Baikov Package - A package for loop-by-loop Baikov Parametrization

Hjalte Frellesvig

Niels Bohr International Academy (NBIA), University of Copenhagen.

September 12, 2024

Dimensionally regulated Feynman Integrals

$$I_{\{a\}} = \int \frac{N(k)}{\prod_i D_i^{a_i}(k)} \prod_{j=1}^{L} \frac{\mathrm{d}^d k_j}{(2\pi)^{d/2}}$$

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- Performing (generalized unitarity) cuts
- Using twisted cohomology and intersection theory
- Unvealing underlying geometries

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The (standard) Baikov representation:

$$I_{\{a\}} = \mathcal{K}\,\mathcal{E}^{\alpha} \int_{\mathcal{C}} \frac{\mathcal{B}(x)^{\beta}\,\mathrm{d}^n x}{\prod_i x_i^{a_i}}$$

The integration variables *are* the propagators!

$\mathcal{B} = \mathrm{Gram}(k_1, \ldots, k_L, p_1, \ldots, p_E)$ $\qquad \beta = (d{-}E{-}L{-}1)/2$

$\mathcal{E} = \mathrm{Gram}(p_1, \ldots, p_E)$ $\qquad \alpha = (E{-}d{+}1)/2$

$L =$ nr. of loops , $\quad E =$ nr. of (independent) external momenta

$n \;=\; L(L{+}1)/2 + LE \;=\;$ nr. of independent scalar products

$$\mathcal{K} \;=\; \frac{\mathcal{J}\,(-i)^L\,\pi^{(L-n)/2}}{\prod_{l=1}^{L} \Gamma\big((d{+}1{-}E{-}l)/2\big)} \qquad \mathcal{J} \sim 2^{L-n}$$

$\mathcal{C} : \left\{ \dfrac{\mathcal{B}}{\mathcal{E}} > 0 \right\}$ $\qquad$ The integration contour

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The *loop-by-loop* Baikov representation:

$$I_{\{a\}} = \mathcal{K} \int_{\mathcal{C}} \frac{\mathrm{d}^{\tilde{n}} x \;\prod_{l=1}^{L} \mathcal{E}_l(x)^{\alpha_l} \mathcal{B}_l(x)^{\beta_l}}{\prod_i x_i^{a_i}}$$

$$\mathcal{B}_l = \mathsf{Gram}(k_l, q_1, \ldots, q_{E_l}) \qquad \beta_l = (d{-}E_l{-}2)/2$$

$$\mathcal{E}_l = \mathsf{Gram}(q_1, \ldots, q_{E_l}) \qquad \alpha_l = (E_l{-}d{+}1)/2$$

$E_l =$ nr. of (independent) momenta external to loop nr. l

$$\tilde{n} \;=\; L + \sum_l E_l \qquad \text{(linear scaling)}$$

$$\mathcal{K} \;=\; \frac{\mathcal{J}\,(-i)^L\,\pi^{(L-\tilde{n})/2}}{\prod_{l=1}^{L} \Gamma\big((d{-}E_l)/2\big)} \qquad \mathcal{J} \;\sim\; 2^{L-\tilde{n}} \qquad \mathcal{C} : \bigcap_l \left\{ \frac{\mathcal{B}_l}{\mathcal{E}_l} > 0 \right\}$$

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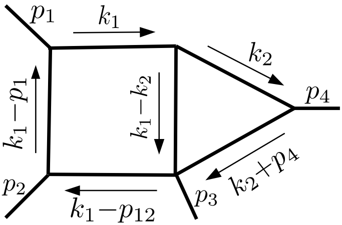
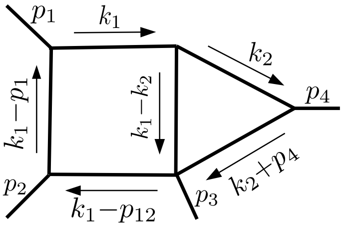
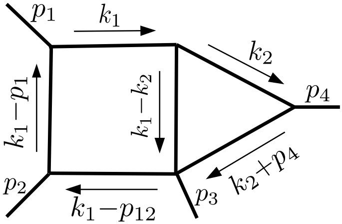
Loop-by-loop examples



