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NNNLO zero-jettiness soft function

Andrey Pikelner | HP^2, Turin
In collaboration with: Daniel Baranowski, Maximilian Delto, Kirill Melnikov and Chen-Yu Wang

Motivation

- Differential calculation require a good handle on IR divergences, many schemes at NNLO
- Slicing scheme seems feasible at N3LO due to complexity of subtraction schemes

$$\sigma(O) = \int_0 \mathrm{d}\tau \frac{\mathrm{d}\sigma(O)}{\mathrm{d}\tau} = \int_0^{\tau_0} \mathrm{d}\tau \frac{\mathrm{d}\sigma(O)}{\mathrm{d}\tau} + \int_{\tau_0} \mathrm{d}\tau \frac{\mathrm{d}\sigma(O)}{\mathrm{d}\tau}$$

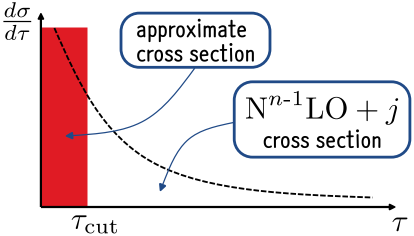
  - q_T subtraction scheme [Catani,Grazzini'07]
  - N-jettiness subtraction scheme [Boughezal et al.'15][Gaunt et al.'15]

- SCET factorization theorem motivates us to consider jettiness as convenient slicing variable

$$\lim_{\tau\to 0} \mathrm{d}\sigma(O) = B_\tau \otimes B_\tau \otimes S_\tau \otimes H_\tau \otimes \mathrm{d}\sigma_{\mathrm{LO}}$$

Slicing scheme ingredients

- Possible to use any lower order calculation with additional jet in the $\tau > \tau_{\text{cut}}$ region



To apply at the NNNLO level:

- Existing NNLO+j calculations
- Many efficient NNLO subtraction schemes

- Approximate cross section in the singular region from the factorisation formula

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\tau} = H_\tau \otimes \{B_\tau\} \otimes \{J_\tau\} \otimes S_\tau \otimes \frac{\mathrm{d}\sigma_0}{\mathrm{d}\tau} + \mathcal{O}(\tau)$$

- Hard function H_τ
- Beam function B_τ, jet function J_τ
- Soft function S_τ

Zero-jettiness measurement function

- For two hard partons with p_a and p_b measurement function $\delta(\tau - \mathcal{T}_0)$ is defined with

$$\mathcal{T}_0 = \sum_{i=1}^{m} \min\left\{\frac{2p_a \cdot k_i}{Q}, \frac{2p_b \cdot k_i}{Q}\right\}$$

- It is possible to rescale $p_a = \frac{\sqrt{s_{ab}}}{2} n, p_b = \frac{\sqrt{s_{ab}}}{2} \bar{n}$ and go to the frame where n and $\bar{n}$ are back-to-back
- Eikonal factors $E(k, l)$ have uniform scaling: rescale integration momenta $q_i = q_i' \frac{Q\tau}{\sqrt{s_{ab}}}$, $q_i \in \{k, l\}$

$$\int [\mathrm{d}^d k]^m [\mathrm{d}^d l]^n \delta(\tau - \mathcal{T}_0) E(k, l) \to \frac{1}{\tau}\left(\frac{s_{ab}}{Q^2 \tau^2}\right)^{\varepsilon(m+n)} \int [\mathrm{d}^d k']^m [\mathrm{d}^d l']^n \delta\left(1 - \sum_{i=1}^{m} \min\{\alpha_i, \beta_i\}\right) E(k', l')$$

Sudakov decomposition

$$k_i = \frac{\alpha_i}{2} n + \frac{\beta_i}{2} \bar{n} + k_{i,\perp}, \quad k_i \cdot n = \beta_i, \quad k_i \cdot \bar{n} = \alpha_i, \quad n \cdot \bar{n} = 2, \quad n^2 = \bar{n}^2 = 0$$

What is actually calculated?

