Wilson loops and amplitudes in $N=4$ Super Yang-Mills

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N=4 Super Yang-Mills

maximal supersymmetric theory (without gravity) conformally invariant, $\beta$ fn. = 0

- spin 1 gluon
- 4 spin 1/2 gluinos
- 6 spin 0 real scalars
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\( 't \text{ Hooft limit: } N_c \to \infty \text{ with } \lambda = g^2 N_c \text{ fixed} \)

- only planar diagrams
**N=4 Super Yang-Mills**

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- ’t Hooft limit: \( N_c \rightarrow \infty \) with \( \lambda = g^2 N_c \) fixed
- only planar diagrams

- **AdS/CFT duality** [Maldacena 97]
  - large-\( \lambda \) limit of 4dim CFT \( \leftrightarrow \) weakly-coupled string theory
  - (aka weak-strong duality)
planar scattering amplitude at strong coupling

\[ \mathcal{M} \sim \exp \left[ \frac{i \sqrt{\lambda}}{2\pi} (\text{Area})_{cl} \right] \]

area of string world-sheet  (classical solution  
(neglect \( O(1/\sqrt{\lambda}) \) corrections )
AdS/CFT duality, amplitudes & Wilson loops

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amplitude has same form as ansatz for MHV amplitudes at weak coupling

\[ M_n = M_n^{(0)} \exp \left[ \sum_{l=1}^{\infty} a^l \left( f^{(l)}(\epsilon) m_n^{(1)}(l\epsilon) + \text{Const}^{(l)} + E_n^{(l)}(\epsilon) \right) \right] \]
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computation ``formally the same as ... the expectation value of a Wilson loop given by a sequence of light-like segments""
MHV amplitudes $\leftrightarrow$ Wilson loops

agreement between $n$-edged Wilson loop and $n$-point MHV amplitude at weak coupling (aka weak-weak duality)
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verified for $n$-edged 1-loop Wilson loop up to 6-edged 2-loop Wilson loop

Brandhuber Heslop Travaglini 07

Drummond Henn Korchemsky Sokatchev 07
Bern Dixon Kosower Roiban Spradlin Vergu Volovich 08
**MHV amplitudes ↔ Wilson loops**

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  Brandhuber Heslop Travaglini 07

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- $n$-edged 2-loop Wilson loops also computed (numerically)
  
  Anastasiou Brandhuber Heslop Khoze Spence Travaglini 09
MHV amplitudes $\Leftrightarrow$ Wilson loops

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verified for $n$-edged 1-loop Wilson loop up to 6-edged 2-loop Wilson loop

$n$-edged 2-loop Wilson loops also computed (numerically)

no amplitudes are known beyond the 6-point 2-loop amplitude
MHV amplitudes in planar $N=4$ SYM

at any order in the coupling, colour-ordered MHV amplitude in $N=4$ SYM can be written as tree-level amplitude times helicity-free loop coefficient

$M_n^{(L)} = M_n^{(0)} m_n^{(L)}$
MHV amplitudes in planar $\mathcal{N}=4$ SYM

at any order in the coupling, colour-ordered MHV amplitude in $\mathcal{N}=4$ SYM can be written as tree-level amplitude times helicity-free loop coefficient

$$M^{(\mathcal{L})}_n = M^{(0)}_n m^{(\mathcal{L})}_n$$

at 1 loop

$$m^{(1)}_n = \sum_{pq} F^{2me}(p, q, P, Q) \quad n \geq 6$$
**MHV amplitudes in planar \( N=4 \) SYM**

- at any order in the coupling, colour-ordered **MHV** amplitude in \( N=4 \) SYM can be written as tree-level amplitude times helicity-free loop coefficient

\[
M_n^{(L)} = M_n^{(0)} m_n^{(L)}
\]

- at 1 loop

\[
m_n^{(1)} = \sum_{pq} F_{2me}^{(1)} (p, q, P, Q) \quad n \geq 6
\]

- at 2 loops, iteration formula for the \( n\)-pt amplitude

\[
m_n^{(2)}(\epsilon) = \frac{1}{2} \left[ m_n^{(1)}(\epsilon) \right]^2 + f^{(2)}(\epsilon) m_n^{(1)}(2\epsilon) + \text{Const}^{(2)} + R
\]

Anastasiou Bern Dixon Kosower 03

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at all loops, ansatz for a resummed exponent

$$m_n^{(L)} = \exp \left[ \sum_{l=1}^{\infty} a^l \left( f^{(l)}(\epsilon) m_n^{(1)}(l\epsilon) + \text{Const}^{(l)} + E_n^{(l)}(\epsilon) \right) \right] + R$$
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\]

