Correlations and quark number susceptibilities in thermodynamic geometry*

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1 Introduction

8. In this talk, we illustrate the geometric nature quark number susceptibilities. Specifically, we shall show that the components of the susceptibility tensor define local pair correlations against the chemical fluctuations.
9. Our consideration follows from the notion of the thermodynamic geometry[4, 5, 6]. Importantly, our framework offers a platform to understand the nature of local correlations and physics of the global phase transitions.
10. From the study of the mixture of gases, black holes in string theory[7, 8, 10], it has been known in diverse contexts that the thermodynamic properties are geometric in nature. This is the main aim to the present talk.
11. Quantum chromodynamics(QCD), on the other hand, is a well celebrated theory of strong interaction. Specifically, the physics of QCD at high temperature, which is known as hot QCD, plays a crucial role in understanding the phase structure of the system.
12. We shall explicate the nature of confinement-deconfinement transitions from the hadronic phase to deconfined phase, or quark-gluon plasma(QGP) phase.
13. This offers a guideline for the physics of strong interactions, when the system is closer to the the phase transition point T_c. Indeed, our proposal supplies both the local and global content of the system in a relatively simpler approach.

14. Here, we analyze the important regime of hot QCD from the set up of the thermodynamic geometry. This notion is explored for the free energy of hot QCD.

15. At the near T_c, our investigation offers possible connections with the existing picture of the quark-number susceptibility tensor [1, 11].

16. To explain the notion of the present consideration, we shall chose the free-energy of QCD of the 2- and 3-flavor hot QCD. Our analysis of the near T_c behavior enjoys the framework of existing quasi-particle theories.

17. We thereby incorporate the thermal fluctuations and study the impact of thermodynamic geometry to the hot QCD.

18. It is worth recalling that the thermodynamic macrostates of an underlying equilibrium chemical space, which is ascribed to the chosen hot QCD cinfiguration, are described by the minima of the underlying effective energy function $F = F(U, V, N)$.

19. For a given energy $F = F(U, V, N)$, the consideration of Weinhold [4] defines an intrinsic Riemannian space, which is spanned by the chemical potentials. In the present case, we find that the metric tensor is given by

$$g_{ij} = \partial_i \partial_j F(\mu_i, T, V, S). \tag{1}$$

20. This metric tensor turns out to be conformal to the Ruppenier metric [5, 6, 7, 8, 9, 10], where the temperature of the system plays the role of the conformal factor.

21. Here, we shall study the behavior of two parameter ensembles, which are associated with the space of chosen two distinct chemical potentials, viz., $\{\mu_1, \mu_2\}$.

2 Quark Number Susceptibility

22. As per the framework of quasi-particle theories of QCD, we hereby investigate the relation between the covariant thermodynamic geometry and the quark-number susceptibility tensor.

23. In fact, both of these pictures have received considerable recently attention, viz., physics of black hole and QCD thermodynamics.

24. In the quasi-particle models [1], let us recall that the quark number susceptibilities are defined as

$$\chi_{ij} \equiv \frac{\partial \mathcal{N}_i}{\partial \mu_j} = \frac{\partial^2 P}{\partial \mu_i \partial \mu_j} = \chi_{ji}. \tag{2}$$

25. Here i, j are the flavor indices, $\mathcal{N}_i$ is the quark number density, and P is the pressure of the associated QCD configurations.

26. From the viewpoints of microscopic field theory, the susceptibility tensor χ is expressed in terms of the associated dressed fermion propagators as a function of the self-energies and fermions helicities [14].

3 Free Energy Near T_C

27. Let's first analyze what is the notion of local and global behavior of an ensemble of quasiparticles. To do so, let's understand how the perspective of the thermodynamic geometry applies to hot QCD backgrounds.

28. Let $\{F, \mu_i, T$ be the set chemical potentials, and temperature, then the free energy can be defined as a functional representation $F(\mu_i, T)$.

29. To offer the physics near T_C, we may choose a particular temperature slices as an intersection of the line $T := T_C^{(0)}$, and thus the underlying space of chemical fluctuation is solely spanned by the chemical potentials $\{\mu_i, T\}$ of the system.

