

Discontinuous Galerkin for linear elasticity of shells

Giuliano Guarino

10/10/2019

An application of the Discontinuous Galerkin method to shells whose mean surface is defined parametrically. The Carrera Unified formulation is used to describe the displacement vector.

INDEX

1 INTRODUCTION	2
2 LINEAR ELASTICITY	3
3 COMPOSITES	6
3.1 Types of materials	6
3.2 Stiffness matrix.....	6
3.3 Rotation of the stiffness matrix	7
3.4 Laminate.....	8
4 PLATES	9
5 DISCONTINUOUS GALERKIN METHOD.....	11
6 SHELLS	14
6.1 Geometric considerations.....	14
6.2 Kinematic hypothesis.....	14
6.3 Computation of QRS matrixes	15
6.4 Work of the volume forces.....	18
6.5 Work of the tractions.....	19
6.6 PVD for shells	21
6.7 Relation between cx and $c\xi$	21
7 CODE IMPLEMENTATION	22
7.1 Input files	22
7.2 Pre-computations	23
7.3 Assembling and solution	24
7.4 Plotting functions	24
7.5 Additional details.....	24
8 RESULTS.....	25
8.1 Isotropic Plate.....	25
8.2 Orthotropic Plate	26
8.3 Laminated Plate	28
8.4 Cylindrical laminated shell.....	29
8.5 Other tests.....	29
9 REMARKS AND FUTURE DEVELOPMENTS	31
9.1 Comparison with FEM	31
9.2 Non-linear analysis.....	31
9.3 Mathematical demonstration	31
10 BIBLIOGRAPHY.....	32

1 INTRODUCTION

Structural theory is a concentration within engineering and in particular aerospace science that investigates how a structure behaves under various loads. In aerospace vehicles, in particular, these loads are a result of their interaction with air, non-stationary manoeuvres, thermal gradients, and engine motion, for example. Under the assumption that these loads are known, the two main points of interests from a structural perspective are the deformation of the body and its stresses. With this information it is possible to determine the efficiency of the structure to resist its loads when compared to maximum allowable values.

Unfortunately, in order to evaluate the stresses and the displacements, it is necessary to solve a complex system of partial differential equations that, except in some cases where specific symmetries occur, is impossible to solve analytically and requires a numerical approach. These methods start from a discretization of the structure by dividing it into a large quantity of elements. For each of these elements is stated an hypothesis regarding the type of functions that represent the behaviour of the unknowns in the element. The unknowns then become the parameters of the functions whose type is supposed to be known.

There are a number of different methods that can be used to solve these problems, each with their own pros and cons. One particular method that has been increasingly used is the Discontinuous Galerkin method. With this method, it has been proven that it is possible to obtain better properties of convergence to zero of the error (that is, the difference between the real values of the stresses and the values computed numerically) when using higher order elements. In other words, the error decreases more rapidly with increasing number of elements.

One of the types of structure that is more frequently used in aerospace industry is the shell. This is due to the fact that, when properly, it can resist some assigned loads with lower overall weight than other bulky structure types. The study of thin shell is simpler than that of completely 3D structures because some hypotheses that simplify the mathematics can be assumed.

The equation of the structural theory are, in general, non-linear. If the hypothesis of small deformation is assumed, the equation can be approximated to linear and the problem solution becomes easier to obtain.

In this work a Discontinuous Galerkin method for linear elasticity of shells is introduced, the results when applied to some well-known problems regarding laminated plates are computed and compared with those from the literature. It is also studied a cylindrical case whose analytical solutions are available. It is shown in these cases that this method is capable to obtain an optimal convergence. Furthermore, it will be shown how the flexibility of the selection of the geometry and the kinematic hypothesis can make the code developed within this work an useful tool for analysis of shell structures.

2 LINEAR ELASTICITY

The background in which this work take place is the so called linear elasticity. The set of equations provided establishes the differential problem through which it is possible to evaluate the displacements of every point of the structure and the stresses in that point. The variables of the differential problem are the strains, the stresses and the displacements. The first two are symmetric tensor of the second order, but in the Voigt notation they become the following vectors:

$$\begin{aligned}\boldsymbol{\sigma} &= [\sigma_{11} \quad \sigma_{22} \quad \sigma_{33} \quad \sigma_{32} \quad \sigma_{31} \quad \sigma_{12}]^T \\ \boldsymbol{\gamma} &= [\gamma_{11} \quad \gamma_{22} \quad \gamma_{33} \quad \gamma_{32} \quad \gamma_{31} \quad \gamma_{12}]^T \\ \mathbf{u} &= [u_1 \quad u_2 \quad u_3]^T\end{aligned}$$

The solution of the problem also necessitates the knowledge of the applied volume forces and traction on the external surface of the body. The applied forces are vector of three component:

$$\begin{aligned}\mathbf{b} &= [b_1 \quad b_2 \quad b_3]^T \\ \mathbf{t} &= [t_1 \quad t_2 \quad t_3]^T\end{aligned}$$

The linear elasticity start from the assumption that the displacements in the structure are small and as a consequence the relationship between the displacements and the strain is linear. When a Cartesian reference system is selected the strains are defined as follows

$$\left\{ \begin{array}{l} \gamma_{11} = \frac{\partial u_1}{\partial x_1} \\ \gamma_{22} = \frac{\partial u_2}{\partial x_2} \\ \gamma_{33} = \frac{\partial u_3}{\partial x_3} \\ \gamma_{32} = \frac{\partial u_3}{\partial x_2} + \frac{\partial u_2}{\partial x_3} \\ \gamma_{31} = \frac{\partial u_3}{\partial x_1} + \frac{\partial u_1}{\partial x_3} \\ \gamma_{12} = \frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2} \end{array} \right.$$

The previous equations can be written more compactly in many ways. Here two different compact notations are introduced. In the following chapters only the second one will be adopted.

$$\boldsymbol{\gamma} = \mathbf{D}^T \mathbf{U}$$

$$\boldsymbol{\gamma} = \mathbf{I}_1 \frac{\partial \mathbf{u}}{\partial x_1} + \mathbf{I}_2 \frac{\partial \mathbf{u}}{\partial x_2} + \mathbf{I}_3 \frac{\partial \mathbf{u}}{\partial x_3}$$

Where the following differential operator and the following matrixes are introduced:

$$\mathbf{D}^T = \begin{bmatrix} \frac{\partial}{\partial x_1} & 0 & 0 & \frac{\partial}{\partial x_2} & 0 & \frac{\partial}{\partial x_3} \\ 0 & \frac{\partial}{\partial x_2} & 0 & \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_3} & 0 \\ 0 & 0 & \frac{\partial}{\partial x_3} & 0 & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_1} \end{bmatrix}$$

$$\mathbf{I}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad \mathbf{I}_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad \mathbf{I}_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The definition of the strains is one of the three sets of formulas that constitute the equilibrium problem together with the equilibrium equations and the constitutive equations. In particular the equilibrium equations link the derivative of the stresses in the three Cartesian directions and the applied volume force:

$$\begin{cases} \frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{21}}{\partial x_2} + \frac{\partial \sigma_{31}}{\partial x_3} = -b_1 \\ \frac{\partial \sigma_{12}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \sigma_{32}}{\partial x_3} = -b_2 \\ \frac{\partial \sigma_{13}}{\partial x_1} + \frac{\partial \sigma_{23}}{\partial x_2} + \frac{\partial \sigma_{33}}{\partial x_3} = -b_3 \end{cases}$$

Introducing the differential operator \mathbf{D} a compact notation can be obtained:

$$\mathbf{D}\boldsymbol{\sigma} = -\mathbf{b}$$

$$\mathbf{D} = \begin{bmatrix} \frac{\partial}{\partial x_1} & 0 & 0 \\ 0 & \frac{\partial}{\partial x_2} & 0 \\ 0 & 0 & \frac{\partial}{\partial x_3} \\ \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_1} & 0 \\ 0 & \frac{\partial}{\partial x_3} & \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_3} & 0 & \frac{\partial}{\partial x_1} \end{bmatrix}$$

In the boundary ∂V_N where a traction \mathbf{t} is applied it also must be valid the following:

$$\mathbf{T}_n^T \boldsymbol{\sigma} = \mathbf{t}$$

$$\mathbf{T}_n^T = \begin{bmatrix} n_1 & 0 & 0 & n_2 & 0 & n_3 \\ 0 & n_2 & 0 & n_1 & n_3 & 0 \\ 0 & 0 & n_3 & 0 & n_2 & n_1 \end{bmatrix}$$

The last set of equations is provided by the relationship between the stresses and the strains. This relation depend on the material of the structure and it is not necessarily linear. In this work it will be assumed a linear relationship between these two quantities:

$$\boldsymbol{\sigma} = \mathbf{c} \boldsymbol{\gamma}$$

It is now possible to combine the previous sets of equations to obtain the strong formulation of the equilibrium problem in the linear approximation:

$$\begin{cases} \boldsymbol{\gamma} = \mathbf{D}^T \mathbf{U} \\ \mathbf{D} \boldsymbol{\sigma} = -\mathbf{b} & \text{in } V \\ \boldsymbol{\sigma} = \mathbf{c} \boldsymbol{\gamma} \end{cases}$$

subjected to two sets of boundary conditions that relate the boundary ∂V_N where tractions are applied and the boundary ∂V_D where displacements are applied:

$$\begin{cases} \mathbf{T}_n^T \boldsymbol{\sigma} = \mathbf{t} & \text{in } \partial V_N \\ \mathbf{u} = \bar{\mathbf{u}} & \text{in } \partial V_D \end{cases}$$

The weak form of the previous equations is called principle of virtual displacements and it is stated as:

$$\int_V \delta \boldsymbol{\gamma}^T \boldsymbol{\sigma} = \int_V \delta \mathbf{u}^T \mathbf{b} + \int_{\partial V} \delta \mathbf{u}^T \mathbf{t}$$

3 COMPOSITES

In the previous chapter it is shown that one of the sets of equations necessary to solve the equilibrium problem is the constitutive equation that relate the stresses and the strains of a material. This relationship can be fully characterized only after a certain amount of standardized test on the selected material. The type of relationship will depend on the material and in some cases it can be simpler than in others. It is than important to understand which are the most common type of materials and how the relation between stresses and strains, that in the linear hypothesis takes the form of a stiffness matrix, is influenced by them.

3.1 TYPES OF MATERIALS

One of the possible classifications of the materials regards the number of symmetries in its mechanical behavior. Since a stiffness matrix relate two vector of 6 components (or two symmetric tensor of the second order) there are in total 36 elements. Since it can be proven that the stiffness matrix must be symmetric and positive defined only 21 of this elements are independent in the more general case. If some symmetries take place than the number of independent elements is reduced. For example the metallic materials usually behaves in the same way in all directions (isotropic) so only two independent parameters are necessary. In the following table the main types of materials, the number of symmetries and the number of independent elements of the matrix are given.

TYPE	NUMBER OF SYMMETRIES	NUMBER OF UNKNOWNNS
ANISOTROPIC	0	21
MONOCLINIC	1	13
ORTHOTROPIC	3	9
TRANSVERSELY ISOTROPIC	$1+\infty$	5
ISOTROPIC	∞	2

The transversely isotropic is a type of material that has one plane of symmetry and infinite ones orthogonal to the previous.

The composite materials is a particular class of materials that has become more and more commons in the last years. This type of materials combines together characteristics of ceramic materials and plastic materials, in particular the purpose of a composite is to take advantage of the high rigidity and resistance of the ceramic materials combining them with polymeric materials in order to reduce the overall fragility. The ceramic material, that comes in a fibrous shape is immersed in a polymeric matrix. Accordingly to the length of the fibers and their orientations in the matrix it is possible to obtain different type of materials typically orthotropic, transversely isotropic or more rarely isotropic. The most common disposition, for structure components, is the composite with long fibers that behaves as an orthotropic material. The fibers can be oriented in the direction of the maximum load or in any direction that would give the structure enough resistance.