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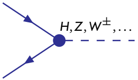
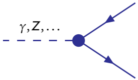
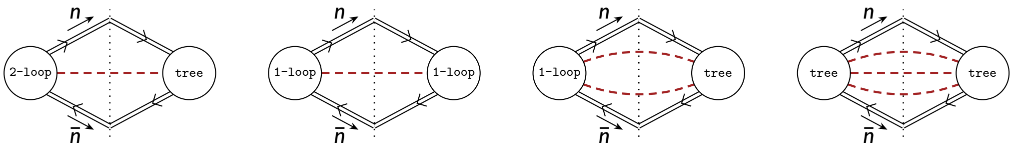
- 0-jettiness in hadronic collisions is equal to Thrust or 2-jettiness in e^+e^- annihilation or Higgs decay





- The limit $\tau \to 0$ corresponds to the soft limit of the squared amplitude - eikonal rules
- Need to include all possible real and virtual corrections to the amplitude squared



- Possible to combine different measurement function terms into unique configurations
- Perform integration over highly non-trivial region - all kinds of divergencies are possible

From measurement function to configurations

- Definition which is more friendly for PS integration generates different configurations

$$\delta\left(1-\sum_{i=1}^{m}\min\{\alpha_i,\beta_i\}\right) = \delta(1-\beta_1-\beta_2-\ldots)\theta(\alpha_1-\beta_1)\theta(\alpha_2-\beta_2)\ldots$$
$$+\delta(1-\beta_1-\alpha_2-\ldots)\theta(\alpha_1-\beta_1)\theta(\beta_2-\alpha_2)\ldots$$

- Configurations can be mapped to the minimal set due to symmetries of Eikonal factor and $\delta(1-\{\alpha,\beta\})$
- RVV single configuration with $\delta(1-k\cdot n)$, trivial PS interation
  - Two-loop soft current is known [Duhr, Gehrmann'13]
- RRV two configurations nn and $n\bar{n}$
  - Emission of gluons and quark pair [Chen et al.'22] [Baranowski et al.'24]
- RRR two configurations nnn and $nn\bar{n}$
  - Same hemisphere gluon emission [Baranowski et al.'22]
  - Different hemispheres configuration $nn\bar{n}$ and quark pair emission in nnn configuration - current work

Laplace space and UV renormalization

- Final unrenormalized result for the NNNLO soft function is a sum over configurations C

$$S^{\text{NNNLO}}_{\tau,B} = \sum_C S^{\text{RVV},C}_{\tau,B} + \sum_C S^{\text{RRV},C}_{\tau,B} + \sum_C S^{\text{RRR},C}_{\tau,B}$$

- For renormalizartion we need NNLO result expanded to higher order in ε [Baranowski'20]

$$S_{\tau,B} = \delta(\tau) + \frac{a_{s,B}}{\tau}\left(\frac{s_{ab}}{Q^2\tau^2}\right)^{\varepsilon} S_1 + \frac{a_{s,B}^2}{\tau}\left(\frac{s_{ab}}{Q^2\tau^2}\right)^{2\varepsilon} S_2 + \frac{a_{s,B}^3}{\tau}\left(\frac{s_{ab}}{Q^2\tau^2}\right)^{3\varepsilon} S_3 + \mathcal{O}\left(a_{s,B}^4\right)$$

- After strong coupling renormalization $a_{s,B} = \mu^{2\varepsilon} Z_{a_s} a_s(\mu)$ and Laplace transform with parameter $\bar{u} = u e^{\gamma_E}$

$$\tilde{S}_B\left(a_s(\mu), L_S\right) = \int_0^\infty d\tau e^{-\tau u} S_{\tau,B}\left(a_{s,B} \to \mu^{2\varepsilon} Z_{a_s} a_s(\mu)\right), \quad L_S = \ln\left(\mu\bar{u}\frac{\sqrt{s_{ab}}}{Q}\right)$$