Loop-by-loop examples



Standard Baikov: $n = L(L+1)/2 + LE. \quad L = 2,\ E = 3 \ \Rightarrow \ n = 9$



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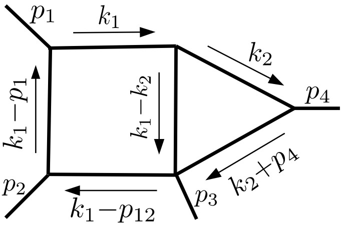
Three extra props needed. Candidates: $(k_1+p_4)^2$, $(k_2-p_1)^2$, $(k_2-p_{12})^2$.



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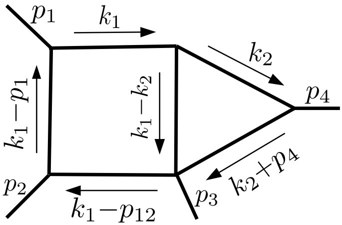

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But why introduce e.g. $k_2 \cdot p_1$? This motivates LBL:

$$E_{k_2} = 2\ ,\ \ E_{k_1} = 3\ ,\ \ \tilde{n} = L + \sum_l E_l = 7$$

Only one extra propagator needed: $(k_1{+}p_4)^2$.

A graphical approach:

First open all vertices to three-point vertices
such that the smallest polygons are as small as possible.

Then remove (i.e. “integrating out”) the loops one by one

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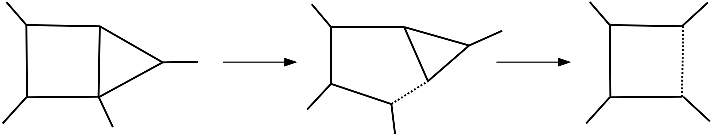

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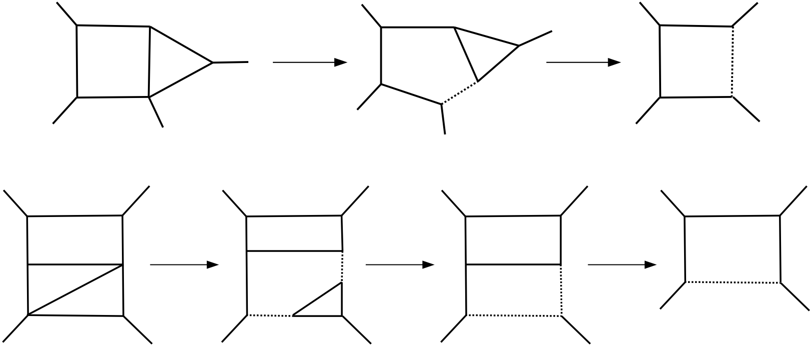

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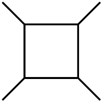

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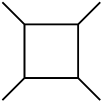