3.2 STIFFNESS MATRIX

In an orthotropic material it is possible to write the stiffness matrix as a function of some constants. Young modulus, Poisson ration and shear modulus. These quantities are not necessarily the same for each direction and in the general case of an orthotropic material there are three of each with a total of 9 independent constants.

$$\mathbf{c} = \begin{bmatrix} \frac{1}{E_1} & -\frac{\nu_{21}}{E_2} & -\frac{\nu_{31}}{E_3} & 0 & 0 & 0 \\ -\frac{\nu_{12}}{E_1} & \frac{1}{E_2} & -\frac{\nu_{32}}{E_3} & 0 & 0 & 0 \\ -\frac{\nu_{13}}{E_1} & -\frac{\nu_{23}}{E_2} & \frac{1}{E_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G_{32}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G_{31}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G_{12}} \end{bmatrix}^{-1}$$

In the previous $\nu_{ij} = \nu_{ji}(E_i/E_j)$. If the material is transversely isotropic in the plane orthogonal to the direction 3 than the following identities hold:

$$\begin{aligned} E_1 &= E_2 \\ \nu_{13} &= \nu_{23} \\ G_{31} &= G_{23} \\ G_{12} &= \frac{E_1}{2(1 + \nu_{12})} \end{aligned}$$

reducing the number of independent constants to 5. If the material is isotropic $E_1 = E_2 = E_3 = E$, $\nu_{12} = \nu_{13} = \nu_{23} = \nu$ and $G_{12} = G_{31} = G_{23} = \frac{E}{2(1+\nu)}$, reducing the number of independent constants to 2.

3.3 ROTATION OF THE STIFFNESS MATRIX

As introduced in the previous section the stiffness matrix is only known in a special reference system of the material. If one is interested in the stiffness matrix in a general reference, it is necessary to rotate it. It can be demonstrated that the rotation of the stiffness matrix from a reference \mathbf{X}_1 to a reference \mathbf{X}_2 , with a rotation matrix \mathbf{R} , can be written as follows:

$$\mathbf{c}_2 = \mathbf{T}_\sigma \mathbf{c}_1 \mathbf{T}_\sigma^T$$

$$\mathbf{R} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\mathbf{T}_\sigma = \begin{bmatrix} a_{11}^2 & a_{12}^2 & a_{13}^2 & 2a_{12}a_{13} & 2a_{11}a_{13} & 2a_{11}a_{12} \\ a_{21}^2 & a_{22}^2 & a_{23}^2 & 2a_{22}a_{23} & 2a_{21}a_{23} & 2a_{21}a_{22} \\ a_{31}^2 & a_{32}^2 & a_{33}^2 & 2a_{32}a_{33} & 2a_{31}a_{33} & 2a_{31}a_{32} \\ a_{21}a_{31} & a_{22}a_{32} & a_{23}a_{33} & a_{22}a_{33} + a_{23}a_{32} & a_{21}a_{33} + a_{23}a_{31} & a_{21}a_{32} + a_{22}a_{31} \\ a_{11}a_{31} & a_{12}a_{32} & a_{13}a_{33} & a_{12}a_{33} + a_{13}a_{32} & a_{11}a_{33} + a_{13}a_{31} & a_{11}a_{32} + a_{12}a_{31} \\ a_{11}a_{21} & a_{12}a_{22} & a_{13}a_{23} & a_{12}a_{23} + a_{13}a_{22} & a_{11}a_{23} + a_{13}a_{21} & a_{11}a_{22} + a_{12}a_{21} \end{bmatrix}$$

3.4 LAMINATE

A laminate is an assembly of layers of composite material. The reason why such an assembly may be a better solution than a simple thicker monolayer is that a long fibers composite is very strong in the direction of the fibers but results quite weak in the orthogonal ones. For such reason it may be advantageous in some cases to juxtapose more layers with different orientations. In such a way it will be given extra strength to more than one direction.

4 PLATES

When the structure considered has one of the three dimension much smaller than the others it is possible to obtain a simplified formulation of the principle of virtual displacements. This is possible after the assumption that the displacements can be expressed as the product of some known functions of the variable through the thickness and some unknowns. This process is also called thickness expansion of the solution or assumption of the kinematic hypothesis. Different kinematic hypothesis will determine different structural theories. If a right hand coordinate system with x_3 oriented perpendicularly to the surface of the laminate and with its origin in the middle point of the thickness is chosen then it is possible to write the displacements as:

$$u_i^{(l)}(x_1, x_2, x_3) = \sum_{\alpha=0}^{N_{ui}} u_{i\alpha}(x_1, x_2) f_{\alpha}(x_3) \quad \forall l = 1, \dots, N_l$$

If the known functions are piecewise in the layers of the laminate, the variable description will be of the type Layer Wise (LW), otherwise it will be of the type Equivalent Single Layer (ESL). The LW descriptions are more accurate than the ESL especially when it comes to the study of laminates but they are not considered in this work. A comparison of the different expansion along the thickness is given in chapter 7.

An ESL theory can be named by the number of variables introduced for every direction of the displacements as in $ED_{N_{u_1}, N_{u_2}, N_{u_3}}$, where E stand for ESL and D denotes the use of the PVD. It is possible to adopt an unique matricial expression for theory as:

$$\mathbf{u} = \mathbf{F} \mathbf{U}$$

where \mathbf{F} contains the known functions of the x_3 variable and \mathbf{U} are the unknown that will depend only on x_1 and x_2 . If the selected order for the expansion of the components of the displacement are N_1, N_2, N_3 , the matrix \mathbf{F} and the vector \mathbf{U} will be:

$$\mathbf{F} = \begin{bmatrix} 1 & x_3^1 & \dots & x_3^{N_{u_1}} & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 1 & x_3^1 & \dots & x_3^{N_{u_2}} & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 1 & x_3^1 & \dots & x_3^{N_{u_3}} \end{bmatrix}$$

$$\mathbf{U} = [U_{1,0} \quad \dots \quad U_{1,N_{u_1}} \quad U_{2,0} \quad \dots \quad U_{2,N_{u_2}} \quad U_{3,0} \quad \dots \quad U_{3,N_{u_3}}]^T$$

The size of \mathbf{U} is $N_u = N_{u_1} + N_{u_2} + N_{u_3} + 3$. Recalling the PVD and integrating along the thickness it is possible to write the PVD in its bidimensional formulation:

$$\int_V \delta \boldsymbol{\gamma}^T \boldsymbol{\sigma} = \int_V \delta \mathbf{u}^T \mathbf{b} + \int_{\partial V} \delta \mathbf{u}^T \mathbf{t}$$

$$\int_{\Omega} \frac{\partial \delta \mathbf{U}^T}{\partial x_{\lambda}} \left(\mathbf{Q}_{\lambda\mu} \frac{\partial \mathbf{U}}{\partial x_{\mu}} + \mathbf{R}_{\lambda 3} \mathbf{U} \right) + \delta \mathbf{U}^T \left(\mathbf{R}_{\lambda 3}^T \frac{\partial \mathbf{U}}{\partial x_{\lambda}} + \mathbf{S}_{33} \mathbf{U} \right) = \int_{\Omega} \delta \mathbf{U}^T \mathbf{B} + \int_{\Omega} \delta \mathbf{U}^T \mathbf{T}$$

In the previous equation the matrixes $\mathbf{Q}_{\lambda\mu}, \mathbf{R}_{\lambda 3}$ and \mathbf{S}_{33} are the generalized stiffness matrixes and are computed as:

$$\mathbf{Q}_{\lambda\mu} = \int_{\tau_b}^{\tau_t} \mathbf{F}_{\lambda} \mathbf{c}_x \mathbf{F}_{\mu} dx_3$$

$$\mathbf{R}_{\lambda 3} = \int_{\tau_b}^{\tau_t} \mathbf{F}_\lambda \mathbf{c}_x \mathbf{F}_3 dx_3$$

$$\mathbf{S}_{33} = \int_{\tau_b}^{\tau_t} \mathbf{F}_3 \mathbf{c}_x \mathbf{F}_3 dx_3$$

The vectors \mathbf{B} and \mathbf{T} are, respectively, the generalized domain force and the generalized boundary force:

$$\mathbf{B} = (\mathbf{F}^T \mathbf{t})_{(x_3=x_3^b)} + (\mathbf{F}^T \mathbf{t})_{(x_3=x_3^t)} + \int_{x_3^b}^{x_3^t} \mathbf{F}^T \mathbf{b} dx_3$$

$$\mathbf{T} = \int_{x_3^b}^{x_3^t} \mathbf{F}^T \mathbf{t} dx_3$$

It is also possible to write the strong form of the bidimensional PVD by integrating by parts and by considering the variational statement must be valid for any $\delta \mathbf{U}$. The strong form is then written as:

$$-\frac{\partial}{\partial x_\lambda} \left(\mathbf{Q}_{\lambda\mu} \frac{\partial \mathbf{U}}{\partial x_\mu} + \mathbf{R}_{\lambda 3} \mathbf{U} \right) + \mathbf{R}_{\lambda 3}^T \frac{\partial \mathbf{U}}{\partial x_\lambda} + \mathbf{S}_{33} \mathbf{U} = \bar{\mathbf{B}} \quad \text{in } \Omega$$

With the following boundary conditions:

$$\begin{cases} n_\lambda \left(\mathbf{Q}_{\lambda\mu} \frac{\partial \mathbf{U}}{\partial x_\mu} + \mathbf{R}_{\lambda 3} \mathbf{U} \right) = \bar{\mathbf{T}} & \text{on } \partial\Omega_N \\ \mathbf{U} = \bar{\mathbf{U}} & \text{on } \partial\Omega_D \end{cases}$$

5 DISCONTINUOUS GALERKIN METHOD

It is possible to transform the problem introduced so far in a system of linear partial differential equations by introducing the auxiliary variable:

$$\boldsymbol{\Sigma}_\lambda = \mathbf{Q}_{\lambda\mu} \frac{\partial \mathbf{U}}{\partial x_\mu} + \mathbf{R}_{\lambda 3} \mathbf{U} \quad \lambda, \mu = 1, 2$$

In such a way it is obtained to the following set of equations:

$$\begin{cases} -\frac{\partial \boldsymbol{\Sigma}_\lambda}{\partial x_\lambda} + \mathbf{R}_{\lambda 3}^T \frac{\partial \mathbf{U}}{\partial x_\lambda} + \mathbf{S}_{33} \mathbf{U} = \bar{\mathbf{B}} \\ \boldsymbol{\Sigma}_\lambda = \mathbf{Q}_{\lambda\mu} \frac{\partial \mathbf{U}}{\partial x_\mu} + \mathbf{R}_{\lambda 3} \mathbf{U} \end{cases} \quad \text{in } \Omega$$

Subjected to the boundary conditions:

$$\begin{cases} n_\lambda \boldsymbol{\Sigma}_\lambda = \bar{\mathbf{T}} & \text{on } \Omega_N \\ \mathbf{U} = \bar{\mathbf{U}} & \text{on } \Omega_D \end{cases}$$

The domain is now divided in N_e elements. It is introduced the space of discontinuous polynomials as:

$$V_{hp} \equiv \left\{ v: \Omega \rightarrow \mathbb{R} \mid v|_{\Omega^{(e)}} \in P_p^{(e)} \forall e = 1, \dots, N_e \right\}$$

where $P_p^{(e)}$ is the space of polynomials functions of degree $p \geq 1$. It is possible to define also the space of polynomials vector fields V_{hp}^d where d denotes the size of the vectors. The weak form of the previous set of equations in a generic element $\Omega^{(e)}$ is obtained by means of the test functions $\mathbf{V}, \boldsymbol{\Gamma}_1, \boldsymbol{\Gamma}_2 \in V_{hp}^{Nu}$:

$$\begin{aligned} \int_{\Omega^{(e)}} \frac{\partial \mathbf{V}^T}{\partial x_\lambda} \boldsymbol{\Sigma}_{h\lambda} + \mathbf{V}^T \left(\mathbf{R}_{\lambda 3}^T \frac{\partial \mathbf{U}_h}{\partial x_\lambda} + \mathbf{S}_{33} \mathbf{U}_h \right) &= \int_{\partial\Omega^{(e)}} \mathbf{V}^T \widehat{\boldsymbol{\Sigma}}_\lambda n_\lambda + \int_{\Omega^{(e)}} \mathbf{V}^T \bar{\mathbf{B}} \\ \int_{\Omega^{(e)}} \boldsymbol{\Gamma}_\lambda^T \boldsymbol{\Sigma}_{h\lambda} &= \int_{\Omega^{(e)}} \boldsymbol{\Gamma}_\lambda^T \left(\mathbf{Q}_{\lambda\mu} \frac{\partial \mathbf{U}_h}{\partial x_\mu} + \mathbf{R}_{\lambda 3} \mathbf{U}_h \right) + \int_{\partial\Omega^{(e)}} (\boldsymbol{\Gamma}_\lambda^T \mathbf{Q}_{\lambda\mu} + \mathbf{V}^T \mathbf{R}_{\mu 3}) (\widehat{\mathbf{U}} - \mathbf{U}_h) n_\mu \end{aligned}$$

$\widehat{\boldsymbol{\Sigma}}_\lambda$ and $\widehat{\mathbf{U}}$ are called the numerical fluxes which are approximation of $\boldsymbol{\Sigma}_\lambda$ and \mathbf{U} on the boundary of the element $\partial\Omega^{(e)}$ and its choice determine the selected DG method. $\boldsymbol{\Sigma}_{h\lambda}$ and \mathbf{U}_h are the solution of the weak formulation.

