Confinement: A Mathematical Perspective

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Quark Gluon Plasma, and the first LHC results

LNF, March 1st - 8th, 2011

Outline

Introduction 1/2

It is a common belief that the structure of hadronic matter at small scales can be interpreted in terms of charged particles (quarks and gluons) which do not appear at large scales because of confinement. The basis for this interpretation are:

- quark model of hadronic spectrum
- parton picture of DIS
- perturbative treatment of QCD

Conceptual problem: One should not attribute a physical interpretation to unobservable gauge fields. There may be another formulation of the theory, in terms of a different set of fields, which yields the same observables. Examples:

- Schwinger model (2d massless QED)
- bosonization of 3d Chern-Simons QED
- Seiberg-Witten dualities in 4d SUSY YM

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The problem can in principle be solved by a version of the Renormalization Group adapted to the Algebraic framework of QFT (AQFT), in which a theory is defined entirely by its algebra of observables.

In this way one obtains a (ultraviolet) Scaling Limit theory which describes particles and symmetries which are regarded as the particles and symmetries of the underlying theory at small scales.

Confinement is then unambiguously defined by comparing the charge structures of the two theories.

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Algebras of Local Observables 1/2

Main assumption of [Haag-Kastler '64]: A QFT is fixed by the knowledge of its local observables

Basic input: assignment of algebras $\mathscr{A}(O)$ of bounded (i.e. with bounded spectrum) operators on the Hilbert space $\mathscr{H}$ to bounded Minkowski space-time regions O, where (hermitian) $A \in \mathscr{A}(O)$ represents an observable measurable in the region O

$\implies$ if O_1 and O_2 spacelike separated, then for all $A_1 \in \mathscr{A}(O_1)$, $A_2 \in \mathscr{A}(O_2)$

$$[A_1, A_2] = 0 \qquad \text{(Einstein causality)}$$

$\Omega \in \mathscr{H}$ vacuum vector, such that $\overline{\bigcup_O \mathscr{A}(O)\Omega} = \mathscr{H}$ (cyclicity of the vacuum)

GNS reconstruction theorem: knowledge of $\langle \Omega, A\Omega \rangle$ for all A allows reconstruction of $\mathscr{A}(O)$, $\mathscr{H}$, Ω (see Streater-Wightman reconstruction for QFT)

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Algebras of Local Observables 2/2

In concrete model: $\phi(x)$ combinations of the basic fields in the Lagrangian which are observable, i.e.

$$[\phi(x), \phi(y)] = 0 \quad \text{if } (x-y)^2 < 0$$

Then (after all cut-offs are removed)

$$\phi(f) := \int dx\, f(x)\phi(x)$$

well defined (unbounded) operator on $\mathscr{H}$ for suitable f.
Define $\mathscr{A}(O)$ as the algebra generated by (bounded) operators $e^{i\phi(f)}$ for supp $f \subset O$.

States and representations

Physical states are represented by linear, positive, normalized expectation functionals $\varphi : \mathscr{A} \to \mathbb{C}$. Examples:

- $\varphi(A) := \langle \Phi, A\Phi \rangle$ with (normalized) $\Phi \in \mathscr{H}$, neutral states, since $\mathscr{H} = \overline{\mathscr{A}\Omega}$
- charged states: $\varphi(A) := \lim_{x \to \infty, x^2 < 0} \langle \Phi_x, A\Phi_x \rangle$ with $\Phi_x \in \mathscr{H}$ representing a charge at the origin and an opposite charge at x ("shifting a compensating charge behind the moon"), can also be induced by vectors in other representations of $\mathscr{A}$

Problem: identify states of interest in particle physics

Definition (DHR selection criterion [Doplicher-Haag-Roberts '71])

A state φ is DHR if $\varphi(A) = \omega(A) := \langle \Omega, A\Omega \rangle$ for all $A \in \mathscr{A}(O_1)$ with O_1 spacelike from some sufficiently large O (i.e. φ is a localized excitation of the vacuum)