- Convinient to consider $\tilde{S}_B$ due to simple multiplicative renormalization in the Laplace space

Renormalization and checks from RG equation

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- Multiplicative renormalization in the Laplace space with L_S dependent renormalization constant

$$\tilde{S}(a_s, L_S) = Z_s(a_s, L_S)\tilde{S}_B(a_s, L_S) = \mathcal{O}\left(\varepsilon^0\right)$$

  - $L_S = \ln\left(\mu \bar{u} \frac{\sqrt{s_{ab}}}{Q}\right)$
  - μ dependence in $a_s(\mu)$ and L_S

- Z_s determined by the pole part of $\tilde{S}_B$ satisfies RG equation

$$\left(\frac{\partial}{\partial L_S} + \beta(a_s)\frac{\partial}{\partial a_s}\right)\ln Z_s(a_s, L_S) = \Gamma_s\left(a_s, L_S\right) = -4\gamma_{cusp}(a_s)L_S - 2\gamma_s(a_s)$$

  - Γ_s is finite
  - Known cusp an.dim γ_{cusp}
  - Known non-cusp an.dim γ_s

- Possible to make prediction for the NNNLO pole part of $\tilde{S}_B$ and therefore for $S_{\tau,B}$ from NNLO result
- Final form of the renormalized NNNLO soft function can be split into constant and L_S dependent parts

$$\ln\left(\tilde{S}(a_s, L_S)\right) = \sum_{i=1}^{\infty}\sum_{j=0}^{2i} C_{ij} a_s^i L_S^j = \ln\left(\tilde{\mathbb{S}}\right) + \sum_{i=1}^{\infty}\sum_{j=1}^{2i} C_{ij} a_s^i L_S^j, \quad \tilde{\mathbb{S}} = \tilde{S}(a_s, 0)$$

Result for NNNLO zero-jettiness soft function

- Eikonal line representation dependence completely factorizes at NNNLO due to Casimir scaling

$$\frac{\ln\left(\tilde{S}\right)}{C_R} = -a_s\pi^2 + a_s^2\left[n_f T_F\left(\frac{80}{81}+\frac{154\pi^2}{27}-\frac{104\zeta_3}{9}\right)-C_A\left(\frac{2140}{80}+\frac{871\pi^2}{54}-\frac{286\zeta_3}{9}-\frac{14\pi^4}{15}\right)\right]$$
$$+a_s^3\left[n_f^2T_F^2\left(\frac{265408}{6561}-\frac{400\pi^2}{243}-\frac{51904\zeta_3}{243}+\frac{328\pi^4}{1215}\right)+n_fT_F\left(C_FX_{FF}+C_AX_{FA}\right)+C_A^2X_{AA}\right]+\mathcal{O}\left(a_s^4\right)$$

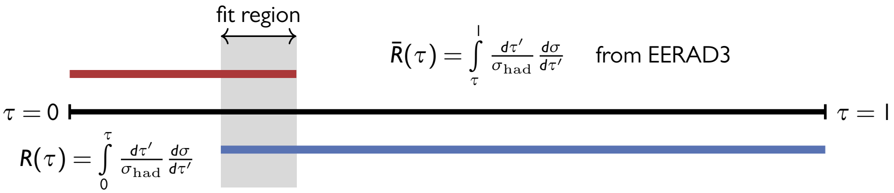
- With $a_s = \frac{\alpha_s}{4\pi}$ and new coefficients calculated numerically with high precision

$$X_{FF} = 68.94258498 \qquad X_{FA} = 839.72385238 \qquad X_{AA} = -753.77578727$$