UseBaikovPackage.nb * - Wolfram Mathematica (NK2)

```
In[1]:= << "/home/hjalte/science/baikovcode/BaikovPackage.m";

In[2]:= Internal = {k};
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In[7]:= components = BaikovLBL[];
        result1 = BaikovCombine[components]
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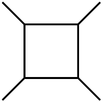

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Out[8]=

$$\frac{1}{\pi^{3/2}\,\mathrm{Gamma}\left[\frac{1}{2}(-3+d)\right]} \, i\, 2^{3-d} \left(-s\,t\,(s+t)\right)^{\frac{4-d}{2}} \left(s^2 t^2 - 2 s t^2 x[1] + t^2 x[1]^2 - 2 s^2 t x[2] + 2 s t x[1] \times x[2] + s^2 x[2]^2 - 2 s t^2 x[3] - 4 s t x[1] \times x[3] - 2 t^2 x[1] \times x[3] + 2 s t x[2] \times x[3] + t^2 x[3]^2 - 2 s^2 t x[4] + 2 s t x[1] \times x[4] - 2 s^2 x[2] \times x[4] - 4 s t x[2] \times x[4] + 2 s t x[3] \times x[4] + s^2 x[4]^2\right)^{\frac{1}{2}(-5+d)}$$

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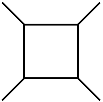

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```
In[9]:= maxcut = Map[(x[#] → 0) &, Range[Length[Propagators]]]

Out[9]= {x[1] → 0, x[2] → 0, x[3] → 0, x[4] → 0}

In[10]:= result2 = PowerExpand[result1 /. maxcut /. {d → 4}]
```

Out[10]= $\frac{i}{2\,\pi^2\,s\,t}$

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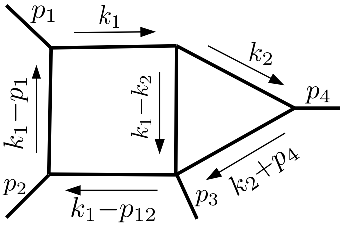
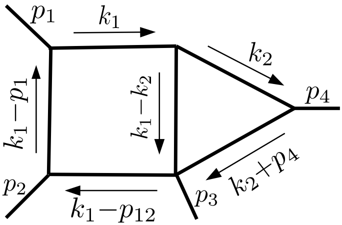

p_1
k_1
k_2
p_4
$k_1 - p_1$
$k_1 - k_2$
p_2
p_3
$k_2 + p_4$
$k_1 - p_{12}$

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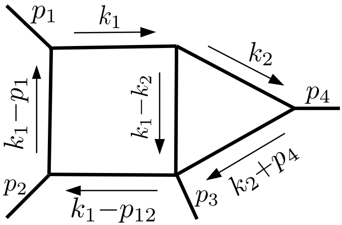


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In[11]:= result2 = PowerExpand[result1 /. maxcut /. {d → 4} /. facpow]
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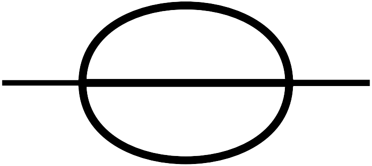
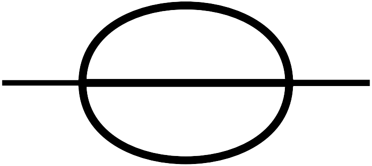
Out[11]= $\dfrac{i}{4\,\pi^3\, s\,(t - x[7])\, x[7]}$





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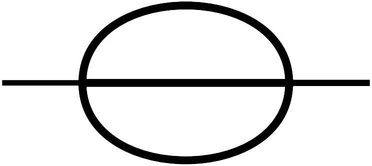

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```

Out[12]= $-\frac{1}{\pi^2 \sqrt{(4\,\mathrm{mm} - z)\, z \left(-\mathrm{mm}^2 + 2\,\mathrm{mm}\, s - s^2 + 2\,\mathrm{mm}\, z + 2\, s\, z - z^2\right)}}$

Cutting edge example

PHYSICAL REVIEW LETTERS **132,** 201602 (2024)

Calabi-Yau Meets Gravity: A Calabi-Yau Threefold at Fifth Post-Minkowskian Order

Hjalte Frellesvig,[1] Roger Morales,[1,2] and Matthias Wilhelm[1]
[1]*Niels Bohr International Academy, Niels Bohr Institute, Copenhagen University, Blegdamsvej 17, 2100 Copenhagen Ø, Denmark*
[2]*Mani L. Bhaumik Institute for Theoretical Physics, University of California at Los Angeles, Los Angeles, California 90095, USA*

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We study geometries occurring in Feynman integrals that contribute to the scattering of black holes in the post-Minkowskian (PM) expansion. These geometries become relevant to gravitational-wave production from binary mergers through the classical conservative potential. At 4PM, a $K3$ surface is known to occur in a three-loop integral, leading to elliptic integrals in the result. In this Letter, we identify a Calabi-Yau threefold in a four-loop integral, contributing at 5PM. The presence of this Calabi-Yau geometry indicates that completely new functions occur in the full analytical results.

