MAX-PLANCK-INSTITUT
FÜR PHYSIK

Triple real contributions to zero-jettiness soft function at N3LO

in collaboration with Daniel Baranowski, Maximilian Delto, Kirill Melnikov and Andrey Pikelner

Based on *2111.13594* & *2204.09459* & *2401.05245* and work in preparation.

Chen-Yu Wang | 2024-09-11 | HP2 2024

Outline

Motivation

- The ever-increasing experimental precision at the LHC and the HL-LHC in the future demands percent level precision from the theoritical side. *ATLAS 2019; CMS 2021*
- On the theoretical side many N3LO calculations and phenomenology results are available.
- Computing differential cross-section requires subtracting infrared divergences in the phase space:
  - Slicing:
    - q_T subtraction scheme *Catani and Grazzini 2007*
    - N-jettiness subtraction scheme *Boughezal, Focke, et al. 2015; Gaunt et al. 2015*
  - Subtraction:
    - CoLoRFull *Somogyi et al. 2005*
    - Antenna *Gehrmann-De Ridder et al. 2005*
    - STRIPPER *Czakon 2010*
    - Nested soft-collinear subtraction *Caola et al. 2017*
    - Local analytic sector subtraction *Magnea et al. 2018*
  - Project-to-Born *Cacciari et al. 2015*
  - ...

Motivation

- To obtain differential cross sections, one can use slicing to extract and cancel infrared divergences properly:

$$\sigma(O) = \int_0 \mathrm{d}\tau \frac{\mathrm{d}\sigma(O)}{\mathrm{d}\tau} = \int_0^{\tau_0} \mathrm{d}\tau \frac{\mathrm{d}\sigma(O)}{\mathrm{d}\tau} + \int_{\tau_0} \mathrm{d}\tau \frac{\mathrm{d}\sigma(O)}{\mathrm{d}\tau}.$$

  - q_T subtraction scheme *Catani and Grazzini 2007*
  - N-jettiness subtraction scheme *Boughezal, Focke, et al. 2015; Gaunt et al. 2015*
- q_T subtraction scheme is available up to N3LO *Li and Zhu 2017; Ebert et al. 2020b; Luo et al. 2020*
- N-jettiness factorization theorem derived in SCET *Stewart et al. 2010a,b*

$$\lim_{\tau\to 0} \mathrm{d}\sigma(O) = B \otimes B \otimes \sum_i J_i \otimes S_N \otimes H \otimes \mathrm{d}\sigma_{\mathrm{LO}} + \mathcal{O}(\tau).$$

  - Beam function B @ N3LO *Ebert et al. 2020a; Baranowski et al. 2023*
  - Jet function J @ N3LO *Banerjee et al. 2018; Brüser et al. 2018*
  - Soft function S_N @ N2LO *Hornig et al. 2011; Kelley et al. 2011; Monni et al. 2011; Boughezal, Liu, et al. 2015; Bell et al. 2018; Campbell et al. 2018; Jin and Liu 2019*

Definition

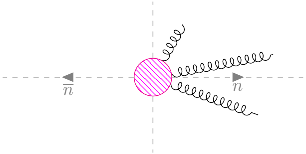
- Zero-jettiness is defined as

$$\tau = \sum_{i=1}^{m} \min_{j\in 1,2} \left[\frac{2q_j \cdot k_i}{Q_j}\right] = \sum_{i=1}^{m} \min\{\alpha_i, \beta_i\}.$$

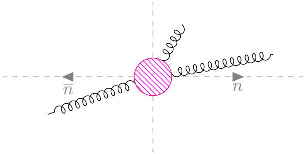
where min(...) can be written out using the **Heaviside θ function**:

$$\delta\left(\tau - \sum_{i=1}^{m} \min\{\alpha_i, \beta_i\}\right) = \delta\left(\tau - \beta_1 - \beta_2 - \ldots\right)\theta(\alpha_1 - \beta_1)\theta(\alpha_2 - \beta_2)\ldots$$
$$+ \delta\left(\tau - \alpha_1 - \beta_2 - \ldots\right)\theta(\beta_1 - \alpha_1)\theta(\alpha_2 - \beta_2)\ldots$$
$$+ \ldots .$$

Sudakov decomposition: $k_i = \frac{\alpha_i}{2} n + \frac{\beta_i}{2}\overline{n} + k_{\perp,i}$, where $\alpha_i = k_i \cdot \overline{n}$, $\beta_i = k_i \cdot n$, and $n \cdot \overline{n} = 2$.