- Soft function constants in $n_f = 5$ QCD required for resummed predictions $(q : C_R \to C_F)$ and $(g : C_R \to C_A)$

$$c_3^{S,q} = -1369.575849 \qquad c_3^{S,g} = -3541.982541$$

Singular region cross section from MC simulation



- Fit in the region, where NNLO MC predictions and approximate factorization prediction overlap
- From the condition $R(\tau) + \bar{R}(\tau) = 1$ and all C_i, G_{ij} except C_3 known

$$R(\tau) = \left(1 + \sum_{k=1}^{\infty} C_k \left(\frac{\alpha_s}{2\pi}\right)^k\right) \exp\left[\sum_{i=1}^{\infty}\sum_{j=1}^{i+1} G_{ij} \left(\frac{\alpha_s}{2\pi}\right)^i \ln^j \frac{1}{\tau}\right]$$

- Missing C_3 in the parametrisation of dijet region for NNLO Thrust [Monni,Gehrmann,Luisoni'11]

$$C_3 = -1050 \pm 180 \pm 500$$

From soft function to singular cross section

- Coefficient $C3$ is determined by constant parts of Hard(H), Jet(J) and Soft(S) functions
  - N3LO hard function is known $c_3^H = 8998.08$ [Abbate,Fickinger,Hoang et al.'10]
  - N3LO jet function known $c_3^J = -128.651$ [Brüser,Liu,Stahlhofen'18]
  - N3LO soft function $c_3^S = -1369.57$ this work
- From C_3 value can determine c_3^S, since all other ingredients are known [Brüser,Liu,Stahlhofen'18]

$$c_3^S = -19988 \pm 1440 \pm 4000$$

- Inverse of the relation with known c_3^S allows C_3 color structures prediction [Monni,Gehrmann,Luisoni'11]

	$n_f^0 N^2$	$n_f^0 N^0$	$n_f^0 N^{-2}$	$n_f^1 N^1$	$n_f^1 N^{-1}$	$n_f^2 N^0$	sum
From c_3^S	2766.05	-60.1237	0.37891	-1581.01	18.4901	133.47	1277.25
Fit	3541 ± 51	-265 ± 8	-71 ± 3	-5078 ± 145	236 ± 7	95 ± 120	-1543 ± 195

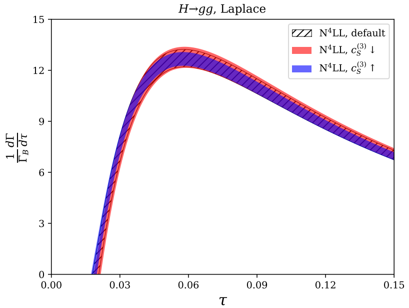
Applications: Thrust resummation

Thrust resummation for α_s determination

- [Becher,Schwartz'08]
  c_2^S numerical fit
- [Abbate,Fickinger,Hoang et al.'10]
  c_3^H known, fitted c_3^J, c_3^S
- [Bell,Lee,Makris et al.'23]
  c_3^H, c_3^J known, attempt to extract c_3^S

Last missing ingredient c_3^S is now available

Higgs decay to quarks/gluons [Ju,Xu,Yang,Zhou'23]



α_s series convergence restored

$$\tilde{s}_g = 1 - 2.36\alpha_s + 1.617\alpha_s^2 - \underbrace{(22.89 \pm 5.67)}_{\text{fit}}\alpha_s^3$$

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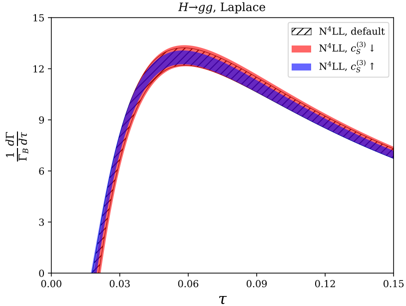

