Towards the Automation of Quarkonium Production Cross Sections with MadGraph

High Precision for Hard Processes 2024

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based on JHEP 07 (2024) 050 [arXiv:2402.19221]
in collaboration with Ajjath A H and H.-S. Shao

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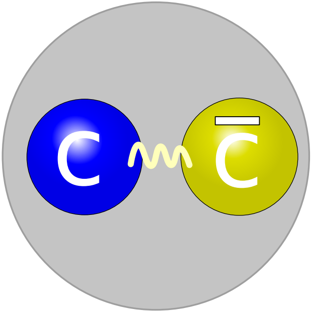
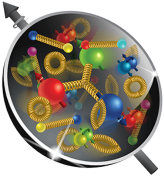
Quarkonium Production

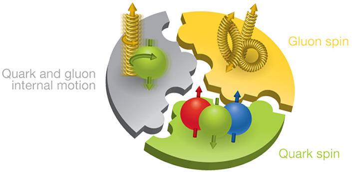


Phenomenological relevance



gluon PDFs
quarkonia are dominantly produced via gluon fusion and give direct access to the momentum distributions of gluons in nuclei and parton correlations in protons

spin puzzle
quarkonia can help to unravel the role of the gluon spin and angular momentum in the nucleon spin



Quarkonium production in NRQCD factorisation

$$\mathrm{d}\sigma(AB \to H + X) = \sum_n \Big(\sum_{a,b,X} \int \mathrm{d}x_a \mathrm{d}x_b \; f_{a/A}(x_a) f_{b/B}(x_b) \;\; \mathrm{d}\hat{\sigma}(ab \to Q\bar{Q}'[n] + X) \Big) \; \langle \mathcal{O}_n^H \rangle$$

Quarkonium production in NRQCD factorisation

$$\mathrm{d}\sigma(AB \to H + X) = \sum_n \Big(\sum_{a,b,X} \int \mathrm{d}x_a \mathrm{d}x_b \underbrace{f_{a/A}(x_a) f_{b/B}(x_b)}_{\text{PDF}} \underbrace{\mathrm{d}\hat{\sigma}(ab \to Q\bar{Q}'[n] + X)}_{\text{partonic cross section}} \Big) \underbrace{\langle \mathcal{O}_n^H \rangle}_{\text{LDME}}$$

Parton Distribution Function (PDF)

- parton distribution functions of partons a and b in the initial hadrons A and B

Partonic cross section

- short distance production of a $Q\bar{Q}'$ pair in a colour representation C, with spin S and orbital angular momentum state L
- spectroscopic notation of Fock states: $n = {}^{2S+1}L_J^{[C]}$

Long Distance Matrix Element (LDME)

- hadronisation of the heavy quark pair into the physical quarkonium state H

Amplitudes with quarkonia

Projection to Fock states

$$a(k_1)b(k_2) \to Q(k_3)\bar{Q}'(k_4) + \ldots$$

$$\boxed{\mathcal{A}(r) = \bar{u}_{\lambda_Q}(k_3)\Gamma(r)v_{\lambda_{\bar{Q}'}}(k_4)}$$

Amplitudes with quarkonia

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↓ colour projection

$$\mathcal{A}_{\{[C]\}}(r) = \sum_{c_3,c_4} \mathbb{P}_C \mathcal{A}(r)$$

Colour projector

$$\mathbb{P}_{C=1} = \frac{\delta^{c_3 c_4}}{\sqrt{N_c}}$$

$$\mathbb{P}_{C=8} = \sqrt{2}\, T^{c_{34}}_{c_4 c_3}$$

Amplitudes with quarkonia

Projection to Fock states

$$a(k_1)b(k_2) \to Q(k_3)\bar{Q}'(k_4) + \dots$$

$$\mathcal{A}(r) = \bar{u}_{\lambda_Q}(k_3)\Gamma(r)v_{\lambda_{\bar{Q}'}}(k_4)$$

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$$\mathcal{A}_{\{[C]\}}(r) = \sum_{c_3,c_4} \mathbb{P}_C \mathcal{A}(r)$$

↓ spin projection

$$\mathcal{A}_{\{[C],S\}}(r) = \sum_{\lambda_Q,\lambda_{\bar{Q}'}} \mathbb{P}_S \mathcal{A}_{\{[C]\}}(r)$$

Spin projector

$$\mathbb{P}_{S=0} = \frac{\bar{v}_{\lambda_{\bar{Q}'}}(k_4)\gamma_5 u_{\lambda_Q}(k_3)}{2\sqrt{2m_Q m_{\bar{Q}'}}}$$

$$\mathbb{P}_{S=1} = \frac{\bar{v}_{\lambda_{\bar{Q}'}}(k_4)\not{\epsilon}^*_{\lambda_s}(K) u_{\lambda_Q}(k_3)}{2\sqrt{2m_Q m_{\bar{Q}'}}}$$

with

$$K^\mu = k_3^\mu + k_4^\mu$$

Amplitudes with quarkonia

Projection to Fock states

$$a(k_1)b(k_2) \to Q(k_3)\bar{Q}'(k_4) + \dots$$

$$\mathcal{A}(r) = \bar{u}_{\lambda_Q}(k_3)\Gamma(r)v_{\lambda_{\bar{Q}'}}(k_4)$$

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↓ spin projection

$$\mathcal{A}_{\{[C],S\}}(r) = \sum_{\lambda_Q,\lambda_{\bar{Q}'}} \mathbb{P}_S \mathcal{A}_{\{[C]\}}(r)$$

