Nf-part of the NNLO virtual correction to electroweak vector boson production using direct numerical integration

based on arXiv:2407.18051

in collaboration with Matilde Vicini

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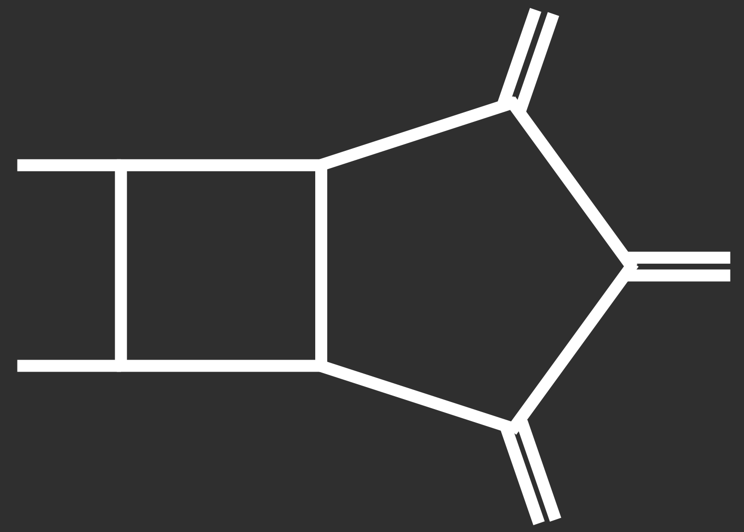
Dario Kermanschah
HP2, 11 September 2024

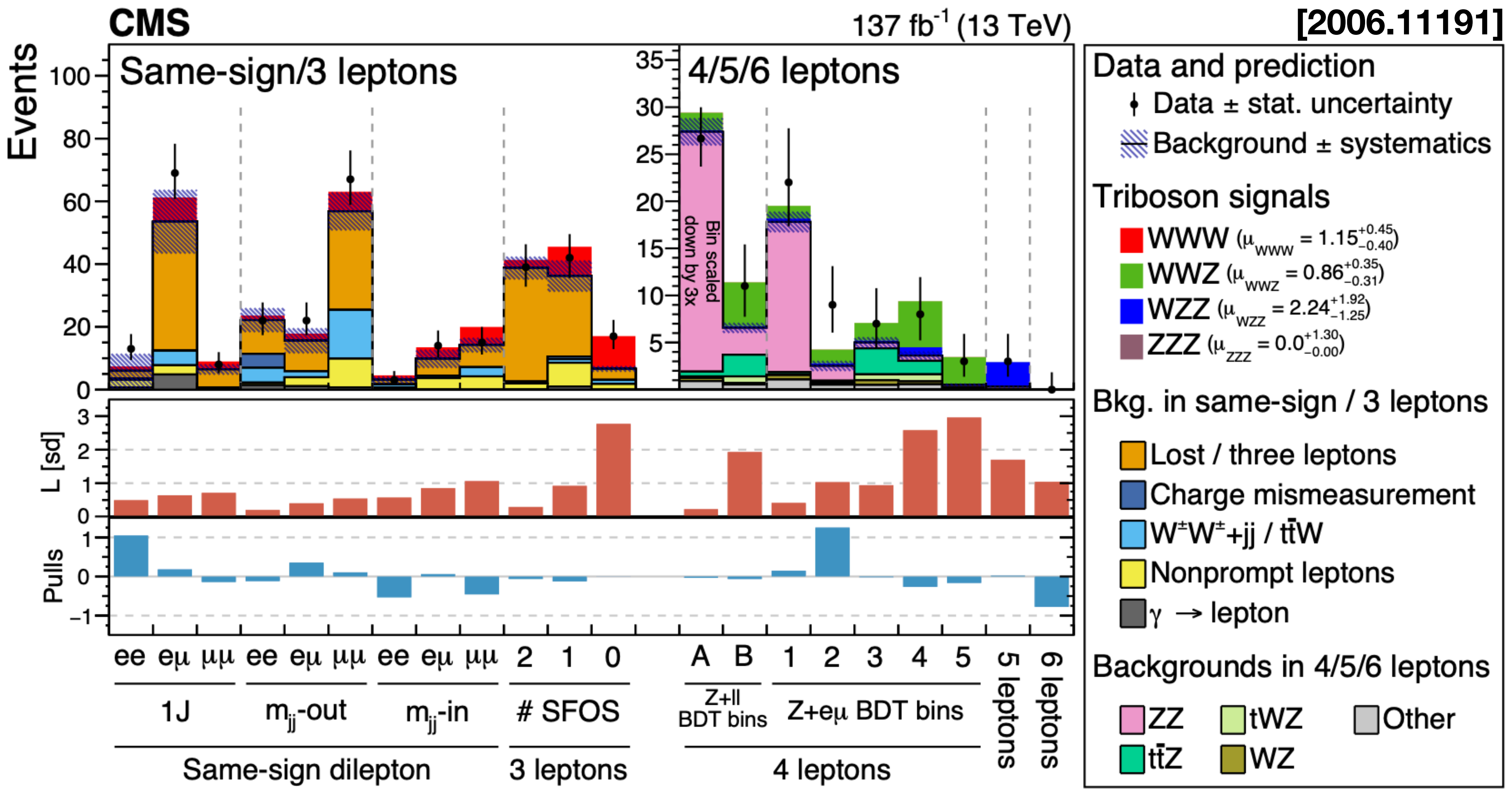
How to conquer multi-scale multi-loop calculations?



- Full two-loop amplitudes beyond 2 → 3 massless particles unavailable
- Overwhelming complexity of IBP reduction & unknown Master Integrals
- NNLO calculations become analytically intractable… resort to numerical methods!

Why vector boson production?

- Uncharted territory: 3 massive bosons at two loops
- Fewer IR singularities: only ISR (no FSR)
- ATLAS and CMS become sensitive to three Z / W production, test quartic gauge-boson couplings & light-quark Yukawa couplings, BSM…



[2006.11191]

Our approach for the two-loop virtual contribution:
Local subtraction & direct numerical integration

Anastasiou, Haindl, Karlen,
Sterman, Venkata, Yang, Zeng
[2403.13712,2212.12162,
2008.12293,1812.03753]

finite remainder: $R^{(2)} = M^{(2)} \underbrace{-\frac{2\beta_0}{\epsilon}M^{(1)}}_{\text{UV renorm.}} \underbrace{-\mathbf{Z}^{(1)}M^{(1)} - \mathbf{Z}^{(2)}M^{(0)}}_{\text{Catani IR poles}}$

$C_{\text{IR\&UV}}$ from local factorisation

see Julia Karlen's talk!

hard scattering amplitude

$$M^{(2)}_{\text{hard}} = M^{(2)} \boxed{-C_{\text{IR\&UV}}}$$

- finite in $D = 4$ dimensions, no dim reg. ($\gamma^5 \dots$)
- integrate numerically with Monte Carlo
- directly in momentum space
- no IBPs, no Master integrals, no sector decomposition

renormalisation & factorisation scheme change

$$\boxed{+C_{\text{IR\&UV}}} - \frac{2\beta_0}{\epsilon}M^{(1)} - \mathbf{Z}^{(1)}M^{(1)} - \mathbf{Z}^{(2)}M^{(0)}$$

calculate analytically in $D = 4 - 2\epsilon$ dimensions

$$= c_1\, M^{(1)}_{\text{hard}} + c_0\, M^{(0)}$$

interfere with tree & integrate over phase space

to get the virtual cross section: $\int \mathrm{d}\Pi \; \sum_{\text{hel.}} |M|^2$

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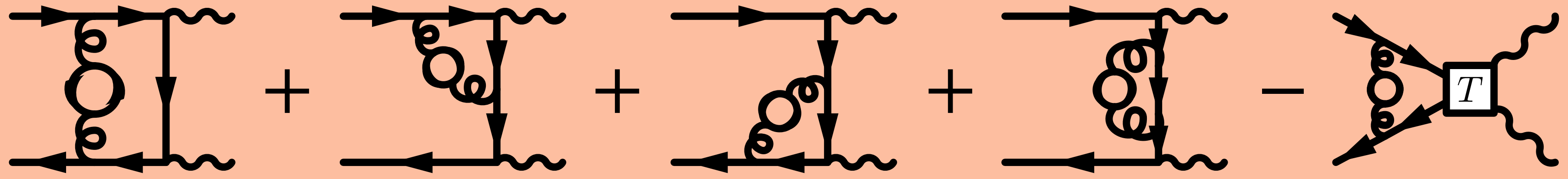
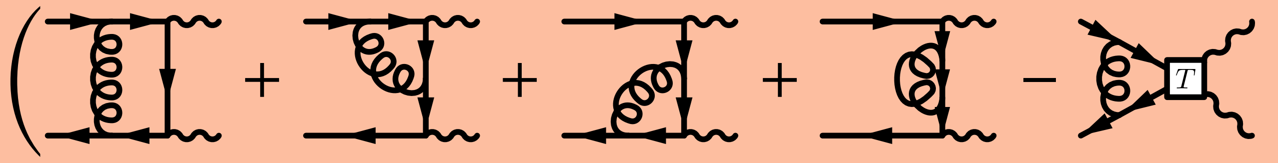
Two-loop N_f-part $\mathscr{M}^{(2,N_f)}_{\mathrm{hard}}$

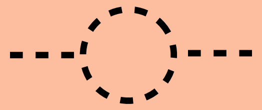


$-C_{\mathrm{UV}}^{(2,N_f)} + \ldots$

$$\sim \underbrace{\left(\frac{1}{l^2(l+k)^2} - \frac{1}{(l^2-M^2)^2}\right)}$$



$- C_{\mathrm{UV}}^{(1)} + \ldots \Big)$

=  $- C_{\mathrm{UV}}^{\mathrm{bubble}}$

$$= \mathscr{M}^{(1)}_{\mathrm{hard}}$$

renormalisation & factorisation scheme change

$$\boxed{+C_{\mathrm{IR\&UV}}} - \frac{2\beta_0}{\epsilon}M^{(1)} - \mathbf{Z}^{(1)}M^{(1)} - \mathbf{Z}^{(2)}M^{(0)}$$

calculate analytically in $D = 4 - 2\epsilon$ dimensions

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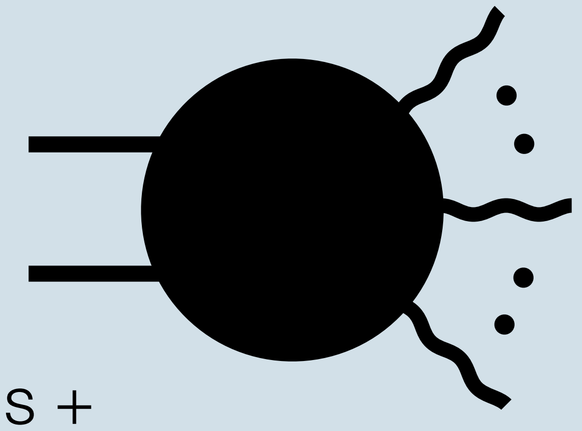
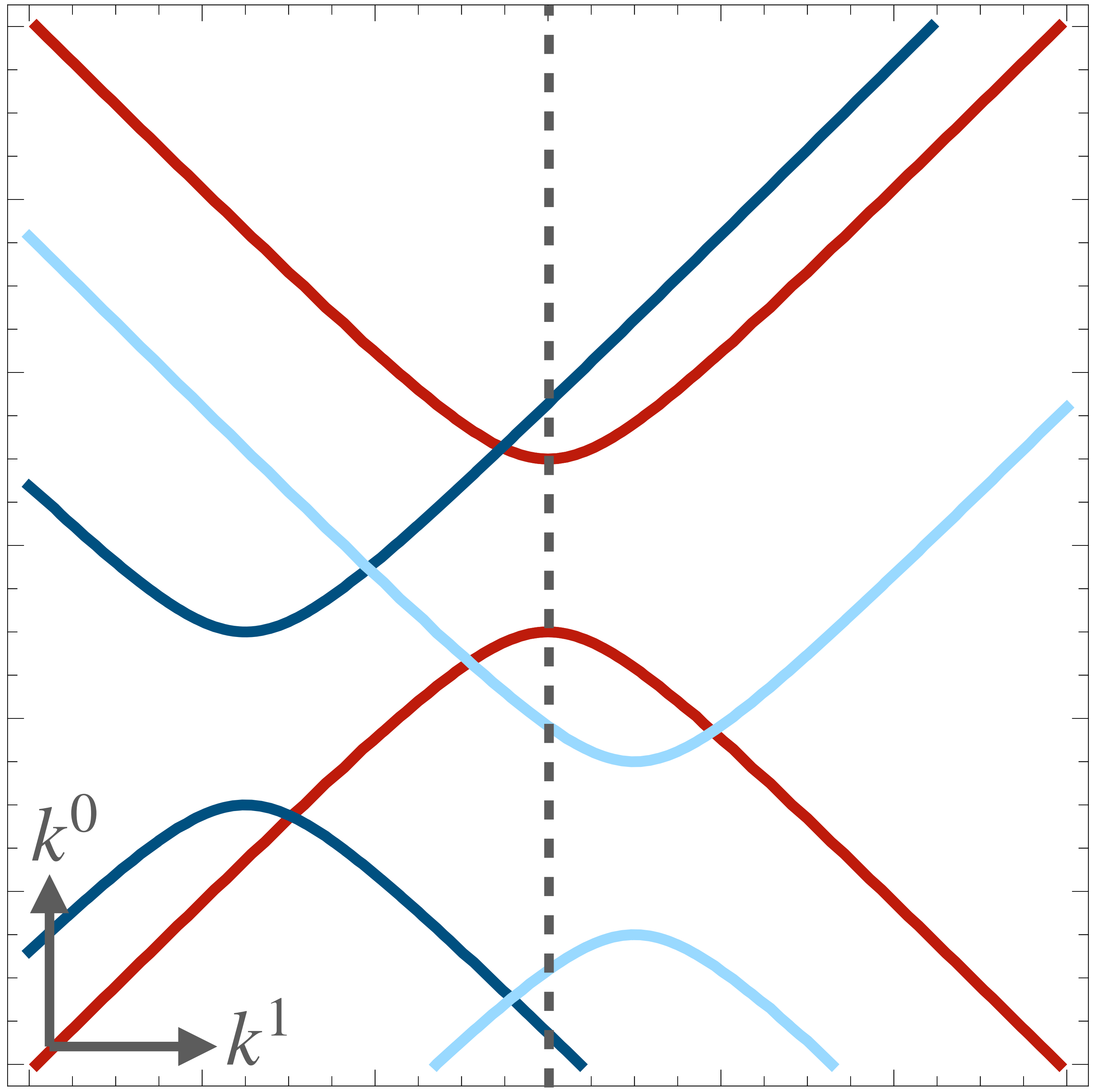
Local singularities of finite loop integrals