DOI: 10.1103/PhysRevLett.132.201602

Introduction.—Following the groundbreaking discovery of gravitational waves [1,2], the inspiral and eventual merger of binary systems of compact astronomical objects such as black holes and neutron stars has become a key object of interest in many branches of physics. The upcoming third-generation gravitational-wave detectors will provide much more and higher-precision data, requiring equally high-precision theoretical predictions for its interpretation [3,4].

Many complementary approaches for the theoretical description of these processes have been developed, ranging from numerical relativity [5–7] to analytical approaches valid in various regions, such as post-Newtonian [8–10], post-Minkowskian [4,11], and self-force [12–15] expansions as well as the effective one-body formalism [16,17].

The post-Minkowskian (PM) expansion treats the dynamics in the inspiraling phase perturbatively in Newton's constant G while maintaining all orders in the velocity, thus accounting for relativistic effects. Since the dynamics of the bound system can also be related to the scattering problem [11], this allows the use of Feynman diagrams and other methods from perturbative quantum field theory (QFT) and scattering amplitudes [18–39], see Refs. [4,40] for an overview, while systematically taking the classical limit $\hbar \to 0$ to retain the classical pieces only. As in QFTs, higher precision thus requires the computation of Feynman integrals with more loops. In particular, the state-of-the-art computation for the gravitational two-body problem currently stands at three loops, corresponding to a 4PM correction for nonspinning black holes [32–35,38,39], as well as including spin-orbit [41,42] and tidal effects [43].

With the objective of calculating Feynman integrals, one task is to characterize the space of functions to which they evaluate. Most Feynman integrals computed to date can be written in terms of multiple polylogarithms [44,45], which are iterated integrals over the Riemann sphere. However, at high loop orders and in cases with non-negligible masses or many physical scales, new special functions start to appear in QFT, involving integrals over nontrivial geometries. These include integrals over elliptic curves, $K3$ surfaces, and higher-dimensional Calabi-Yau manifolds; see Ref. [46] for a recent review. In particular, various L-loop families of Feynman integrals have been identified that involve Calabi-Yau manifolds of dimensions growing linearly with the loop order L [47–56].

Up to two loops (3PM order), the results in the PM expansion are expressible in terms of polylogarithms. However, at three loops they contain products of complete elliptic integrals, which stem from a $K3$ surface [32,33,57]. In this Letter, we initiate an analysis beyond the current state of the art, finding that at four loops a new geometry appears—a Calabi-Yau threefold. This is the first instance that this type of geometry appears in integrals relevant for the scattering and inspiral of black holes, and it indicates that completely new functions are needed for the full analytical result at 5PM order.

In order to detect geometries in Feynman integrals, we use two complementary approaches: differential equations [58] and leading singularities [59]. While leading singularities can, in principle, be calculated using any parametric representation of the Feynman integral as well as the loop-momentum-space representation, we find that a loop-by-loop Baikov representation [60,61] is particularly

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Classifying post-Minkowskian geometries for gravitational waves via loop-by-loop Baikov

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ABSTRACT: We use the loop-by-loop Baikov representation to investigate the geometries in Feynman integrals contributing to the classical dynamics of a black-hole two-body system in the post-Minkowskian expansion of general relativity. These geometries determine the spaces of functions to which the corresponding Feynman diagrams evaluate. As a proof of principle, we provide a full classification of the geometries appearing up to three loops, i.e. fourth post-Minkowskian order, for all diagrams relevant to the conservative as well as the dissipative dynamics, finding full agreement with the literature. Moreover, we show that the non-planar top topology at four loops, which is the most complicated sector with respect to integration-by-parts identities, has an algebraic leading singularity and thus can only depend on non-trivial geometries through its subsectors.