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Doplicher-Roberts reconstruction 1/2

One would like to have a more explicit description of DHR states

Theorem (DR reconstruction [Doplicher-Roberts '90])

- *There exist local field algebras $\mathscr{F}(O) \supset \mathscr{A}(O)$ generated by charge carrying Bose/Fermi (bounded) operators*
- *There exists a compact global gauge group G acting on $\mathscr{F}$ such that $\mathscr{A}(O)$ is the invariant part of $\mathscr{F}(O)$*
- *The Hilbert space $\mathscr{K} := \overline{\mathscr{F}\Omega}$ is made by vectors inducing all DHR states, and if φ is localized in O there exist $\psi_1, \dots \psi_d \in \mathscr{F}(O)$ orthogonal isometries ($\psi_j^* \psi_k = \delta_{jk}$) transforming like a d-dimensional representation of G such that*

$$\varphi(A) = \sum_j \langle \Omega, \psi_j A \psi_j^* \Omega \rangle$$

- *$\mathscr{F}$ and G are unique*

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Doplicher-Roberts reconstruction 2/2

Coherent subspaces of $\mathscr{K}$ (superselection sectors) are spanned by vectors $\psi_j^*\Omega$ for a fixed irreducible representation of G.
They provide representations of $\mathscr{A}$ disjoint from the fundamental one (on $\mathscr{H}$) and represent the possible "values" of the charges of physical states.

In concrete models:

- $\mathscr{F}(O)$ generated by $e^{i\phi(f)}$ for all fields in the Lagrangian
- G fixed by the Lagrangian
- Bose/Fermi alternative put in by assumption

Main result of AQFT superselection theory: all this structure is fixed by the knowledge of the local observables.

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Conventional approach

Conventional approach to the Renormalization Group:

- Pass from ϕ to renormalized field $\phi_\lambda(x) = Z_\lambda \phi(\lambda x)$
- Renormalization constants Z_λ fixed by requiring e.g. $\langle \Omega, \phi_\lambda(f)\phi_\lambda(f)\Omega\rangle \sim$ const as $\lambda \to 0$
- in favourable cases (e.g. asymptotically free theories)

$$\exists \lim_{\lambda\to 0} \langle \Omega, \phi_\lambda(x_1)\dots\phi_\lambda(x_n)\Omega\rangle = \langle \Omega_0, \phi_0(x_1)\dots\phi_0(x_n)\Omega_0\rangle$$

- new field ϕ_0 defines the scaling limit of the theory

Problems:

- in general $Z_\lambda \to 0$ due to singular ultraviolet behaviour
- need to have detailed information on short-distance behaviour of Z_λ to control the limit (solvable in favourable cases through RG equations and perturbative methods)
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Scaling Algebras

Algebraic approach [Buchholz-Verch '95]:

- Z_λ not needed
- based only on observables and model independent

Main idea: what really matters is that c and $\hbar$ are kept fixed in the rescaling $\implies$ one should only consider observables that at scale λ transfer to states 4-momentum of order λ^{-1} (uncertainty principle)
Typical example:

$$\underline{A}_\lambda = \int dx\, g(x) U(\lambda x) e^{i\phi_\lambda(f)} U(\lambda x)^*, \quad \operatorname{supp} f + \operatorname{supp} g \subset O$$

with Z_λ arbitrary.
For the "wrong" Z_λ, $\underline{A}_\lambda$ converges (in correlation functions) to a multiple of the identity for $\lambda \to 0$
Scaling algebras $\underline{\mathfrak{A}}(O)$ generated by such functions $\lambda \to \underline{A}_\lambda$

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Scaling Limit

Consider state on the scaling algebra

$$\underline{\omega}_0(\underline{A}) := \lim_{\lambda \to 0} \langle \Omega, \underline{A}_\lambda \Omega \rangle$$

(technical problem: limit may exist only along subsequences)