30. An analysis of the free energy of hot QCD, near the T_C, can herewith be traded off as a function of the quark chemical potentials. The consideration of Ref.[22] shows that the resulting temperature T_C of the system is given by the following the relation

$$\frac{T_c(\mu)}{T_c^0} = 1 + \sum_{i=1}^{2} \tilde{a}_i \frac{\mu_i^2}{(T_c^0)^2}, \tag{3}$$

where, $\tilde{a}_1 = \tilde{a}_2 = -0.07$ for the 2-flavor case and $\tilde{a}_1 = \tilde{a}_2 = -0.114$ for the 3-flavor QCD.

31. In the small chemical potential limit ($\beta|\mu| << 1$), we find that the free energy appears only up to the quartic terms in the $\beta\mu$. Thus, the underlying expression can be written as

$$F = T^4\Big(a_0 + c_1\frac{\mu_1^2}{T^2} + c_2\frac{\mu_2^2}{T^2} + c_4^1\frac{\mu_1^4}{T^4} + c_4^2\frac{\mu_2^4}{T^4}\Big) + T^4 O(\mu^6/T^6). \tag{4}$$

32. Considering the fact that high temperature behavior of QCD can be mapped into an ensemble of effective quasi-particles. We find interesting dimensions of the hot QCD[25].

33. In general, the free energy reduces to the following

$$\begin{aligned} F(\mu_1, \mu_2): \;\; = \;\; & a_0(b + a_1\mu_1^2 + a_2\mu_2^2)^4 + (b + a_1\mu_1^2 + a_2\mu_2^2)^2 \\ & \times(c_1\mu_1^2 + c_2\mu_2^2) + c_4^{(1)}\mu_1^4 + c_4^{(2)}\mu_2^4, \end{aligned} \tag{5}$$

where $b = T_c^0$, $a_1 = \tilde{a_1}/b$ and $a_2 = \tilde{a_2}/b$.

34. Both the a_1 and a_2 have the dimension of b^{-1}(GeV). Hereby, the free energy is measured in the units of GeV^4.

35. The various coefficients appearing above are chosen [22, 25] as follows:

For 2-flavor QCD,

$$a_0 = -\frac{8\pi^2}{45}\Big(1 + \frac{42}{32}\Big), c_1 = c_2 = 1/2, c_4^1 \equiv c_4^2 = -\frac{1}{4\pi^2}. \tag{6}$$

For 3-flavor QCD,

$$a_0 = -\frac{8\pi^2}{45}\Big(1 + \frac{63}{32}\Big), c_1 = 1, c_2 = 1/2, c_4^1 = -\frac{1}{2\pi^2}, c_4^2 = -\frac{1}{4\pi^2}. \tag{7}$$

36. The free energy with an incorporation of the thermal fluctuations takes the following form

$$\begin{aligned} F(\mu_1,\mu_2): &= a_0+\frac{c_1\mu_1^2+c_2\mu_2^2}{(b+a_1\mu_1^2+a_2\mu_2^2)^2} \\ &\quad +\frac{1}{2}\ln(\frac{c_1\mu_1^2+c_2\mu_2^2}{(b+a_1\mu_1^2+a_2\mu_2^2)^2}). \end{aligned} \tag{8}$$

37. If the components of the thermodynamic metric tensor are measured in the units of GeV^2, then the thermodynamic curvature is measured in the units of GeV^4, whenever the chemical potentials μ_i are in the units of GeV.

4 Intrinsic Thermodynamic Geometry

38. As per the definition of Weinhold geometry, the components of the intrinsic metric tensor [4] are,

$$g_{\mu_1\mu_1}=\frac{\partial^2 F}{\partial\mu_1^2},\quad g_{\mu_1\mu_2}=\frac{\partial^2 F}{\partial\mu_1\partial\mu_2},\quad g_{\mu_2\mu_2}=\frac{\partial^2 F}{\partial\mu_2^2}. \tag{9}$$