- Remainder function

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Bern Dixon Smirnov 05
ansatz for MHV amplitudes in planar $N=4$ SYM

\[ M_n = M_n^{(0)} \left[ 1 + \sum_{L=1}^{\infty} a^L m_n^{(L)}(\epsilon) \right] \]

\[ = M_n^{(0)} \exp \left[ \sum_{l=1}^{\infty} a^l \left( f^{(l)}(\epsilon)m_n^{(1)}(l\epsilon) + Const^{(l)} + E_n^{(l)}(\epsilon) \right) \right] \]

coupling \( a = \frac{\lambda}{8\pi^2}(4\pi e^{-\gamma})\epsilon \)

\[ \lambda = g^2 N \quad \text{'t Hooft parameter} \]

\[ f^{(l)}(\epsilon) = \frac{\hat{\gamma}_K^{(l)}}{4} + \epsilon \frac{l}{2} \hat{G}^{(l)} + \epsilon^2 f_2^{(l)} \]

\[ E_n^{(l)}(\epsilon) = O(\epsilon) \]

\[ \hat{\gamma}_K^{(l)} \quad \text{cusp anomalous dimension, known to all orders of } a \]

\[ \hat{G}^{(l)} \quad \text{collinear anomalous dimension, known through } O(a^4) \]

ansatz generalises the iteration formula for the 2-loop $n$-pt amplitude $m_n^{(2)}$

\[ m_n^{(2)}(\epsilon) = \frac{1}{2} \left[ m_n^{(1)}(\epsilon) \right]^2 + f^{(2)}(\epsilon)m_n^{(1)}(2\epsilon) + Const^{(2)} + O(\epsilon) \]
Factorisation of a multi-leg amplitude in QCD

\[
\mathcal{M}_N(p_i/\mu, \epsilon) = \sum_L S_{NL}(\beta_i \cdot \beta_j, \epsilon) H_L \left( \frac{2p_i \cdot p_j}{\mu^2}, \frac{(2p_i \cdot n_i)^2}{n_i^2 \mu^2} \right) \prod_i \frac{J_i \left( \frac{(2p_i \cdot n_i)^2}{n_i^2 \mu^2}, \epsilon \right)}{J_i \left( \frac{2(\beta_i \cdot n_i)^2}{n_i^2}, \epsilon \right)}
\]

\[
p_i = \beta_i Q_0 / \sqrt{2}
\]

\text{value of } Q_0 \text{ is immaterial in } S, J

to avoid double counting of soft-collinear region (IR double poles),
\[ J \] removes eikonal part from \[ J_i \], which is already in \[ S \]
\[ J_i / J \] contains only single collinear poles

Mueller 1981
Sen 1983
Botts Sterman 1987
Kidonakis Oderda Sterman 1998
Catani 1998
Tejeda-Yeomans Sterman 2002
Kosower 2003
Aybat Dixon Sterman 2006
Becher Neubert 2009
Gardi Magnea 2009
N = 4 SYM in the planar limit

colour-wise, the planar limit is trivial:
  can absorb $S$ into $J_i$

each slice is square root of Sudakov form factor

\[ M_n = \prod_{i=1}^{n} \left[ M^{[gg \to 1]} \left( \frac{S_{i,i+1}}{\mu^2}, \alpha_s, \epsilon \right) \right]^{1/2} h_n \left( \{p_i\}, \mu^2, \alpha_s, \epsilon \right) \]

\[ \beta \text{ fn = 0 } \Rightarrow \text{ coupling runs only through dimension } \bar{\alpha}_s(\mu^2) \mu^{2\epsilon} = \bar{\alpha}_s(\lambda^2) \lambda^{2\epsilon} \]

Sudakov form factor has simple solution

\[ \ln \left[ \Gamma \left( \frac{Q^2}{\mu^2}, \alpha_s(\mu^2), \epsilon \right) \right] = -\frac{1}{2} \sum_{n=1}^{\infty} \left( \frac{\alpha_s(\mu^2)}{\pi} \right)^n \left( \frac{-Q^2}{\mu^2} \right)^{-n\epsilon} \left[ \frac{\gamma_{K}^{(n)}}{2n^2\epsilon^2} + \frac{G^{(n)}(\epsilon)}{n\epsilon} \right] \]

\( \Rightarrow \) IR structure of \( N = 4 \) SUSY amplitudes

\[ \text{Magnea Sterman 90} \]
\[ \text{Bern Dixon Smirnov 05} \]
Brief history of the ansatz

the ansatz checked for the 3-loop 4-pt amplitude
2-loop 5-pt amplitude

Bern Dixon Smirnov 05
Cachazo Spradlin Volovich 06
Bern Czakon Kosower Roiban Smirnov 06
Brief history of the ansatz

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the ansatz fails on 2-loop 6-pt amplitude

Bern Dixon Kosower Roiban Spradlin Vergu Volovich 08
Alday Maldacena 07; Bartels Lipatov Sabio-Vera 08
Brief history of the ansatz

the ansatz checked for the 3-loop 4-pt amplitude

2-loop 5-pt amplitude

the ansatz fails on 2-loop 6-pt amplitude

at 2 loops, the remainder function characterises the deviation from the ansatz

\[ R_n^{(2)} = m_n^{(2)}(\epsilon) - \frac{1}{2} \left[ m_n^{(1)}(\epsilon) \right]^2 - f^{(2)}(\epsilon) m_n^{(1)}(2\epsilon) - \text{Const}^{(2)} \]

\[ R_6^{(2)} \text{ known numerically} \]

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Bern Dixon Kosower Roiban Spradlin Vergu Volovich 08
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Monday, March 7, 2011
Wilson loops & Ward identities

$N=4$ SYM is invariant under $SO(2,4)$ conformal transformations
**Wilson loops & Ward identities**

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The Wilson loops fulfill conformal Ward identities.
Wilson loops & Ward identities

- \( N=4 \) SYM is invariant under \( SO(2,4) \) conformal transformations
- the Wilson loops fulfill conformal Ward identities
- the solution of the Ward identity for special conformal boosts is given by the finite parts of the BDS ansatz + \( R \)
Wilson loops & Ward identities