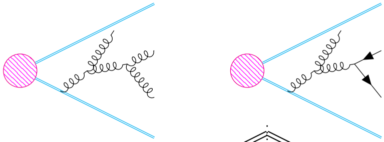


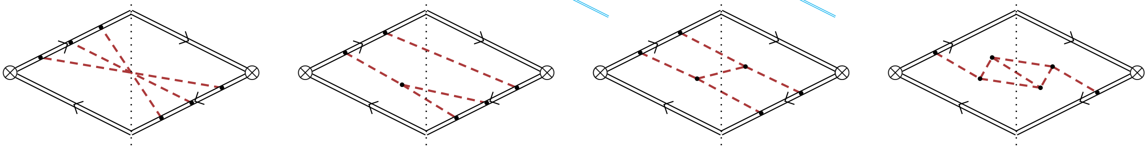
nnn configuration



$nn\overline{n}$ configuration

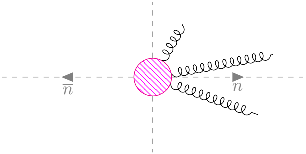
Triple real corrections

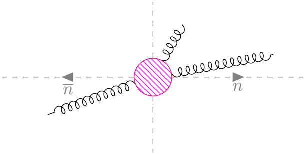




A $\qquad B \sim \frac{1}{k_1 \cdot k_2}$ $\qquad C \sim \frac{1}{(k_1 \cdot k_2)(k_1 \cdot k_3)}$ $\qquad D \sim \frac{1}{(k_1 + k_2 + k_3)^2}$

- Integrate over two configurations:





nnn configuration

$\delta\left(\tau - \beta_1 - \beta_2 - \beta_3\right)\theta(\alpha_1 - \beta_1)\theta(\alpha_2 - \beta_2)\theta(\alpha_3 - \beta_3)$

$nn\overline{n}$ configuration

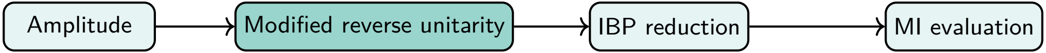
$\delta\left(\tau - \beta_1 - \beta_2 - \alpha_3\right)\theta(\alpha_1 - \beta_1)\theta(\alpha_2 - \beta_2)\theta(\beta_3 - \alpha_3)$

RRR: Procedures

Amplitude → Modified reverse unitarity → IBP reduction → MI evaluation

- Amplitude: generate from scratch $S = \sum_i C_i I_i$.
  - $ggg = ggg + gc\bar{c}$, coincides with known expression in physical gauge [Catani,Colferai,Torrini'19]
  - $gq\bar{q}$ in agreement with [Del Duca,Duhr,Haindl,Liu'23]

RRR: Procedures



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- Generate IBP system: $\int \mathrm{d}^d k \frac{\partial}{\partial k_\mu}\left[p_\mu \frac{1}{\prod_i D_i}\right] = 0.$ [Chetyrkin,Tkachov'81]

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  - Reverse unitarity: transform δ functions to denominators [Anastasiou,Melnikov'02]

$$\delta(p^2 - m^2) = \frac{1}{2\pi} \left[\frac{i}{p^2 - m^2 + i\varepsilon} - \frac{i}{p^2 - m^2 - i\varepsilon} \right].$$

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  - **Modified to also include θ functions**:

$$\int \mathrm{d}^d k_i \frac{\partial}{\partial k_{1\mu}}\left[v_\mu f(k_i)\theta_1(\ldots)\ldots\right] = 0, \qquad \frac{\partial}{\partial \alpha_1}\theta(\alpha_1 - \beta_1) = \delta(\alpha_1 - \beta_1).$$

Modified reverse unitarity

- IBP identities can be constructed for properly regularized integrals:

$$\int \mathrm{d}^d k_i \, \frac{\partial}{\partial k_{1\mu}} \left[v_\mu f(k_i) \theta_1(\ldots) \theta_2(\ldots) \right] = 0, \qquad \frac{\partial}{\partial \alpha_1} \theta(\alpha_1 - \beta_1) = \delta(\alpha_1 - \beta_1),$$

  which generates two kinds of contributions:

$$\int \mathrm{d}^d k_i \, \left\{ \frac{\partial}{\partial k_{1\mu}} \left[v_\mu f(k_i) \right] \theta_1 \theta_2 + f'(k_i) \delta_1 \theta_2 \right\} = 0$$

- The **homogenous** term corresponds to the normal IBP identites without θ functions.
- The **inhomogenous** term introduces new families and requires **partial fraction decomposition**.
  - Auxiliary families with δ functions in place of θ functions are required.