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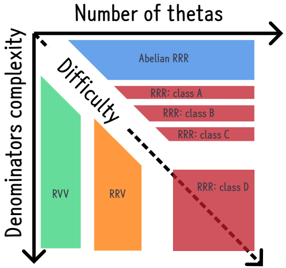
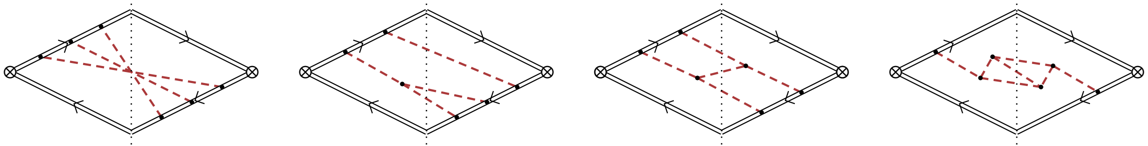
Relative complexity of ingredients



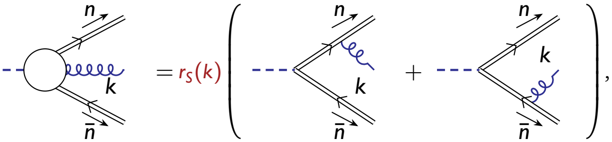
- For each soft emission we have one θ-function in the measurement function making integration more complicated
- Most complicated denominators in RRR case make direct integration impossible
- Most complicated one-loop sub-integrals in the RRV case make direct integration impossible
- Unregulated divergencies in the RRR case



A $\quad B \sim \frac{1}{k_1 \cdot k_2}$ $\quad C \sim \frac{1}{(k_1 \cdot k_2)(k_1 \cdot k_3)}$ $\quad D \sim \frac{1}{(k_1 + k_2 + k_3)^2}$

RVV corrections

Two-loop corrections $r_S^{(2)}$ to single gluon emission soft current are known exactly in ε [Duhr, Gehrmann'13]

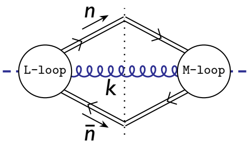


$$r_S(k) = 1 + \sum_{l=1}^{\infty} A_s^l \left[\frac{-(n\cdot\bar{n})}{2(k\cdot n)(k\cdot\bar{n})} \right]^{l\varepsilon} r_S^{(l)}$$

Two contributions from different hemisphere emissions need to be integrated, $S_g^{(3)} = s_{2,0} + s_{1,1} + s_{0,2}$

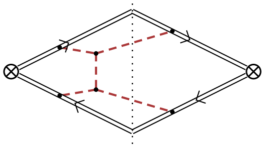
$$s_{l,m} = \int \frac{\mathrm{d}^d k}{(2\pi)^{d-1}} \delta^+(k^2) \left[\delta(1-k\cdot n)\theta(k\cdot\bar{n}-k\cdot n) + \delta(1-k\cdot\bar{n})\theta(k\cdot n - k\cdot\bar{n})\right] w_{L,M}(k)$$

$w_{L,M}(k) = \mathrm{Re}\left[J_L^\dagger(k) J_M(k)\right] =$



- Linear propagators only
- Factorisation of k-dependent part of soft current

One-loop corrections with double emission



- RRV squared amplitudes generated from scratch
- Results for one-loop soft current are known [Zhu'20][Czakon et al.'22]
- RRV result for gg final state is known [Chen,Feng,Jia,Liue'22]
- Recalculation including $q\bar{q}$ final state [Baranowski et al.'24]

Multi-loop calculations inspired approach

- Reduction to the minimal set of master integrals
- Differential equations from IBP reduction

Modified reverse unitarity

- In dimensional regularisation system of IBP equation can be constructed by integration under integral sign

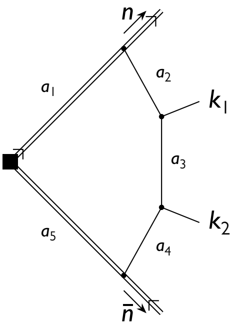
$$\int \mathrm{d}^d l \frac{\partial}{\partial l_\mu}\left[v_\mu \cdot f(\{l\})\right], \qquad \frac{\partial}{\partial k\cdot\bar{n}}\theta(k\cdot\bar{n} - k\cdot n) = \delta(k\cdot\bar{n} - k\cdot n)$$