39. We shall illustrate that the symmetry property of the χ_{ij} and g_{ij} remains the same.

40. The System is well-defined as long is a positive determinant of the metric tensor

$$\|g\|=F_{\mu_1\mu_1}F_{\mu_2\mu_2}-F_{\mu_1\mu_2}^2 \tag{10}$$

41. Next, the global correlation is defined by the scalar curvature

$$\begin{aligned} R &= -\frac{1}{2}(F_{\mu_1\mu_1}F_{\mu_2\mu_2}-F_{\mu_1\mu_2}^2)^{-2}(F_{\mu_2\mu_2}F_{\mu_1\mu_1\mu_1}F_{\mu_1\mu_2\mu_2} \\ &\quad +F_{\mu_1\mu_2}F_{\mu_1\mu_1\mu_2}F_{\mu_1\mu_2\mu_2}+F_{\mu_1\mu_1}F_{\mu_1\mu_1\mu_2}F_{\mu_2\mu_2\mu_2} \\ &\quad -F_{\mu_1\mu_2}F_{\mu_1\mu_1\mu_1}F_{\mu_2\mu_2\mu_2}-F_{\mu_1\mu_1}F_{\mu_1\mu_2\mu_2}^2 \\ &\quad -F_{\mu_2\mu_2}F_{\mu_1\mu_1\mu_2}^2). \end{aligned} \tag{11}$$

42. Interestingly, the relation between thermodynamic scalar curvature and thermodynamic curvature tensor of two dimensional intrinsic surface $(M_2(R),g)$ (see for details [7]) is given by

$$R=\frac{2}{\|g\|}R_{\mu_1\mu_2\mu_1\mu_2}. \tag{12}$$

43. More clearly, the scalar curvature for two component systems can be thought of as the square of the correlation length at some given QCD transition temperature $T_C(\mu_1,\mu_2)$ to be $R(\mu_1,\mu_2)\sim\xi^2$, where the $\xi(T_C)$ identified as the correlation length of the corresponding system. Physically, we find that the thermodynamic curvature corresponds to the nature of the correlation present in the statistical system.

5 2-flavor QCD

5.1 Leading Order

44. Considering Eq.5 in the neighborhood of T_C and substituting for T_C as function of μ_1 and μ_2, one obtains the following expression:

$$\begin{aligned} F(\mu_1,\mu_2): &= a_0(b+a_1\mu_1^2+a_2\mu_2^2)^4+(b+a_1\mu_1^2+a_2\mu_2^2)^2 \\ &\quad\times(c_1\mu_1^2+c_2\mu_2^2). \end{aligned} \tag{13}$$

45. Employing the formula Eq. 9, the following polynomial expression arise for the components of the metric tensor

$$\begin{aligned} g_{\mu_1,\mu_1} &= 1.12\mu_2^4+5.60\mu_1^4+6.72\mu_1^2\mu_2^2-6.48\mu_1^4\mu_2^2 \\ &\quad -3.89\mu_1^2\mu_2^4-2.21\mu_1^2-0.74\mu_2^2+0.13 \\ &\quad -3.02\mu_1^6-0.43\mu_2^6, \\ g_{\mu_1,\mu_2} &= -1.48\mu_1\mu_2+4.48\mu_1^3\mu_2+4.48\mu_1\mu_2^3 \\ &\quad -2.59\mu_1^5\mu_2-2.59\mu_1\mu_2^5-5.18\mu_1^3\mu_2^3, \\ g_{\mu_2,\mu_2} &= 1.12\mu_1^4+5.60\mu_2^4+6.72\mu_1^2\mu_2^2-6.48\mu_1^2\mu_2^4 \\ &\quad -3.89\mu_1^4\mu_2^2-2.21\mu_2^2-0.74\mu_1^2+0.13 \\ &\quad -3.02\mu_2^6-0.43\mu_1^6. \end{aligned} \tag{14}$$

46. As announced earlier, it is hereby evident that the components of metric tensor indeed satisfy

    (i) $g_{\mu_1,\mu_1}(\mu_1,\mu_2)=g_{\mu_2,\mu_2}(\mu_2,\mu_1)$

    (ii) $g_{\mu_1,\mu_2}(\mu_1,\mu_2)=g_{\mu_1,\mu_2}(\mu_2,\mu_1)$.

47. In fact, the symmetry of the components of the metric tensor *viz.* symmetry under the exchange of the chemical potentials μ_1 and μ_2 shows fundamental symmetry properties of microscopic ensemble, e.g., parity symmetry under the change of the sing of μ_1 and μ_2.

48. The determinant of the metric tensor remains non-zero in the limit of the small chemical potential limit defines a non-degenerate thermodynamic geometry near T_C.

49. Namely, the stability of the system is determined from the determinant of the metric tensor. It can be expressed as following polynomial

$$g(\mu_1,\mu_2)=\sum_{k,l=0|k+l\leq 6}^{6} a_{k,l}^A\mu_1^{2k}\mu_2^{2l} \tag{15}$$

50. Employing the formula displayed in the Eq.12, it turns out that the thermodynamic scalar curvature takes the following expression

$$R(\mu_1,\mu_2)=-\frac{4}{g^2}\sum_{k,l=0|k+l\leq 7}^{7} b_{k,l}^A\mu_1^{2k}\mu_2^{2l}. \tag{16}$$

51. It turns out that the physically interesting domain near T_C arises, when both the chemical potentials lie in the range of 0.0 to 0.2.