KEYWORDS: Scattering Amplitudes, Black Holes, Classical Theories of Gravity, Differential and Algebraic Geometry

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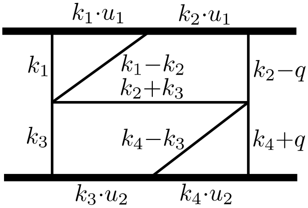
Cutting edge example



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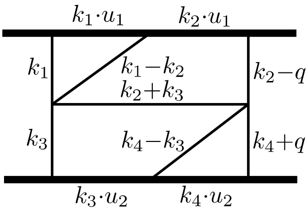

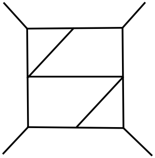

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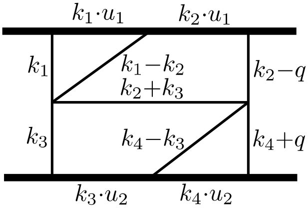
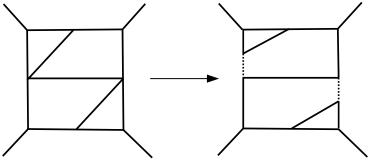
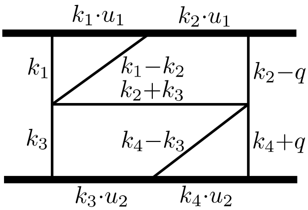
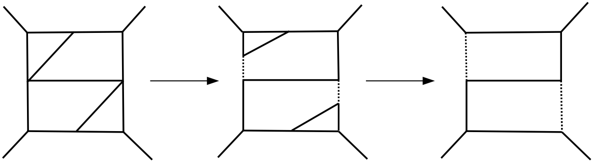




$$P_{12} = (k_3+q)^2\ , \quad P_{13} = k_2^2$$





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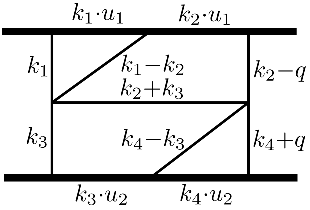
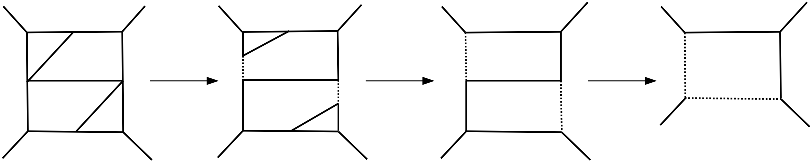
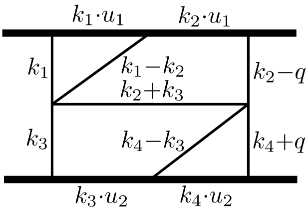