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the solution of the Ward identity for special conformal boosts is given by the finite parts of the BDS ansatz + \( R \)

for \( n = 4, 5 \), \( R \) is a constant

for \( n \geq 6 \), \( R \) is an unknown function of conformally invariant cross ratios
Wilson loops & Ward identities

N=4 SYM is invariant under $SO(2,4)$ conformal transformations

the Wilson loops fulfill conformal Ward identities

the solution of the Ward identity for special conformal boosts is given by the finite parts of the BDS ansatz + $R$

for $n = 4, 5$, $R$ is a constant
for $n \geq 6$, $R$ is an unknown function of conformally invariant cross ratios

for $n = 6$, the conformally invariant cross ratios are

$$u_1 = \frac{x_{13}^2 x_{46}^2}{x_{14}^2 x_{36}^2} \quad u_2 = \frac{x_{24}^2 x_{15}^2}{x_{25}^2 x_{14}^2} \quad u_3 = \frac{x_{35}^2 x_{26}^2}{x_{36}^2 x_{25}^2}$$

$x_i$ are variables in a dual space s.t. $p_i = x_i - x_{i+1}$

thus $x_{k,k+r}^2 = (p_k + \ldots + p_{k+r-1})^2$
Wilson loops

\[ W[C_n] = \text{Tr} \ \mathcal{P} \ \exp \left[ i g \int d\tau \dot{x}^\mu(\tau) A_\mu(x(\tau)) \right] \]

closed contour \( C_n \) made by light-like external momenta

\[ p_i = x_i - x_{i+1} \]

Alday Maldacena 07
Wilson loops

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closed contour \( \mathcal{C}_n \) made by light-like external momenta

non-Abelian exponentiation theorem: vev of Wilson loop as an exponential, allows us to compute the log of \( W \)

\[ \langle W[\mathcal{C}_n] \rangle = 1 + \sum_{L=1}^{\infty} a^L W_n^{(L)} = \exp \sum_{L=1}^{\infty} a^L w_n^{(L)} \]

through 2 loops

\[ w_n^{(1)} = W_n^{(1)} \quad w_n^{(2)} = W_n^{(2)} - \frac{1}{2} \left( W_n^{(1)} \right)^2 \]
Wilson loops

\[ W[C_n] = \text{Tr} \mathcal{P} \exp \left[ ig \oint d\tau \dot{x}^{\mu}(\tau) A_\mu(x(\tau)) \right] \]

closed contour \( C_n \) made by light-like external momenta

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\[ w_n^{(1)} = W_n^{(1)} \quad w_n^{(2)} = W_n^{(2)} - \frac{1}{2} \left( W_n^{(1)} \right)^2 \]

relation between 1 loop amplitudes & Wilson loops

\[ w_n^{(1)} = \frac{\Gamma(1 - 2\epsilon)}{\Gamma^2(1 - \epsilon)} m_n^{(1)} = m_n^{(1)} - n \frac{\zeta_2}{2} + \mathcal{O}(\epsilon) \]

\[ p_i = x_i - x_{i+1} \]

Alday Maldacena 07

Gatheral 83
Frenkel Taylor 84

Brandhuber Heslop Travaglini 07
Wilson loops

Wilson loops fulfill a Ward identity for special conformal boosts. The solution is the BDS ansatz + $R$. 

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Wilson loops

Wilson loops fulfill a Ward identity for special conformal boosts; the solution is the BDS ansatz + $R$

at 2 loops

\[ w^{(2)}_n(\epsilon) = f^{(2)}_{WL}(\epsilon) w^{(1)}_n(2\epsilon) + C^{(2)}_{WL} + R^{(2)}_{n, WL} + O(\epsilon) \]

with \[ f^{(2)}_{WL}(\epsilon) = -\zeta_2 + 7\zeta_3 \epsilon - 5\zeta_4 \epsilon^2 \]

(to be compared with \[ f^{(2)}(\epsilon) = -\zeta_2 - \zeta_3 \epsilon - \zeta_4 \epsilon^2 \] for the amplitudes)

\[ R_{4, WL} = R_{5, WL} = 0 \]
**Wilson loops**

Wilson loops fulfill a Ward identity for special conformal boosts; the solution is the BDS ansatz + $R$

at 2 loops

\[
w_n^{(2)}(\epsilon) = f_W^{(2)}(\epsilon) w_n^{(1)}(2\epsilon) + C_{WL}^{(2)} + R_{n, WL}^{(2)} + \mathcal{O}(\epsilon)
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with

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(to be compared with $f^{(2)}(\epsilon) = -\zeta_2 - \zeta_3 \epsilon - \zeta_4 \epsilon^2$ for the amplitudes)

\[R_{4, WL} = R_{5, WL} = 0\]

$R_{n, WL}^{(2)}$ arbitrary function of conformally invariant cross ratios

\[u_{ij} = \frac{x_{i+1}^2 x_{i+1,j+1}^2}{x_{ij}^2 x_{i+1,j+1}^2} \quad \text{with} \quad x_{k,k+r}^2 = (p_k + \cdots + p_{k+r-1})^2\]
Wilson loops

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$$R^{(2)}_{4, WL} = R^{(2)}_{5, WL} = 0$$