Modified reverse unitarity

$$\frac{\partial}{\partial k_{1\mu}}\left[v_\mu f(k_i)\right]\theta_1\theta_2 + f'(k_i)\delta_1\theta_2 = 0$$

nn configuration	$n\bar{n}$ configuration
$\theta\theta$	$\theta\theta$

- $f_i = \theta$ or δ:

  nn configuration: $\delta\left(\tau - \beta_1 - \beta_2\right) f_1(\alpha_1 - \beta_1) f_2(\alpha_2 - \beta_2)$

  $n\bar{n}$ configuration: $\delta\left(\tau - \beta_1 - \alpha_2\right) f_1(\alpha_1 - \beta_1) f_2(\beta_2 - \alpha_2)$

- Starting from the amplitude with measure $\theta\theta$,

Modified reverse unitarity

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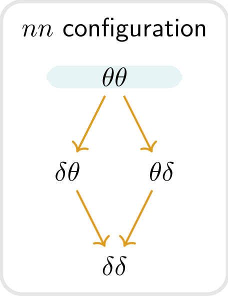

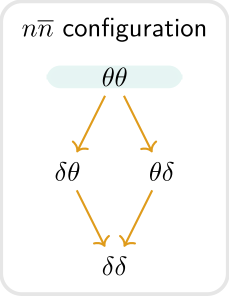

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- Starting from the amplitude with measure $\theta\theta$,
  - **IBP identities** connect measures with fewer θ functions.

Modified reverse unitarity

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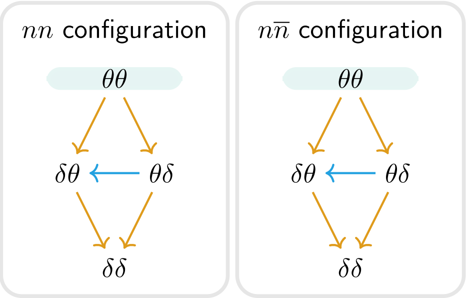

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Modified reverse unitarity

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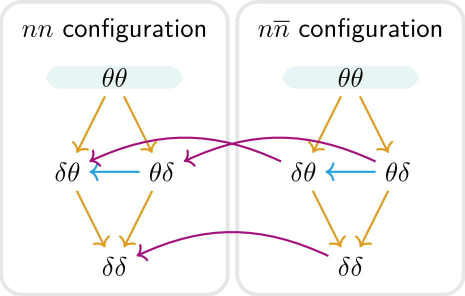

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- More **symmetry relations** between configurations.

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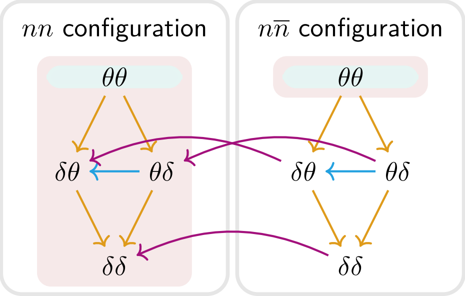

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- Starting from the amplitude with measure $\theta\theta$,
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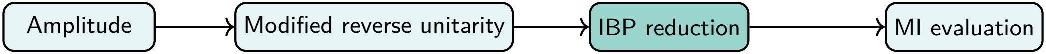
- More **symmetry relations** between configurations.

