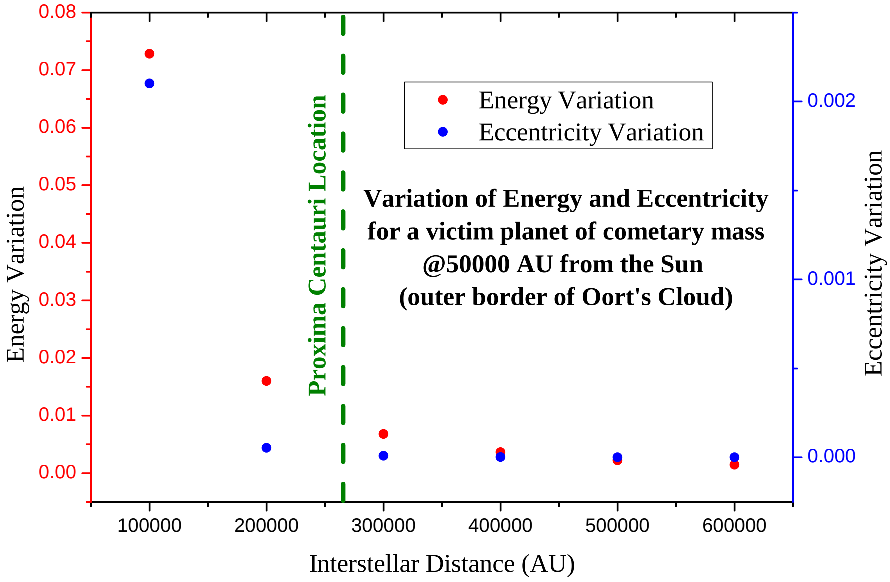

Fig. 6: Variations in kinetic energy and eccentricity for some interstellar distances inside the blue-shaded region of the previous graph, for a cometary mass at 50 000 AU from the center of the stellar system.

This graph demonstrates that the procedure we adopted in order to calculate the region of high-efficiency tidal coupling works remarkably well.

5 Conclusions

To summarize the work performed during this short-term research, the first period was dedicated to the achievement of a deeper understanding of the mathematical model developed, first from an analytical and then from a simulation-based point of view. In particular, the final outcome of this introductory step has been the simulation environment used to execute all the simulations. The first step has been to apply this simulation environment, along with the mathematical background, to the real matter of interest, the planetary simulations. In the course of this step, a mathematical description of the tidal interaction, based on a two-dimensional space of parameters, has been devised.

6 Acknowledgments

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