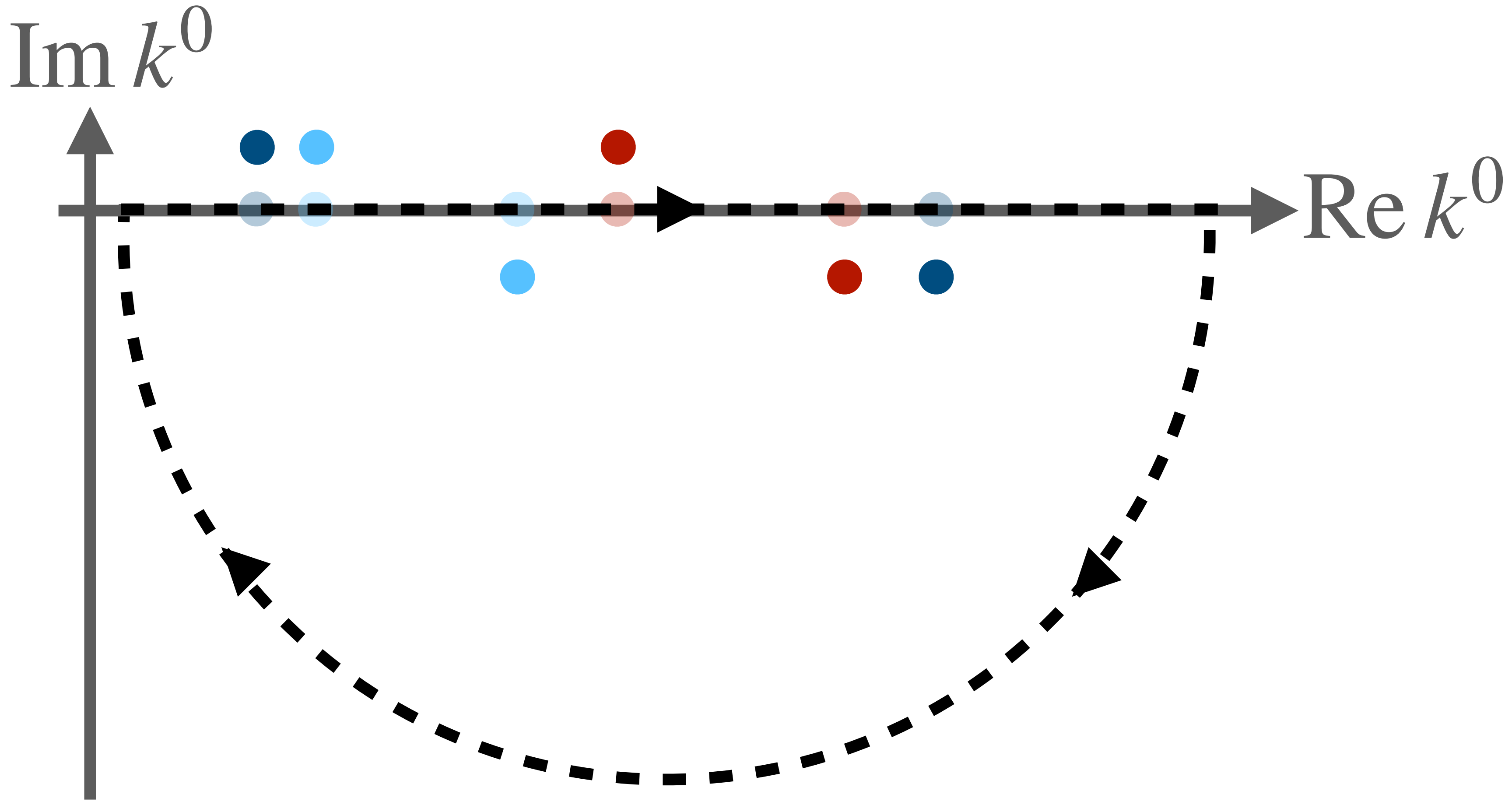


$$M_{\text{hard}} = \underset{\substack{\text{Feyn. diagrams +} \\ \text{local IR \& UV CTs}}}{\sum}$$

$$= \lim_{\epsilon \to 0} \int [\mathrm{d}^4 k] \sum \cdots \frac{\cdots}{q_i^2 - m_i^2 + \mathrm{i}\epsilon} \cdots$$

- ↯ poles in the integration domain
- ✔ causal prescription
- ⚠ implement causal prescription for numerical integration
  → analytic integration over k^0



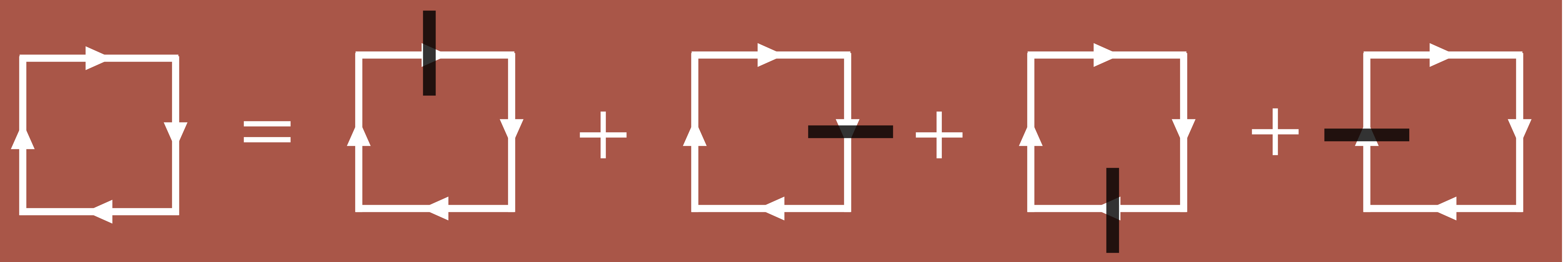


Loop integrals are rational functions in the energy component of the loop momentum
→ integrate using the residue theorem: from D to $D-1$ integration dimensions per loop

Catani, Rodrigo et al. [0804.3170], ETHZ [1906.06138], Mainz [1902.02135], Valencia [2001.03564, 2010.12971]

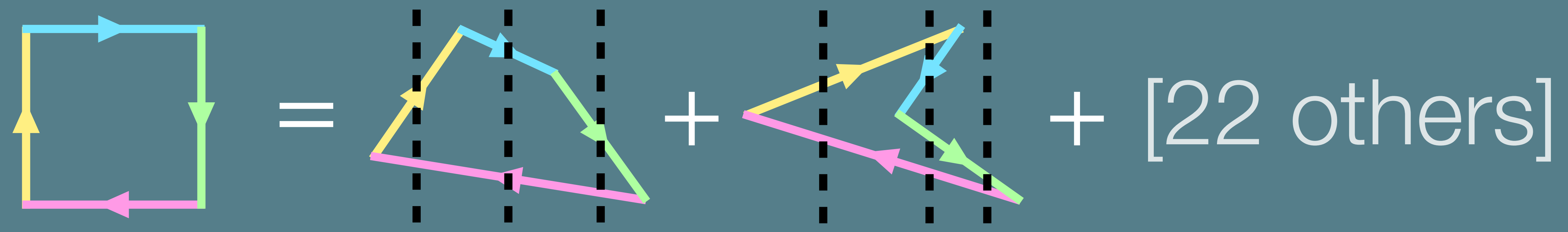
Loop-Tree Duality

compact expression but problematic spurious singularities and derivatives for raised propagators



Time-Ordered Perturbation Theory

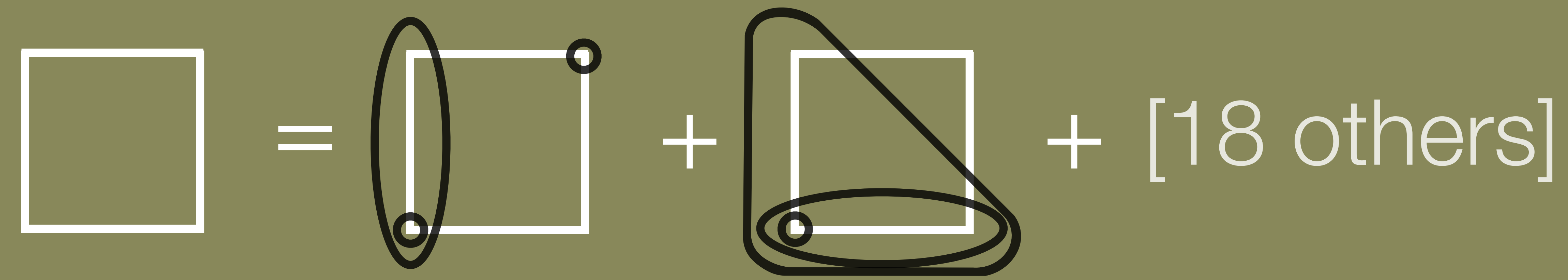
still some spurious singularities and more terms



Capatti [2211.09653]

Cross-Free Family representation

no spurious singularities

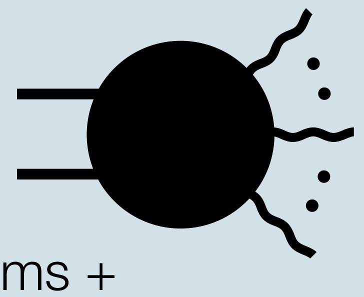


+ many others …

ETHZ [2009.05509], Mainz [2208.01060], Stony Brook [2309.13023], Valencia [2006.11217, 2112.09028, 2103.09237, 2102.05062]

Threshold singularities

$$M_{\mathrm{hard}} = \sum$$



Feyn. diagrams +
local IR & UV CTs

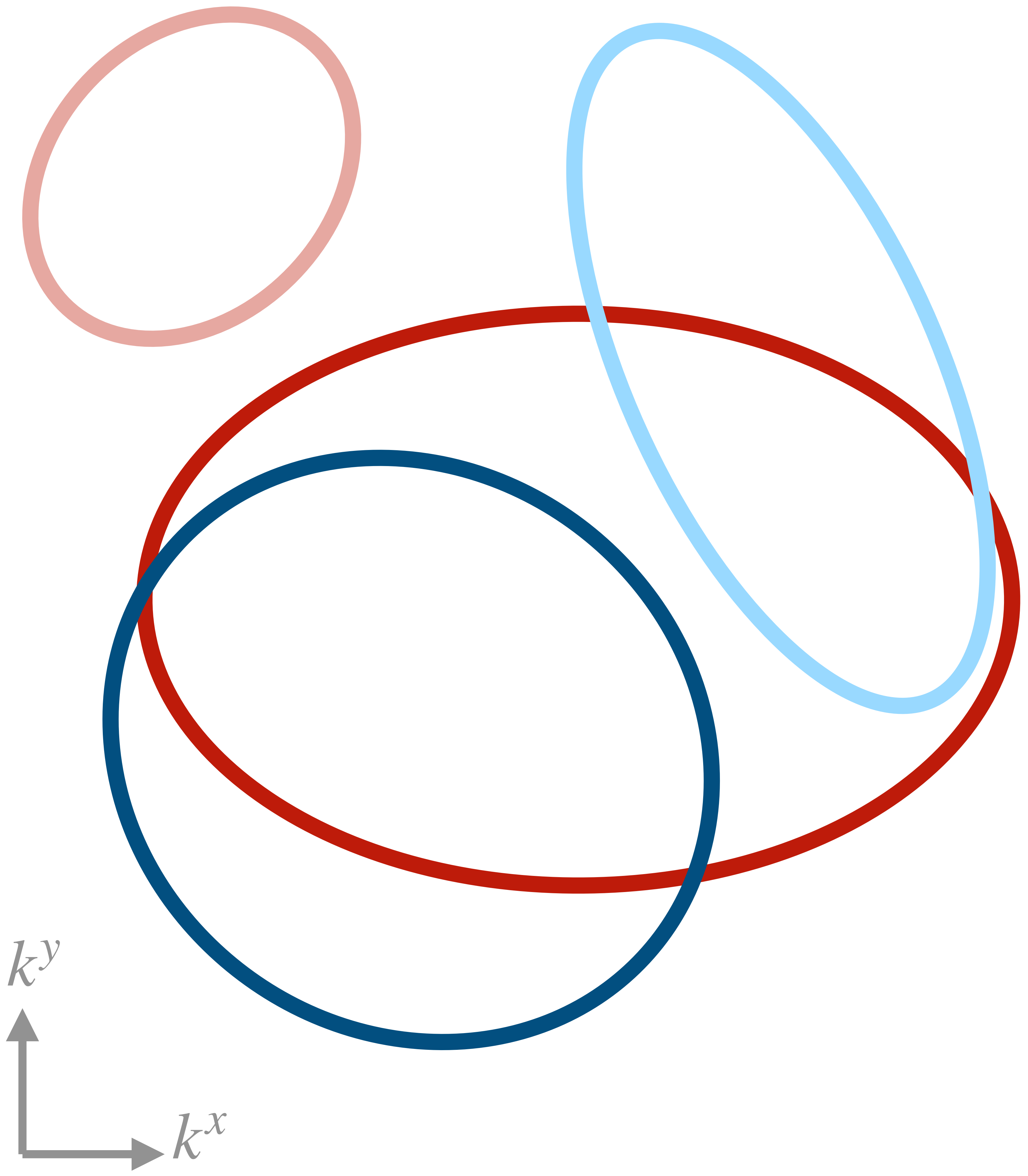
$$= \lim_{\epsilon\to0} \int [\mathrm{d}^4 k] \sum \cdots \frac{\cdots}{q_i^2 - m_i^2 + \mathrm{i}\epsilon} \cdots$$