$R^{(2)}_{n, WL}$ arbitrary function of conformally invariant cross ratios

$$u_{ij} = \frac{x_{ij+1}^2 x_{i+1j}^2}{x_{ij}^2 x_{i+1j+1}^2}$$

with

$$x_{k,k+r} = (p_k + \cdots + p_{k+r-1})^2$$

duality Wilson loop $\leftrightarrow$ MHV amplitude is expressed by

$$R^{(2)}_{n, WL} = R^{(2)}_n$$
Brief history of 2-loop Wilson loops

4-edged Wilson loop
5-edged Wilson loop
6-edged Wilson loop (numeric)

6-edged Wilson loop (analytic)

$n$-edged Wilson loop (numeric)

Drummond Henn Korchemsky Sokatchev 07
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Anastasiou Brandhuber Heslop Khoze Spence Travaglini 09
Duhr Smirnov VDD 09
Anastasiou Brandhuber Heslop Khoze Spence Travaglini 09

checked that $R_n = R_n(u_{ij})$

checked multi-collinear limits
Collinear limits of Wilson loops

collinear limit $a \parallel b$

$R_6 \rightarrow 0$  $R_7 \rightarrow R_6$  $R_n \rightarrow R_{n-1}$

Anastasiou Brandhuber Heslop Khoze Spence Travaglini 09
Collinear limits of Wilson loops

**collinear limit**  \( a||b \)

\[
R_6 \rightarrow 0 \quad R_7 \rightarrow R_6 \quad R_n \rightarrow R_{n-1}
\]

**triple collinear limit**  \( a||b||c \)

\[
R_6 \rightarrow R_6 \quad R_7 \rightarrow R_6 \quad R_8 \rightarrow R_6 + R_6 \quad R_n \rightarrow R_{n-2} + R_6
\]

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Collinear limits of Wilson loops

- **collinear limit** \( a \parallel b \)
  \[
  R_6 \rightarrow 0 \quad R_7 \rightarrow R_6 \quad R_n \rightarrow R_{n-1}
  \]
- **triple collinear limit** \( a \parallel b \parallel c \)
  \[
  R_6 \rightarrow R_6 \quad R_7 \rightarrow R_6 \quad R_8 \rightarrow R_6 + R_6 \quad R_n \rightarrow R_{n-2} + R_6
  \]
- **quadruple collinear limit** \( a \parallel b \parallel c \parallel d \)
  \[
  R_7 \rightarrow R_7 \quad R_8 \rightarrow R_7 \quad R_9 \rightarrow R_6 + R_7 \quad R_n \rightarrow R_{n-3} + R_7
  \]
Collinear limits of Wilson loops

Collinear limit $a||b$

\[ R_6 \to 0 \quad R_7 \to R_6 \quad R_n \to R_{n-1} \]

Triple collinear limit $a||b||c$

\[ R_6 \to R_6 \quad R_7 \to R_6 \quad R_8 \to R_6 + R_6 \quad R_n \to R_{n-2} + R_6 \]

Quadruple collinear limit $a||b||c||d$

\[ R_7 \to R_7 \quad R_8 \to R_7 \quad R_9 \to R_6 + R_7 \quad R_n \to R_{n-3} + R_7 \]

$(k+1)$-ple collinear limit $i_1||i_2||\cdots||i_{k+1}$

\[ R_n \to R_{n-k} + R_{k+4} \]

$(n-4)$-ple collinear limit $i_1||i_2||\cdots||i_{n-4}$

\[ R_{n-1} \to R_{n-1} \quad R_n \to R_{n-1} \]

$(n-3)$-ple collinear limit $i_1||i_2||\cdots||i_{n-3}$

\[ R_n \to R_n \]
Collinear limits of Wilson loops

**Collinear limit** \( a || b \)

\[
R_6 \to 0 \quad R_7 \to R_6 \quad R_n \to R_{n-1}
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**Triple collinear limit** \( a || b || c \)

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R_6 \to R_6 \quad R_7 \to R_6 \quad R_8 \to R_6 + R_6 \quad R_n \to R_{n-2} + R_6
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**(k+1)-ple collinear limit** \( i_1 || i_2 || \cdots || i_{k+1} \)

\[
R_n \to R_{n-k} + R_{k+4}
\]

**(n-4)-ple collinear limit** \( i_1 || i_2 || \cdots || i_{n-4} \)

\[
R_{n-1} \to R_{n-1} \quad R_n \to R_{n-1}
\]

**(n-3)-ple collinear limit** \( i_1 || i_2 || \cdots || i_{n-3} \)

\[
R_n \to R_n
\]

Thus \( R_n \) is fixed by the \((n-3)-ple\) collinear limit.
Quasi-multi-Regge limit of hexagon Wilson loop

6-pt amplitude in the qmR limit of a pair along the ladder

\[ y_3 \gg y_4 \simeq y_5 \gg y_6; \quad |p_3\perp| \simeq |p_4\perp| \simeq |p_5\perp| \simeq |p_6\perp| \]

the conformally invariant cross ratios are

\[ u_{36} = \frac{x_{13}^2 x_{46}^2}{x_{14}^2 x_{36}^2} = \frac{s_{12}s_{45}}{s_{123}s_{345}} \]
\[ u_{14} = \frac{x_{24}^2 x_{15}^2}{x_{25}^2 x_{14}^2} = \frac{s_{23}s_{56}}{s_{234}s_{123}} \]
\[ u_{25} = \frac{x_{35}^2 x_{26}^2}{x_{36}^2 x_{25}^2} = \frac{s_{34}s_{61}}{s_{234}s_{345}} \]
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\begin{align*}
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\end{align*}
\]

the cross ratios are all \( O(1) \)