Applying GNS reconstruction theorem to state $\underline{\omega}_0$ one obtains new local algebras $\mathscr{A}_0(O)$, on new Hilbert space $\mathscr{H}_0$, with new vacuum vector Ω_0: the scaling limit theory

Morally, $\mathscr{A}_0(O)$ is generated by $e^{i\phi_0(f)}$ with supp $f \subset O$

Again: Scaling limit theory is fixed by the knowledge of the local observables at scale $\lambda = 1$

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Scaling limit of charged fields

Field scaling algebra $\underline{\mathfrak{F}}$ and scaling limit field net $\mathscr{F}_0$ defined in analogy to $\underline{\mathfrak{A}}$, $\mathscr{A}_0$ [D'Antoni-M-Verch '04]

$\exists G_0 = G/N_0$ acting on $\mathscr{F}_0$ such that $\mathscr{A}_0(O)$ is the invariant part of $\mathscr{F}_0(O)$

General situation:

$$\begin{tikzcd}
 & \mathscr{A} \arrow[dl, "\mathrm{SL}"'] \arrow[dr, "\mathrm{DR}"] & & \\
\mathscr{A}_0 \arrow[dr, "\mathrm{DR}"'] & & & \mathscr{F} \arrow[dl, "\mathrm{SL}"] \\
 & \mathscr{F}^{(0)} & \supseteq \mathscr{F}_0 &
\end{tikzcd}$$

$\mathscr{A}_0$ may describe "more" charges than those obtained by scaling limit of charges of $\mathscr{A}$. These should be confined.

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Preservation of charges and confinement

Problem: find a canonical way of identifying charges of $\mathscr{A}$ which are preserved in the limit

Theorem ([D'Antoni-M-Verch '04])

The preserved charges are those associated to orthogonal isometries $\psi_j(\lambda) \in \mathscr{F}(\lambda O)$ such that $\psi_j(\lambda)^*\Omega$ has energy scaling as λ^{-1}

This means that preserved charges are pointlike (their localization only requires energy according to uncertainty principle, no internal structure) $\Rightarrow$ they survive in the limit
This gives an intrinsic notion of confinement

Definition ([D'Antoni-M-Verch '04])

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The model

Schwinger model:

- 2d QED with massless fermions
- algebra of observables $\mathscr{A}$ generated by 2d neutral free field ϕ of mass $m > 0$ [Lowenstein-Swieca '71]
- no charged states $\Rightarrow \mathscr{F} = \mathscr{A}$ [Fröhlich-Morchio-Strocchi '79]
- interpreted as confinement of fermions: 2d electric potential rises linearly
- interpretation questionable form the point of view of observables

Scaling algebra

Consider scaling algebra elements

$$\underline{A}_\lambda^{(j)} = \int d^2y\, g_j(y) \exp\Big(i \int d^2x\, f_j(x-y) Z_{j,\lambda} \phi(\lambda x)\Big)$$

One has, if $Z_{j,\lambda} \to Z_{j,0}$ and $\sum_j Z_{j,0} \hat{f}_j(0) = 0$,

$$\langle \Omega, \underline{A}_\lambda^{(1)} \dots \underline{A}_\lambda^{(n)} \Omega \rangle \sim \int \dots \int \prod_j d^2y_j\, g_j(y_j) \eta(y_1, \dots, y_n) \times$$

$$\exp\Big(-\pi \int \frac{dp}{2|p|} \Big| \sum_j Z_{j,0} e^{i(|p|y_j^0 - py_j^1)} \hat{f}_j(|p|, p) \Big|^2 - \pi |\log \lambda| \Big| \sum_j Z_{j,\lambda} \hat{f}_j(0) \Big|^2 \Big)$$

The presence of the $\log \lambda$ factor makes clear that the rate of convergence of $Z_{j,\lambda}$ matters: if $|Z_{j,\lambda} - Z_{j,0}| \sim |\log \lambda|^{-1/2}$ one gets a nontrivial limit interpretable as the contribution of a classical gaussian variable not visibile in the conventional approach, where $Z_{j,\lambda} = 1$