$$= \lim_{\epsilon\to0} \int [\mathrm{d}^3 \vec{k}] \sum \cdots \frac{\cdots}{E_1 + E_2 - \mathrm{i}\epsilon} \cdots$$

$$E_i = \sqrt{\vec{q}_i^{\,2} + m_i^2}$$

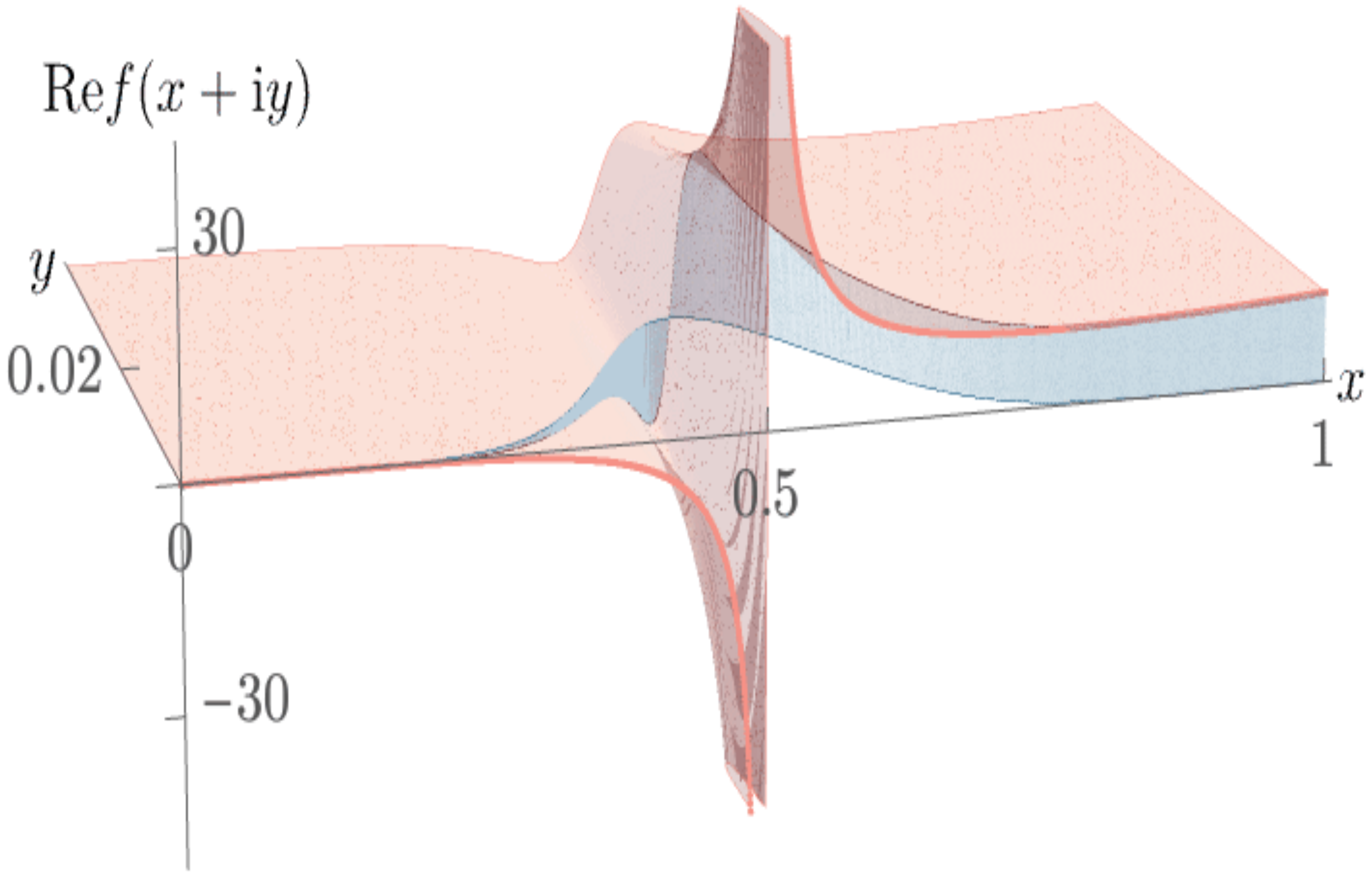
- ↯ poles in the integration domain
- ✔ causal prescription
- ⚠ implement causal prescription for numerical integration

→ Same problems? Yes but fewer integration dimensions & fewer integrand singularities in compact region!



contour deformation

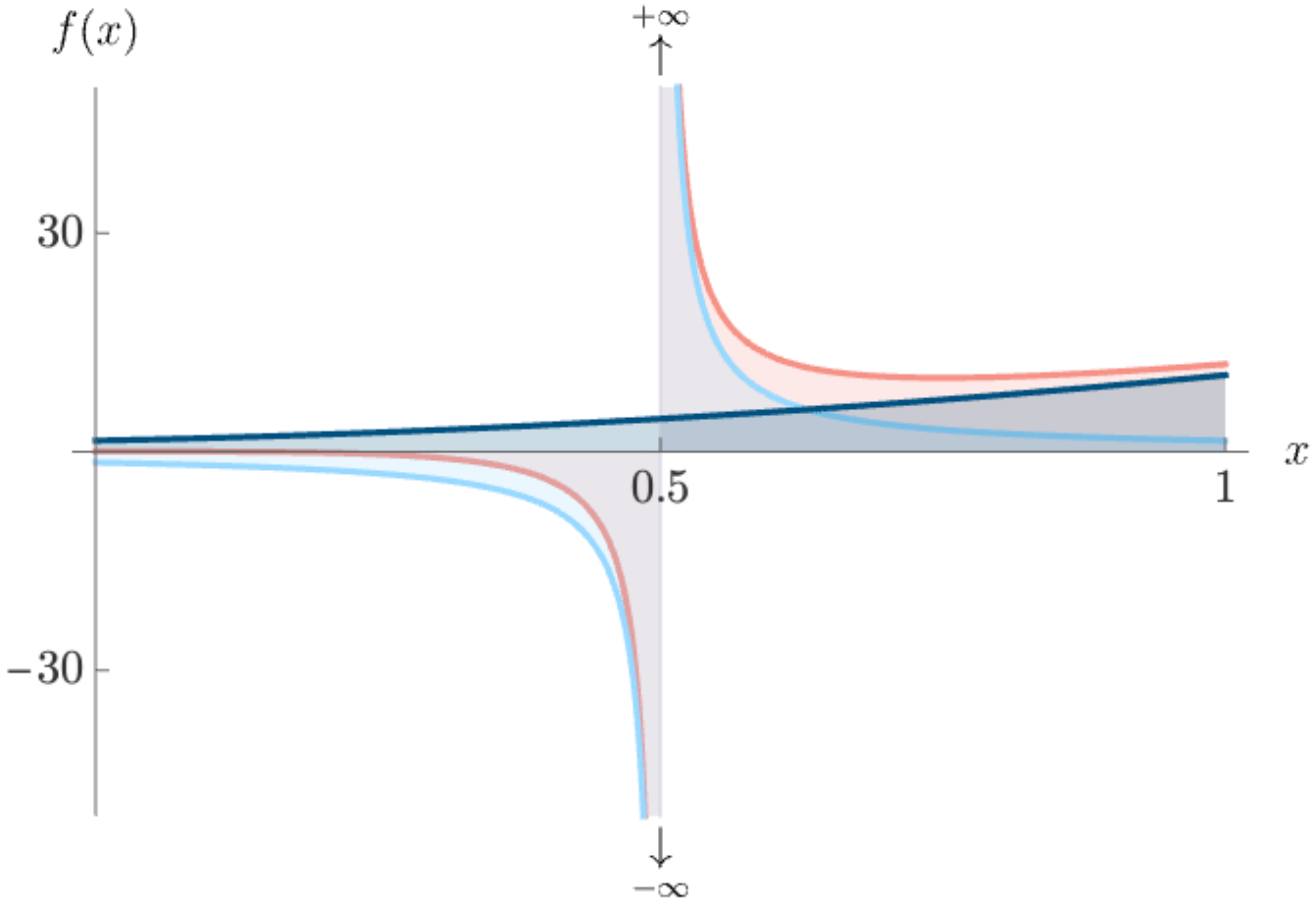
$$\mathbb{R} \to \mathbb{C}$$



$$\frac{1}{1000000}\sum_{i=1}^{1000000} \mathrm{Re} f(z(x_i)) j_z(x_i) = 4.9948$$

subtraction

$$\frac{1}{x - x_0 + \mathrm{i}\epsilon} = \mathrm{PV}\frac{1}{x - x_0} - \mathrm{i}\pi\delta(x_0)$$



$$\frac{1}{1000000}\sum_{i=1}^{1000000} (f(x_i) - f_{\mathrm{ct}}(x_i)) = 5.0008$$

DK: [2110.06869]
DK, Matilde Vicini [2407.21511]

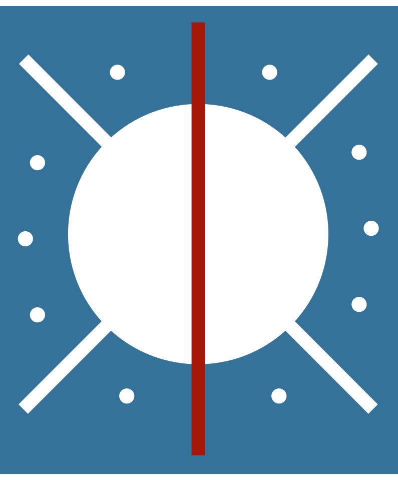
Subtraction of threshold singularities

around a threshold the integrand behaves as

$$\mathscr{J} \sim \frac{\mathrm{Res}_i\, \mathscr{J}}{|\vec{k}| - k_i \pm \mathrm{i}\epsilon} \quad \to \quad \mathrm{CT}_i \quad \text{threshold counterterm}$$

$$\mathrm{Re}\, I = \int [\mathrm{d}^3\vec{k}] \left(\mathscr{J} - \sum_i \mathrm{CT}_i \right) \qquad \text{dispersive part}$$