It is evident that the boundary of the whole domain is the union of the external boundaries of the elements. For this reason it is possible to write that $\partial\Omega_D = \cup_{e=1}^{N_e} \partial\Omega_D^{(e)}$, $\partial\Omega_N = \cup_{e=1}^{N_e} \partial\Omega_N^{(e)}$ and $\Omega_D = \cup_{e=1}^{N_e} \Omega_D^{(e)}$. In a similar way it is possible to define the union of the internal boundaries as $\partial\Omega_I = \cup_{e=1}^{N_e} \partial\Omega_I^{(e)}$ that coincides with the union of internal edges of the mesh. The *average* and *jump* operators are also defined:

$$\{\bullet\}^{(i)} = \frac{1}{2} (\bullet^{(e)} + \bullet^{(e')})$$

$$[[\cdot]]_{\lambda}^{(i)} = n_{\lambda}^{(e)} \cdot^{(e)} + n_{\lambda}^{(e')} \cdot^{(e')}$$

where the superscript (i) denotes the internal edge identified as the intersection of $\Omega^{(e)}$ and $\Omega^{(e')}$. Using the previous and considering that $n_{\lambda}^{(e)} = -n_{\lambda}^{(e')}$ it is possible to demonstrate that the following identity holds:

$$\sum_{e=1}^{N_e} \int_{\partial\Omega^{(e)}} \mathbf{a}^T \mathbf{b} n_{\lambda} = \int_{\partial\Omega_{hl}} \{\mathbf{a}\}^T [[\mathbf{b}]]_{\lambda} + [[\mathbf{a}]]_{\lambda}^T \{\mathbf{b}\} + \int_{\partial\Omega_{hD} \cup \partial\Omega_{hN}} \mathbf{a}^T \mathbf{b} n_{\lambda}$$

This equation is used to rewrite the weak formulation of the governing equations after the sum over all the elements. It is then obtained the following DG formulation for the whole domain:

$$\begin{aligned} \int_{\Omega_h} \frac{\partial \mathbf{V}^T}{\partial x_{\lambda}} \boldsymbol{\Sigma}_{h\lambda} + \mathbf{V}^T \left(\mathbf{R}_{\lambda 3}^T \frac{\partial \mathbf{U}_h}{\partial x_{\lambda}} + \mathbf{S}_{33} \mathbf{U}_h \right) \\ = \int_{\partial\Omega_{hl}} \{\mathbf{V}\}^T [[\boldsymbol{\Sigma}_{\lambda}]]_{\lambda} + [[[\mathbf{V}]]]_{\lambda}^T \{\boldsymbol{\Sigma}_{\lambda}\} + \int_{\partial\Omega_{hD} \cup \partial\Omega_{hN}} \mathbf{V}^T \hat{\boldsymbol{\Sigma}} n_{\lambda} + \int_{\Omega_h} \mathbf{V}^T \bar{\mathbf{B}} \\ \int_{\Omega_h} \boldsymbol{\Gamma}_{\lambda}^T \boldsymbol{\Sigma}_{h\lambda} = \int_{\Omega_h} \boldsymbol{\Gamma}_{\lambda}^T \left(\mathbf{Q}_{\lambda\mu} \frac{\partial \mathbf{U}_h}{\partial x_{\mu}} + \mathbf{R}_{\lambda 3} \mathbf{U}_h \right) \\ + \int_{\partial\Omega_{hl}} \{\boldsymbol{\Gamma}_{\lambda}^T \mathbf{Q}_{\lambda\mu} + \mathbf{V}^T \mathbf{R}_{\mu 3}\} [[\hat{\mathbf{U}} - \mathbf{U}_h]]_{\mu} + [[[\boldsymbol{\Gamma}_{\lambda}^T \mathbf{Q}_{\lambda\mu} + \mathbf{V}^T \mathbf{R}_{\mu 3}]]]_{\mu} \{\hat{\mathbf{U}} - \mathbf{U}_h\} \\ + \int_{\partial\Omega_{hD} \cup \partial\Omega_{hN}} (\boldsymbol{\Gamma}_{\lambda}^T \mathbf{Q}_{\lambda\mu} + \mathbf{V}^T \mathbf{R}_{\mu 3}) (\hat{\mathbf{U}} - \mathbf{U}_h) n_{\mu} \end{aligned}$$

It is now possible to write the numerical fluxes $\hat{\mathbf{U}}$ and $\hat{\boldsymbol{\Sigma}}$. In this work it is chosen an Interior Penalty formulation and the fluxes are specified as:

$$\hat{\mathbf{U}} = \begin{cases} \{\mathbf{U}_h\} & \text{on } \partial\Omega_{hl} \\ \bar{\mathbf{U}} & \text{on } \partial\Omega_{hD} \\ \mathbf{U}_h & \text{on } \partial\Omega_{hN} \end{cases}$$

$$\hat{\boldsymbol{\Sigma}} = \begin{cases} \left\{ \mathbf{Q}_{\lambda\mu} \frac{\partial \mathbf{U}_h}{\partial x_{\mu}} + \mathbf{R}_{\lambda 3} \mathbf{U}_h \right\} - \mu [[[\mathbf{U}_h]]]_{\lambda} & \text{on } \partial\Omega_{hl} \\ \mathbf{Q}_{\lambda\mu} \frac{\partial \mathbf{U}_h}{\partial x_{\mu}} + \mathbf{R}_{\lambda 3} \mathbf{U}_h - \mu (\mathbf{U}_h - \bar{\mathbf{U}}) n_{\mu} & \text{on } \partial\Omega_{hD} \\ n_{\lambda} \hat{\boldsymbol{\Sigma}} = \bar{\mathbf{T}} & \text{on } \partial\Omega_{hN} \end{cases}$$

in the previous μ is the penalty parameter. Setting $\boldsymbol{\Gamma}_{\lambda} \equiv \partial \mathbf{V} / \partial x_{\lambda}$ and using the flux variables it is possible to obtain the *primal form* of the proposed method:

$$B_{IP}(\mathbf{V}, \mathbf{U}_h) = R_{IP}(\mathbf{V}, \bar{\mathbf{B}}, \bar{\mathbf{T}}, \bar{\mathbf{U}})$$

$$\begin{aligned}
B_{IP}(\mathbf{V}, \mathbf{U}_h) &= \int_{\Omega_h} \frac{\partial \mathbf{V}^T}{\partial x_\lambda} \left(\mathbf{Q}_{\lambda\mu} \frac{\partial \mathbf{U}_h}{\partial x_\mu} + \mathbf{R}_{\lambda 3} \mathbf{U}_h \right) + \mathbf{V}^T \left(\mathbf{R}_{\lambda 3}^T \frac{\partial \mathbf{U}}{\partial x_\lambda} + \mathbf{S}_{33} \mathbf{U}_h \right) \\
&\quad - \int_{\partial\Omega_{hI}} [[\mathbf{V}]]_\lambda^T \left\{ \mathbf{Q}_{\lambda\mu} \frac{\partial \mathbf{U}_h}{\partial x_\mu} + \mathbf{R}_{\lambda 3} \mathbf{U}_h \right\} + \left\{ \frac{\partial \mathbf{V}^T}{\partial x_\lambda} \mathbf{Q}_{\lambda\mu} + \mathbf{V}^T \mathbf{R}_{\mu 3}^T \right\} [[\mathbf{U}_h]]_\mu \\
&\quad - \int_{\partial\Omega_{hD}} n_\lambda \mathbf{V}^T \left(\mathbf{Q}_{\lambda\mu} \frac{\partial \mathbf{U}_h}{\partial x_\mu} + \mathbf{R}_{\lambda 3} \mathbf{U}_h \right) + \left(\frac{\partial \mathbf{V}^T}{\partial x_\lambda} \mathbf{Q}_{\lambda\mu} + \mathbf{V}^T \mathbf{R}_{\mu 3}^T \right) \mathbf{U}_h n_\mu \\
&\quad + \int_{\partial\Omega_{hI}} \mu [[\mathbf{V}]]_\lambda^T [[\mathbf{U}_h]]_\lambda + \int_{\partial\Omega_{hD}} \mu \mathbf{V}^T \mathbf{U}_h
\end{aligned}$$

$$\begin{aligned}
R_{IP}(\mathbf{V}, \bar{\mathbf{B}}, \bar{\mathbf{T}}, \bar{\mathbf{U}}) &= \int_{\Omega_h} \mathbf{V}^T \bar{\mathbf{B}} - \int_{\partial\Omega_{hN}} \mathbf{V}^T \bar{\mathbf{T}} - \int_{\partial\Omega_{hD}} \left(\frac{\partial \mathbf{V}^T}{\partial x_\lambda} \mathbf{Q}_{\lambda\mu} + \mathbf{V}^T \mathbf{R}_{\lambda\mu}^T \right) \bar{\mathbf{U}} n_\mu \\
&\quad + \int_{\partial\Omega_{hD}} \mu \mathbf{V}^T \bar{\mathbf{U}}
\end{aligned}$$

6 SHELLS

A shell is a thin structure whose mean surface is not flat. It is possible to bring back the study of this type of structure to those of the flat laminates by using a transformation of the coordinates of the mean surface to an auxiliary reference system. This reference system identify on the shell the so called curvilinear coordinates whose basis is local and will be defined in the following section. It is convenient in the case of the shells to write the kinematic hypothesis in this auxiliary reference system. In section 1 some helpful geometric relationships are shown. In the sections 2 to 5 it is shown how the quantities in the PVD for plates are redefined for the case of the shells.

