Operator algebras, KMS Thermal states, Hawking radiation, Unruh effect.

Daniele Guido

Università di Roma Tor Vergata

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Outline

KMS boundary condition

In Quantum Field Theory, observable quantities are described by (self-adjoint) operators acting on a Hilbert space, and form a noncommutative algebra.

Time evolution is described by a 1-parameter group of automorphisms:

$$A \mapsto \alpha_t(A) := e^{itH} A e^{-itH}.$$

A thermal equilibrium state Ω at inverse temperature β is an invariant state ($e^{itH}\Omega = \Omega$) which satisfies the KMS-boundary condition. For all pairs of observables A, B, the function

$$F_{A,B}(t) = (\Omega, A\alpha_t(B)\Omega)$$

extends to an analytic function on the strip $\{0 < |\Im z| < \beta\}$, and

$$F_{A,B}(t + i\beta) = (\Omega, \alpha_t(B)A\Omega).$$

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Tomita-Takesaki Theory

Let $\mathcal{A}$ be an algebra of operators (closed in the weak topology) on the Hilbert space $\mathcal{H}$, Ω a unit vector such that
(1) $A\Omega = 0 \Rightarrow A = 0$ for $A \in \mathcal{A}$ (separating vector),
(2) the set $\{A\Omega : A \in \mathcal{A}\}$ is dense in $\mathcal{H}$ (cyclic vector).
For any $A \in \mathcal{A}$, set $S : A\Omega \mapsto A^*\Omega$ (anti-linear operator).
Decompose S as $S = J\Delta^{1/2}$, where $\Delta = S^*S$ (positive op.), $J = S\Delta^{-1/2}$ (anti-unitary op.)

Theorem (Tomita-Takesaki)

(1) $\{JAJ : A \in \mathcal{A}\}$ coincides with the commutant of $\mathcal{A}$,
(2) for all $t \in \mathbb{R}$, $\sigma_t(A) := \Delta^{-it}A\Delta^{it} \in \mathcal{A}$ (σ_t is a 1-parameter group of automorphisms)
(3) Ω is a KMS-state for σ_t on $\mathcal{A}$, with temperature 1.

Comments

The KMS-boundary condition

$$F_{A,B}(t) = (\Omega, A\alpha_t(B)\Omega), \qquad F_{A,B}(t+i\beta) = (\Omega, \alpha_t(B)A\Omega)$$

is compatible with the trivial evolution ($\alpha_t(A) = A$) when
(1) the algebra $\mathcal{A}$ is commutative,
(2) the state is a trace state.
In these to cases, the T.T. evolution σ_t is trivial. Otherwise,

Remark

The algebra $\mathcal{A}$ and the state Ω generate an intrinsic time-evolution, for which Ω is a temperature state.

M. Tomita, *Quasi-standard von Neumann algebras*, 1967.

M. Takesaki, *Tomita's theory of modular Hilbert algebras and its applications*, LNM 128, Springer 1970.

A proof of the KMS boundary cond. for T.T. evolution

The proof of the Tomita-Takesaki Theorem is quite hard. However, the boundary condition can be easily checked:

$$
\begin{aligned}
(\Omega, A\,\sigma_{t+i}(B)\Omega) &= (\Omega, A\,\Delta^{-i(t+i)}B\Omega)\\
&= (\Omega, A\,\Delta\,\sigma_t(B)\Omega)\\
&= (\Delta^{1/2}A^*\Omega, \Delta^{1/2}\sigma_t(B)\Omega)\\
&= (J\Delta^{1/2}\sigma_t(B)\Omega, J\Delta^{1/2}A^*\Omega)\\
&= (S\sigma_t(B)\Omega, SA^*\Omega)\\
&= (\sigma_t(B^*)\Omega, A\Omega)\\
&= (\Omega, \sigma_t(B)A\Omega)
\end{aligned}
$$

Outline

Time-evolution may be considered as a space-time symmetry, and in some cases we may expect such symmetry to be unitarily implemented on the Hilbert space, thus giving a 1-parameter group of automorphisms. It is natural to ask when such a 1-parameter group can be reconstructed by the T.T. Theorem.