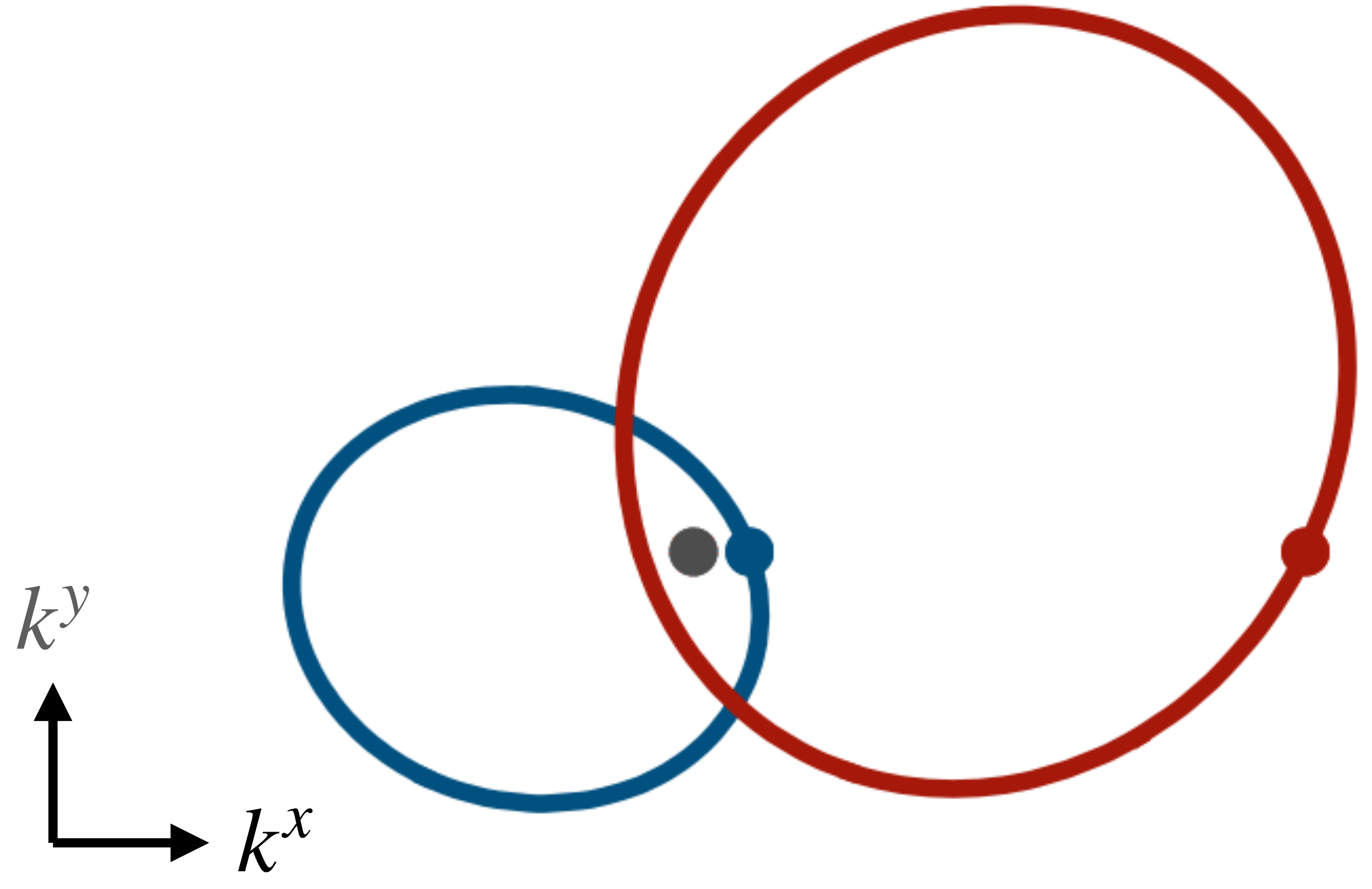
$$\int [\mathrm{d}^3\vec{k}]\, \mathrm{CT}_i = \mathrm{i}\pi \int \mathrm{d}\Pi\, \mathrm{Res}_i\, \mathscr{J} \quad =$$



phase space integral

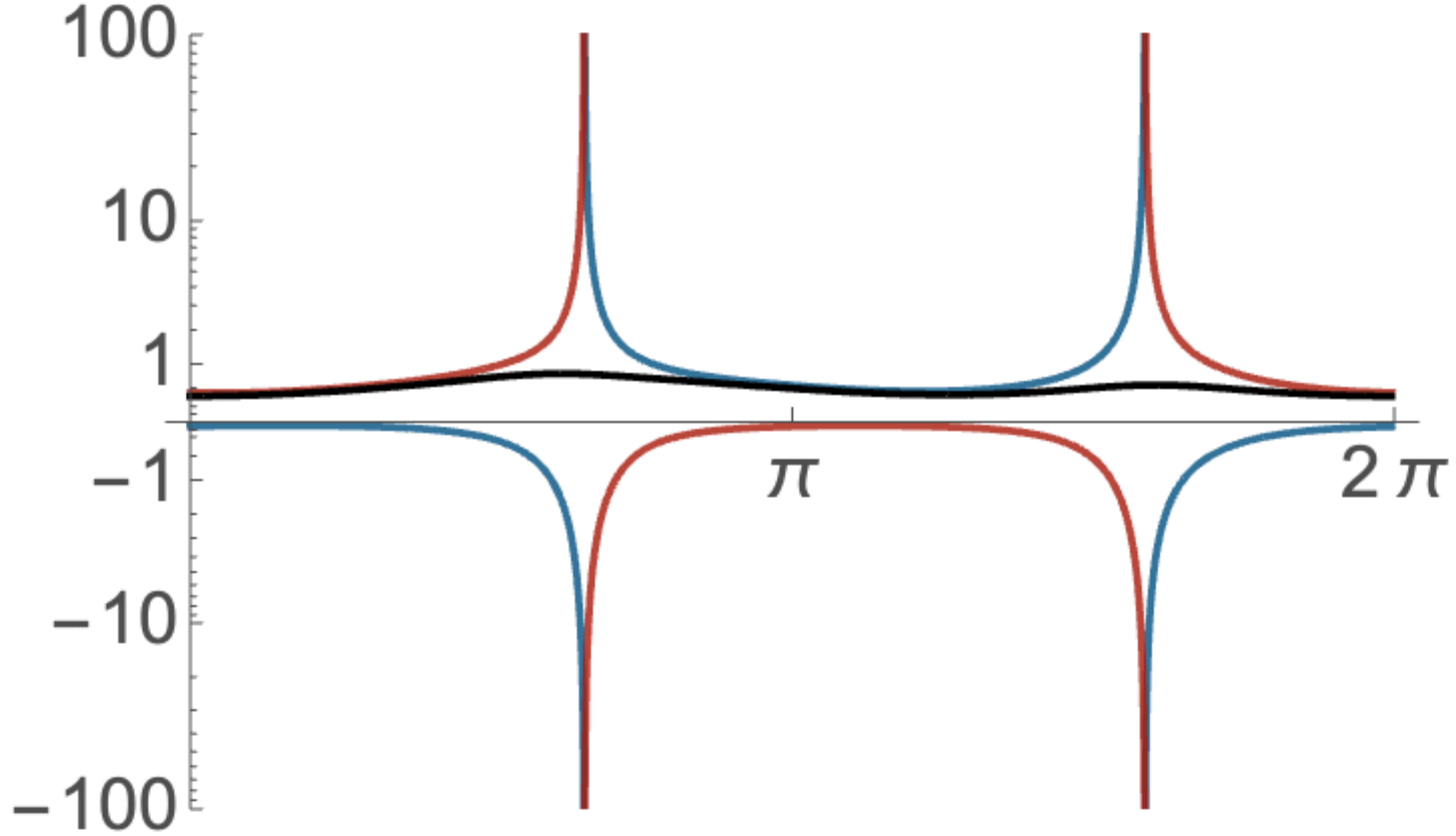
$$\mathrm{Im}\, I = \pi \int \mathrm{d}\Pi \sum_i \mathrm{Res}_i\, \mathscr{J} \qquad \text{absorptive part}$$

local optical theorem: all singularities incl. IR cancel



parameterisation aligns singularities

$$\mathrm{Res}_{\circ}\, \mathscr{J} + \mathrm{Res}_{\circ}\, \mathscr{J}$$



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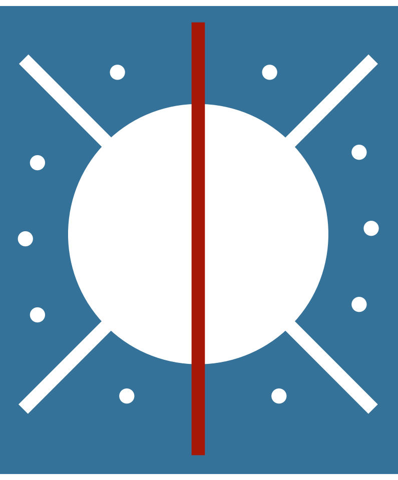

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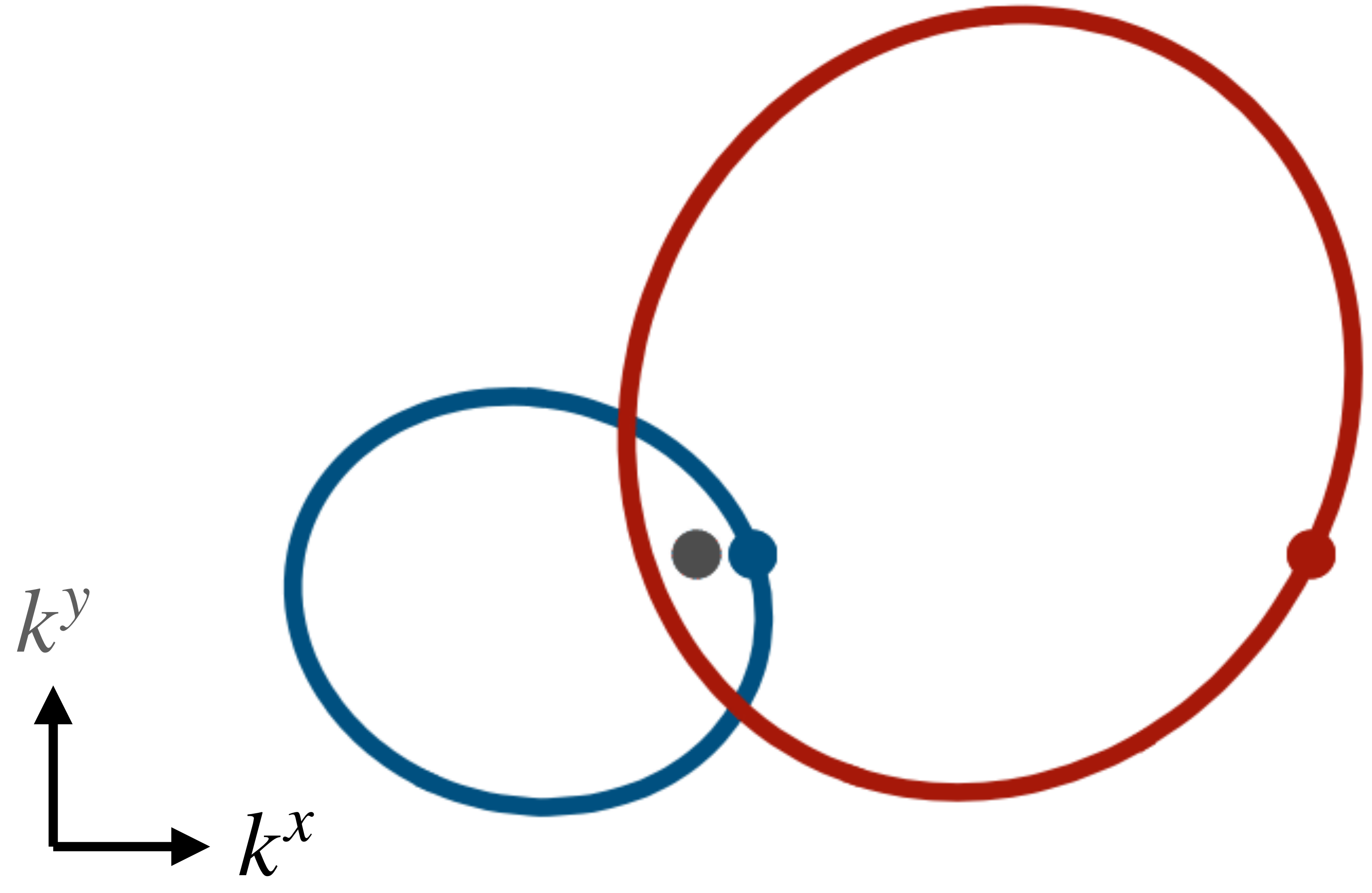