- Generate the IBP system manually.
- Solve the system with `Kira` and `FireFly` (`reduce_user_defined_system`). [Klappert et al.'21]

RRR: Procedures



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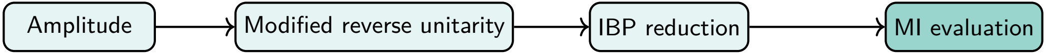

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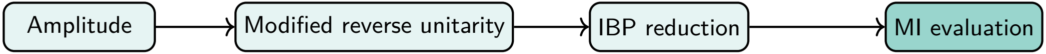

- IBP reduction with `Kira` and `FireFly` (`reduce_user_defined_system`). [Klappert et al.'21]
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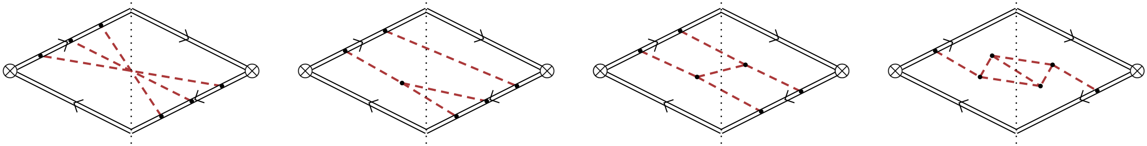
- Direct integration: **Subtract divergence at integrand level and then integrate with** `HyperInt`.



A $\quad B \sim \frac{1}{k_1 \cdot k_2}$ $\quad C \sim \frac{1}{(k_1 \cdot k_2)(k_1 \cdot k_3)}$ $\quad D \sim \frac{1}{(k_1 + k_2 + k_3)^2}$

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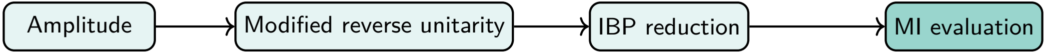


- IBP reduction with `Kira` and `FireFly` (`reduce_user_defined_system`). [Klappert et al.'21]
- Master integral evaluation: $S = \sum_i C'_i I'_i$.
    - Direct integration: **Subtract divergence at integrand level and then integrate with `HyperInt`.**
    - Solve differential equations w.r.t. auxiliary parameters:

$$\frac{1}{(k_1+k_2+k_3)^2} \sim \frac{1}{2k_1\cdot k_2 + 2k_2\cdot k_3 + 2k_3\cdot k_1}$$

$\Longrightarrow$ **add a mass-like auxiliary parameter m^2 and evaluate with DE**

$$\frac{1}{(k_1+k_2+k_3)^2 + m^2}.$$

Evaluation of $1/k_{123}^2$ integrals

- The problematic denominator $1/k_{123}^2$

$$\frac{1}{(k_1+k_2+k_3)^2} \sim \frac{1}{2k_1\cdot k_2 + 2k_2\cdot k_3 + 2k_3\cdot k_1}$$

  involves 3 dot products $\Longrightarrow$ **add a mass-like auxiliary parameter m^2 and take the limit $m^2 \to \infty$**

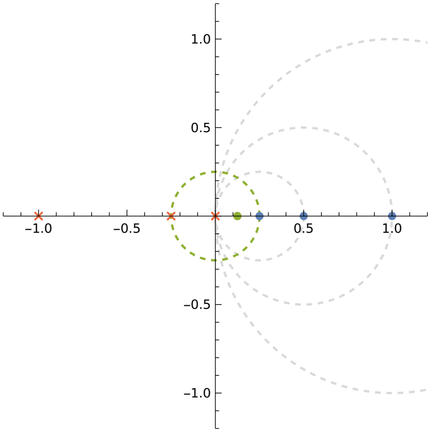
$$\partial_{m^2} J(\varepsilon, m^2) = M(\varepsilon, m^2) J(\varepsilon, m^2), \qquad I_i(\varepsilon) = \lim_{m^2\to 0} J_i(\varepsilon, m^2) = \lim_{m^2\to 0} \int \mathrm{d}\Phi \, \frac{1}{k_{123}^2 + m^2} \frac{\cdots}{\cdots}.$$

- Solve the differential equations **numerically** from $m^2 \to \infty$ to $m^2 \to 0$ [Liu et al.'18][Chen et al.'22]
- System size:
  - ~ 150 integrals for nnn configuration.
  - ~ 650 integrals for $nn\bar{n}$ configuration.