\( \rightarrow R_6 \) does not change its functional dependence on the \( u \)'s
Quasi-multi-Regge limit of hexagon Wilson loop

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\[ u_{25} = \frac{x_{35}^2 x_{26}^2}{x_{36}^2 x_{25}^2} = \frac{s_{34}s_{61}}{s_{234}s_{345}} \]

the cross ratios are all \( O(1) \)

\( \rightarrow R_6 \) does not change its functional dependence on the \( u \)'s

\( R_6 \) is invariant under the qmR limit of a pair along the ladder

Duhr Glover Smirnov VDD 08
Quasi-multi-Regge limit of $n$-sided Wilson loop

7-pt amplitude in the qmR limit of a triple along the ladder

$$y_3 \gg y_4 \simeq y_5 \simeq y_6 \gg y_7; \quad |p_3\perp| \simeq |p_4\perp| \simeq |p_5\perp| \simeq |p_6\perp| \simeq |p_7\perp|$$

7 cross ratios, which are all $O(1)$

$R_7$ is invariant under the qmR limit of a triple along the ladder
Quasi-multi-Regge limit of $n$-sided \textbf{Wilson} loop

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7 cross ratios, which are all $O(1)$

$R_7$ is invariant under the qmR limit of a triple along the ladder

\[ y_3 \gg y_4 \simeq \cdots \simeq y_{n-1} \gg y_n; \quad |p_3 \perp| \simeq \cdots \simeq |p_n \perp| \]

Duhr Smirnov VDD 09
Quasi-multi-Regge limit of Wilson loops

$L$-loop Wilson loops are Regge exact

\[
    w_n^{(L)}(\epsilon) = f_{WL}^{(L)}(\epsilon) w_n^{(1)}(L\epsilon) + C_{WL}^{(L)} + R_{n, WL}^{(L)}(u_{ij}) + O(\epsilon)
\]

Drummond Korchemsky Sokatchev 07
Duhr Smirnov VDD 09
Quasi-multi-Regge limit of Wilson loops

L-loop Wilson loops are Regge exact

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Quasi-multi-Regge limit of Wilson loops

$L$-loop Wilson loops are Regge exact

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\[ \ln(s_{ij}) + \text{Li}_2(1 - u_{ij}) \]
Quasi-multi-Regge limit of Wilson loops

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\[ \ln(s_{ij}) + \text{Li}_2(1 - u_{ij}) \]

u's are invariant in the qmRk

Drummond Korchemsky Sokatchev 07
Duhr Smirnov VDD 09

Monday, March 7, 2011
Quasi-multi-Regge limit of Wilson loops

*L-loop Wilson* loops are *Regge* exact

\[ w_n^{(L)}(\epsilon) = f_{WL}^{(L)}(\epsilon) w_n^{(1)}(L\epsilon) + C_{WL}^{(L)} + R_{n, WL}^{(L)}(u_{ij}) + O(\epsilon) \]

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\[ \text{log's are not power suppressed} \]

Drummond Korchemsky Sokatchev 07
Duhr Smirnov VDD 09

Monday, March 7, 2011
Quasi-multi-Regge limit of Wilson loops

$L$-loop Wilson loops are Regge exact

\[ w_n^{(L)}(\epsilon) = f_W^{(L)}(\epsilon) w_n^{(1)}(L\epsilon) + C_{WL}^{(L)} + R_{n, WL}^{(L)}(u_{ij}) + \mathcal{O}(\epsilon) \]

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\[ \ln(s_{ij}) + \text{Li}_2(1 - u_{ij}) \]

\( u \)'s are invariant in the qmRk

\[ \text{log's are not power suppressed} \]

we may compute the Wilson loop in qmRk

the result will be correct in general kinematics !!!
Diagrams of 2-loop Wilson loops

- Hard diagram
- Curtain diagram
- Cross diagram
- Y diagram
- Factorised cross diagram

Each diagram yields an integral, similar to a Feynman-parameter integral.
Computing 2-loop *Wilson* loops

cusp diagrams are given by cross and Y diagrams with gluons attaching to consecutive sides
Computing 2-loop Wilson loops

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most difficult diagrams to compute are hard diagrams

$f_H$ has $1/\varepsilon^2$ singularities if $Q_1 = Q_2 = 0$, $Q_3 \neq 0$

it has $1/\varepsilon$ singularities if $Q_1 = 0$, $Q_2$, $Q_3 \neq 0$

it is finite if $Q_1$, $Q_2$, $Q_3 \neq 0$

e.g. for $n=6$, the most difficult diagram is

$f_H(p_1, p_3, p_5; p_4, p_6, p_2)$ which is finite
Computing 2-loop Wilson loops

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e.g. for \( n=6 \), the most difficult diagram is

\[ f_H(p_1, p_3, p_5; p_4, p_6, p_2) \] which is finite

most general hard diagram has \( Q_1^2, Q_2^2, Q_3^2 \neq 0 \); it occurs for \( n \geq 9 \)
Wilson loops: analytic calc