6.1 GEOMETRIC CONSIDERATIONS

The mean surface of the shell \mathbf{x}_0 can be parameterised as a function of ξ_1 and ξ_2 :

$$\mathbf{x}_0 = \begin{bmatrix} x_{10} \\ x_{20} \\ x_{30} \end{bmatrix} = \begin{bmatrix} x_{10}(\xi_1, \xi_2) \\ x_{20}(\xi_1, \xi_2) \\ x_{30}(\xi_1, \xi_2) \end{bmatrix}$$

It is than possible to write the full transformation from the domain in coordinates ξ_1, ξ_2, ξ_3 to the volume of the shell in coordinates x_1, x_2, x_3 as it follows:

$$\mathbf{n}_3 = \mathbf{n}_3(\xi_1, \xi_2) = \left(\frac{\partial \mathbf{x}_0}{\partial \xi_1} \times \frac{\partial \mathbf{x}_0}{\partial \xi_2} \right) \left| \frac{\partial \mathbf{x}_0}{\partial \xi_1} \times \frac{\partial \mathbf{x}_0}{\partial \xi_2} \right|^{-1}$$

$$\mathbf{x} = \mathbf{x}_0(\xi_1, \xi_2) + \xi_3 \mathbf{n}_3(\xi_1, \xi_2)$$

where \mathbf{n}_3 is the unit vector in the direction of ξ_3 . The range of variation of ξ_3 is $[\tau_b \ \tau_t]$. In such a way the transformation of coordinates is of the following type: $\Omega_\xi \times [\tau_b \ \tau_t] \rightarrow S_0 \times [\tau_b \ \tau_t]$. It is possible at this point to define also \mathbf{n}_1 and \mathbf{n}_2 :

$$\mathbf{n}_1 = \mathbf{n}_1(\xi_1, \xi_2, \xi_3) = \frac{\partial \mathbf{x}}{\partial \xi_1} \left| \frac{\partial \mathbf{x}}{\partial \xi_1} \right|^{-1} = \left(\frac{\partial \mathbf{x}_0}{\partial \xi_1} + \xi_3 \frac{\partial \mathbf{n}_3}{\partial \xi_1} \right) \left| \frac{\partial \mathbf{x}_0}{\partial \xi_1} + \xi_3 \frac{\partial \mathbf{n}_3}{\partial \xi_1} \right|^{-1}$$

$$\mathbf{n}_2 = \mathbf{n}_2(\xi_1, \xi_2, \xi_3) = \frac{\partial \mathbf{x}}{\partial \xi_2} \left| \frac{\partial \mathbf{x}}{\partial \xi_2} \right|^{-1} = \left(\frac{\partial \mathbf{x}_0}{\partial \xi_2} + \xi_3 \frac{\partial \mathbf{n}_3}{\partial \xi_2} \right) \left| \frac{\partial \mathbf{x}_0}{\partial \xi_2} + \xi_3 \frac{\partial \mathbf{n}_3}{\partial \xi_2} \right|^{-1}$$

A vector \mathbf{v}_ξ in the local basis $\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3$ can be rotated in the canonical base by means of the matrix \mathbf{R} :

$$\mathbf{R} = \mathbf{R}(\xi_1, \xi_2, \xi_3) = [\mathbf{n}_1 \ \mathbf{n}_2 \ \mathbf{n}_3]$$

$$\mathbf{v}_x = \mathbf{R} \mathbf{v}_\xi$$

6.2 KINEMATIC HYPOTHESIS

In the case of shells it is more convenient to write the kinematic hypothesis in the local basis $\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3$ in the following way:

$$\mathbf{u}_\xi = \begin{bmatrix} u_{\xi 1} \\ u_{\xi 2} \\ u_{\xi 3} \end{bmatrix} = \mathbf{F}(\xi_3) \cdot \mathbf{U}(\xi_1, \xi_2)$$

In such a way it is possible to separate the variation along the thickness to the variation in the surface. The displacements in the reference system \mathbf{x} can be evaluated by:

$$\mathbf{u}_x = \begin{bmatrix} u_{x1} \\ u_{x2} \\ u_{x3} \end{bmatrix} = \mathbf{R} \mathbf{u}_\xi$$

It is interesting to notice that the basis $\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3$ can also be not orthogonal. If so, it will be a consequence of the not orthogonality of \mathbf{n}_1 and \mathbf{n}_2 . Both vectors are in fact always orthogonal to \mathbf{n}_3 because of its definition.

6.3 COMPUTATION OF QRS MATRIXES

The matrix QRS can be computed in two different ways accordingly to the reference system used to write the principle of the virtual. The 2 possible reference systems are \mathbf{x} and ξ each one referring to a different basis. As a consequence, the domain integral of the principle of virtual displacements can be written in the 2 following ways:

$$\int_V \delta \gamma_x^T \sigma_x dV$$

$$\int_V \delta \gamma_\xi^T \sigma_\xi dV$$

In the latter case, the reference system has to be orthogonal or the integral will not represent the work of internal forces. On the other hand in the first integral the reference system \mathbf{x} is always Cartesian.

Reference system \mathbf{x}

The strains and the stresses are both written in a reference \mathbf{x} . The relation between γ_x and \mathbf{u}_x is the following:

$$\gamma_x = \mathbf{I}_1 \frac{\partial \mathbf{u}_x}{\partial x_1} + \mathbf{I}_2 \frac{\partial \mathbf{u}_x}{\partial x_2} + \mathbf{I}_3 \frac{\partial \mathbf{u}_x}{\partial x_3}$$

With the following matrixes:

$$\mathbf{I}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad \mathbf{I}_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad \mathbf{I}_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Substituting the expression for \mathbf{u}_x and changing the derivatives in x_i with derivatives in ξ_i it is possible to write:

$$\gamma_x = \mathbf{I}_i \frac{\partial \mathbf{u}_x}{\partial x_i} = \mathbf{I}_i \frac{\partial (\mathbf{R} \mathbf{u}_\xi)}{\partial \xi_j} \frac{\partial \xi_j}{\partial x_i}$$

The first derivative in this expression can be written as:

$$\frac{\partial (\mathbf{R} \mathbf{u}_\xi)}{\partial \xi_1} = \frac{\partial \mathbf{R}}{\partial \xi_1} \mathbf{F} \mathbf{U} + \mathbf{R} \mathbf{F} \frac{\partial \mathbf{U}}{\partial \xi_1}$$

$$\frac{\partial(\mathbf{R} \mathbf{u}_\xi)}{\partial \xi_2} = \frac{\partial \mathbf{R}}{\partial \xi_2} \mathbf{F} \mathbf{U} + \mathbf{R} \mathbf{F} \frac{\partial \mathbf{U}}{\partial \xi_2}$$

$$\frac{\partial(\mathbf{R} \mathbf{u}_\xi)}{\partial \xi_3} = \frac{\partial \mathbf{R}}{\partial \xi_3} \mathbf{F} \mathbf{U} + \mathbf{R} \frac{\partial \mathbf{F}}{\partial \xi_3} \mathbf{U}$$

It is possible to define some auxiliary matrix as:

$$\mathbf{F}_0 = \mathbf{I}_i \left(\frac{\partial \xi_j}{\partial x_i} \frac{\partial \mathbf{R}}{\partial \xi_j} \mathbf{F} + \mathbf{R} \frac{\partial \mathbf{F}}{\partial \xi_3} \right)$$

$$\mathbf{F}_1 = \mathbf{I}_i \left(\frac{\partial \xi_1}{\partial x_i} \mathbf{R} \mathbf{F} \right)$$

$$\mathbf{F}_2 = \mathbf{I}_i \left(\frac{\partial \xi_2}{\partial x_i} \mathbf{R} \mathbf{F} \right)$$

In such way the vector of the strains can be related to the generalized displacements \mathbf{U} as follow:

$$\boldsymbol{\gamma}_x = \mathbf{F}_0 \mathbf{U} + \mathbf{F}_1 \frac{\partial \mathbf{U}}{\partial \xi_1} + \mathbf{F}_2 \frac{\partial \mathbf{U}}{\partial \xi_2}$$

Recalling the following relationship between $\boldsymbol{\sigma}_x$ and $\boldsymbol{\gamma}_x$:

$$\boldsymbol{\sigma}_x = \mathbf{c}_x \boldsymbol{\gamma}_x$$

where \mathbf{c}_x is the local stiffness matrix in the reference system \mathbf{x} , it is than possible to write the integral on V of the PVD:

$$\begin{aligned} \int_V \delta \boldsymbol{\gamma}_x^T \boldsymbol{\sigma}_x dV = \int \left(\delta \mathbf{U} \mathbf{F}_0 + \frac{\partial \delta \mathbf{U}}{\partial \xi_1} \mathbf{F}_1 + \frac{\partial \delta \mathbf{U}}{\partial \xi_2} \mathbf{F}_2 \right) \mathbf{c}_x \left(\mathbf{F}_0 \mathbf{U} + \mathbf{F}_1 \frac{\partial \mathbf{U}}{\partial \xi_1} \right. \\ \left. + \mathbf{F}_2 \frac{\partial \mathbf{U}}{\partial \xi_2} \right) |\mathbf{J}_{x\xi}| d\xi_1 d\xi_2 d\xi_3 \end{aligned}$$

The matrixes $\mathbf{Q}_{\lambda\mu}$, $\mathbf{R}_{\lambda 3}$, \mathbf{S}_{33} can be defined here as:

$$\mathbf{Q}_{\lambda\mu} = \int_{\tau_b}^{\tau_t} \mathbf{F}_\lambda \mathbf{c}_x \mathbf{F}_\mu |\mathbf{J}_{x\xi}| d\xi_3$$

$$\mathbf{R}_{\lambda 3} = \int_{\tau_b}^{\tau_t} \mathbf{F}_\lambda \mathbf{c}_x \mathbf{F}_3 |\mathbf{J}_{x\xi}| d\xi_3$$

$$\mathbf{S}_{33} = \int_{\tau_b}^{\tau_t} \mathbf{F}_3 \mathbf{c}_x \mathbf{F}_3 |\mathbf{J}_{x\xi}| d\xi_3$$

obtaining the following formulation for the domain integral in the PVD:

$$\int_{\Omega_\xi} \frac{\partial \delta \mathbf{U}^T}{\partial \xi_\lambda} \left(\mathbf{Q}_{\lambda\mu} \frac{\partial \mathbf{U}}{\partial \xi_\mu} + \mathbf{R}_{\lambda 3} \mathbf{U} \right) + \delta \mathbf{U}^T \left(\mathbf{R}_{\lambda 3}^T \frac{\partial \mathbf{U}}{\partial \xi_\lambda} + \mathbf{S}_{33} \mathbf{U} \right)$$

Reference system ξ

It is possible through the procedure described in Reddy et al. prove that the relation between $\boldsymbol{\gamma}_\xi$ and \boldsymbol{u}_ξ in the curvilinear reference system ξ :

$$\left\{ \begin{array}{l} \gamma_{11} = \frac{1}{A_1} \left(\frac{\partial u_1}{\partial \xi_1} + \frac{u_2}{a_2} \frac{\partial a_1}{\partial \xi_2} + \frac{a_1}{R_1} u_3 \right) \\ \gamma_{22} = \frac{1}{A_2} \left(\frac{\partial u_2}{\partial \xi_2} + \frac{u_1}{a_1} \frac{\partial a_2}{\partial \xi_1} + \frac{a_2}{R_2} u_3 \right) \\ \gamma_{33} = \frac{\partial u_3}{\partial \xi_3} \\ \gamma_{32} = \frac{1}{A_2} \frac{\partial u_3}{\partial \xi_2} + A_2 \frac{\partial}{\partial \xi_3} \left(\frac{u_2}{A_2} \right) \\ \gamma_{31} = \frac{1}{A_1} \frac{\partial u_3}{\partial \xi_1} + A_1 \frac{\partial}{\partial \xi_3} \left(\frac{u_1}{A_1} \right) \\ \gamma_{12} = \frac{A_2}{A_1} \frac{\partial}{\partial \xi_1} \left(\frac{u_2}{A_2} \right) + \frac{A_1}{A_2} \frac{\partial}{\partial \xi_2} \left(\frac{u_1}{A_1} \right) \end{array} \right.$$

In the previous relations the variables A_i and a_i have been introduced. These variables are obtained in the following way:

$$A_i = a_i \left(1 + \frac{\xi_3}{R_i} \right) \quad a_i = \sqrt{\frac{\partial \boldsymbol{x}_0^T}{\partial \xi_i} \frac{\partial \boldsymbol{x}_0}{\partial \xi_i}}$$

$$\frac{1}{R_1} = -\frac{L}{a_1^2} \quad \frac{1}{R_2} = -\frac{N}{a_2^2}$$

$$L = \boldsymbol{n}_3^T \frac{\partial \boldsymbol{x}_0^2}{\partial \xi_1^2} \quad N = \boldsymbol{n}_3^T \frac{\partial \boldsymbol{x}_0^2}{\partial \xi_2^2}$$