Evaluation of $1/k_{123}^2$ integrals

$$J = \int \mathrm{d}\Phi_{\delta\theta\theta}^{nnn} \frac{1}{(k_{123}^2 + m^2)(k_2 \cdot \overline{n})}$$

m^2 plane



- **Boundary conditions** at $m^2 \to \infty$ involves several regions as the Heaviside functions allow α_i to be large:

$$J|_{m^2\to\infty} = \begin{cases} (m^2)^0 & \text{with } \alpha_1, \alpha_2, \alpha_3 \ll m^2 \\ (m^2)^{-\varepsilon} & \text{with } \alpha_1, \alpha_i \ll m^2, \text{ while } \alpha_j \sim m^2 \\ (m^2)^{-2\varepsilon} & \text{with } \alpha_1 \ll m^2, \text{ while } \alpha_2, \alpha_3 \sim m^2 \end{cases}$$

- The problematic denominator simplifies at the boundary:

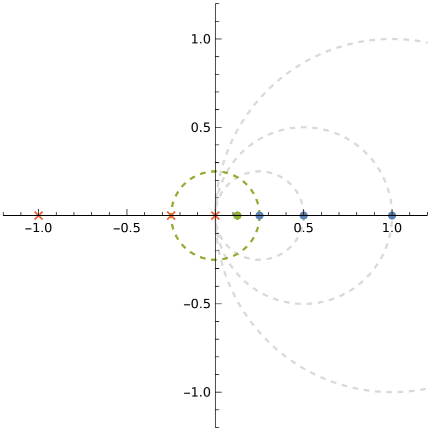
$$k_{123}^2 + m^2 \sim \begin{cases} m^2 & (m^2)^0 \\ \alpha_j(\beta_1 + \beta_i) + m^2 & (m^2)^{-\varepsilon} \\ (\alpha_2 + \alpha_3)\beta_1 + 2k_2 \cdot k_3 + m^2 & (m^2)^{-2\varepsilon} \end{cases}$$

- All ~ 100 boundary conditions can be evaluated by direct integration.

Evaluation of $1/k_{123}^2$ integrals

$$J = \int \mathrm{d}\Phi_{\delta\theta\theta}^{nnn} \frac{1}{(k_{123}^2 + m^2)(k_2 \cdot \overline{n})}$$

m^2 plane



- Analytic continuate to the neighborhood of **physical point** $m^2 = 0$.
- Consistency checks:
  - Integrals should be real on the positive real axis.
  - Integrals should not diverge at singularities on the right half plane.
- Matching at the **physical point** $m^2 = 0$:

$$J = \sum_{i,j,k} c_{ijk}(\varepsilon)(m^2)^{i+j\varepsilon} \ln^k m^2$$

- I corresponds to $\lim_{m^2 \to 0} J(\varepsilon, m^2) = c_{000}(\varepsilon)$.
- Finally we reconstruct the analytical expression.

RRR: (Planned) Procedures

Amplitude → Modified reverse unitarity → IBP reduction → MI evaluation

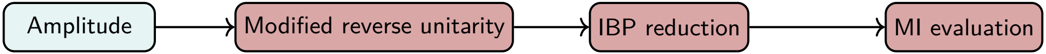
- Amplitude: generate from scratch $S = \sum_i C_i I_i$.
- Generate IBP system using **modified reversed unitarity**:

$$\int \mathrm{d}^d k_i \, \frac{\partial}{\partial k_{1\mu}} \left[v_\mu f(k_i) \theta_1(\ldots) \ldots \right] = 0, \qquad \frac{\partial}{\partial \alpha_1} \theta(\alpha_1 - \beta_1) = \delta(\alpha_1 - \beta_1).$$

- IBP reduction with `Kira` and `FireFly` (`reduce_user_defined_system`).
- Master integral evaluation: $S = \sum_i C_i' I_i'$.
  - Direct integration with `HyperInt`.
  - Solve **differential equations w.r.t. auxiliary parameters**:

$$\partial_{m^2} J(\varepsilon, m^2) = M(\varepsilon, m^2) J(\varepsilon, m^2), \qquad I_i(\varepsilon) = \lim_{m^2 \to 0} J_i(\varepsilon, m^2) = \lim_{m^2 \to 0} \int \mathrm{d}\Phi \, \frac{1}{k_{123}^2 + m^2} \frac{\cdots}{\cdots}.$$

Unregulated integrals



- IBP reduction: unregulated integrals in the IBP system $\Longrightarrow$ wrong reduction!
  - **Additional analytic regulator** $\left(\prod_{i=1}^{3} \min\{\alpha_i, \beta_i\}\right)^{\nu}$ **is required.**
  - Can we get rid of it?
- IBP system generation needs to be fixed.
- Master integral evaluation is also affected as the differential equation is also obtained through the IBP reduction.
  - How does the additional regulator ν interact with the auxiliary parameter m^2?