1. Use Mellin-Barnes (MB) representation of the Feynman-parameter integrals: replace each denominator by a contour integral

\[
\frac{1}{(A + B)^\lambda} = \frac{1}{\Gamma(\lambda)} \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} dz \frac{\Gamma(-z) \Gamma(\lambda + z)}{\Gamma(z)} \frac{A^z}{B^{\lambda+z}}
\]

integral turns into a sum of residues

\[
\text{Res}_{z=-n} \Gamma(z) = \frac{(-1)^n}{n!}
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\]

2. Use Regge exactness in the qmR limit: retain only leading behaviour (i.e. leading residues) of the integral

leading residue
Wilson loops: analytic calc

3. Use Regge exactness again: iterate the qmR limit \( n \) times, by taking the \( n \) cyclic permutations of the external legs
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Wilson loops: analytic calc

3. Use Regge exactness again: iterate the qmR limit $n$ times, by taking the $n$ cyclic permutations of the external legs

4. Sum remaining towers of residues

\[ \sum_{n=1}^{\infty} \frac{u^n}{n} = -\ln(1 - u) \]

\[ \sum_{n=1}^{\infty} \frac{u^n}{n^k} = \text{Li}_k(u) \]
Wilson loops: analytic calc

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\]

in general, get nested harmonic sums \( \rightarrow \) Goncharov polylogarithms

\[
\sum_{n_1=1}^{\infty} \frac{u_1^{n_1}}{n_1^{m_1}} \sum_{n_2=1}^{n_1-1} \cdots \sum_{n_k=1}^{n_{k-1}-1} \frac{u_k^{n_k}}{n_k^{m_k}} = (-1)^k G \left( \frac{0, \ldots, 0, 1}{m_1-1, u_1}, \ldots, 0, \ldots, 0, \frac{1}{u_1 \ldots u_k}; 1 \right)
\]
Analytic 2-loop 6-edged Wilson loop

- compute 2-loop 6-edged Wilson loop

- in MB representation of the integrals in general kinematics, get up to 8-fold integrals
Analytic 2-loop 6-edged Wilson loop

compute 2-loop 6-edged Wilson loop

in MB representation of the integrals in general kinematics, get up to 8-fold integrals

after procedure in qmR limit, at most 3-fold integrals in fact, only one 3-fold integral, which comes from $f_H(p_1, p_3, p_5; p_4, p_6, p_2)$

\[
\int_{-i\infty}^{+i\infty} \int_{-i\infty}^{+i\infty} \int_{-i\infty}^{+i\infty} \frac{dz_1}{2\pi i} \frac{dz_2}{2\pi i} \frac{dz_3}{2\pi i} (z_1 z_2 + z_2 z_3 + z_3 z_1) u_1^{\tilde{z}_1} u_2^{\tilde{z}_2} u_3^{\tilde{z}_3} \\
\times \Gamma(-z_1)^2 \Gamma(-z_2)^2 \Gamma(-z_3)^2 \Gamma(z_1 + z_2) \Gamma(z_2 + z_3) \Gamma(z_3 + z_1)
\]

the result is in terms of Goncharov polylogarithms

$$G(a, \vec{w}; z) = \int_0^{\tilde{z}} \frac{dt}{t - a} G(\vec{w}; t), \quad G(a; z) = \ln \left(1 - \frac{z}{a}\right)$$
Analytic 2-loop 6-edged Wilson loop

compute 2-loop 6-edged Wilson loop

in MB representation of the integrals in general kinematics, get up to 8-fold integrals

after procedure in qmR limit, at most 3-fold integrals in fact, only one 3-fold integral, which comes from

\[ \int_{-i\infty}^{+i\infty} \int_{-i\infty}^{+i\infty} \int_{-i\infty}^{+i\infty} \frac{dz_1}{2\pi i} \frac{dz_2}{2\pi i} \frac{dz_3}{2\pi i} (z_1 z_2 + z_2 z_3 + z_3 z_1) u_1^{\bar{z}_1} u_2^{\bar{z}_2} u_3^{\bar{z}_3} \]

\[ \times \Gamma (-z_1)^2 \Gamma (-z_2)^2 \Gamma (-z_3)^2 \Gamma (z_1 + z_2) \Gamma (z_2 + z_3) \Gamma (z_3 + z_1) \]

the result is in terms of Goncharov polylogarithms

\[ G(a, \bar{\omega}; z) = \int_0^{\bar{z}} \frac{dt}{t - a} G(\bar{\omega}; t), \quad G(a; z) = \ln \left(1 - \frac{\bar{z}}{a}\right) \]

the remainder function \( R_6^{(2)} \) is given in terms of \( O(10^3) \) Goncharov polylogarithms \( G(u_1, u_2, u_3) \)
the remainder function $R_{6}^{(2)}$ is explicitly dependent on the cross ratios $u_1, u_2, u_3$
2-loop 6-edged remainder function $R_6^{(2)}$

- the remainder function $R_6^{(2)}$ is explicitly dependent on the cross ratios $u_1, u_2, u_3$

- it is symmetric in all its arguments
  (in general it’s symmetric under cyclic permutations and reflections)
2-loop 6-edged remainder function $R_6^{(2)}$

- The remainder function $R_6^{(2)}$ is explicitly dependent on the cross ratios $u_1, u_2, u_3$.
- It is symmetric in all its arguments (in general it’s symmetric under cyclic permutations and reflections).
- It is of uniform, and intrinsic, transcendental weight 4.