5.2 Non Vanishing $\{c_4^1, c_4^2\}$

52. Considering the free energy given in Eq. 5, and employing Eq. 9 for the non vanishing c_4^1, c_4^2 as defined in Eq. 6, the following expression are obtained for the components of the metric tensor

$$\begin{aligned}
g_{\mu_1,\mu_1} &= 6.72\mu_1^2\mu_2^2 - 2.52\mu_1^2 - 0.74\mu_2^2 + 0.13 + 5.60\mu_1^4 \\
&\quad +1.12\mu_2^4 - 3.02\mu_1^6 - 0.43\mu_2^6 - 6.48\mu_1^4\mu_2^2 \\
&\quad -3.89\mu_1^2\mu_2^4, \\
g_{\mu_1,\mu_2} &= -1.48\mu_1\mu_2 + 4.48\mu_1^3\mu_2 + 4.48\mu_1\mu_2^3 - 2.59\mu_1^5\mu_2 \\
&\quad -2.59\mu_1\mu_2^5 - 5.18\mu_1^3\mu_2^3, \\
g_{\mu_2,\mu_2} &= 6.72\mu_1^2\mu_2^2 - 2.52\mu_2^2 - 0.74\mu_1^2 + 0.13 + 5.60\mu_2^4 \\
&\quad +1.12\mu_1^4 - 3.02\mu_2^6 - 0.43\mu_1^6 - 6.48\mu_1^2\mu_2^4 \\
&\quad -3.89\mu_1^4\mu_2^2. \qquad (17)
\end{aligned}$$

53. We see further that the symmetry($\mu_i \to -\mu_i$) and the exchange symmetry of the thermodynamic geometry remain conserved, as mentioned in the previous case.

54. In this case, the determinant of the metric tensor turns out to be a polynomial function in $\{\mu_1, \mu_2\}$, which in a compact notation, is given by

$$g(\mu_1, \mu_2) = \sum_{k,l|k+l=0\leq 6}^{6} a_{k,l}^B \ \mu_1^{2l}\mu_2^{2k} \qquad (18)$$

55. Interestingly, the determinant of the metric tensor remains non-zero in the small chemical potential limit and thus defines a non-degenerate thermodynamic geometry near T_C.

56. The underlying thermodynamic curvature can be written as

$$R(\mu_1, \mu_2) = -\frac{4}{g^2} \sum_{k,l=0|k+l\leq 7}^{7} b_{k,l}^B \ \mu_1^{2k}\mu_2^{2l}. \qquad (19)$$

5.3 Thermal Fluctuations

57. We hereby discuss the thermodynamic geometry of hot QCD with an inclusion of the thermal fluctuations about an equilibrium configuration.

58. In this case, the corresponding free energy of the two flavor QCD reads

$$\begin{aligned}
F(\mu_1, \mu_2) &= a_0 + \frac{c_1\mu_1^2 + c_2\mu_2^2}{(b + a_1\mu_1^2 + a_2\mu_2^2)^2} \\
&\quad + \frac{1}{2}\ln\left(\frac{c_1\mu_1^2 + c_2\mu_2^2}{(b + a_1\mu_1^2 + a_2\mu_2^2)^2}\right). \qquad (20)
\end{aligned}$$

59. Employing the formulae of the metric tensor as displayed in Eqs.9, we find that the components of the metric tensor in the scale of 10^{10} are given by