Unregulated integrals

- Not all integrals appearing during IBP reduction are regulated dimensionally.

$$J = \int \frac{\mathrm{d}\Phi_3 \delta(\tau - \beta_{123})\theta(\alpha_1 - \beta_1)\delta(\alpha_2 - \beta_2)\theta(\alpha_3 - \beta_3)}{(k_1 \cdot k_3)(\alpha_1 + \alpha_2)\alpha_3\beta_1}$$
$$\propto \int \left(\prod_{i=1}^{3} \mathrm{d}\Omega_i^{d-2}\right) \mathrm{d}\xi_1 \mathrm{d}\xi_3 \mathrm{d}\beta_1 \mathrm{d}\beta_2 \mathrm{d}\beta_3 \left(\frac{\beta_1^2\beta_3^2}{\xi_1\xi_3}\right)^{-\varepsilon} \beta_2^{-2\varepsilon} \frac{\delta(1-\beta_{123})}{(\xi_1 + \xi_3 + 2\sqrt{\xi_1\xi_3}\cos\theta_{13})(\beta_1 + \beta_2\xi_1)\beta_3\beta_1}$$

where $\xi_i = \beta_i/\alpha_i$. In the following region

$$\xi_1 \sim \xi_3 \sim \beta_1 \sim \lambda \ll 1,$$

let $\lambda = \xi_1 + \xi_3 + \beta_1$, we have

$$J \sim \int \left(\prod_{i=1}^{3} \mathrm{d}\Omega_i^{d-2}\right) \mathrm{d}\xi_1 \mathrm{d}\xi_3 \mathrm{d}\beta_1 \mathrm{d}\beta_2 \mathrm{d}\beta_3 \delta(1 - \xi_1 - \xi_3 - \beta_1) \int_0 \frac{\mathrm{d}\lambda}{\lambda} \left(\frac{\lambda^2}{\lambda\lambda}\right)^{-\varepsilon} \times \cdots$$

Integration over λ diverges.

Unregulated integrals

- Not all integrals appearing during IBP reduction are regulated dimensionally.

$$J = \int \frac{\mathrm{d}\Phi_3 \delta(\tau - \beta_{123})\theta(\alpha_1 - \beta_1)\delta(\alpha_2 - \beta_2)\theta(\alpha_3 - \beta_3)}{(k_1 \cdot k_3)(\alpha_1 + \alpha_2)\alpha_3\beta_1} (\beta_1\beta_2\beta_3)^{\nu}$$

$$\propto \int \left(\prod_{i=1}^{3} \mathrm{d}\Omega_i^{d-2}\right) \mathrm{d}\xi_1 \mathrm{d}\xi_3 \mathrm{d}\beta_1 \mathrm{d}\beta_2 \mathrm{d}\beta_3 \left(\frac{\beta_1^2\beta_3^2}{\xi_1\xi_3}\right)^{-\varepsilon} \beta_2^{-2\varepsilon} \frac{\delta(1-\beta_{123})(\beta_1\beta_2\beta_3)^{\nu}}{(\xi_1 + \xi_3 + 2\sqrt{\xi_1\xi_3}\cos\theta_{13})(\beta_1 + \beta_2\xi_1)\beta_3\beta_1}$$

where $\xi_i = \beta_i/\alpha_i$. In the following region

$$\xi_1 \sim \xi_3 \sim \beta_1 \sim \lambda \ll 1,$$

let $\lambda = \xi_1 + \xi_3 + \beta_1$, we have

$$J \sim \int \left(\prod_{i=1}^{3} \mathrm{d}\Omega_i^{d-2}\right) \mathrm{d}\xi_1 \mathrm{d}\xi_3 \mathrm{d}\beta_1 \mathrm{d}\beta_2 \mathrm{d}\beta_3 \delta(1 - \xi_1 - \xi_3 - \beta_1) \int_0 \frac{\mathrm{d}\lambda}{\lambda} \left(\frac{\lambda^2}{\lambda\lambda}\right)^{-\varepsilon} \lambda^{\nu} \times \cdots$$

Integration over λ regulated by ν.