Transcendental weights:

- $w(\ln x) = w(\pi) = 1$  
- $w(\text{Li}_2(x)) = w(\pi^2) = 2$
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Anastasiou Brandhuber Heslop Khoze Spence Travaglini 09
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Straightforward computation
- QmR kinematics make it technically feasible.
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straightforward computation
qmR kinematics make it technically feasible

finite answer, but in intermediate steps many divergences
output is punishingly long
our result has been simplified and given in terms of polylogarithms

\[
R_{6, WL}^{(2)}(u_1, u_2, u_3) = \sum_{i=1}^{3} \left( L_4(x_i^+, x_i^-) - \frac{1}{2} \text{Li}_4(1 - 1/u_i) \right) \\
- \frac{1}{8} \left( \sum_{i=1}^{3} \text{Li}_2(1 - 1/u_i) \right)^2 + \frac{J_4^4}{24} + \frac{\pi^2}{12} J^2 + \frac{\pi^4}{72}
\]
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\[ R_{6, WL}^{(2)}(u_1, u_2, u_3) = \sum_{i=1}^{3} \left( L_4(x_i^+, x_i^-) - \frac{1}{2} \text{Li}_4(1 - 1/u_i) \right) \]

where

\[ x_i^\pm = u_i x^\pm \quad x^\pm = \frac{u_1 + u_2 + u_3 - 1 \pm \sqrt{\Delta}}{2u_1u_2u_3} \quad \Delta = (u_1 + u_2 + u_3 - 1)^2 - 4u_1u_2u_3 \]

\[ L_4(x^+, x^-) = \sum_{m=0}^{3} \frac{(-1)^m}{(2m)!!} \log(x^+x^-)^m (\ell_{4-m}(x^+) + \ell_{4-m}(x^-)) + \frac{1}{8!!} \log(x^+x^-)^4 \]

\[ \ell_n(x) = \frac{1}{2} (\text{Li}_n(x) - (-1)^n \text{Li}_n(1/x)) \quad J = \sum_{i=1}^{3} (\ell_1(x_i^+) - \ell_1(x_i^-)) \]
our result has been simplified and given in terms of polylogarithms

\[ R^{(2)}_{6,WL}(u_1, u_2, u_3) = \sum_{i=1}^{3} \left( L_4(x_i^+, x_i^-) - \frac{1}{2} \text{Li}_4(1 - 1/u_i) \right) \]

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not a new, independent, computation just a manipulation of our result
our result has been simplified and given in terms of polylogarithms

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not a new, independent, computation
just a manipulation of our result
answer is short and simple
introduces the theory of motives in TH physics
Symbols

Fn. $F$ of $\deg(F) = n$ : fn. with log cuts, s.t. $\text{Disc} = 2\pi i \times f$, with $w(f) = n-1$
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$\text{deg}(\text{const}) = 0 \Rightarrow \text{deg}(\pi) = 0$

$\ln x$ : cut along $[-\infty, 0]$ with $\text{Disc} = 2\pi i \Rightarrow \text{deg}(\ln x) = 1$

$\text{Li}_2(x)$ : cut along $[1, \infty]$ with $\text{Disc} = -2\pi i \ln x \Rightarrow \text{deg}(\text{Li}_2(x)) = 2$
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Take a fn. defined as an iterated integral

\[
T_k = \int_a^b d\ln R_1 \circ \cdots \circ d\ln R_k
\]

the symbol is

\[
\text{Sym}[T_k] = R_1 \otimes \cdots \otimes R_k
\]

defined on the tensor product of the group of rational functions, modulo constants

\[
\cdots \otimes R_1 R_2 \otimes \cdots = \cdots \otimes R_1 \otimes \cdots + \cdots \otimes R_2 \otimes \cdots
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Symbols

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$R_i$ rational functions

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\]

$\text{Sym}[\ln x] = x$ \hspace{1cm} $\text{Sym}[\operatorname{Li}_2(x)] = -(x - 1) \otimes x$

$T_k = \int_a^b d\ln R_1 \circ \cdots \circ d\ln R_k$

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Symbols

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defined on the tensor product of the group of rational functions, modulo constants

$\cdots \otimes R_1 R_2 \otimes \cdots = \cdots \otimes R_1 \otimes \cdots + \cdots \otimes R_2 \otimes \cdots$

Sym[$\ln x$] = $x$ \quad Sym[Li$_2$(x)] = $-(x - 1) \otimes x$

take $f, g$ with $\deg(f) = \deg(g) = n$ and $\text{Sym}[f] = \text{Sym}[g]$ then $f-g = h$ with $\deg(h) = n - 1$
Symbols

Fn. $F$ of $\deg(F) = n$ : fn. with log cuts, s.t. $\text{Disc} = 2\pi i \times f$, with $w(f) = n - 1$

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take a fn. defined as an iterated integral $T_k = \int_a^b d\ln R_1 \circ \cdots \circ d\ln R_k$

the symbol is $\text{Sym}[T_k] = R_1 \otimes \cdots \otimes R_k$

defined on the tensor product of the group of rational functions, modulo constants

$\cdots \otimes R_1 R_2 \otimes \cdots = \cdots \otimes R_1 \otimes \cdots + \cdots \otimes R_2 \otimes \cdots$

$\text{Sym}[\ln x] = x$ $\text{Sym}[\text{Li}_2(x)] = -(x - 1) \otimes x$

take $f, g$ with $\deg(f) = \deg(g) = n$ and $\text{Sym}[f] = \text{Sym}[g]$

then $f - g = h$ with $\deg(h) = n - 1$

a symbol determines a polynomial of uniform degree up to a constant
**Z_n symmetric regular hexagons**