$$\begin{aligned}
g_{\mu_1,\mu_1} &= \{-0.17(\mu_1^2-\mu_2^2)-1.33(\mu_2^{10}-\mu_1^{10}) \\
&+9.96\mu_2^8-36.26\mu_1^8-3.55\mu_2^4-5.82\mu_1^4 \\
&+2.67(\mu_2^6\mu_1^4-\mu_1^6\mu_2^4)-102.80(\mu_2^2\mu_1^4-0.5\mu_1^2\mu_2^4) \\
&-78.93\mu_1^4\mu_2^4-36.27\mu_1^8-51.40\mu_1^6-5.82\mu_1^4 \\
&-1.33\mu_1^{10}-6.38\mu_2^6\mu_1^2-98.86\mu_2^2\mu_1^6 \\
&-4.00(\mu_2^2\mu_1^8-\mu_1^2\mu_2^8)-9.37\mu_1^2\mu_2^2\} \\
&\times(-203+340\mu_1^2+340\mu_2^2)^{-4}(\mu_1^2+\mu_2^2)^{-2}, \\
g_{\mu_1,\mu_2} &= 2\mu_1\mu_2\{1.13(\mu_1^2+\mu_2^2)+23.12(\mu_2^6+\mu_1^6) \\
&+1.33(\mu_1^8+\mu_2^8)+51.40(0.50\mu_1^4+\mu_2^4) \\
&+69.36(\mu_2^2\mu_1^4+\mu_1^2\mu_2^4)8.01(\mu_1^4+\mu_2^4) \\
&+5.34(\mu_1^2\mu_2^6+\mu_1^6\mu_2^2)+51.40\mu_1^2\mu_2^2-0.17\} \\
&\times(-203+340\mu_1^2+340\mu_2^2)^{-4}(\mu_1^2+\mu_2^2)^{-2}, \\
g_{\mu_2,\mu_2} &= \{-0.17(\mu_2^2-\mu_1^2)-1.33(\mu_1^{10}-\mu_2^{10}) \\
&+9.96\mu_1^8-36.26\mu_1^8-3.55\mu_2^4-5.82\mu_1^4 \\
&+2.67(\mu_1^6\mu_2^4-\mu_2^6\mu_1^4)-102.80(\mu_1^2\mu_2^4 \\
&-0.5\mu_2^2\mu_1^4)-78.93\mu_2^4\mu_1^4-36.27\mu_2^8 \\
&-51.40\mu_2^6-5.82\mu_2^4-1.33\mu_2^{10}-6.38\mu_1^6\mu_2^2 \\
&-98.86\mu_1^2\mu_2^6-4.00(\mu_1^2\mu_2^8-\mu_2^2\mu_1^8) \\
&-9.37\mu_2^2\mu_1^2\}\times(\mu_1^2+\mu_2^2)^{-2} \\
&(-203+340\mu_1^2+340\mu_2^2)^{-4}. \qquad (21)
\end{aligned}$$

60. We see that the symmetry of the metric tensor remains preserved under the thermal fluctuations.

61. In this case, we find the following positive definite expressions for the determinant of thermodynamic metric tensor

$$g(\mu_1,\mu_2)=4\frac{\sum_{k,l=0|k+l\leq 7}^{7}a_{l,k}^{C}\ \mu_1^{2k}\mu_2^{2l}}{(\mu_1^2+\mu_2^2))^2(0.20-0.34(\mu_1^2-\mu_2^2))^7}. \qquad (22)$$

62. The thermodynamic curvature takes the form of the ratio of two polynomial, which can be written as

$$\begin{aligned}
R(\mu_1,\mu_2) &= -\frac{4}{g^2}(0.20-0.34(\mu_1^2+\mu_2^2) \\
&\times\sum_{k,l=0|k+l\leq 12}^{12}b_{k,l}^{C}\ \mu_1^{2k}\mu_2^{2l}, \qquad (23)
\end{aligned}$$

where the coefficients $\{b_{k,l}^{C}\}$ of the polynomial expression appearing in the numerator may easily be tracked from the corresponding expression of the free energy.

63. We find that the thermal contributions are dominant in the range $0.1 < \mu_i < 0.2$ of the chemical potentials.

64. In general, for any quasi-particle free energy $F(\mu_1, \mu_2) := \sum_{i,j} a_{i,j} \mu_1^i \mu_2^j$ satisfying $a_{i,j} = a_{j,i}$, we find that the associated determinant and scalar curvature of the intrinsic Riemannian manifold, when considered as functions of the quark chemical potentials, are symmetric functions $g(\mu_1, \mu_2) = g(\mu_2, \mu_1)$ and $R(\mu_1, \mu_2) = R(\mu_2, \mu_1)$.

6 3-flavor QCD

6.1 Leading Order

65. As mentioned in 2-flavor cases, let's analyze the behavior of 3-flavor QCD free energy with or without an inclusion of the thermal corrections. As a function of the quark chemical potentials, we shall systematically offer the covariant metric tensor, determinant of the metric tensor and scalar curvature of the 3-flavor QCD thermodynamics.