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Scaling limit and charged states 1/2

Scaling limit of $\mathscr{A}$ [Buchholz '96]:

- algebra $\mathscr{A}_0 = \mathscr{W}_0 \otimes \mathscr{C}_0$, with $\mathscr{W}_0$ (Weyl) algebra generated by $e^{i\phi_0(f)}$, ϕ_0 2d massless free field, and $\mathscr{C}_0$ a suitable abelian algebra
- vacuum (on $\mathscr{W}_0$)

$$\langle \Omega_0, e^{i\phi_0(f)}\Omega_0\rangle = \begin{cases} \exp\Big[-\pi\int_{\mathbb{R}} \frac{dp}{2|p|}|\hat f(|p|,p)|^2\Big] & \text{if } \hat f(0) = 0 \\ 0 & \text{if } \hat f(0) \neq 0 \end{cases}$$

(non-regular because of infrared divergence)

Charged states on $\mathscr{A}_0$

$$\omega_q(e^{i\phi_0(f)}) = e^{iL(f)}\langle \Omega_0, e^{i\phi_0(f)}\Omega_0\rangle$$

where

- $L(f) = -\sqrt{\frac{\pi}{2}}\int_{\mathbb{R}} dx(h(x) - h(-x))\int_{-\infty}^{x} dy\,\rho(y)$
- $\hat h(p) = \frac{1}{2}[\hat f(|p|,p) + \hat f(-|p|,p)]$
- ρ function with support in $[-r, r]$ and such that $\int_{\mathbb{R}} dx\,\rho(x) = q$

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Scaling limit and charged states 2/2

Properties of ω_q:

- ω_q is "localized": for supp f in the left/right spacelike complement of $[-r, r] \times \{0\}$

$$L(f) = \mp\pi q\hat{f}(0) \Rightarrow \omega_q(e^{i\phi_0(f)}) = \langle\Omega_0, e^{i\phi_0(f)}\Omega_0\rangle$$

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Scaling limit and charged states 2/2

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Local Algebras and Pointlike Fields

It is a quite general situation [Bostelmann '05] that there exist functionals σ_j on $\mathscr{A}$ and pointlike fields ϕ_j, $j = 0, 1, 2, ...$ such that for all $\gamma > 0$ there exists N such that for all $A \in \mathscr{A}(O_r)$ and all states σ of energy less than E

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and typically $\sigma_j(A) \sim r^{[\phi_j]}$

The ϕ_j are the pointlike fields locally associated to the local algebras and can be obtained as suitable limits of observables $A_r \in \mathscr{A}(O_r)$ as $r \to 0$

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Theorem ([Bostelmann-D'Antoni-M '09])

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- *with $\underline{\phi}_\lambda := \sum_j \sigma_j(\underline{A}_\lambda)\phi_j = \sum_j Z_{j,\lambda}\phi_j$, $\phi_0 := \lim_{\lambda\to 0}\underline{\phi}_\lambda$ (in correlation functions) is a pointlike field locally associated to $\mathscr{A}_0$*
- *one has, e.g. for the 2-point function:*

$$\langle \Omega_0, \phi_0(x)\phi_0'(x')\Omega_0\rangle = \lim_{\lambda\to 0}\sum_{j,k} Z_{j,\lambda}Z'_{k,\lambda}\langle\Omega, \phi_j(\lambda x)\phi_k(\lambda x')\Omega\rangle$$

It follows:

- Mutliplicative renormalization of pointlike fields follows from first principles and the renormalization constants Z_λ are automatically provided by the scaling algebra framework
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- can be used to formulate an intrinsic notion of charge confinement, not relying on attaching a physical interpretation to unobservable quantities, that can be tested in concrete models
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Outlook:

- study more realistic, interacting models in 2d (work in progress with Bostelmann and Lechner)
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