- IBP for integrals with θ-functions generate new auxiliary topologies, partial fractioning required

$$\frac{\theta(k\cdot\bar{n} - k\cdot n)}{(k\cdot\bar{n})^a(k\cdot n)^b} \to \frac{\delta(k\cdot\bar{n} - k\cdot n)}{(k\cdot\bar{n})^a(k\cdot n)^b}$$

  - RRR $\underbrace{\theta\theta\theta}_{\text{Level 3}} \to \underbrace{\delta\theta\theta + \theta\delta\theta + \theta\theta\delta}_{\text{Level 2}} \to \underbrace{\delta\delta\theta + \delta\theta\delta + \theta\delta\delta}_{\text{Level 1}} \to \underbrace{\delta\delta\delta}_{\text{Level 0}}$
  - RRV $\underbrace{\theta\theta}_{\text{Level 2}} \to \underbrace{\delta\theta + \theta\delta}_{\text{Level 1}} \to \underbrace{\delta\delta}_{\text{Level 0}}$

RRV master integrals calculation



- Number of MIs after IBP reduction of both configurations in RRV case

$\delta\delta$	$\delta\theta + \theta\delta$	$\theta\theta$
8	36	15

- Direct integration possible, except pentagon and box with $a_3 = 0$
- DE in auxiliary parameters for most complicated integrals

Original integrals from DE solution

Different strategy compared to RRR case: instead of $I = \lim_{z\to z_0} J(z)$ now $I = \int \mathrm{d}z J(z)$

RRV master integrals from differential equations

- For $\delta\delta$ integrals we introduce auxiliary paremeter x and solve DE system $\partial_x J(x) = M(\varepsilon, x) J(x)$

$$I_{\delta\delta} = \int \mathrm{d}\,(k_1 \cdot k_2) f(k_1 \cdot k_2) = \int_0^1 \mathrm{d}x \int \mathrm{d}\,(k_1 \cdot k_2)\, \delta(k_1 \cdot k_2 - \frac{x}{2}) f(k_1 \cdot k_2) = \int_0^1 J(x) \mathrm{d}x$$

- For $\delta\theta$ and $\theta\delta$ we use integral representation for θ-function and solve DE system $\partial_z J(z) = M(\varepsilon, z) J(z)$

$$\theta(b-a) = \int_0^1 b\delta(zb-a)\mathrm{d}z, \quad I_{\delta\theta} = \int_0^1 J(z)\mathrm{d}z$$

- For $\theta\theta$ integrals PDE system in two variables z_1, z_2, no IBP reduction with θ-functions needed

$$I_{\theta\theta} = \int_0^1 \mathrm{d}z_1 \int_0^1 \mathrm{d}z_2 J(z_1, z_2)$$

Differential equations in canonical form

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- For all auxiliary integrals it is possible to find alternative basis of integrals, such ε dependence of the DE system matrix factorizes completely: $M(\varepsilon) \to \varepsilon A$
- Straightforward solution for integrals in canonicaal basis in terms of GPLs
- Simpler boundary conditions fixing due to known general form of expansion near singular points

$$g(z) = z^{a_1+b_1\varepsilon}\left(c_1 + \mathcal{O}(z)\right) + z^{a_2+b_2\varepsilon}\left(c_2 + \mathcal{O}(z)\right) + \ldots$$

- Construction of subtraction terms to remove endpoint singularities in final integration

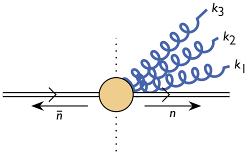
$$\int_0^1 J(z)\mathrm{d}z = \int_0^1 \underbrace{\left[J(z) - z^{a_i+b_i\varepsilon} j_0(z) - (1-z)^{a_k+b_k\varepsilon} j_1(z)\right]}_{\varepsilon\text{—expanded}} \mathrm{d}z + \int_0^1 \underbrace{\left(z^{a_i+b_i\varepsilon} j_0(z) - (1-z)^{a_k+b_k\varepsilon} j_1(z)\right)}_{\varepsilon\text{—exact}} \mathrm{d}z$$