If we consider fields on the Minkowski space-time *a la* Wightman, the time evolution is implemented by the unitaries e^{itH}, the energy operator H being positive. This implies that $t \to e^{itH}$ extends to the half-plane $\Im z > 0$, hence the vacuum state satisfies the KMS condition for 0 temperature. However, β being infinite, this cannot cannot be related with the T.T. Theorem.

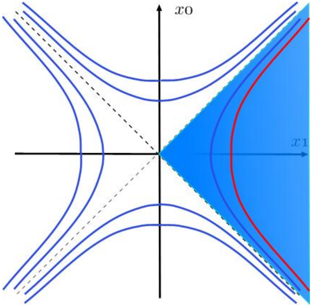
QFT and local algebras

Given a (scalar) Wightman field $\phi(x)$ on the Minkowski space-time, we may define (under some technical conditions) the *local observable algebras*

$$\mathcal{A}(\mathcal{O}) = \text{alg}\left\{ \exp\left(i \int \phi(x) f(x) dx \right) \text{supp}\, f \subset \mathcal{O} \right\}$$

where $\mathcal{O}$ is a space-time region. A classical result says that the vacuum Ω is a cyclic and separating vector for local algebras, therefore T.T. construction may be applied.

Wedge regions and boosts



We consider the wedge-shaped region (in blu) $W = \{|x_0| < x_1\}$ and the Lorentz boosts

$$\Lambda_W(t) = \begin{pmatrix} \cosh t & -\sinh t & 0 & 0 \\ -\sinh t & \cosh t & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

which leave W invariant.

Consider the net of local observable algebras $\mathcal{O} \to \mathcal{A}(\mathcal{O})$ generated by a Wightman field, and the unitary action of the Poincaré group $g \to U(g)$. Let $S = J\Delta$ the Tomita operator for the algebra $\mathcal{A}(W)$ and the vacuum vector Ω. Then,

Theorem (Bisognano-Wichmann, 1975)

The vacuum Ω is a thermal equilibrium state at inverse temperature 2π for the evolution given by the boosts Λ_W. More precisely $\Delta^{it} = U(\Lambda_W(2\pi t))$.

Outline

1. A historical coincidence: Thermal states and Operator Algebras
2. The Bisognano-Wichmann Theorem
3. Thermalization
4. Conformal dethermalization on de Sitter space

Unruh effect

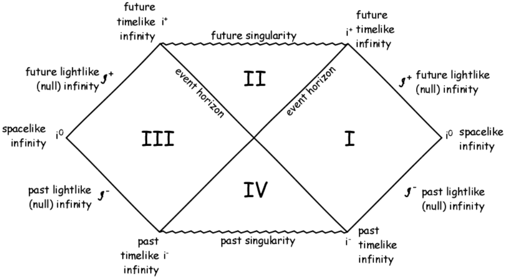
The Bisognano-Wichman theorem means that a uniformly accelerated observer, whose time-evolution is described by the boosts, feels the vacuum as a temperature state. This may be seen as a proof of the Unruh effect in this framework, namely of the fact that accelerated observers measure a radiation.

Conformal thermalization for future cones.

In the Minkowski space-time, wedges and future cones are conformally equivalent, with the boosts corresponding to dilations. Therefore, in a conformally invariant theory, the T.T. flow associated with the algebra $\mathcal{A}(V_+)$ of the future cone and the vacuum is implemented by dilations. By the Tomita-Takesaki theorem, Ω is KMS for $\mathcal{A}(V_+)$ and the evolution given by the dilations $D(t)$.
If the proper time of the observer corresponding to the geodesic curve $D(t)x$ makes it a wordline, such inertial observer detects a temperature.

The Hawking effect

The Hawking effect, i.e. the gravitational counterpart of the Unruh effect, says that a free falling observer in presence of a black hole is also measures a (Hawking) temperature.



For spacetimes with bifurcated Killing horizon such as the Schwarzchild space-time, the vacuum is KMS at the Hawking temperature $\frac{\kappa}{2\pi}$ (κ being the surface gravity) for the evolutions associated with free falling observers.