$$P_{12} = (k_3+q)^2 \,, \quad P_{13} = k_2^2 \,, \quad P_{14} = k_2{\cdot}u_2$$



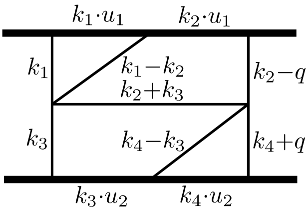

```
In[1]:= << "/home/hjalte/science/baikovcode/BaikovPackage.m";
        facpow = {x1_ ^ x2_ :> Factor[x1] ^ Factor[x2]};

In[3]:= Internal = {k2, k3, k4, k1};
        External = {u1, u2, q};
        Propagators = {k1 ^ 2, k3 ^ 2, (k1 - k2) ^ 2, (k2 + k3) ^ 2, (k4 - k3) ^ 2, (k2 - q) ^ 2, (k4 + q) ^ 2, 2 * u1 * k1, 2 * u1 * k2, 2 * u2 * k3, 2 * u2 * k4};
        PropagatorsExtra = {(k3 + q) ^ 2, k2 ^ 2, 2 * k2 * u2};
        Replacements = {u1 ^ 2 → 1, u2 ^ 2 → 1, u1 * u2 → y, u1 * q → 0, u2 * q → 0, q ^ 2 → -1};

In[8]:= components = BaikovLBL[];
        result1 = BaikovCombine[components];

In[10]:= maxcut = Map[(x[#] → 0) &, Range[Length[Propagators]]];
         result2 = PowerExpand[result1 /. maxcut /. {d → 4} /. {y → (s ^ 2 + 1) / (2 * s)} /. facpow]
```

Cutting edge example

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        PropagatorsExtra = {(k3 + q)^2, k2^2, 2*k2*u2};
        Replacements = {u1^2 → 1, u2^2 → 1, u1*u2 → y, u1*q → 0, u2*q → 0, q^2 → -1};

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```

Out[11]=
$$s \Big/ \Big(64\,\pi^6 \sqrt{x[12]}\,\sqrt{x[13]}\,\sqrt{1 - 2 s^2 + s^4 + 2 x[13] - 4 s^2 x[13] + 2 s^4 x[13] + x[13]^2 - 2 s^2 x[13]^2 + s^4 x[13]^2 + 4 s^2 x[14]^2}$$
$$\sqrt{-4 x[12] \times x[13] - 4 x[12]^2 x[13] - 4 x[12]\, x[13]^2 + x[14]^2 + 2 x[12]\, x[14]^2 + x[12]^2 x[14]^2}\Big)$$

Cutting edge example

Out[11]= $$s \Big/ \left(64\,\pi^6 \sqrt{x[12]}\,\sqrt{x[13]}\,\sqrt{1 - 2 s^2 + s^4 + 2\,x[13] - 4 s^2\,x[13] + 2 s^4\,x[13] + x[13]^2 - 2 s^2\,x[13]^2 + s^4\,x[13]^2 + 4 s^2\,x[14]^2}\;\sqrt{-4\,x[12] \times x[13] - 4\,x[12]^2\,x[13] - 4\,x[12]\,x[13]^2 + x[14]^2 + 2\,x[12]\,x[14]^2 + x[12]^2\,x[14]^2}\right)$$

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Out[11]=

$$s \Big/ \left(64\,\pi^6 \sqrt{x[12]}\,\sqrt{x[13]}\,\sqrt{1-2 s^2+s^4+2\,x[13]-4 s^2\,x[13]+2 s^4\,x[13]+x[13]^2-2 s^2\,x[13]^2+s^4\,x[13]^2+4 s^2\,x[14]^2}\,\sqrt{-4\,x[12]\times x[13]-4\,x[12]^2\,x[13]-4\,x[12]\,x[13]^2+x[14]^2+2\,x[12]\,x[14]^2+x[12]^2\,x[14]^2}\right)$$

In[12]:=

```
ratsq = 1 - 2 s^2 + s^4 + 2 x[13] - 4 s^2 x[13] + 2 s^4 x[13] + x[13]^2 - 2 s^2 x[13]^2 + s^4 x[13]^2 + 4 s^2 x[14]^2;
rsols1 = Solve[ratsq == 0, x[14]];
rsols = Map[(#[[1, 2]]) &, rsols1];
x14rhs = rsols[[1]] - (rsols[[2]] - rsols[[1]]) * (1 - t14) ^2 / (4 * t14);
```