Triple real emissions

Recalculated input for eikonal factor with partial fractioning and topology mapping

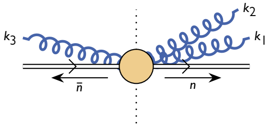
- $ggg = ggg + gc\bar{c}$, coincides with known expression in physical gauge [Catani,Colferai,Torrini'19]
- $gq\bar{q}$ in agreement with [Del Duca,Duhr,Haindl,Liu'23]

Same hemisphere



$\delta(\tau - \beta_1 - \beta_2 - \beta_3)$

Different hemispheres



$\delta(\tau - \beta_1 - \beta_2 - \alpha_3)$

- Same hemisphere result for ggg final state is known [Baranowski et al.'22]

DE for RRR integrals with auxiliary mass

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- Integrals for both *nnn* and $nn\bar{n}$ configurations with denominator $1/k_{123}^2$ are difficult to calculate
- Integrals are single scale, auxiliary parameter needed to construct DE system $I \to J(m^2)$
- Our solution is to make the most complicated propagator massive $\frac{1}{(k_1+k_2+k_3)^2} \to \frac{1}{(k_1+k_2+k_3)^2+m^2}$
- Calculation of boundary conditions simplifies in the limit $m^2 \to \infty$
- Result for integrals of our interest from the solution for $J(m^2)$ in the limit $m^2 \to 0$

Difficulties of the chosen strategy

- Both points $m^2 \to 0$ and $m^2 \to \infty$ are singular points of the DE system
- Solution of the DE for integrals with massive denominator possible only numerically

Details of the DE solution

- Larger DE system size with ~ 650 equations for $nn\bar{n}$ configuration compared to ~ 150 for nnn
- Needed to calculate all contributing regions into boundary conditions in the $m^2 \to \infty$ limit

$\sim (m^2)^0$	$\sim (m^2)^{-\varepsilon}$	$\sim (m^2)^{-2\varepsilon}$
$1/m^2$	$\alpha_i \sim m^2$	$\alpha_i, \alpha_j \sim m^2$

- For each large parameter $\alpha_i \sim m^2$ we remove $\theta \Longrightarrow$ additional IBP reduction of BC integrals possible
- Numerical solution of the DE system as a sequence of series expansions [Liu et al.'18][Chen et al.'22]

Direct integration of MIs and boundary constants

- We have calculated ~ 130 integrals without $1/k_{123}^2$ denominator and ~ 100 boundary conditions by direct integration with `HyperInt` [Panzer'15]
- Summary of used techniques
  1. Change variables to satisfy all constraints from δ and θ functions
  2. Perform as many integrations as possible in terms of ${}_2F_1$ and F_1 functions with known argument transforms
  3. Do remaining integrations in terms of ${}_pF_q$ functions if possible
  4. For final integral representation with minimal number of integrations and minimal set of divergencies we construct subtraction terms
  5. Integrand with all divergencies subtracted is expanded in ε and integrated term by term with `HyperInt`
  6. Subtraction terms are integrated in the same way

Conclusion

- Developed techniques
  - For efficient reduction for integrals with Heaviside θ-functions applicable for phase-space integrals with loops and additional regulators
  - For calculation of the most complicated RRR integrals by solving differential equations in auxuliary parameter numerically with high precision and calculated all needed boundary conditions

- Obtained new results
  - Recalculated one-loop corrections for two gluon emission contribution to the soft function together with a new part for the quark pair emission
  - High precision numerical result for triple emission with quarks and gluons in the final state in the different hemispheres contribution

- Final result for renormalized NNNLO zero-jettiness soft function
  - Application to $\mathbf{e}^+\mathbf{e}^- \to 2j$ and Higgs decay Thrust resummation
  - Last missing ingredient for application of jettiness-variable as slicing variable at NNNLO

Thank you for attention!