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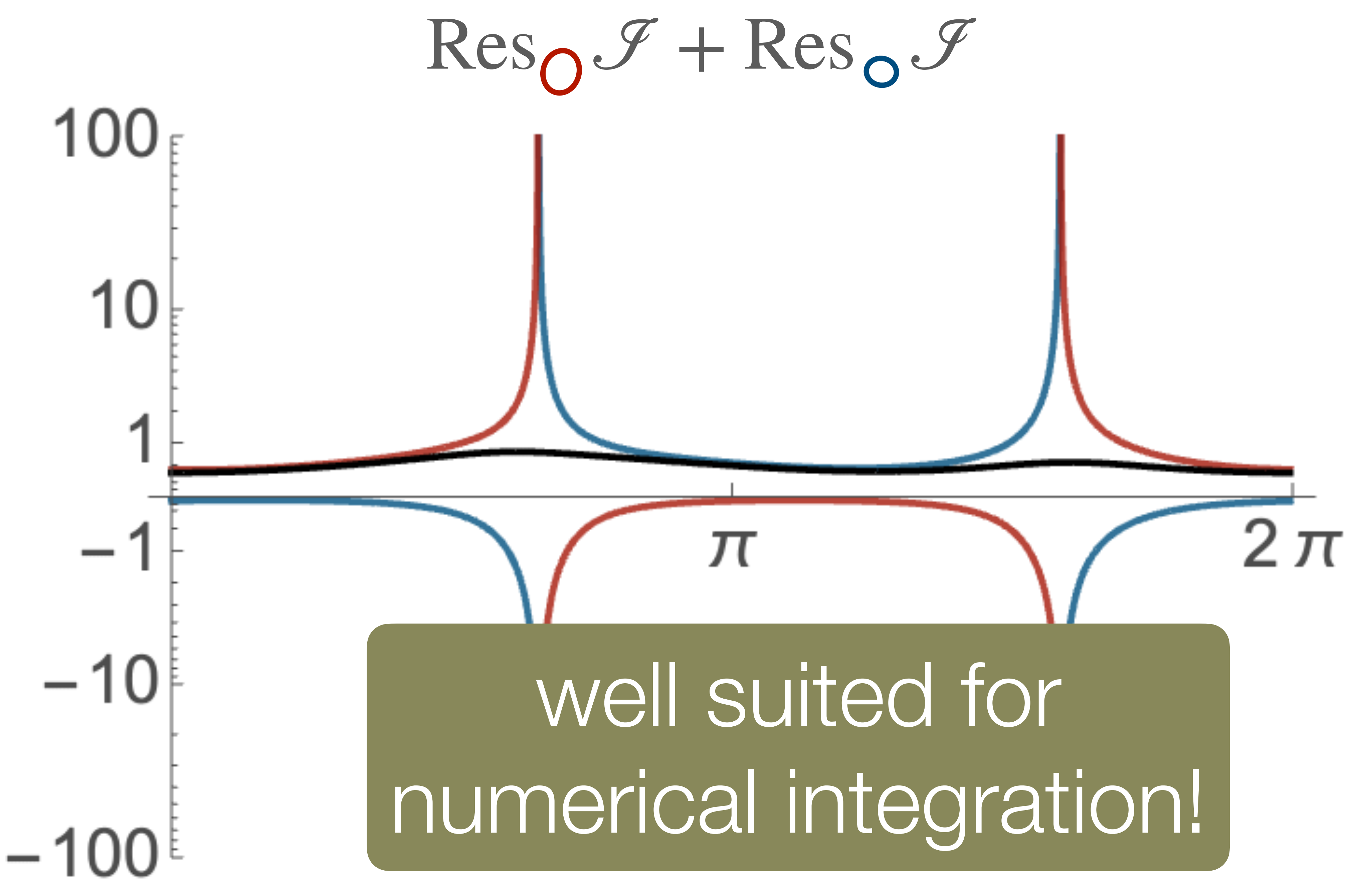

well suited for numerical integration!

DK: [2110.06869]
DK, Matilde Vicini [2407.21511]

Subtraction of threshold singularities

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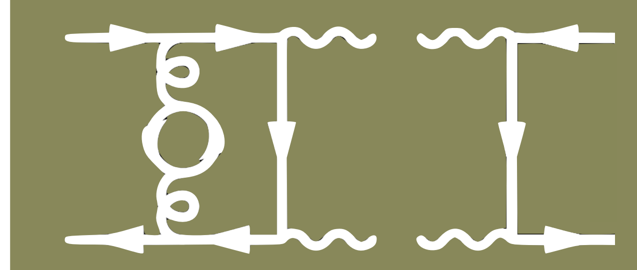
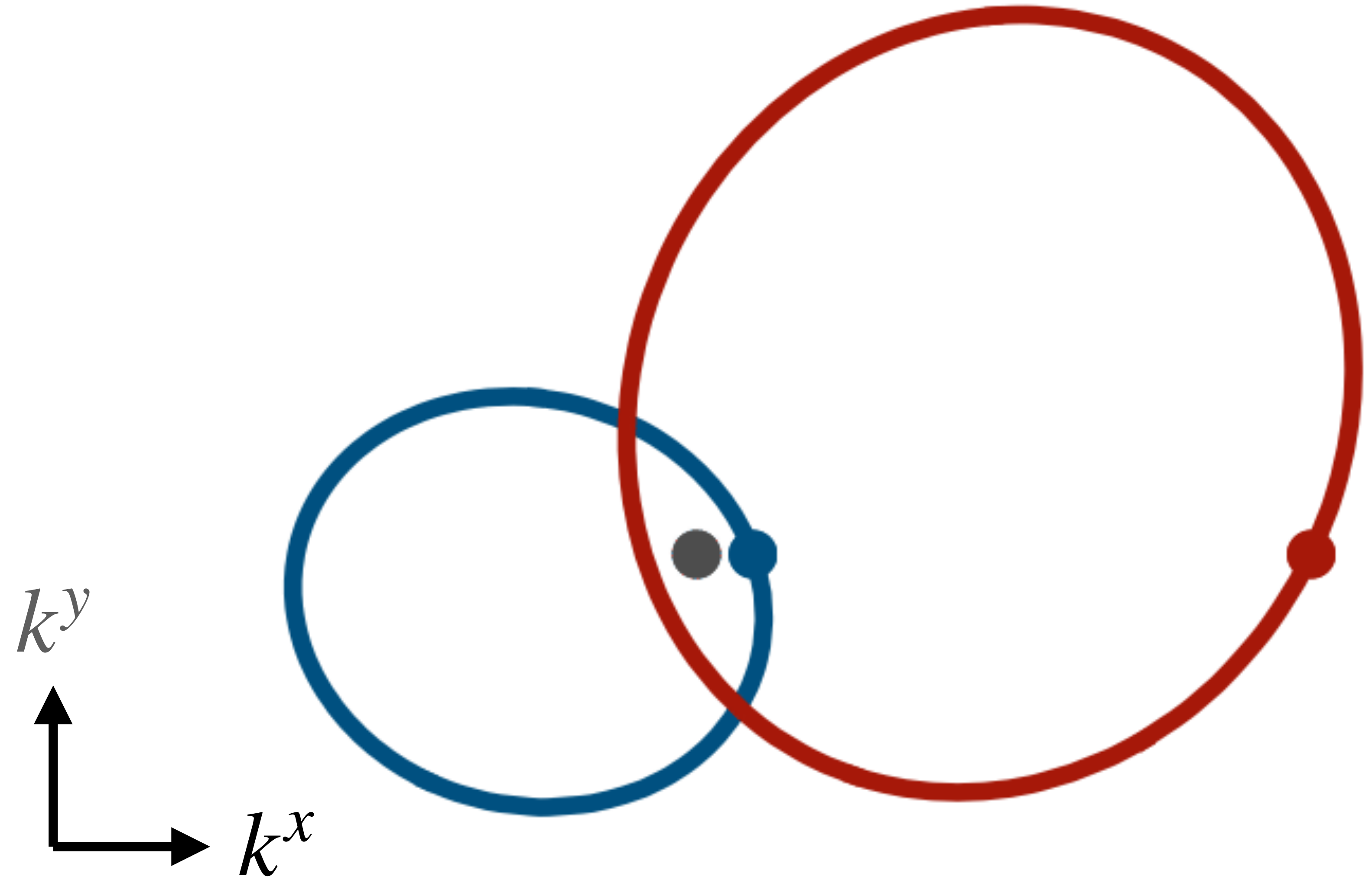
$$\mathcal{I} \sim \frac{\text{Res}_i\, \mathcal{I}}{|\vec{k}| - k_i \pm i\epsilon} \to \text{CT}_i$$ threshold counterterm

$$\text{Re}\, I = \int [\text{d}^3\vec{k}] \left(\mathcal{I} - \sum_i \text{CT}_i \right)$$ dispersive part

we will only need
the dispersive part!

$$\int \text{d}\Pi\, \text{d}^3\vec{k}\, \text{d}^3\vec{l} \sum_{\text{hel.}} 2\, \text{Re} \Big[\qquad + \cdots \Big]$$





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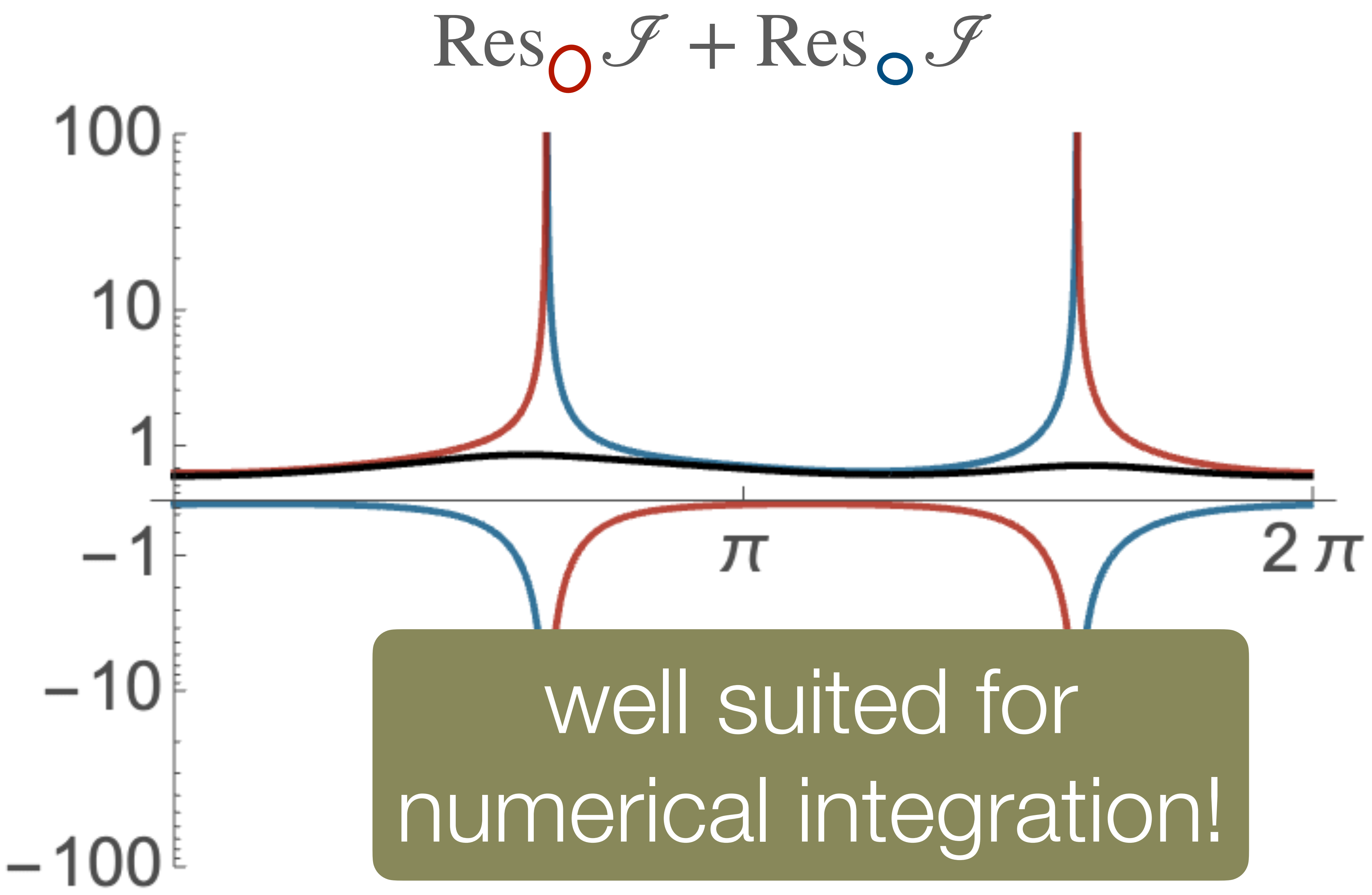
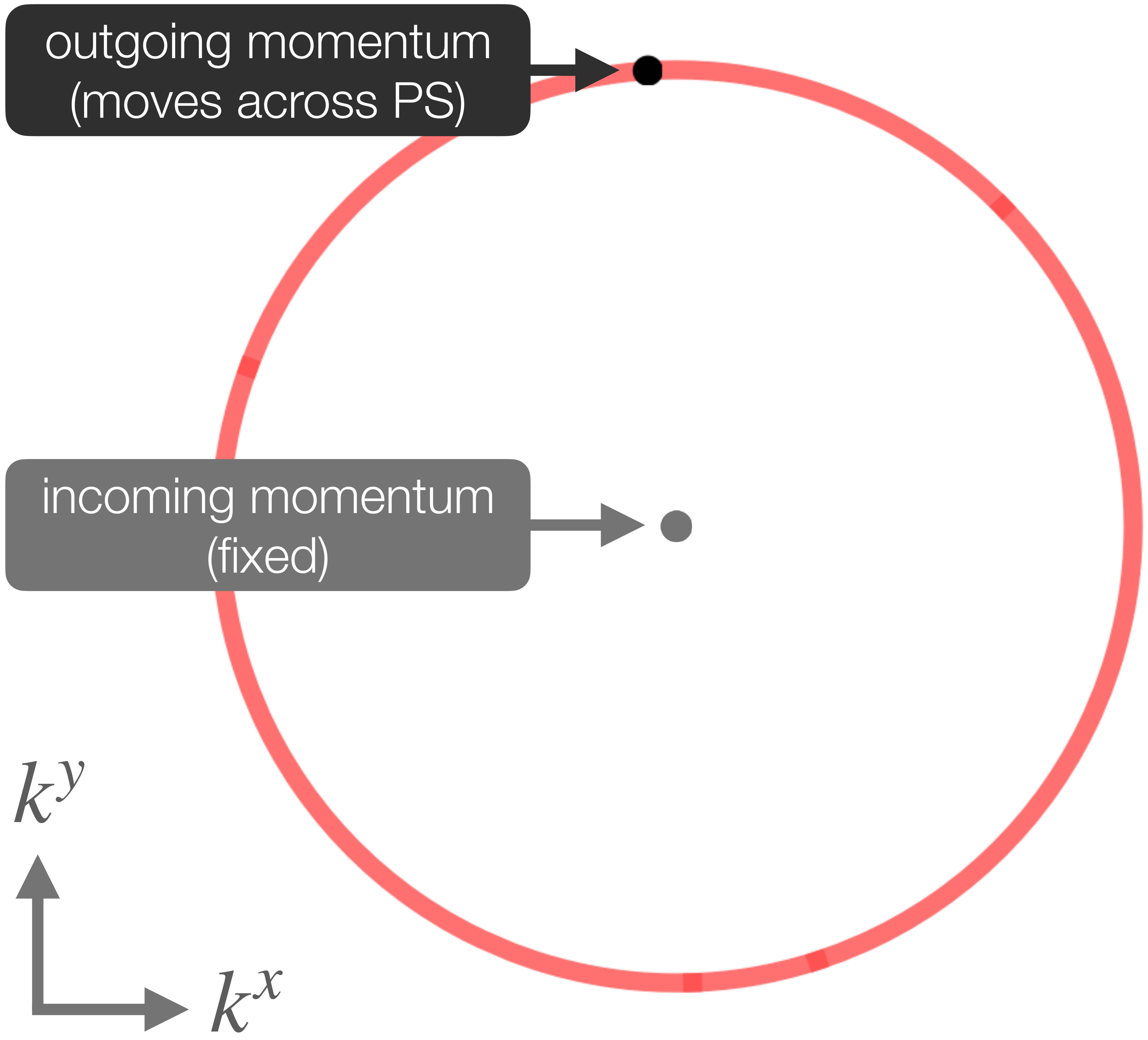


well suited for
numerical integration!