Cutting edge example

Out[11]=
$$s\Big/\left(64\pi^6\sqrt{x[12]}\sqrt{x[13]}\sqrt{1-2s^2+s^4+2x[13]-4s^2x[13]+2s^4x[13]+x[13]^2-2s^2x[13]^2+s^4x[13]^2+4s^2x[14]^2}\right.$$
$$\left.\sqrt{-4x[12]\times x[13]-4x[12]^2x[13]-4x[12]x[13]^2+x[14]^2+2x[12]x[14]^2+x[12]^2x[14]^2}\right)$$

```
In[12]:= ratsq = 1 - 2 s^2 + s^4 + 2 x[13] - 4 s^2 x[13] + 2 s^4 x[13] + x[13]^2 - 2 s^2 x[13]^2 + s^4 x[13]^2 + 4 s^2 x[14]^2;
rsols1 = Solve[ratsq == 0, x[14]];
rsols = Map[(#[[1, 2]]) &, rsols1];
x14rhs = rsols[[1]] - (rsols[[2]] - rsols[[1]]) * (1 - t14) ^2 / (4 * t14);

In[16]:= PowerExpand[Sqrt[ratsq] /. {x[14] → x14rhs} /. facpow]
```

Out[16]=
$$\frac{i\,(-1+s)\,(1+s)\,(-1+t14)\,(1+t14)\,(1+x[13])}{2\,t14}$$

Out[11]=

$$\frac{s}{64\,\pi^6\,\sqrt{x[12]}\,\sqrt{x[13]}\,\sqrt{1-2\,s^2+s^4+2\,x[13]-4\,s^2\,x[13]+2\,s^4\,x[13]+x[13]^2-2\,s^2\,x[13]^2+s^4\,x[13]^2+4\,s^2\,x[14]^2}\,\sqrt{-4\,x[12]\times x[13]-4\,x[12]^2\,x[13]-4\,x[12]\,x[13]^2+x[14]^2+2\,x[12]\,x[14]^2+x[12]^2\,x[14]^2}}$$

In[12]:=

```
ratsq = 1 - 2 s^2 + s^4 + 2 x[13] - 4 s^2 x[13] + 2 s^4 x[13] + x[13]^2 - 2 s^2 x[13]^2 + s^4 x[13]^2 + 4 s^2 x[14]^2;
rsols1 = Solve[ratsq == 0, x[14]];
rsols = Map[(#[[1, 2]]) &, rsols1];
x14rhs = rsols[[1]] - (rsols[[2]] - rsols[[1]]) * (1 - t14) ^2 / (4 * t14);
```

In[16]:=

```
PowerExpand[Sqrt[ratsq] /. {x[14] → x14rhs} /. facpow]
```

Out[16]=

$$\frac{i\,(-1+s)\,(1+s)\,(-1+t14)\,(1+t14)\,(1+x[13])}{2\,t14}$$

In[17]:=

```
result3 = Factor[PowerExpand[D[x14rhs, t14] * result2 /. {x[14] → x14rhs} /. facpow]];
```

In[18]:=

```
t14rhs = t3 / Sqrt[x[12]] / Sqrt[x[13]];
```

In[19]:=

```
result4 = PowerExpand[Factor[D[t14rhs, t3] * result3 /. {t14 → t14rhs} /. {x[12] → t1, x[13] → t2}] /. facpow];
```

Cutting edge example

Out[11]= $s \Big/ \left(64\,\pi^6 \sqrt{x[12]}\,\sqrt{x[13]}\,\sqrt{1-2 s^2+s^4+2 x[13]-4 s^2 x[13]+2 s^4 x[13]+x[13]^2-2 s^2 x[13]^2+s^4 x[13]^2+4 s^2 x[14]^2}\,\sqrt{-4 x[12]\times x[13]-4 x[12]^2 x[13]-4 x[12]\,x[13]^2+x[14]^2+2 x[12]\,x[14]^2+x[12]^2 x[14]^2}\right)$

```
In[12]:= ratsq = 1 - 2 s^2 + s^4 + 2 x[13] - 4 s^2 x[13] + 2 s^4 x[13] + x[13]^2 - 2 s^2 x[13]^2 + s^4 x[13]^2 + 4 s^2 x[14]^2;
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In[16]:= PowerExpand[Sqrt[ratsq] /. {x[14] → x14rhs} /. facpow]
```