66. In this case, the components of the metric tensor reduce to

$$\begin{aligned}
g_{\mu_1,\mu_1} &= -7.59\mu_1^2 - 2.30\mu_2^2 + 0.26 + 33.76\mu_1^4 \\
&\quad +6.08\mu_2^4 - 32.50\mu_1^6 + 38.51\mu_1^2\mu_2^2 \\
&\quad -69.65\mu_1^4\mu_2^2 - 41.79\mu_1^2\mu_2^4 - 4.64\mu_2^6, \\
g_{\mu_1,\mu_2} &= -4.60\mu_1\mu_2 + 25.67\mu_1^3\mu_2 + 24.33\mu_1\mu_2^3 \\
&\quad -27.86\mu_1^5\mu_2 - 55.72\mu_1^3\mu_2^3 - 27.86\mu_1\mu_2^5, \\
g_{\mu_2,\mu_2} &= -6.22\mu_1^2 - 2.30\mu_2^2 + 0.22 + 6.41\mu_1^4 \\
&\quad +28.75\mu_2^4 - 4.64\mu_1^6 + 36.50\mu_1^2\mu_2^2 \\
&\quad -41.79\mu_1^4\mu_2^2 - 69.65\mu_1^2\mu_2^4 - 32.50\mu_2^6. \qquad (24)
\end{aligned}$$

67. The components of the metric tensor at zero chemical potentials take the values $g_{\mu_1,\mu_1} = 0.26$, $g_{\mu_2,\mu_2} = 0.22$ and $g_{\mu_1,\mu_2} = 0$, which in effect respectively describe the diagonal and off diagonal quark susceptibility tensors.

68. We find that the determinant of the metric tensor reduces to the value of $g = 0.06$ at zero chemical potentials.

69. Whilst, the scalar curvature vanishes at this point and thus the underlying statistical system approaches a non-interacting system.

70. On the other hand, the determinant of the metric for this 3 flavor case is as follows

$$g(\mu_1, \mu_2) = \sum_{k,l=0|k+l\leq 6}^{6} \tilde{a}^A_{k,l}\ \mu_1^{2l} \mu_2^{2k} \qquad (25)$$

71. In this case, we see that $g(\mu_1, \mu_2) \neq g(\mu_2, \mu_1)$, which follows from the fact that the 3-flavor QCD free energy (displayed in Eq.5) is not symmetric under the exchange in the chemical potentials.

72. Interestingly, the same remains true for the associated curvature, which may be seen from

$$R(\mu_1, \mu_2) = -\frac{4}{g^2} \sum_{k,l=0|k+l\leq 7}^{7} \tilde{b}^A_{k,l}\ \mu_1^{2k} \mu_2^{2l} \tag{26}$$

73. In this case, it turns out that the determinant of the metric tensor as a function of the chemical potentials has two lines of minima and the corresponding maximum lying in between the minima, which occurs when one of the chemical potentials reaches a non-zero constant value, while the other vanishes.

74. There are two distinct bumps of varying heights instead of a continuous curves of macroscopic interactions. The height of the bumps depends on the domain chosen in the (u, v) space. The significance of these bumps is that they signify a nontrivial thermodynamical interactions in the 3-flavor QCD system.

6.2 Non Vanishing $\{c_4^1, c_4^2\}$

75. Let's now investigate role of the the thermodynamic geometry corresponding to the free energy as given by the Eq.4.

76. We find that the components of the metric tensor are

$$\begin{aligned}
g_{\mu_1,\mu_1} &= -38.51\mu_1^2\mu_2^2 + 6.08\mu_2^4 - 69.65\mu_1^4\mu_2^2 \\
&\quad -8.20\mu_1^2 - 2.30\mu_2^2 + 0.61 + 33.76\mu_1^4 \\
&\quad -41.79\mu_1^2\mu_2^4 - 32.50\mu_1^6 - 4.64\mu_2^6, \\
g_{\mu_1,\mu_2} &= -4.60\mu_1\mu_2 + 25.67\mu_1^3\mu_1 + 24.33\mu_1\mu_2^3 \\
&\quad -27.86\mu_1^5\mu_2 - 55.72\mu_1^3\mu_2^3 - 27.86\mu_1\mu_1^5, \\
g_{\mu_2,\mu_2} &= -36.51\mu_1^2\mu_2^2 + 28.75\mu_2^4 - 41.79\mu_1^4\mu_2^2 \\
&\quad -2.30\mu_1^2 - 6.53\mu_2^2 + 0.22 + 3.42\mu_1^4 \\
&\quad -69.65\mu_1^2\mu_2^4 - 4.64\mu_1^6 - 32.50\mu_2^6.
\end{aligned} \tag{27}$$

77. In this case, the conclusion to be drawn in the range of the chemical potentials as considered in the previous case remains the same, except the fact that the number of bumps is increased. This indicates relatively stronger interactions in the internal space of quark chemical potentials.