Unregulated integrals

- Requires 3 real emissions to produce non-vanishing contribution.
- Additional analytic regulator to the integration measure $\mathrm{d}\Phi_3 \left(\prod_{i=1}^{3} \min\{\alpha_i, \beta_i\}\right)^{\nu}$ is required.
- IBP identities need to be modified as

$$\frac{\partial}{\partial \beta_1} \beta_1^{\nu} = \beta_1^{\nu} \frac{\nu}{\beta_1}.$$

Similar to the modified reverse unitarity:

$$\frac{\partial}{\partial k_{1\mu}} \left[v_\mu f(k_i)\right] \theta_1\theta_2\theta_3\beta_1^{\nu}\beta_2^{\nu}\beta_3^{\nu} + f'(k_i)\delta_1\theta_2\theta_3\beta_1^{\nu}\beta_2^{\nu}\beta_3^{\nu} + \nu\frac{f''(k_i)}{\beta_1}\theta_1\theta_2\theta_3\beta_1^{\nu}\beta_2^{\nu}\beta_3^{\nu} = 0$$

$\Longrightarrow$ more auxiliary families & partial fraction decomposition.

- IBP reduction is more challenging:
  - Possible for the amplitude reduction with a good basis choice.
  - Hard to obtain differential equation with full dependence of ε, ν, and m^2.

Unregulated integrals: Analytic regulator

- Modified IBP identities:

$$\frac{\partial}{\partial k_{1\mu}}\left[v_\mu f(k_i)\right]\theta_1\theta_2\theta_3\beta_1^\nu\beta_2^\nu\beta_3^\nu + f'(k_i)\delta_1\theta_2\theta_3\beta_1^\nu\beta_2^\nu\beta_3^\nu + \nu\frac{f''(k_i)}{\beta_1}\theta_1\theta_2\theta_3\beta_1^\nu\beta_2^\nu\beta_3^\nu = 0$$

$\Longrightarrow$ more auxiliary families & partial fraction decomposition.

- IBP reduction is more challenging: a good basis choice is required.
- The limit of auxiliary parameter $m^2 \to 0$ should be taken before $\nu \to 0$:

$$\partial_{m^2} J(\varepsilon,\nu,m^2) = M(\varepsilon,\nu,m^2)J(\varepsilon,\nu,m^2)$$

$$, I_i(\varepsilon) = \lim_{\nu\to 0}\lim_{m^2\to 0} J_i(\varepsilon,\nu,m^2) = \lim_{\nu\to 0}\lim_{m^2\to 0}\sum_{i,j,k,l} c_{ijk}(\varepsilon,\nu)(m^2)^{i+j\varepsilon+k\nu}\ln^l m^2.$$

- Can we get rid of the regulator while obtaining a correct IBP reduction?

Unregulated integrals: Filtering out unregulated integrals

- Not all integrals appearing during IBP reduction are regulated dimensionally.

$$J = \int \frac{\mathrm{d}\Phi_3 \delta(\tau - \beta_{123})\theta(\alpha_1 - \beta_1)\delta(\alpha_2 - \beta_2)\theta(\alpha_3 - \beta_3)}{(k_1 \cdot k_3)(\alpha_1 + \alpha_2)\alpha_3\beta_1}$$

$$\propto \int \left(\prod_{i=1}^{3} \mathrm{d}\Omega_i^{d-2}\right) \mathrm{d}\xi_1 \mathrm{d}\xi_3 \mathrm{d}\beta_1 \mathrm{d}\beta_2 \mathrm{d}\beta_3 \left(\frac{\beta_1^2\beta_3^2}{\xi_1\xi_3}\right)^{-\varepsilon} \beta_2^{-2\varepsilon} \frac{\delta(1-\beta_{123})}{(\xi_1 + \xi_3 + 2\sqrt{\xi_1\xi_3}\cos\theta_{13})(\beta_1 + \beta_2\xi_1)\beta_3\beta_1}$$

where $\xi_i = \beta_i/\alpha_i$. In the following region

$$\xi_1 \sim \xi_3 \sim \beta_1 \sim \lambda \ll 1,$$

let $\lambda = \xi_1 + \xi_3 + \beta_1$, we have

$$J \sim \int \left(\prod_{i=1}^{3} \mathrm{d}\Omega_i^{d-2}\right) \mathrm{d}\xi_1 \mathrm{d}\xi_3 \mathrm{d}\beta_1 \mathrm{d}\beta_2 \mathrm{d}\beta_3 \delta(1 - \xi_1 - \xi_3 - \beta_1) \int_0 \frac{\mathrm{d}\lambda}{\lambda} \left(\frac{\lambda^2}{\lambda\lambda}\right)^{-\varepsilon} \times \cdots$$

Integration over λ diverges.