Regular hexagons are characterised by

\[ x_{13}^2 = x_{24}^2 = x_{35}^2 = x_{46}^2 = x_{51}^2 = x_{62}^2; \quad x_{14}^2 = x_{25}^2 = x_{36}^2 \]

\[ u_{36} = \frac{x_{13}^2 x_{46}^2}{x_{14}^2 x_{36}^2} = \frac{s_{12}s_{45}}{s_{123}s_{345}} \]

\[ u_{14} = \frac{x_{24}^2 x_{15}^2}{x_{25}^2 x_{14}^2} = \frac{s_{23}s_{56}}{s_{234}s_{123}} \]

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$Z_n$ symmetric regular hexagons

regular hexagons are characterised by

$$x_{13}^2 = x_{24}^2 = x_{35}^2 = x_{46}^2 = x_{51}^2 = x_{62}^2; \quad x_{14}^2 = x_{25}^2 = x_{36}^2$$

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At strong coupling, remainder function is obtained from "minimal area surfaces in AdS$_5$ which end on a null polygonal contour at the boundary". One gets "integral equations which determine the area as a function of the shape of the polygon. The equations are identical to those of the Thermodynamics Bethe Ansatz. The area is given by the free energy of the TBA system. The high temperature limit of the TBA system can be exactly solved"

$$R_{6}^{\text{strong}} (u, u, u) = \frac{\pi}{6} - \frac{1}{3\pi} \phi^2 - \frac{3}{8} \left( \ln^2 (u) + 2 \text{Li}_2^2 (1 - u) \right)$$

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Remainder function at weak and strong coupling

compare remainder functions at weak and strong coupling introducing coefficients in the strong coupling result and try to curve fit the 2 results

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\[ c_1 = 0.263\pi^3 \quad c_2 = 0.860\pi^2 \quad c_3 = -\frac{\pi^2}{12} c_2 \]

Alday Gaiotto Maldacena 09
Brandhuber Heslop Khoze Travaglini 09
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the 2 curves are strikingly similar
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\[ R_{6, WL}^{(2)} \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right) = -\frac{105}{64} \zeta_3 \log 2 - \frac{5}{64} \log^4 2 + \frac{5}{64} \pi^2 \log^2 2 - \frac{15}{8} \text{Li}_4 \left( \frac{1}{2} \right) + \frac{17\pi^4}{2304} \]

uniform, and intrinsic, weight 4
One-loop amplitude squared

the 2-loop $n$-pt amplitude is

$$m_n^{(2)}(\epsilon) = \frac{1}{2} \left[ m_n^{(1)}(\epsilon) \right]^2 + f^{(2)}(\epsilon) m_n^{(1)}(2\epsilon) + \text{Const}^{(2)} + R$$

what about that?
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- use the vev masses as regulators

\textbf{Alday Henn Plefka Schuster 09}
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Monday, March 7, 2011
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not practical for phenomenology (where DR rules the waves)

Alday Henn Plefka Schuster 09

Monday, March 7, 2011
Amplitudes in **twistor** space

**twistors** live in the fundamental irrep of **SO(2,4)**

any point in **dual** space corresponds to a line in **twistor** space

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2-loop **$n$-pt** MHV amplitudes can be written as sum of pentaboxes in **twistor** space

\[ m_n^{(2)} = \frac{1}{2} \sum_{i<j<k<l<i} \]

Arkani-Hamed Bourjaily Cachazo Trnka10

Monday, March 7, 2011
8-edged *Wilson* loop in $\text{AdS}_3$

- at strong coupling, Alday & Maldacena have considered $2n$-sided polygons embedded into the boundary of $\text{AdS}_3$

- $2n$-sided remainder function depends on $2(n-3)$ variables
8-edged Wilson loop in AdS$_3$

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for the octagon, the remainder function is

$$R_{8, WL}^{strong} = -\frac{1}{2} \ln (1 + \chi^-) \ln \left( 1 + \frac{1}{\chi^+} \right) + \frac{7\pi}{6}$$

$$+ \int_{-\infty}^{+\infty} dt \frac{|m| \sinh t}{\tanh(2t + 2i\phi)} \ln \left( 1 + e^{-2\pi|m| \cosh t} \right)$$

where

$$\chi^+ = e^{2\pi \text{Im} m} \quad \chi^- = e^{-2\pi \text{Re} m} \quad m = |m|e^{i\phi}$$

Alday Maldacena 09
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Duhr Smirnov VDD 10
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2-loop $2n$-sided polygon $R$ conjectured through collinear limits Heslop Khoze 10

proven through OPE Gaiotto Maldacena Sever Vieira 10

Monday, March 7, 2011
Conclusions

Planar $N=4$ SYM is a great lab where to test comparisons between strong and weak couplings

features weak-strong duality and weak-weak duality

Wilson loops are the ideal quantities to perform those comparisons
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- first (and so far only) analytic computation of a 2-loop remainder function

- analytic comparison between the 2-loop 6-edged Wilson loop at weak and strong couplings

- in a particular kinematic set-up, defined on the boundary of AdS$_3$
  first analytic computation of the 2-loop 8-edged Wilson loop at weak coupling;
  2-loop Wilson loops in AdS$_3$ are now completely solved
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more is to come ... stay tuned!

Monday, March 7, 2011