78. In the range of the chemical potentials $(u, v) = (0.2, 0.2)$, the determinant of the metric tensor, when viewed as a function of (u, v) is observed to be relatively flatter wiith respect to the previous case.

79. The determinant of the metric tensor turns out to be polynomial of the following explicit form

$$g(\mu_1,\mu_2)=\sum_{k,l=0|k+l\leq 6}^{6}\tilde{a}^{B}_{k,l}\ \mu_1^{2l}\mu_2^{2k} \tag{28}$$

80. We herewith see that $g(\mu_1,\mu_2)\neq g(\mu_2,\mu_1)$. This follows from the fact that the free energy with the $(\frac{\mu}{T})^4$ contributions, as displayed in the Eq.4, remains a non-symmetric function under the exchange of the chemical potentials.

81. In fact, we find that the same outcomes remains true for the associated curvature

$$R(\mu_1,\mu_2)=-\frac{4}{g^2}\sum_{k,l=0|k+l\leq 7}^{7}\tilde{b}^{B}_{k,l}\ \mu_1^{2k}\mu_2^{2l} \tag{29}$$

82. As in the above cases, when the free energy is treated as a function of the chemical potentials, we see that the thermodynamic space spanned by the chemical potentials remains completely regular on the intrinsic manifold, except at the points where the determinant of the metric tensor vanishes.

83. The intrinsic space spanned by the chemical potentials thus turns out to be a well-defined and stable statistical configuration. In the domain of physical interests, the determinant of the associated metric tensor remains positive definite.

84. For a range of $\{\mu_1,\mu_2\}$, we see that the underlying diagonal quark susceptibility tensor does not remain the same at zero chemical potentials.

85. In particular, we find $g_{\mu_1,\mu_1}=0.61, g_{\mu_2,\mu_2}=0.22$ and $g_{\mu_1,\mu_2}=0$. In fact, the determinants of the metric tensor and the scalar curvature of the Gaussian fluctuations possess the same physical behavior under the $(\frac{\mu}{T})^4$-contributions, which continues even at the zero chemical potentials.

6.3 Thermal Fluctuations

86. We us finally discuss the thermodynamic properties of 3-flavor QCD when the thermal fluctuations are taken into the consideration. We thus offer the conditions such that the underlying system can be treated as a well-defined statistical ensemble.

87. A straightforward computation yields that the components of the metric tensor in the scale of 10^{10} read to the following simple formulas

$$\begin{aligned} g_{\mu_1,\mu_1} &= -2(-197+578\mu_1^2+578\mu_2^2)^{-4}(2\mu_1^2+\mu_2^2)^{-2} \\ &\{7.50\mu_2^6+3.80\mu_2^8+0.30\mu_1^2-0.15\mu_2^2 \\ &-2.56\mu_2^4+182.75\mu_1^4\mu_2^4-28.69\mu_1^2\mu_2^4 \\ &+78.13\mu_1^6\mu_2^4-22.32\mu_1^{10}-416.12\mu_1^8 \end{aligned}$$