Unregulated integrals: Filtering out unregulated integrals

- Not all integrals appearing during IBP reduction are regulated dimensionally.

$$J = \int \frac{\mathrm{d}\Phi_3 \delta(\tau - \beta_{123})\theta(\alpha_1 - \beta_1)\delta(\alpha_2 - \beta_2)\theta(\alpha_3 - \beta_3)}{(k_1 \cdot k_3)(\alpha_1 + \alpha_2)\alpha_3\beta_1} \frac{1}{\alpha_3}$$

$$\propto \int \left(\prod_{i=1}^{3} \mathrm{d}\Omega_i^{d-2}\right) \mathrm{d}\xi_1 \mathrm{d}\xi_3 \mathrm{d}\beta_1 \mathrm{d}\beta_2 \mathrm{d}\beta_3 \left(\frac{\beta_1^2\beta_3^2}{\xi_1\xi_3}\right)^{-\varepsilon} \beta_2^{-2\varepsilon} \frac{\delta(1-\beta_{123})\xi_3/\beta_3}{(\xi_1 + \xi_3 + 2\sqrt{\xi_1\xi_3}\cos\theta_{13})(\beta_1 + \beta_2\xi_1)\beta_3\beta_1}$$

where $\xi_i = \beta_i/\alpha_i$. In the following region

$$\xi_1 \sim \xi_3 \sim \beta_1 \sim \lambda \ll 1,$$

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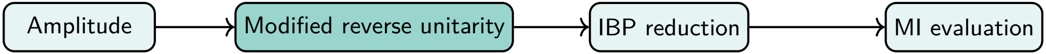
Integration over λ regulated by additional power of denominators.

Unregulated integrals: Filtering out unregulated integrals

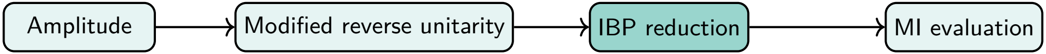
- Observation: unregularized integrals only appear in the IBP system not in the amplitude, and not many integrals are unregulated.
- Filter the regulated IBP system:
  - Generate IBP relations with the analytic regulator.
  - Filter away all unregulated integrals in the IBP system. $\Longleftarrow$ **requires proper power counting**
  - Set the regulator to 0.
- **Correct and fast** reduction.

RRR: Procedures



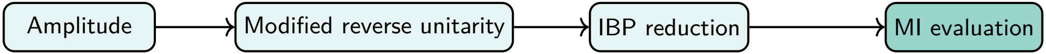
- Generate IBP system:
  - Generate IBP relations with the analytic regulator.
  - Filter away all unregulated integrals in the IBP system and set the regulator to 0.

RRR: Procedures



- Generate IBP system:
  - Generate IBP relations with the analytic regulator.
  - Filter away all unregulated integrals in the IBP system and set the regulator to 0.
- IBP reduction:
  - Maintains similar performance as the unregulated system.
  - Constructing differential equation is efficient.

RRR: Procedures



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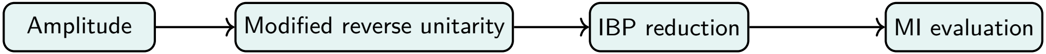
  - Constructing differential equation is efficient.
- Solving differential equation in the same way as unregulated system.

RRR: Procedures



- Generate IBP system:
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- IBP reduction:
  - Maintains similar performance as the unregulated system.
  - Constructing differential equation is efficient.
- Solving differential equation in the same way as unregulated system.
- We obtained high precision numerical result for the triple real corrections to zero-jettiness soft function at N3LO.

Conclusion

- N3LO QCD corrections are crucial to the percent level phenomenology at LHC and HL-LHC.
- We obtained high precision numerical result for the triple real corrections to zero-jettiness soft function at N3LO.
- Efficient **reduction techniques for integrals with Heaviside θ functions** applicable for phase-space integrals with loops and additional regulators.
- Powerful method for evaluating complicated RRR integrals by **solving differential equations in auxiliary parameter numerically** with high precision and calculated all needed boundary conditions.