Thresholds
one-loop & two-loop Nf amplitude

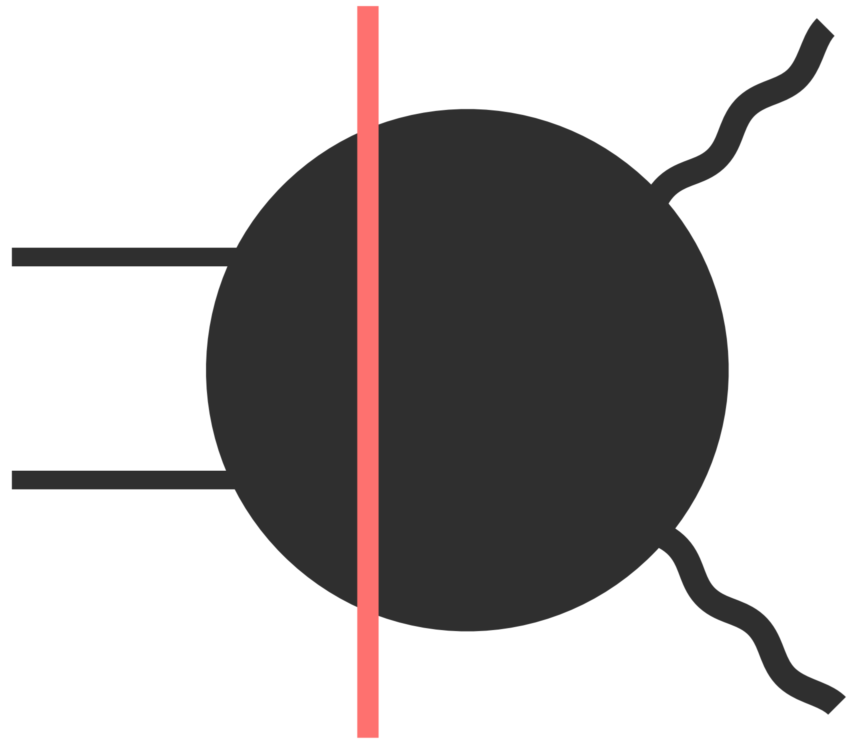
$q\bar{q} \to \gamma\gamma$



outgoing momentum
(moves across PS)

incoming momentum
(fixed)

corresponding Cutkosky cuts



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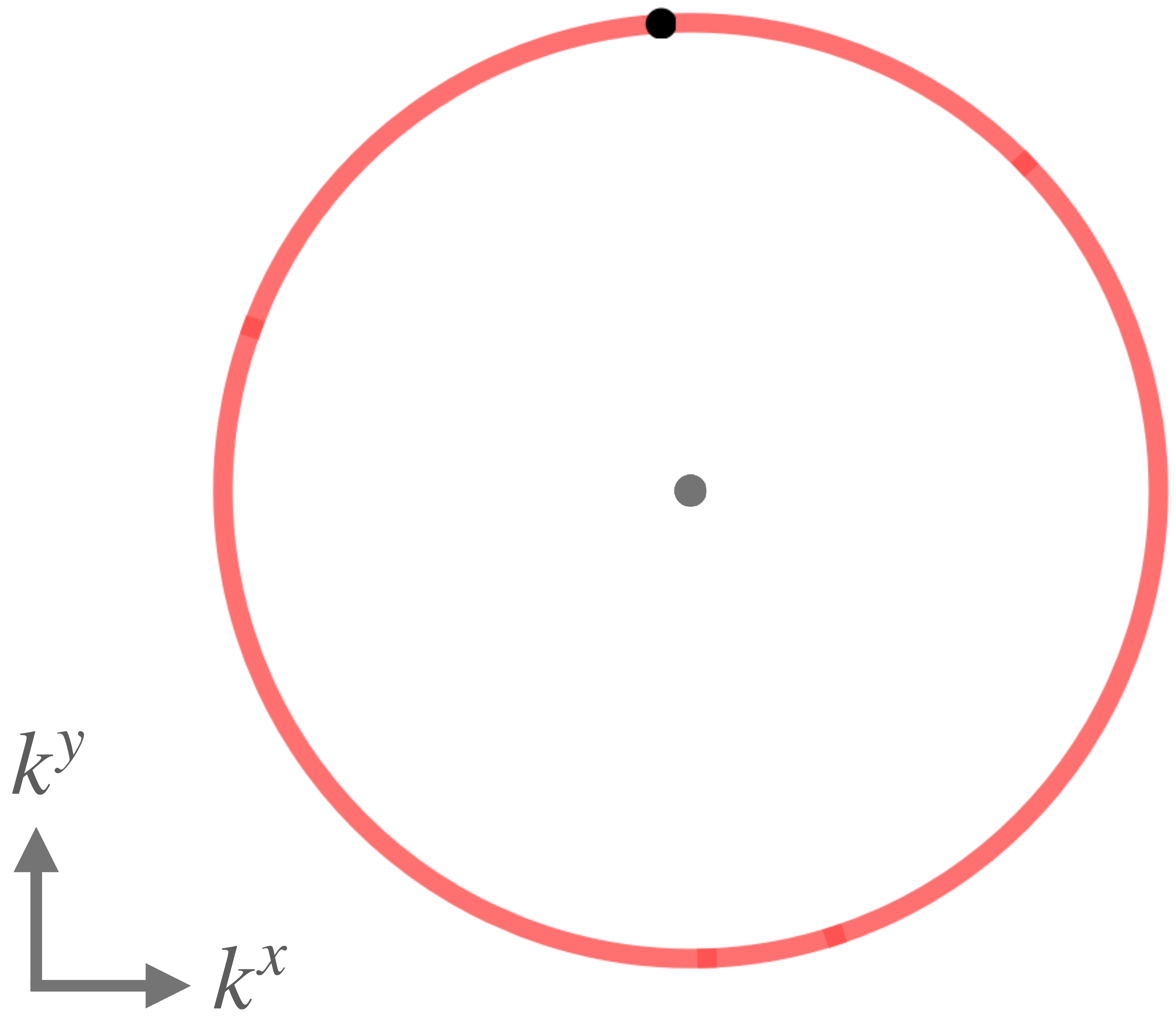

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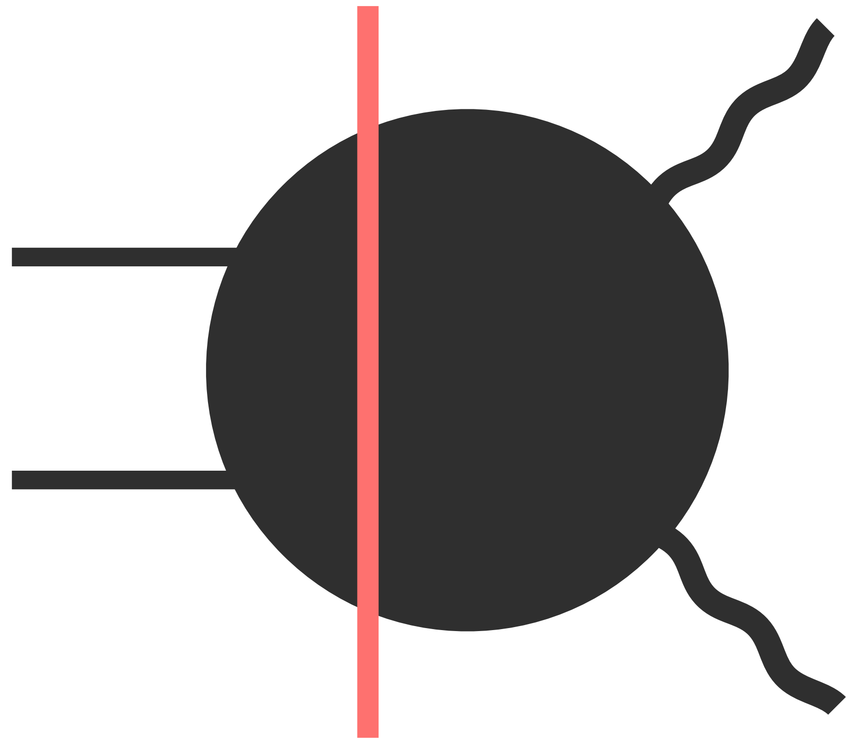
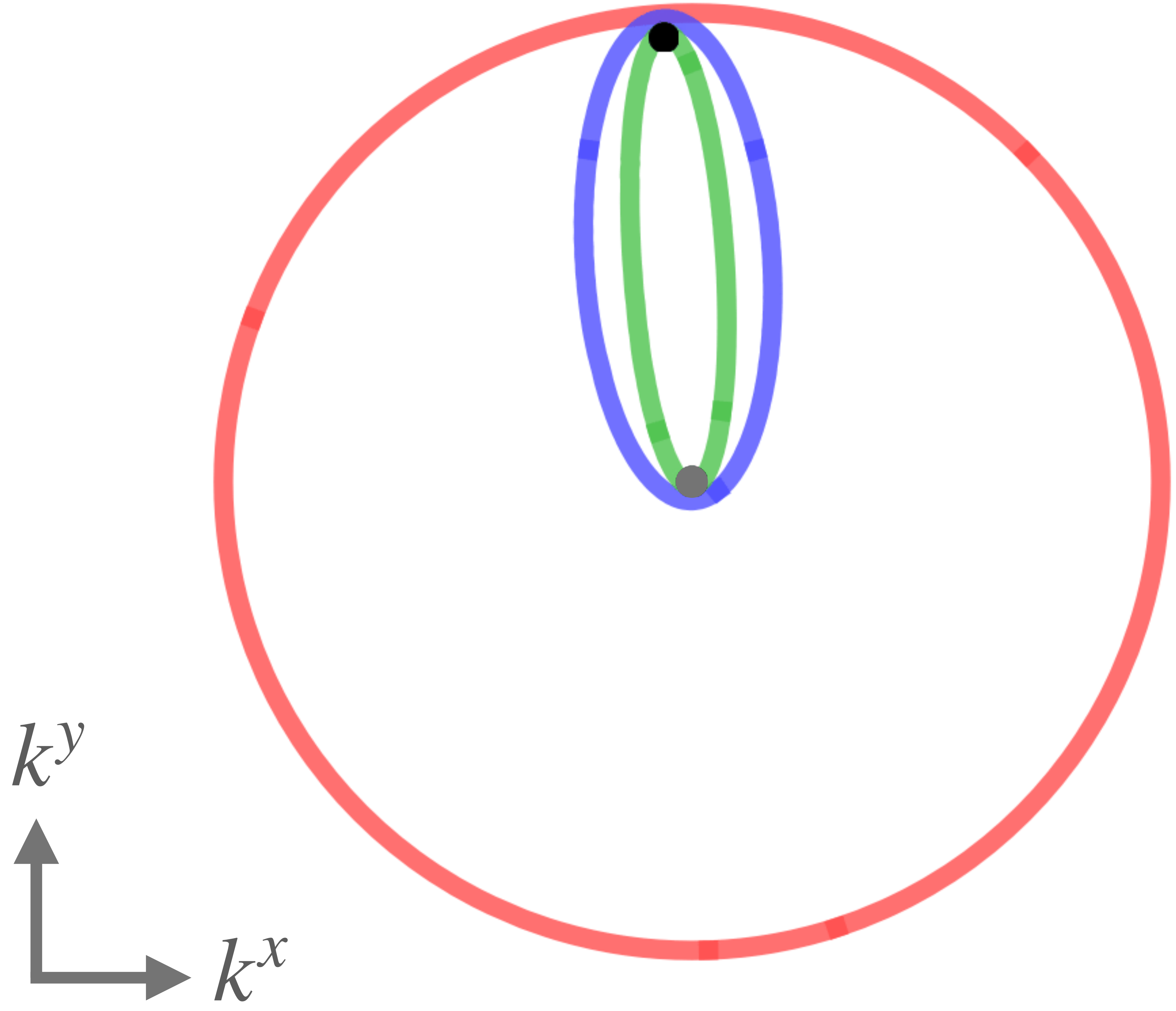