$$
\begin{aligned}
& -343.63\mu_1^6 - 11.16\mu_1^8\mu_2^2 - 91.30\mu_1^6\mu_2^2 \\
& -19.06\mu_1^2\mu_2^2 - 274.74\mu_1^4\mu_2^2 - 20.83\mu_1^4 \\
& +62.18\mu_1^2\mu_2^6 + 100.45\mu_1^4\mu_2^6 + 33.48\mu_1^2\mu_2^8\}, \\
g_{\mu_1,\mu_1} &= 4\mu_1\mu_2(-197 + 578\mu_1^2 + 578\mu_2^2)^{-4}(2\mu_1^2 + \mu_2^2)^{-2} \\
& \{7.61\mu_2^6 + 1.77\mu_1^2 + 1.77\mu_2^2 + 27.68\mu_2^4 \\
& +78.13\mu_1^4\mu_2^4 + 107.83\mu_1^2\mu_2^4 + 33.48\mu_1^8 \\
& +385.68\mu_1^6 + 89.29\mu_1^6\mu_2^2 + 126.27\mu_1^2\mu_2^2 \\
& +385.68\mu_1^4\mu_2^2 + 143.05\mu_1^4 + 22.32\mu_1^2\mu_2^6 \\
& -0.15\}, \\
g_{\mu_2,\mu_2} &= (-197 + 578\mu_1^2 + 578\mu_2^2)^{-4}(2\mu_1^2 + \mu_2^2)^{-2} \\
& \{80.72\mu_2^6 + 107.83\mu_2^8 + 0.30\mu_2^2 - 0.15\mu_2^2 \\
& -11.16\mu_2^{10} + 6.53\mu_2^4 + 1911.89\mu_1^4\mu_2^4 \\
& +368.99\mu_1^2\mu_2^4 + 66.97\mu_1^4\mu_2^6 - 69.97\mu_1^{10} \\
& -340.04\mu_1^8 + 75.53\mu_1^6 - 100.54\mu_1^8\mu_2^2 \\
& +1278.78\mu_1^6\mu_2^2 + 17.29\mu_1^2\mu_2^2 + 437.31\mu_1^4\mu_2^2 \\
& +15.52\mu_1^4 + 801.80\mu_1^2\mu_2^6 + 178.58\mu_1^4\mu_2^6 \\
& +89.29\mu_1^8\mu_2^2\}. \qquad (30)
\end{aligned}
$$

88. The determinant of the metric tensor is given by the following polynomial

$$
\begin{aligned}
g(\mu_1,\mu_2) &= \Big((0.197 - 0.578(\mu_1^2 + \mu_2^2))^7(\mu_1^2 + 0.50\mu_2^2)^2\Big) \\
& \times \sum_{k,l=0|k+l\leq 12}^{12} \tilde{b}_{k,l}^{C}\ \mu_1^{2k}\mu_2^{2l}, \qquad (31)
\end{aligned}
$$

89. In this case, we see that the determinant of the metric tensor plotted against the chemical potentials acquires two finite size bumps which were absent in the case without thermal corrections.

90. Hereby, we find that the domain of the interaction has shifted towards the origin, as in the previously treated case of 2-flavor QCD. We observe that there are two bumps of the interactions present in the limit of chemical potentials.

91. Interestingly, the strength of interaction depends on an exact location of the point chosen in the internal space. For example, in the small limit of the constituent chemical potentials, we notice that the interaction could be as high as 10^4 or as small as 10^2.

92. In this case, it turns out further that the thermodynamic curvature can be exprssed as the ratio of two polynomial expressions. Specifically, we find that the underlying thermodynamic scalar curvature is given by

$$
\begin{aligned}
R(\mu_1,\mu_2) &= -\frac{4}{g^2}(0.197 - 0.578(\mu_1^2 + \mu_2^2)) \\
& \sum_{\{k,l=0|k+l\leq 12\}}^{12} \tilde{b}_{k,l}^{C}\ \mu_1^{2l}\mu_2^{2k}, \qquad (32)
\end{aligned}
$$

where the coefficients $\{\tilde{b}_{k,l}\}$ appearing in the numerator of the scalar curvature may suitably be defined from the corresponding free energy funtion.

93. Thus, in this case, we see that one can always find, from the general considerations of the intrinsic Riemannian geometry, as mentioned in the case of 2-flavor QCD, that, for a given polynomial free energy, after including the logarithmic contributions, there exist certain bump(s) in the intrinsic Reimannian space of the chemical potentials.

7 Conclusion and Outlook

94. We have examined the role of the intrinsic geometry for the 2-flavor and 3-flavor QCD thermodynamic configurations.

95. The free energy, near the critical temperature, describes thermodynamic geometric behavior and possesses the symmetry of the quark susceptibilities. Specifically, we have demonstrated that the components of the thermodynamic metric correspond to the microscopic quark number susceptibility tensor.

96. The physics of fluctuations describes properties of the underlying statistical system, e.g. crossover/ transition near the critical temperature.

97. Based on quasi-particle theory, our analysis is consistent with the Hard Thermal Loop and finite temperature resummed Hard Thermal and Dense Loops in hot QCD.

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