The previous relationship can be rewritten in the following way by introducing the matrixes $\boldsymbol{I}_0, \boldsymbol{I}_1, \boldsymbol{I}_2, \boldsymbol{I}_3$:

$$\boldsymbol{\gamma}_\xi = \boldsymbol{I}_0 \boldsymbol{u}_\xi + \boldsymbol{I}_1 \frac{\partial \boldsymbol{u}_\xi}{\partial \xi_1} + \boldsymbol{I}_2 \frac{\partial \boldsymbol{u}_\xi}{\partial \xi_2} + \boldsymbol{I}_3 \frac{\partial \boldsymbol{u}_\xi}{\partial \xi_3}$$

$$\boldsymbol{I}_0 = \begin{bmatrix} 0 & \frac{1}{A_1 a_2} \frac{\partial a_1}{\partial \xi_2} & \frac{a_1}{A_1 R_1} \\ \frac{1}{A_2 a_1} \frac{\partial A_2}{\partial \xi_1} & 0 & \frac{a_2}{A_2 R_2} \\ 0 & 0 & 0 \\ 0 & -\frac{1}{A_2} \frac{\partial A_2}{\partial \xi_3} & 0 \\ -\frac{1}{A_1} \frac{\partial A_1}{\partial \xi_3} & 0 & 0 \\ \frac{1}{A_1 A_2} \frac{\partial A_2}{\partial \xi_1} & -\frac{1}{A_1 A_2} \frac{\partial A_1}{\partial \xi_2} & 0 \end{bmatrix} \quad \boldsymbol{I}_1 = \begin{bmatrix} \frac{1}{A_1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1}{A_1} \\ 0 & \frac{1}{A_1} & 0 \end{bmatrix}$$

$$\mathbf{I}_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{A_2} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1}{A_2} \\ 0 & 0 & 0 \\ \frac{1}{A_2} & 0 & 0 \end{bmatrix} \quad \mathbf{I}_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

By means of the kinematic hypothesis $\mathbf{u}_\xi = \mathbf{F}(\xi_3) \mathbf{U}(\xi_1, \xi_2)$ it is possible to write the strains in the following way:

$$\boldsymbol{\gamma}_\xi = \mathbf{I}_0 \mathbf{F} \mathbf{U} + \mathbf{I}_1 \mathbf{F} \frac{\partial \mathbf{U}}{\partial \xi_1} + \mathbf{I}_2 \mathbf{F} \frac{\partial \mathbf{U}}{\partial \xi_2} + \mathbf{I}_3 \frac{\partial \mathbf{F}}{\partial \xi_3} \mathbf{U}$$

The following matrixes can be defined:

$$\mathbf{F}_1 = \mathbf{I}_1 \mathbf{F}$$

$$\mathbf{F}_2 = \mathbf{I}_2 \mathbf{F}$$

$$\mathbf{F}_0 = \mathbf{I}_0 \mathbf{F} + \mathbf{I}_3 \frac{\partial \mathbf{F}}{\partial \xi_3}$$

Using the relationship between $\boldsymbol{\sigma}_\xi = \mathbf{c}_\xi \boldsymbol{\gamma}_\xi$, where \mathbf{c}_ξ is the local stiffness matrix in the reference system ξ , it is than possible to define the matrixes $\mathbf{Q}_{\lambda\mu}$, $\mathbf{R}_{\lambda 3}$ and \mathbf{S}_{33} as it follows:

$$\int_V \delta \boldsymbol{\gamma}_\xi^T \boldsymbol{\sigma}_\xi dV = \int \left(\delta \mathbf{U} \mathbf{F}_0 + \frac{\partial \delta \mathbf{U}}{\partial \xi_1} \mathbf{F}_1 + \frac{\partial \delta \mathbf{U}}{\partial \xi_2} \mathbf{F}_2 \right) \mathbf{c}_\xi \left(\mathbf{F}_0 \mathbf{U} + \mathbf{F}_1 \frac{\partial \mathbf{U}}{\partial \xi_1} + \mathbf{F}_2 \frac{\partial \mathbf{U}}{\partial \xi_2} \right) |J_{x\xi}| d\xi_1 d\xi_2 d\xi_3$$

$$\mathbf{Q}_{\lambda\mu} = \int_{\tau_b}^{\tau_t} \mathbf{F}_\lambda \mathbf{c}_\xi \mathbf{F}_\mu |J_{x\xi}| d\xi_3$$

$$\mathbf{R}_{\lambda 3} = \int_{\tau_b}^{\tau_t} \mathbf{F}_\lambda \mathbf{c}_\xi \mathbf{F}_3 |J_{x\xi}| d\xi_3$$

$$\mathbf{S}_{33} = \int_{\tau_b}^{\tau_t} \mathbf{F}_3 \mathbf{c}_\xi \mathbf{F}_3 |J_{x\xi}| d\xi_3$$

It is than obtained the same expression as before that is true in this case only in a orthogonal system ξ .

$$\int_{\Omega_\xi} \frac{\partial \delta \mathbf{U}^T}{\partial \xi_\lambda} \left(\mathbf{Q}_{\lambda\mu} \frac{\partial \mathbf{U}}{\partial \xi_\mu} + \mathbf{R}_{\lambda 3} \mathbf{U} \right) + \delta \mathbf{U}^T \left(\mathbf{R}_{\lambda 3}^T \frac{\partial \mathbf{U}}{\partial \xi_\lambda} + \mathbf{S}_{33} \mathbf{U} \right)$$

6.4 WORK OF THE VOLUME FORCES

The work of the volume forces can be written in the reference \mathbf{x} or in the reference ξ . In the latter case it is necessary to check if the curvilinear coordinates are orthogonal.

$$\begin{aligned}\int_V \delta \mathbf{u}_x^T \mathbf{b}_x dV &= \int \delta \mathbf{U}^T \mathbf{F}^T \mathbf{R}^T \mathbf{b}_x |\mathbf{J}_{x\xi}| d\xi_1 d\xi_2 d\xi_3 \\ &= \int_{\Omega_\xi} \delta \mathbf{U}^T \left(\int_{\xi_3^b}^{\xi_3^t} \mathbf{F}^T \mathbf{R}^T \mathbf{b}_x |\mathbf{J}_{x\xi}| d\xi_3 \right) d\xi_1 d\xi_2\end{aligned}$$

$$\int_V \delta \mathbf{u}_\xi^T \mathbf{b}_\xi dV = \int \delta \mathbf{U}^T \mathbf{F}^T \mathbf{b}_\xi |\mathbf{J}_{x\xi}| d\xi_1 d\xi_2 d\xi_3 = \int_{\Omega_\xi} \delta \mathbf{U}^T \left(\int_{\xi_3^b}^{\xi_3^t} \mathbf{F}^T \mathbf{b}_\xi |\mathbf{J}_{x\xi}| d\xi_3 \right) d\xi_1 d\xi_2$$

Here it is more clear why the reference ξ has to be orthogonal. The two previous expressions are equal only if:

$$\mathbf{b}_\xi = \mathbf{R}^T \mathbf{b}_x$$

But, since $\mathbf{b}_x = \mathbf{R} \mathbf{b}_\xi$, the previous is true only if $\mathbf{R}^{-1} = \mathbf{R}^T$ that means that both coordinate systems are orthogonal.

6.5 WORK OF THE TRACTIONS

The work of the tractions can be written as follows in the reference system \mathbf{x} or ξ :

$$\begin{aligned}\int_{\partial V} \delta \mathbf{u}_x^T \mathbf{t}_x d\partial V &= \int_{\partial V} \delta \mathbf{U}^T \mathbf{F}^T \mathbf{R}^T \mathbf{t}_x d\partial V \\ \int_{\partial V} \delta \mathbf{u}_\xi^T \mathbf{t}_\xi d\partial V &= \int_{\partial V} \delta \mathbf{U}^T \mathbf{F}^T \mathbf{t}_\xi d\partial V\end{aligned}$$

The 6 external surfaces of the domain are obtained by substituting in the expression for the transformation from $\xi \rightarrow \mathbf{x}$ the value of the component of ξ that characterizes that surface:

$$\begin{aligned}\mathbf{S}_{\xi_1^b} &= \mathbf{x}_{(\xi_1=\xi_1^b)} = \mathbf{x}_0(\xi_1=\xi_1^b) + \xi_3 \mathbf{n}_3(\xi_1=\xi_1^b) \\ \mathbf{S}_{\xi_1^t} &= \mathbf{x}_{(\xi_1=\xi_1^t)} = \mathbf{x}_0(\xi_1=\xi_1^t) + \xi_3 \mathbf{n}_3(\xi_1=\xi_1^t) \\ \mathbf{S}_{\xi_2^b} &= \mathbf{x}_{(\xi_2=\xi_2^b)} = \mathbf{x}_0(\xi_2=\xi_2^b) + \xi_3 \mathbf{n}_3(\xi_2=\xi_2^b) \\ \mathbf{S}_{\xi_2^t} &= \mathbf{x}_{(\xi_2=\xi_2^t)} = \mathbf{x}_0(\xi_2=\xi_2^t) + \xi_3 \mathbf{n}_3(\xi_2=\xi_2^t) \\ \mathbf{S}_{\xi_3^b} &= \mathbf{x}_{(\xi_3=\xi_3^b)} = \mathbf{x}_0 + \xi_3^b \mathbf{n}_3 \\ \mathbf{S}_{\xi_3^t} &= \mathbf{x}_{(\xi_3=\xi_3^t)} = \mathbf{x}_0 + \xi_3^t \mathbf{n}_3\end{aligned}$$

The infinitesimal elements of the surfaces can be computed as:

$$\begin{aligned}d\mathbf{S}_{\xi_1^b} &= \left| \frac{\partial \mathbf{S}_{\xi_1^b}}{\partial \xi_2} \times \frac{\partial \mathbf{S}_{\xi_1^b}}{\partial \xi_3} \right| d\xi_2 d\xi_3 \\ d\mathbf{S}_{\xi_1^t} &= \left| \frac{\partial \mathbf{S}_{\xi_1^t}}{\partial \xi_2} \times \frac{\partial \mathbf{S}_{\xi_1^t}}{\partial \xi_3} \right| d\xi_2 d\xi_3 \\ d\mathbf{S}_{\xi_2^b} &= \left| \frac{\partial \mathbf{S}_{\xi_2^b}}{\partial \xi_3} \times \frac{\partial \mathbf{S}_{\xi_2^b}}{\partial \xi_1} \right| d\xi_3 d\xi_1 \\ d\mathbf{S}_{\xi_2^t} &= \left| \frac{\partial \mathbf{S}_{\xi_2^t}}{\partial \xi_3} \times \frac{\partial \mathbf{S}_{\xi_2^t}}{\partial \xi_1} \right| d\xi_3 d\xi_1 \\ d\mathbf{S}_{\xi_3^b} &= \left| \frac{\partial \mathbf{S}_{\xi_3^b}}{\partial \xi_1} \times \frac{\partial \mathbf{S}_{\xi_3^b}}{\partial \xi_2} \right| d\xi_1 d\xi_2\end{aligned}$$

$$d\mathbf{S}_{\xi_3^t} = \left| \frac{\partial \mathbf{S}_{\xi_3^t}}{\partial \xi_1} \times \frac{\partial \mathbf{S}_{\xi_3^t}}{\partial \xi_2} \right| d\xi_1 d\xi_2$$

The work of the external forces in the reference \mathbf{x} is:

$$\begin{aligned} & \int_{\partial V} \delta \mathbf{u}_x^T \mathbf{t}_x d\partial V = \int_{\partial V} \delta \mathbf{U}^T \mathbf{F}^T \mathbf{R}^T \mathbf{t}_x d\partial V = \\ & = \int_{\Omega_\xi} \delta \mathbf{U}^T \left(\left(\mathbf{F}^T \mathbf{R}^T \mathbf{t}_x \left| \frac{\partial \mathbf{S}_{\xi_3^b}}{\partial \xi_1} \times \frac{\partial \mathbf{S}_{\xi_3^b}}{\partial \xi_2} \right| \right)_{(\xi_3=\xi_3^b)} \right. \\ & \quad \left. + \left(\mathbf{F}^T \mathbf{R}^T \mathbf{t}_x \left| \frac{\partial \mathbf{S}_{\xi_3^b}}{\partial \xi_1} \times \frac{\partial \mathbf{S}_{\xi_3^b}}{\partial \xi_2} \right| \right)_{(\xi_3=\xi_3^t)} \right) d\xi_1 d\xi_2 + \\ & \int_{\partial \Omega_\xi} \delta \mathbf{U}^T \left(\int_{\xi_3^b}^{\xi_3^t} \left(\mathbf{F}^T \mathbf{R}^T \mathbf{t}_x \left| \frac{\partial \mathbf{S}_{\xi_n^b}}{\partial \xi_t} \times \frac{\partial \mathbf{S}_{\xi_n^b}}{\partial \xi_3} \right| \right) d\xi_3 \right) d\xi_t \end{aligned}$$