Out[16]= $\frac{i\,(-1+s)\,(1+s)\,(-1+t14)\,(1+t14)\,(1+x[13])}{2\,t14}$

```
In[17]:= result3 = Factor[PowerExpand[D[x14rhs, t14] * result2 /. {x[14] → x14rhs} /. facpow]];

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In[19]:= result4 = PowerExpand[Factor[D[t14rhs, t3] * result3 /. {t14 → t14rhs} /. {x[12] → t1, x[13] → t2}] /. facpow];

In[20]:= prefac = I * s / (32 * Pi ^ 6);
         p8eks = (1 + t1) ^ 2 * (1 + t2) ^ 2 * (t1 * t2 + t3 ^ 2) ^ 2 * (1 - s ^ 2) ^ 2 + 64 * t1 ^ 2 * t2 ^ 2 * (1 + t1 + t2) * t3 ^ 2 * s ^ 2;
         result5 = prefac / Sqrt[p8eks]
```

Out[22]= $\frac{i\,s}{32\,\pi^6 \sqrt{64\,s^2\,t1^2\,t2^2\,(1+t1+t2)\,t3^2+\left(1-s^2\right)^2 (1+t1)^2 (1+t2)^2 \left(t1\,t2+t3^2\right)^2}}$

```
In[23]:= Simplify[result4 - result5]
```

Out[23]= 0

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Cutting edge example

Out[11]= $s\Big/\left(64\,\pi^6\,\sqrt{x[12]}\,\sqrt{x[13]}\,\sqrt{1-2\,s^2+s^4+2\,x[13]-4\,s^2\,x[13]+2\,s^4\,x[13]+x[13]^2-2\,s^2\,x[13]^2+s^4\,x[13]^2+4\,s^2\,x[14]^2}\;\sqrt{-4\,x[12]\times x[13]-4\,x[12]^2\,x[13]-4\,x[12]\,x[13]^2+x[14]^2+2\,x[12]\,x[14]^2+x[12]^2\,x[14]^2}\right)$

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In[12]:= ratsq = 1 - 2 s^2 + s^4 + 2 x[13] - 4 s^2 x[13] + 2 s^4 x[13] + x[13]^2 - 2 s^2 x[13]^2 + s^4 x[13]^2 + 4 s^2 x[14]^2;
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In[16]:= PowerExpand[Sqrt[ratsq] /. {x[14] → x14rhs} /. facpow]
```

Out[16]= $\dfrac{i\,(-1+s)\,(1+s)\,(-1+t14)\,(1+t14)\,(1+x[13])}{2\,t14}$

```
In[17]:= result3 = Factor[PowerExpand[D[x14rhs, t14] * result2 /. {x[14] → x14rhs} /. facpow]];

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In[19]:= result4 = PowerExpand[Factor[D[t14rhs, t3] * result3 /. {t14 → t14rhs} /. {x[12] → t1, x[13] → t2}] /. facpow];

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         result5 = prefac / Sqrt[p8eks]
```

Out[22]= $\dfrac{i\,s}{32\,\pi^6\,\sqrt{64\,s^2\,t1^2\,t2^2\,(1+t1+t2)\,t3^2+\left(1-s^2\right)^2\,(1+t1)^2\,(1+t2)^2\,\left(t1\,t2+t3^2\right)^2}}$

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In[23]:= Simplify[result4 - result5]

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```

Calabi-Yau meets Gravity: A Calabi-Yau three-fold at fifth post-Minkowskian order

Discussion

The *Baikov representation* is good for many things

For that reason I have made the Baikov Package

https://github.com/HjalteFrellesvig/BaikovPackage

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A publication will follow soon (this year?)

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Thank you for listening!

Hjalte Frellesvig

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