Thresholds
one-loop & two-loop Nf amplitude

$$q\bar{q} \to \gamma^*\gamma^*$$



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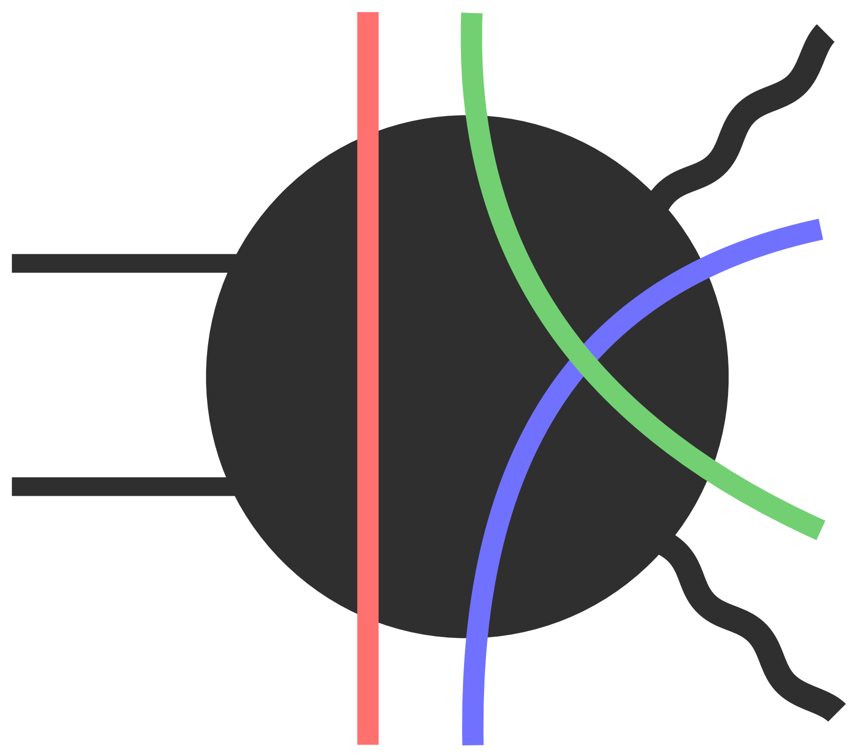

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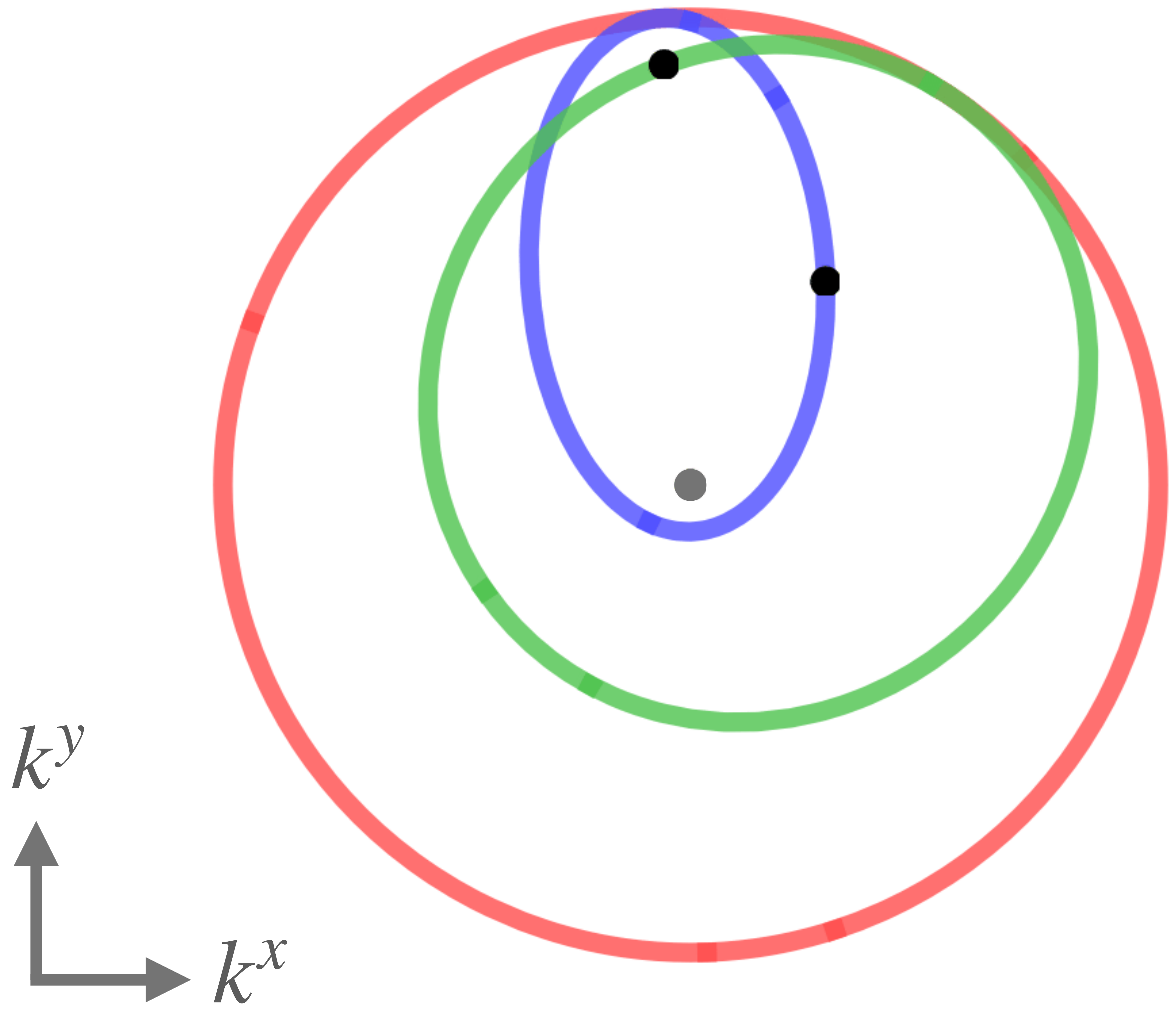
$q\bar{q} \to \gamma\gamma\gamma$



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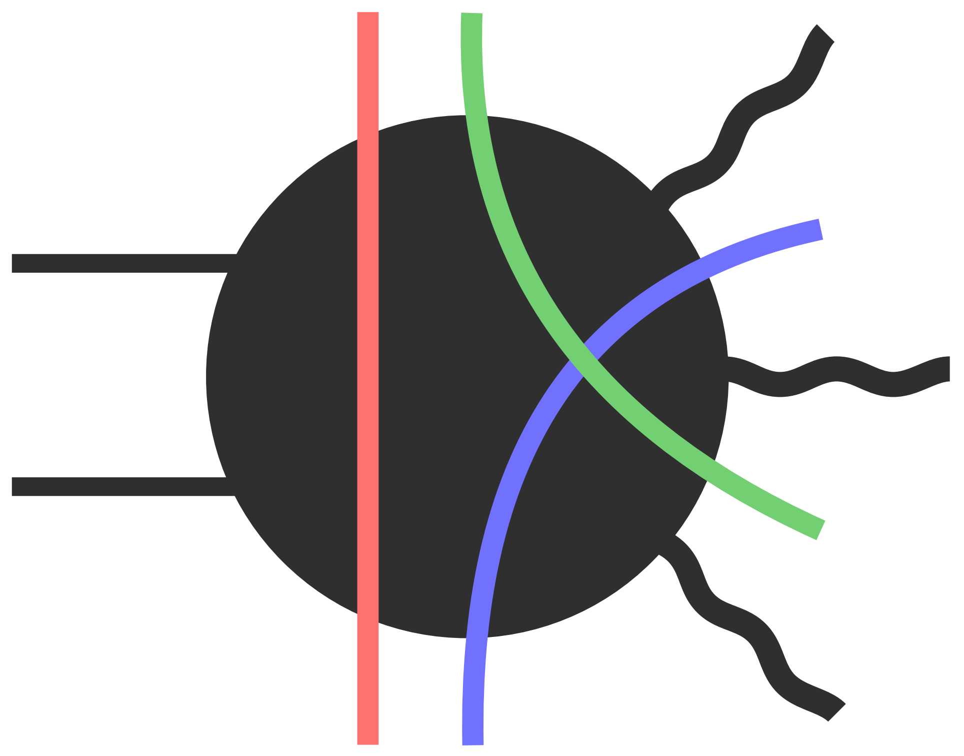
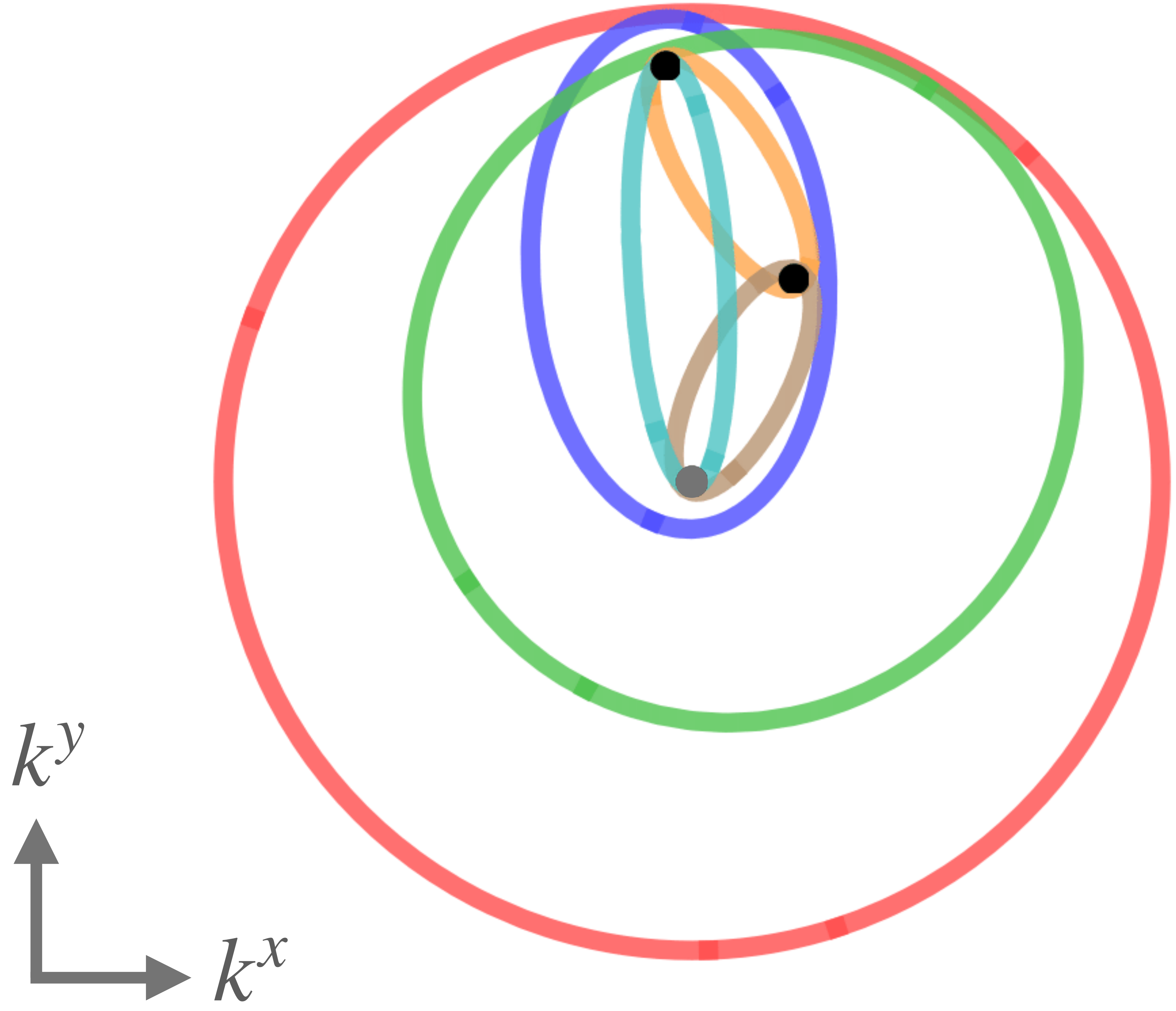