The Gibbons-Hawking effect

According to this cosmological effect, a spacetime $\mathcal{M}$ with repulsive (i.e. positive) cosmological constant has certain similarities with a black hole spacetime. $\mathcal{M}$ is expanding so rapidly that, if γ is a freely falling observer in $\mathcal{M}$, there are regions of $\mathcal{M}$ that are inaccessible to γ, even if it waits indefinitely long. The past of the world line of γ is a proper subregion $\mathcal{N}$ of $\mathcal{M}$. The boundary $\mathfrak{H}$ of $\mathcal{N}$ is a cosmological event horizon for γ.

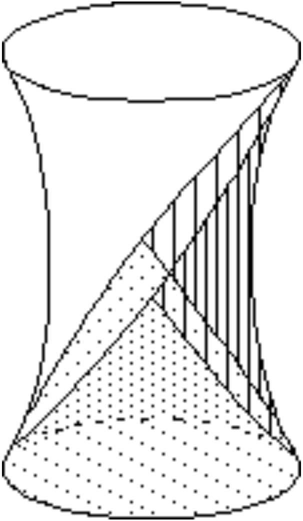
As in the black hole case, one argues that γ detects a temperature related to the surface gravity of $\mathfrak{H}$. This is a quantum effect described by quantum fields on $\mathcal{M}$; heuristically: spontaneous particle pairs creation happens on $\mathfrak{H}$, negative energy particles may tunnel into the inaccessible region, the others contribute to the thermal radiations.

The spherically symmetric, complete vacuum solution of Einstein equation with cosmological constant $\Lambda > 0$ is dS^d, the d-dimensional de Sitter spacetime, which may be defined as the submanifold of the ambient Minkowski spacetime $\mathbb{R}^{d+1}$

$$x_0^2 - x_1^2 - \cdots - x_d^2 = -\rho^2$$

where the de Sitter radius is $\rho = \sqrt{\frac{(d-1)(d-2)}{2\Lambda}}$.

de Sitter universe



The past of the world line of γ, called Steady State Universe, is a proper subregion $\mathcal{N}$ of $\mathcal{M}$ (the whole marked area). The boundary $\mathfrak{H}$ of $\mathcal{N}$ is a cosmological event horizon for γ. The world line γ determines a causally complete (wedge-like) region W (the striped area). There is a one-prameter group of geometric transformations (boosts) σ_t preserving W such that the vacuum is KMS with Gibbons-Hawking temperature $\frac{1}{2\pi\rho}$ for $(\mathcal{A}(W), \sigma_t)$.

Outline

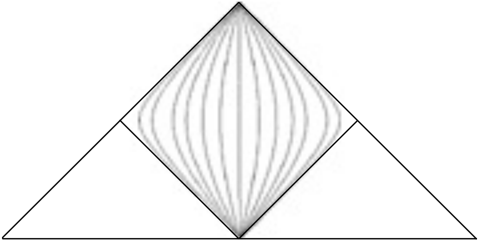


Fig. 1. The flow lines of the isometric evolution λ in the wedge contained in the steady-state universe.

Theorem

Let W be the wedge in dS^d causally generated by a geodesic observer γ. Then, the local isometric evolution λ corresponding to the de Sitter metric is indeed global, there exists one-parameter unitary group U on the Hilbert space implementing λ and the vacuum is a thermal state at the Gibbons-Hawking temperature w.r.t. U.

Converse Unruh effect

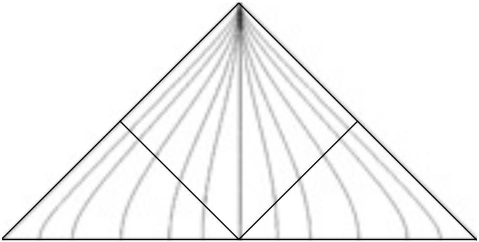


Fig. 2. The flow lines of the dethermalizing evolution μ in the steady-state universe.

Theorem

For a conformally covariant theory, the local isometric evolution μ corresponding to the flat metric is unitarily implemented, namely there exists a one-parameter unitary group V on the Hilbert space such $V(t)$ implements μ_t for $t > 0$, and the vacuum is a ground state w.r.t. V.

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