And in the reference ξ :

$$\begin{aligned} & \int_{\partial V} \delta \mathbf{u}_\xi^T \mathbf{t}_\xi d\partial V = \int_{\partial V} \delta \mathbf{U}^T \mathbf{F}^T \mathbf{t}_\xi d\partial V = \\ & = \int_{\Omega_\xi} \delta \mathbf{U}^T \left(\left(\mathbf{F}^T \mathbf{t}_\xi \left| \frac{\partial \mathbf{S}_{\xi_3^b}}{\partial \xi_1} \times \frac{\partial \mathbf{S}_{\xi_3^b}}{\partial \xi_2} \right| \right)_{(\xi_3=\xi_3^b)} + \left(\mathbf{F}^T \mathbf{t}_\xi \left| \frac{\partial \mathbf{S}_{\xi_3^b}}{\partial \xi_1} \times \frac{\partial \mathbf{S}_{\xi_3^b}}{\partial \xi_2} \right| \right)_{(\xi_3=\xi_3^t)} \right) d\xi_1 d\xi_2 + \\ & \int_{\partial \Omega_\xi} \delta \mathbf{U}^T \left(\int_{\xi_3^b}^{\xi_3^t} \left(\mathbf{F}^T \mathbf{t}_\xi \left| \frac{\partial \mathbf{S}_{\xi_n^b}}{\partial \xi_t} \times \frac{\partial \mathbf{S}_{\xi_n^b}}{\partial \xi_3} \right| \right) d\xi_3 \right) d\xi_t \end{aligned}$$

It is possible to combine the work of the external forces by using two functions \mathbf{B} and \mathbf{T} , defined as:

$$\begin{aligned} \mathbf{B} & = \left(\mathbf{F}^T \mathbf{R}^T \mathbf{t}_x \left| \frac{\partial \mathbf{S}_{\xi_3^b}}{\partial \xi_1} \times \frac{\partial \mathbf{S}_{\xi_3^b}}{\partial \xi_2} \right| \right)_{(\xi_3=\xi_3^b)} + \left(\mathbf{F}^T \mathbf{R}^T \mathbf{t}_x \left| \frac{\partial \mathbf{S}_{\xi_3^t}}{\partial \xi_1} \times \frac{\partial \mathbf{S}_{\xi_3^t}}{\partial \xi_2} \right| \right)_{(\xi_3=\xi_3^t)} \\ & \quad + \int_{\xi_3^b}^{\xi_3^t} \mathbf{F}^T \mathbf{R}^T \mathbf{b}_x |\mathbf{J}_{x\xi}| d\xi_3 \\ \mathbf{T} & = \int_{\xi_3^b}^{\xi_3^t} \left(\mathbf{F}^T \mathbf{R}^T \mathbf{t}_x \left| \frac{\partial \mathbf{S}_{\xi_n^b}}{\partial \xi_t} \times \frac{\partial \mathbf{S}_{\xi_n^b}}{\partial \xi_3} \right| \right) d\xi_3 \end{aligned}$$

Or in the curvilinear system ξ :

$$\begin{aligned} \mathbf{B} & = \left(\mathbf{F}^T \mathbf{t}_\xi \left| \frac{\partial \mathbf{S}_{\xi_3^b}}{\partial \xi_1} \times \frac{\partial \mathbf{S}_{\xi_3^b}}{\partial \xi_2} \right| \right)_{(\xi_3=\xi_3^b)} + \left(\mathbf{F}^T \mathbf{t}_\xi \left| \frac{\partial \mathbf{S}_{\xi_3^t}}{\partial \xi_1} \times \frac{\partial \mathbf{S}_{\xi_3^t}}{\partial \xi_2} \right| \right)_{(\xi_3=\xi_3^t)} \\ & \quad + \int_{\xi_3^b}^{\xi_3^t} \mathbf{F}^T \mathbf{b}_\xi |\mathbf{J}_{x\xi}| d\xi_3 \\ \mathbf{T} & = \int_{\xi_3^b}^{\xi_3^t} \left(\mathbf{F}^T \mathbf{t}_\xi \left| \frac{\partial \mathbf{S}_{\xi_n^b}}{\partial \xi_t} \times \frac{\partial \mathbf{S}_{\xi_n^b}}{\partial \xi_3} \right| \right) d\xi_3 \end{aligned}$$

6.6 PVD FOR SHELLS

In the previous sections it has been shown how it is possible to modify the definitions of the matrixes $\mathbf{Q}_{\lambda\mu}$, $\mathbf{R}_{\lambda 3}$, \mathbf{S}_{33} and of the external applied forces \mathbf{B} and \mathbf{T} in order to obtain the same weak formulation as the case of plates. In this case the derivatives of the unknown functions \mathbf{U} are respect the curvilinear coordinates ξ_1 and ξ_2 but the solution of the problem is the same as the one described in the previous chapter:

$$\int_{\Omega_\xi} \frac{\partial \delta \mathbf{U}^T}{\partial \xi_\lambda} \left(\mathbf{Q}_{\lambda\mu} \frac{\partial \mathbf{U}}{\partial \xi_\mu} + \mathbf{R}_{\lambda 3} \mathbf{U} \right) + \delta \mathbf{U}^T \left(\mathbf{R}_{\lambda 3}^T \frac{\partial \mathbf{U}}{\partial \xi_\lambda} + \mathbf{S}_{33} \mathbf{U} \right) = \int_{\Omega_\xi} \delta \mathbf{U}^T \mathbf{B} + \int_{\partial \Omega_\xi} \delta \mathbf{U}^T \mathbf{T}$$

6.7 RELATION BETWEEN \mathbf{c}_x AND \mathbf{c}_ξ

In this work only orthotropic materials have been used. For such material the stiffness matrix can be written with respect to the elastic constants:

$$\mathbf{c} = \begin{bmatrix} \frac{1}{E_1} & -\frac{\nu_{21}}{E_2} & -\frac{\nu_{31}}{E_3} & 0 & 0 & 0 \\ -\frac{\nu_{12}}{E_1} & \frac{1}{E_2} & -\frac{\nu_{32}}{E_3} & 0 & 0 & 0 \\ -\frac{\nu_{13}}{E_1} & -\frac{\nu_{23}}{E_2} & \frac{1}{E_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G_{12}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G_{31}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G_{12}} \end{bmatrix}^{-1}$$

The principal reference system of the material in which the previous matrix is defined is orthogonal and it is assumed here that it can be obtained by a rotation along the ξ_3 axis of the local basis defined as follow:

$$\begin{aligned} \mathbf{m}_1 &= \mathbf{n}_1 \\ \mathbf{m}_3 &= \mathbf{n}_3 \\ \mathbf{m}_2 &= \mathbf{n}_3 \times \mathbf{n}_1 \end{aligned}$$

Using this local basis instead of $\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3$ ensures that the local basis will always be orthogonal. The rotation matrix from the basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ and the local basis of the principal direction of the material is:

$$\mathbf{R} = \mathbf{R}_m \mathbf{R}_p$$

$$\mathbf{R}_m = [\mathbf{m}_1 \quad \mathbf{m}_2 \quad \mathbf{m}_3] \quad \mathbf{R}_p = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

In such a way it is possible to rotate the stiffness matrix using the rotation rule introduced in chapter 3.

7 CODE IMPLEMENTATION

In this chapter the implementation of the theories described so far will take place. Firstly, the input data is described and the conventions chosen for the various input files are detailed. Secondly, it is provided some information about the routines of the so called pre-computation group. Thirdly, it will be detailed the assembling and the solution of the linear system that derives from the differential problem. Finally, the description of the plotting functions and some additional details are provided.

7.1 INPUT FILES

The information related to the simulation is introduced via six input files. Each one contains the quantitative data related to the different aspects of the analysis. In the next paragraphs the various input files are described individually.

Geometry

This file contains the parametric definition of the mean surface. The independent variables used are called ξ_1 , ξ_2 , ξ_3 . It is possible to define the mean surface using all the functions accepted by the Matlab Symbolic Math Toolbox[®]. Here there are two examples of possible geometries definitions:

Cylindrical surface:

- 1) X_10: xi_2
- 2) X_20:cos(xi_1)
- 3) X_30: sin(xi_1)

Flat surface:

- 1) X_10: xi_1
- 2) X_20:xi_2
- 3) X_30:0

In this files it is also specified which are the limits of the auxiliary variable ξ_1 and ξ_2 .

Laminate

This file contains the information related to the materials used. They can be defined multiple materials and how they are combined into the finite laminate. It is assumed that the laminate is the same in every point of the shell. The angle of lamination is referred to the first curvilinear coordinate for a rotation positive accordingly to the curvilinear axis orthogonal to the mean surface.

The first part of the file is about the material description. It is provided the name of the material the progressive ID number, the type and the elastic constants for each of them.

The second part relates the combination of the previously defined materials into the laminate. It is described the sequence of materials, their thickness and their orientation.

Displacements

Here it is selected the expansion of the solution through the thickness and along the surface. It can be chosen the type of polynomials for the expansion and their degree. To what concerns the thickness expansion it can be selected a different degree for the different components of the displacement. It is also important to note that the expansion relates the displacement in the local curvilinear coordinate.

Boundary

This file contains the boundary conditions in terms of applied displacements, applied forces and mixed boundary information. All of this input can be in the Cartesian reference system or in the curvilinear one but they have to be function of the curvilinear coordinates. In the mixed boundary it can be selected the sequence of assigned generalized displacements and assigned generalized tractions as a sequence of 1 and 0 respectively. So for example the next statement relate the surface correspondent to $\xi_1 = \xi_{1b}$, in this surface the first three and the last three components of the generalized displacement belong to $\partial\Omega_D$ while the 4th, 5th and 6th belong to $\partial\Omega_N$

Surface Xi1b :1,1,1,0,0,0,1,1,1

Mesh

In this file the indications for the discretization of the domain are given. In particular it is specified the number of Mesh element in both direction of the domain. The mesh refers to the auxiliary variables ξ_1 and ξ_2 . Rectangular elements are used.

Solution

Here the settings for the solution of the problem are given. First, there are all the options for the computation of the matrixes $\mathbf{Q}_{\lambda\mu}$, $\mathbf{R}_{\lambda\beta}$ and \mathbf{S}_{33} : the reference system, the integration along each layer, the shear factor and the plane state option. It can be selected also whether or not computing the matrixes in just one point and using that value everywhere else. This option can speed up the code when it can be established that the matrixes are constants. In the second part the number of points for the integration is the thickness, the element and the edge is chosen. Finally, the parameters adopted for the method are introduced.

7.2 PRE-COMPUTATIONS

To this category belong all those functions that execute the preliminary computations necessary for the assembling of the final linear system. These computations include the generation of the mesh, the selection of the Gaussian points, the building of the function that contain the kinematic hypothesis and the functions related to the geometric transformations.

PrecomputationMesh

This function create the discretization of the domain by defining the elements and their connection to edges and nodes.

PrecomputationGauss

This function computes the Gaussian points for the different types of integration, i.e. integration through the layers, integration through the thickness, integration along one edge and integration in the surface.

PrecomputationGeometric

This function build symbolically the variables necessary for the transformation from one reference system to the other.

PrecomputationDisplacements

This function create symbolically the through-the-thickness and in the surface expansion of the solution. It computes also the derivatives of the previous quantities in the curvilinear variables.

PrecomputationOthers

In this routine all the pre-calculations that not belong to the previous category are included.