Thresholds
one-loop & two-loop Nf amplitude

$q\bar{q} \rightarrow \gamma^*\gamma^*\gamma^*$



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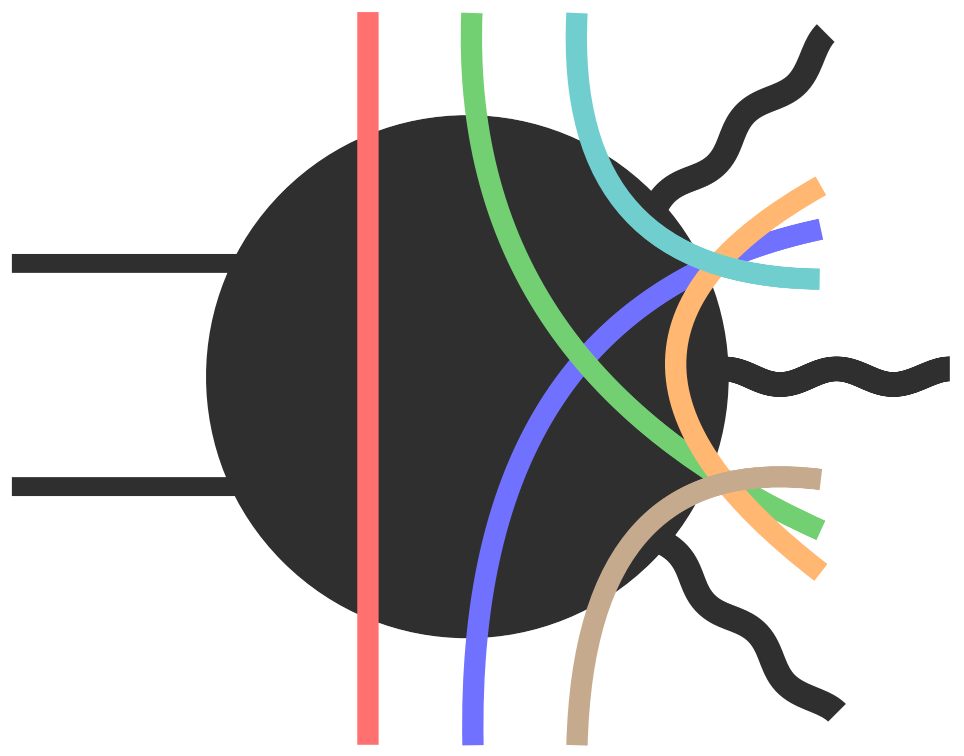

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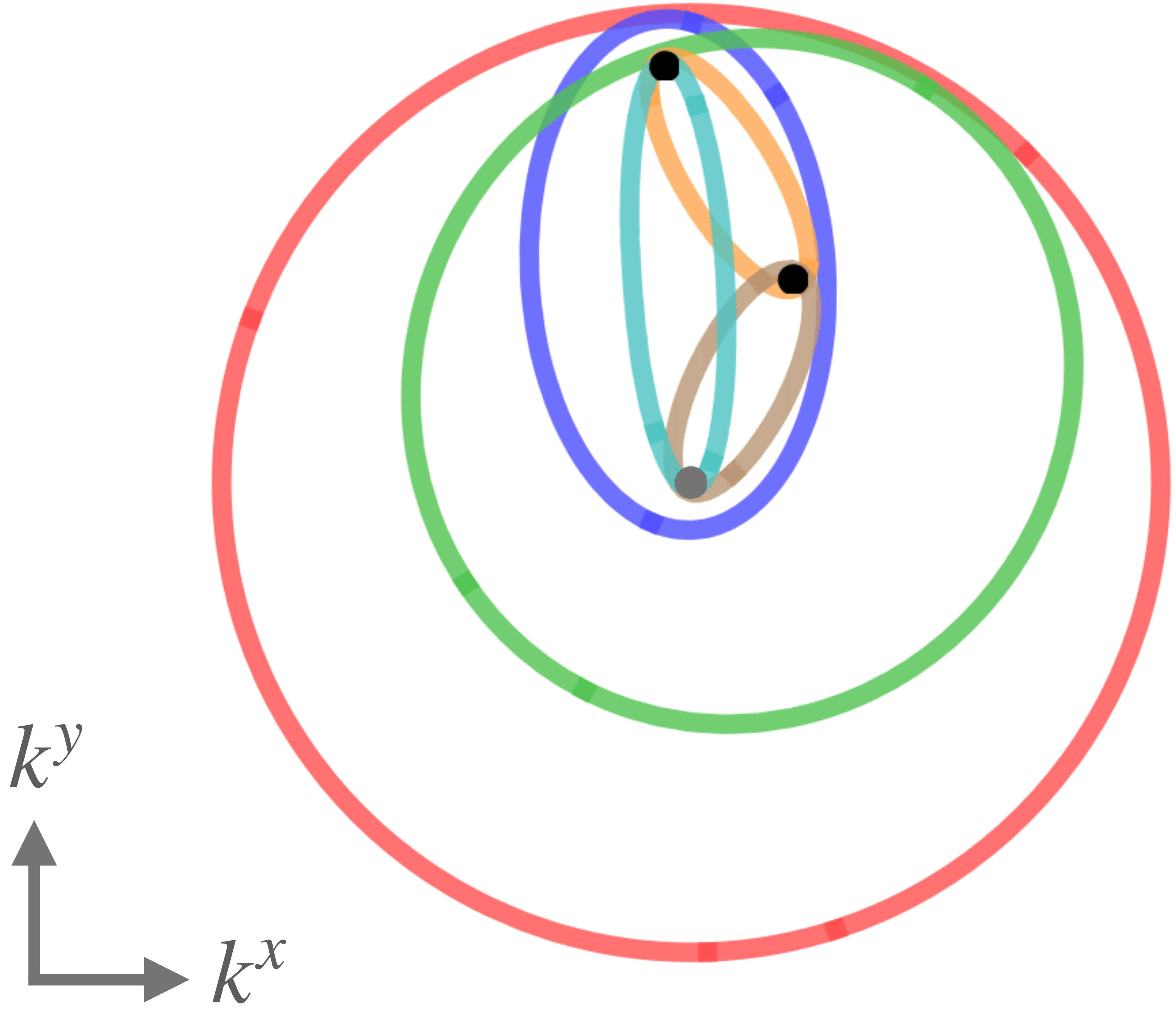

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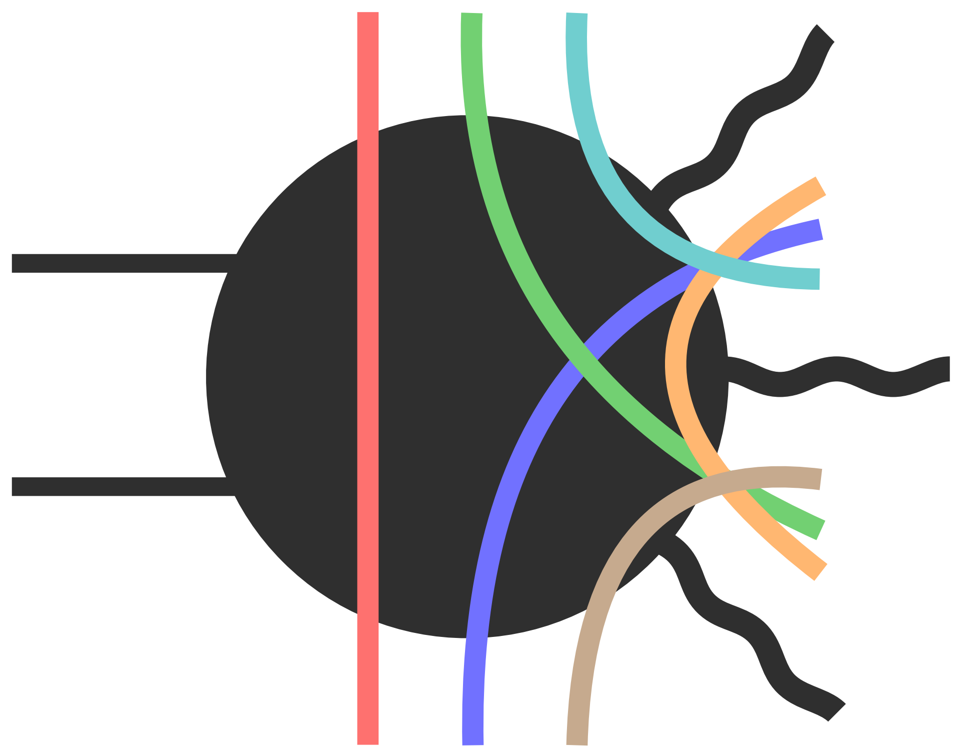
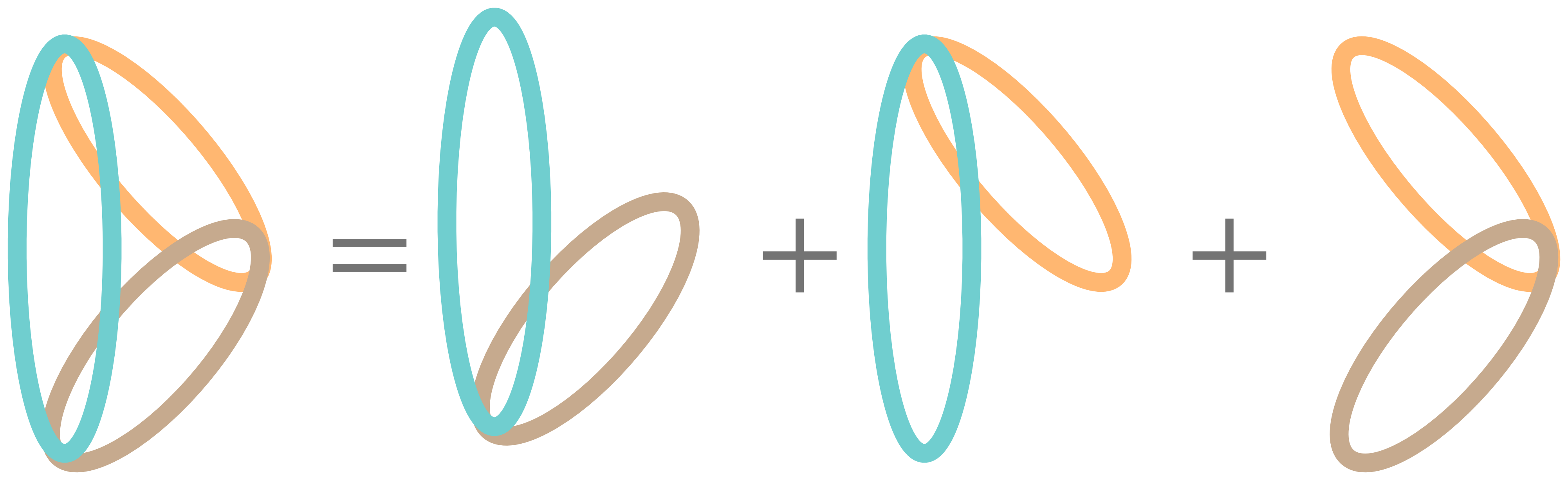


overlapping thresholds: multi-channelling



same pipeline & same computer with 24 cores

NLO and NNLO-Nf virtual cross sections

DK, Matilde Vicini [2407.18051]
numerical integration over loop & phase space
summed over helicities and convoluted with PDFs

	Order	Result [pb]	Δ[%]	total time#
$pp \to \gamma\gamma$	NLO	5.2851 ± 0.0164 e-01	0.3	10 min
	NNLO-Nf	-6.1475 ± 0.0349 e-02	0.6	1 h 30 min
$pp \to \gamma^*\gamma^*$	NLO	4.3172 ± 0.0089 e-01	0.2	2 min
	NNLO-Nf	-3.6943 ± 0.0322 e-02	0.9	40 min
$p_d p_d \to ZZ$	NLO	7.0067 ± 0.0159 e-01	0.2	4 min
	NNLO-Nf	-5.9363 ± 0.0520 e-02	0.9	1 h 30 min
$pp \to \gamma\gamma\gamma$	NLO	1.4874 ± 0.0140 e-04	0.9	2 h 30 min
	NNLO-Nf	-2.5460 ± 0.0237 e-05	0.9	1 day
$pp \to \gamma^*\gamma^*\gamma^*$	NLO	1.4692 ± 0.0144 e-04	1.0	2h 45 min
	NNLO-Nf	-1.4301 ± 0.0137 e-05	1.0	4 days
$p_d p_d \to Z\gamma_1^*\gamma_2^*$	NLO	2.4600 ± 0.0210 e-04	0.9	1 day 12 h
	NNLO-Nf	-2.5301 ± 0.0229 e-05	0.9	1 month

#potential for optimization!

masses? no prob!*

masses? no prob!*

NLO in BLHA
NNLO-Nf in $\overline{\text{MS}}$

NLO cross checked interferences with OpenLoops and cross sections with MadGraph

in agreement with `FivePoint Amplitudes-cpp` Abreu, De Laurentis, Ita, Klinkert, Page, Sotnikov [2305.17056]

$\times 3!$

new!

*additional thresholds have to be considered

Summary & Outlook

☑ Nf-contribution to NNLO virtual cross section for 3 massive vector boson production

☑ First NNLO calculation for the LHC using numerical integration over loop & phase space

Local IR factorisation
& UV renormalisation

Analytic loop energy integration
LTD, CFF, TOPT, …

Threshold
subtraction

flexible and robust framework suited for automation

☐ apply these techniques to the full NNLO virtual contribution

☐ combine with real radiation

☐ processes with colorful final state

☐ …