7.3 ASSEMBLING AND SOLUTION

The assembling of the linear solving system necessitate 8 routines in total, divided into 3 groups. The first group of routines includes 8 function related to the evaluation of the generalized stiffness matrixes, the generalized applied forces and the assigned generalized displacements:

- QRS
- GeneralizedBoundaryForce
- GeneralizedVolumeForce
- AssignedDisplacements

The second group of routines are dedicated to the integrations involved in the DG method and related to the internal edges, the external edges and the domain of the elements:

- IntegralElement
- IntegralInternalEdge
- IntegralExternalEdge

This three integration routines make use of the previous group of functions and of the Gaussian points. The output of this routines are the contributions to the stiffness matrix and to the right hand side of the linear system. While this contributions are being computed they are assembled together by the assembling routine the create the final linear system. The linear system is then solved by the Matlab internal solver.

7.4 PLOTTING FUNCTIONS

In order to visualize the results some functions dedicated to plotting the results are used. This function all start from the solution of the linear system to be able to compute the domain variables in the various points of the shell. These values are then used to create graphs and plots. The two main plotting functions are called Shape and Plot. The first one is used to display the original and the deformed configurations of the volume and of the mean surface, the second is used to plot the selected variable in the selected sequence of points.

7.5 ADDITIONAL DETAILS

All the information computed by pre-computation group of functions is saved in an object of the class ModelData together with the data provided by the reading of the input files. An object of this class has the following properties:

- Geometry
- Laminate
- Mesh
- Displacements
- Boundary
- Solution
- Results

8 RESULTS

In this chapter they will be shown the solution obtained with the algorithm introduces so far to some specific problems that can be solved analytically. Four different cases are considered. In the first three it is considered a flat plate while in the last one a cylindrical shell. The materials used for each test are described in the relative section.

The first test is the evaluation of the error computed as follow:

$$e(\mathbf{U}_h) = \frac{\|\mathbf{U}_h - \mathbf{U}_{exact}\|_{\infty}}{\|\mathbf{U}_{exact}\|_{\infty}}$$

The error is computed for different polynomial expansion in the surface and for different mesh sizes. In such a way is it possible to observe the convergence of the error.

The second test (only for the plates) consists in a comparison of the along the thickness behavior for three different theories ($FSDT, ED_{222}, ED_{444}$) to what regard the dimensionless displacements \bar{u}_1 in the point $(0, L/2)$ and \bar{u}_3 in the point $(L/2, L/2)$. The dimensionless displacements are computed as:

$$\bar{u}_1 = u_1 \left(\frac{\tau^2 E_r}{L^3 q} \right)$$

$$\bar{u}_3 = u_3 \left(\frac{t^3 E_r}{L^4 q} \right)$$

In all tests the plate has a thickness of $0.1 L$, the boundary condition are those relative to the simply supported plate. The geometry is defined as:

$$\begin{aligned} x_{10} &= \xi_1 \\ x_{20} &= \xi_2 \\ x_{30} &= 0 \\ \xi_1 &\in [0, a] \\ \xi_2 &\in [0, b] \\ a &= b = L = 1 \text{ m} \end{aligned}$$

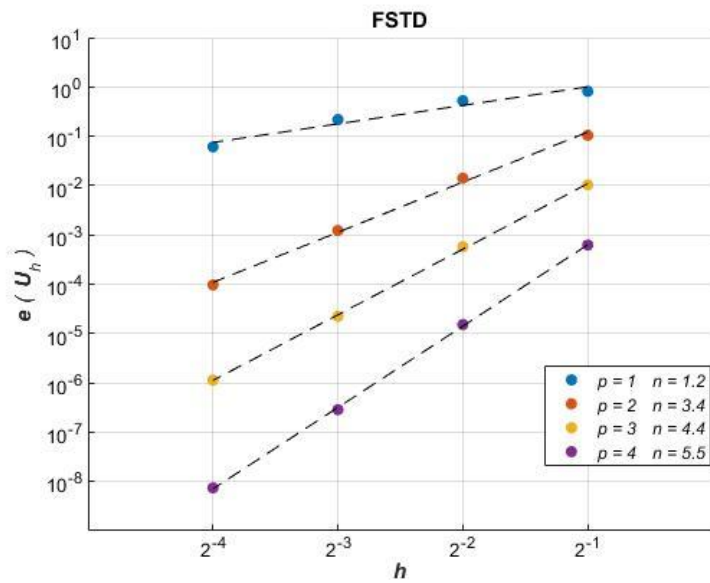
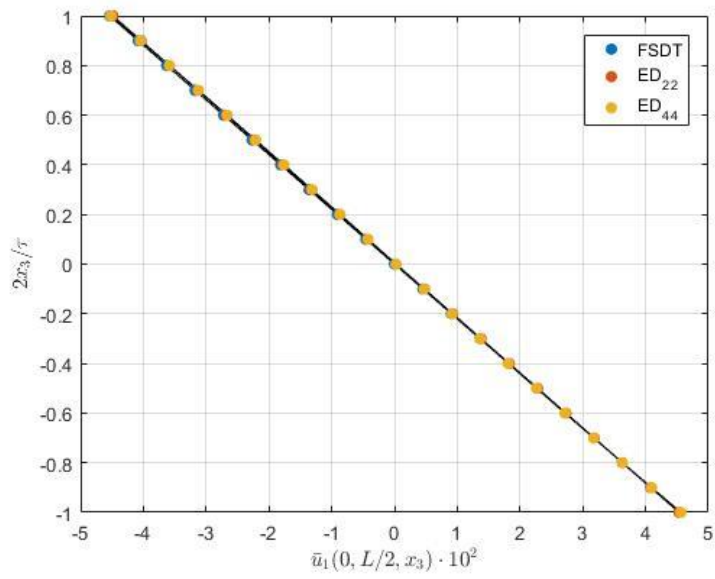
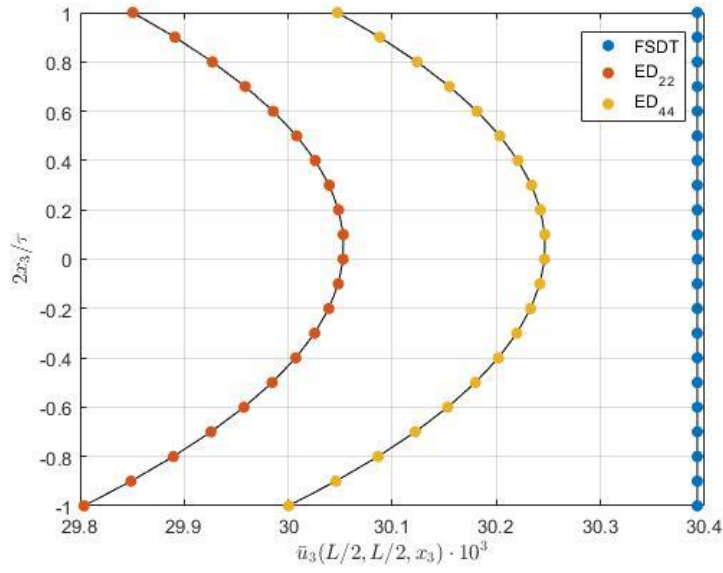
The load is perpendicular to the surface and varies accordingly to:

$$\begin{aligned} t_3 &= q \sin(\pi \xi_1) \sin(\pi \xi_2) \\ q &= 1 \text{ N/m} \end{aligned}$$

8.1 ISOTROPIC PLATE

The isotropic material has the following elastic constants:

- $E/E_r = 1$
- $\nu = 0.25$

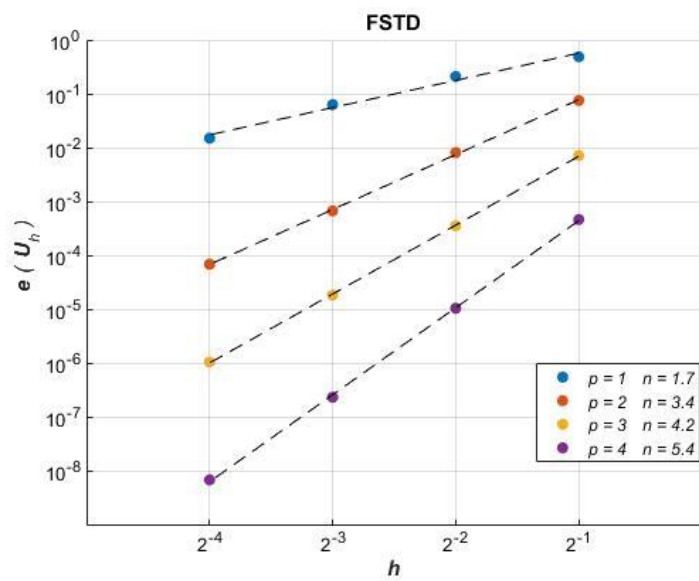
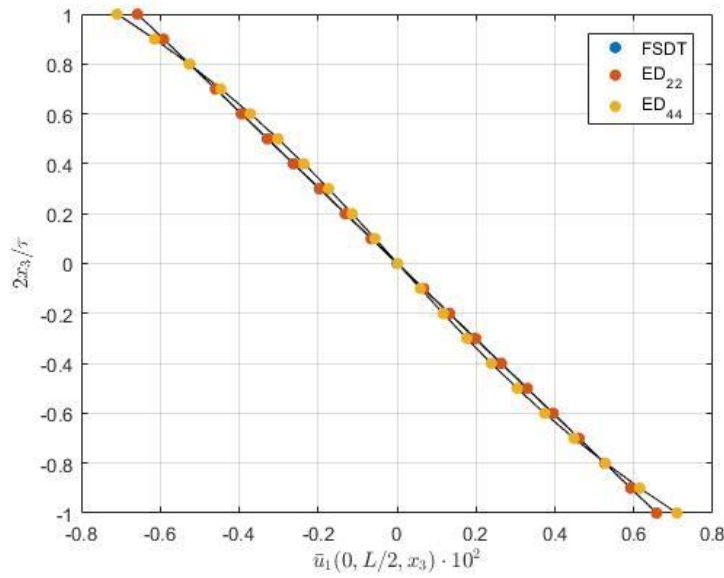
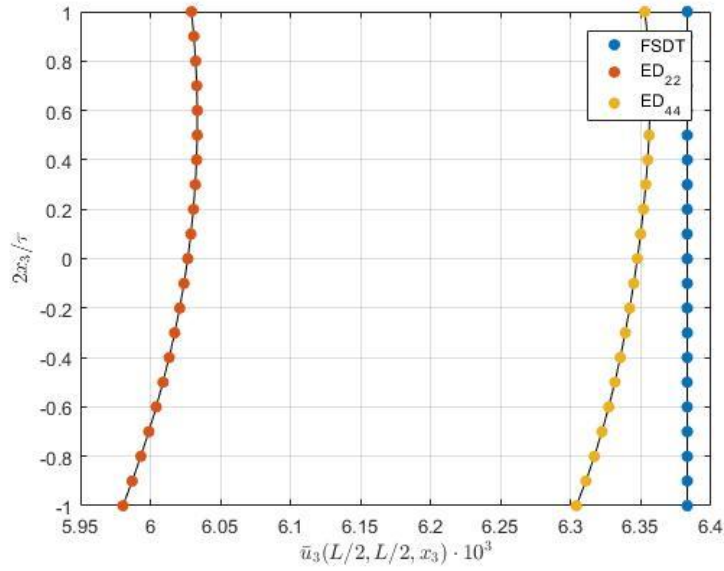


8.2 ORTHOTROPIC PLATE

The orthotropic material has the following elastic constants:

- $E_3/E_r = E_2/E_r = 1$

- $E_1/E_r = 25$
- $\nu_{12} = \nu_{13} = \nu_{23} = 0.25$

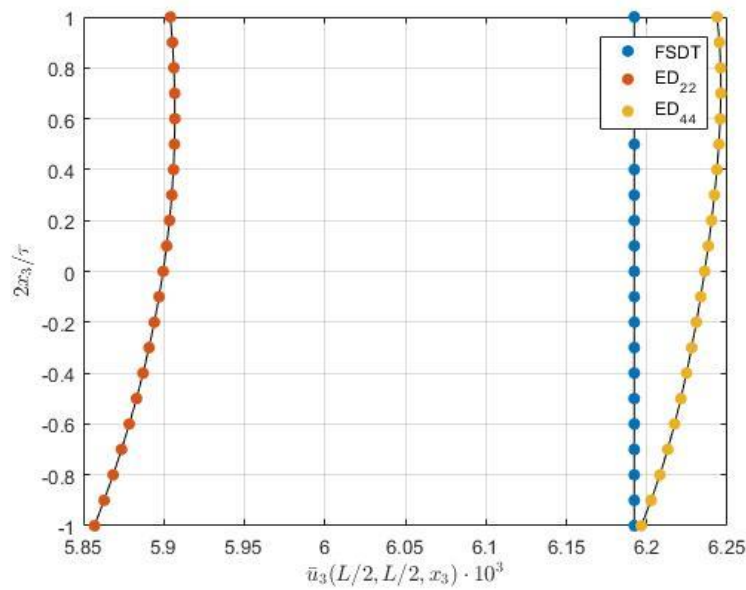
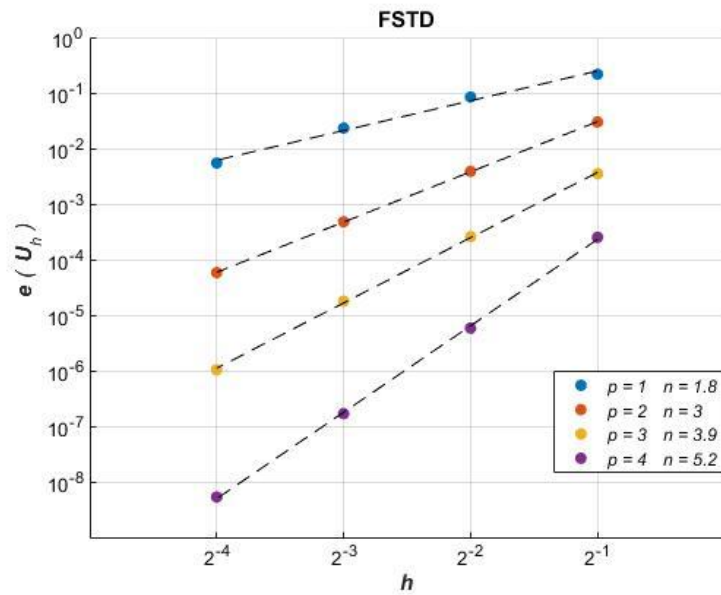


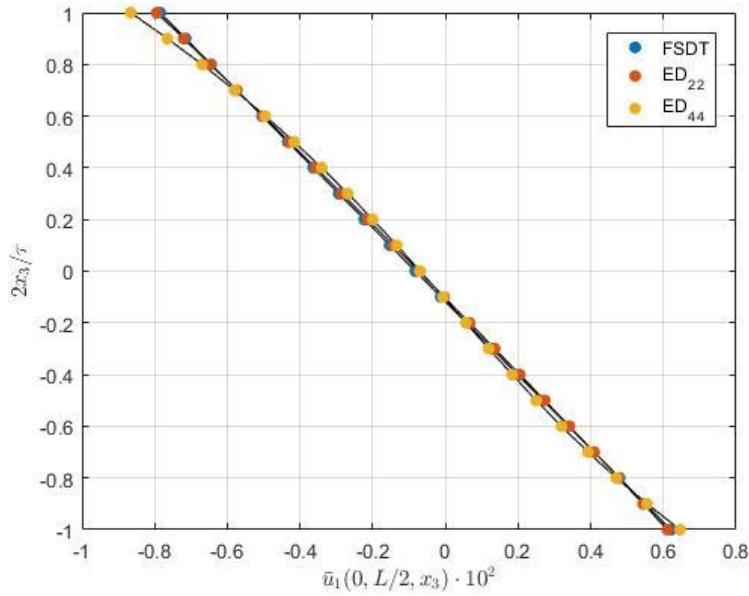
8.3 LAMINATED PLATE

The material used here is the same as the one in the previous section and it is arranged in a laminate whose sequence of layers is:

$$[0,90]_4$$

Each layer has a thickness of 0.0125.



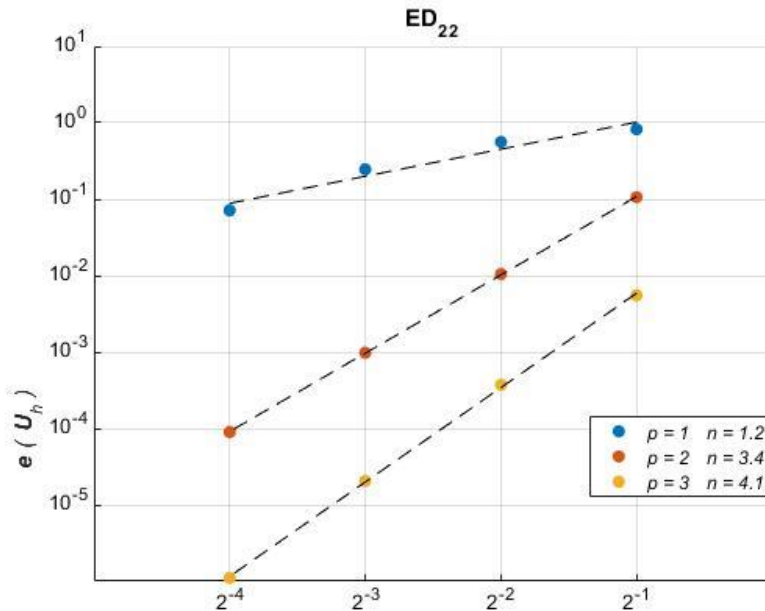


8.4 CYLINDRICAL LAMINATED SHELL

In the test for the cylindrical shell the geometry is defined as:

$$\begin{aligned}
 x_{10} &= R_0 \sin(\xi_1) \\
 x_{20} &= \xi_2 \\
 x_{30} &= R_0 \cos(\xi_1) \\
 \xi_1 &\in [0, \pi] \\
 \xi_2 &\in [0, L] \\
 R_0 &= 1m \\
 L &= 4m
 \end{aligned}$$

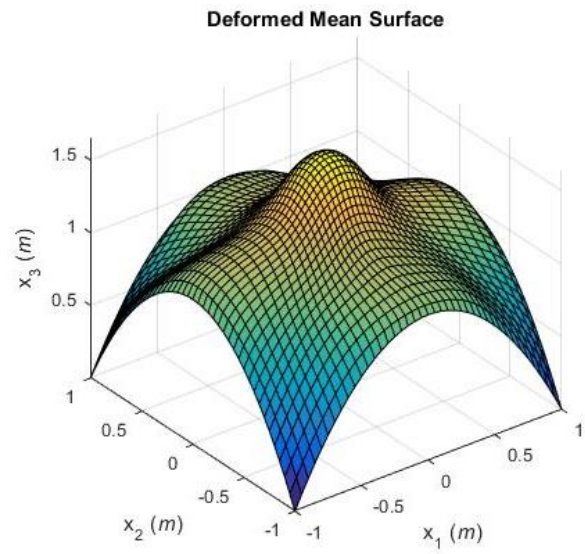
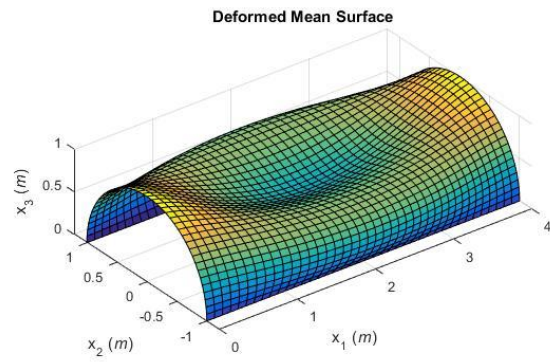
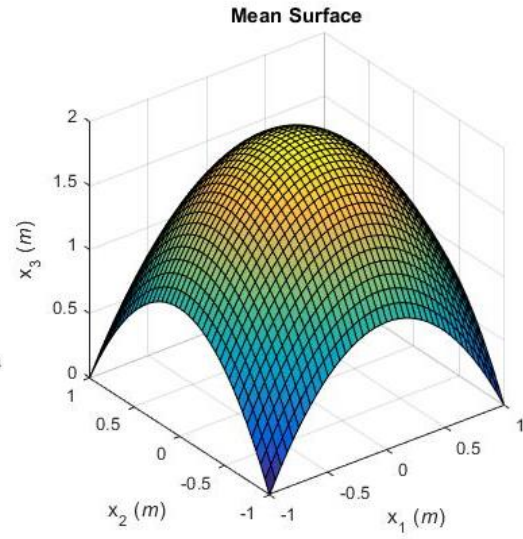
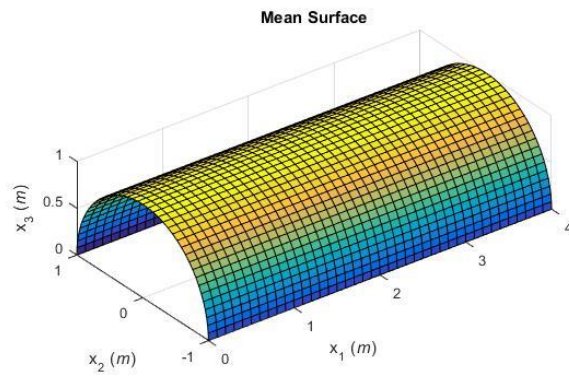
The material used here is the same laminate used in the previous test.



8.5 OTHER TESTS

The following images show two different geometries with fixed boundary under some loads. The material is the same as the previous case. The loads applied are defined in the same way for both geometries:

$$t_3 = -1 \sin(\pi\xi_1/a) \sin(\pi\xi_2/b)$$



9 REMARKS AND FUTURE DEVELOPMENTS

In this work it has been extended the DG method for linear elasticity of laminated plates to the case of shells. The plate itself can be considered a shell with zero curvature. The results for three cases for which the solution exists have been checked and the convergence properties of the algorithm have been proven. The results already in the literature have been here reproduced. It has also been studied a cylindrical shell and the results have been compared to the exact solutions.

Possible further developments of this project are introduced in the next paragraphs:

Comparison with FEM

When an analytical solution of the problem does not exist, in order to prove the algorithm it is necessary to compare the solution with that obtained with another well-proven method. In this framework it would be very interesting to compare the results obtained with the method presented here with those obtained with classical 3D FEM.

Non-linear analysis

The usage of linear elasticity equations is appropriate only under the assumption of small displacements. When the loads are high enough to cause the displacements field to not respect the previous hypothesis the non-linear theory should be used. The use of non-linear elasticity equations requires a more complex definition of the fluxes and a more complex non-linear equation system will be obtained. The non-linear system will require an iterative approach with some deep modifies to the code..

Mathematical demonstration

The comparison of the results obtained with this method with the exact solutions available for some specific cases formally does not certify with mathematical rigor that the method works for every possible case. A rigorous demonstration would be necessary for more properly asses the efficiency of the method introduced.

10 BIBLIOGRAPHY

- 1) V. Gulizzi, I. Benedetti, A. Milazzo (2019): **An implicit mesh discontinuous Galerkin formulation for higher-order plate theories**, Mechanics of Advanced Materials and Structures, DOI: 10.1080/15376494.2018.1516258
- 2) J.N. Reddy (2007): **Theory and analysis of elastic plates and shells**. 2nd ed., Philadelphia: Taylor & Francis, ISBN: 9780429127601
- 3) D.N. Arnold, F. Brezzi, B. Cockburn, L.D. Marini (2002): **Unified analysis of discontinuous Galerkin methods for elliptic problems**, SIAM J. Numer. Anal.. 39. 10.1137/S0036142901384162
- 4) A.T. Eyck, A. Lew (2006): **Discontinuous Galerkin methods for non-linear elasticity**, International Journal for Numerical Methods in Engineering. 67. 1204 - 1243. 10.1002/nme.1667
- 5) T. Belytschko, W.K Liu, B. Moran (2000), **Nonlinear Finite Elements for Continua and Structures**, John Wiley & Sons
- 6) Reddy, J.N. (1997), **Mechanics of Laminated Composite Plates, Theory and Analysis**, CRC Press