↓ orbital angular momentum projection

$$\mathcal{A}_{\{[C],S,L\}}(r) = \left[\left(\varepsilon^{\mu,*}_{\lambda_l}(K)\frac{d}{dq^\mu}\right)^L \mathcal{A}_{\{[C],S\}}(r)\right]_{q=0}$$

Orbital angular momentum

$$k_3^\mu = \frac{m_Q}{m_Q + m_{\bar{Q}'}}K^\mu + q^\mu$$

$$k_4^\mu = \frac{m_{\bar{Q}'}}{m_Q + m_{\bar{Q}'}}K^\mu - q^\mu$$

Amplitudes with quarkonia

Projection to Fock states

$$a(k_1)b(k_2) \to Q(k_3)\bar{Q}'(k_4) + \dots$$

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↓ orbital angular momentum projection

$$\mathcal{A}_{\{[C],S,L\}}(r) = \left[\left(\varepsilon^{\mu,*}_{\lambda_l}(K) \frac{d}{dq^\mu} \right)^L \mathcal{A}_{\{[C],S\}}(r) \right]_{q=0}$$

↓ total angular momentum projection

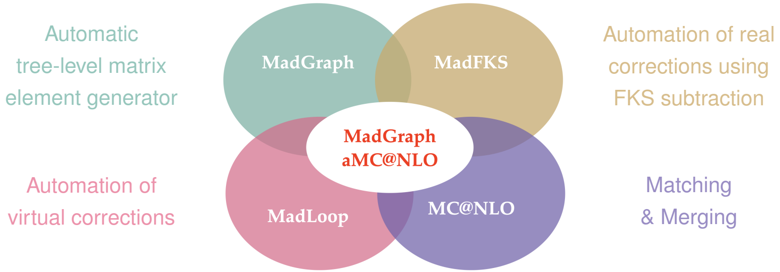
$$\mathcal{A}_{\{[C],S,L,J\}}(r) = \sum_{\lambda_s,\lambda_l} \underbrace{\langle J,\lambda_j | L,\lambda_l; S,\lambda_s \rangle}_{\text{Clebsch–Gordan coefficient}} \mathcal{A}_{\{[C],S,L\}}(r)$$

Quarkonia in MadGraph

MadGraph5_aMC@NLO [Alwall *et al.* 2014]

Automated cross sections for particle colliders

MadGraph5_aMC@NLO is an open-source software that provides a complete set of tools for the simulation and analysis of particle collisions up to next-to-leading order precision in QED and QCD.

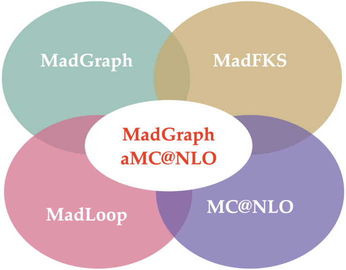


$$\sigma^{\mathrm{NLO}} = \int \mathrm{d}\phi_n \, \mathcal{B} + \int \mathrm{d}\phi_n \, \mathcal{V} + \int \mathrm{d}\phi_{n+1} \, \mathcal{R}$$

MadGraph

Born-level contributions

Automatic tree-level matrix element generator



$$\sigma^{\mathrm{NLO}} = \int \mathrm{d}\phi_n\, \mathcal{B} + \int \mathrm{d}\phi_n\, \mathcal{V} + \int \mathrm{d}\phi_{n+1}\, \mathcal{R}$$

Quarkonia in MadGraph5

in collaboration with A. Colpani Serri, C. Flett, J.-P. Lansberg, O. Mattelaer, H.-S. Shao

Status:

Automated tree-level amplitude generation:

$\mathcal{A}(ab \to Q\bar{Q}'[n] + X) = \mathbb{P}_J \mathbb{P}_L \mathbb{P}_S \mathbb{P}_C \mathcal{A}(ab \to Q\bar{Q}' + X)$

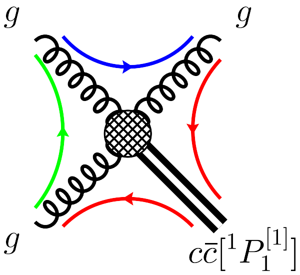
- colour projectors $\mathbb{P}_C$ for colour singlets ✓ and colour octets ✓
- spin projectors $\mathbb{P}_S$ for spin 0 ✓ and spin 1 ✓ states
- orbital $\mathbb{P}_L$ and total angular momentum projectors $\mathbb{P}_J$ for P-wave states ✗

LO cross sections with an arbitrary number of S-wave quarkonia and an arbitrary number of elementary particles can be computed.

Next:

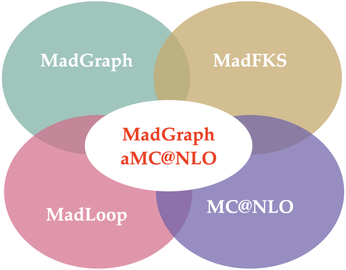
Extension to $\underbrace{\text{P-wave}}_{\text{technical challenges}}$ quarkonia and $\underbrace{\text{NLO}}_{\text{conceptual challenges}}$

NLO Quarkonium Production (Real Emission)



MadFKS

Real corrections

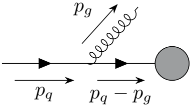


Automation of real corrections using FKS subtraction

$$\sigma^{\mathrm{NLO}} = \int \mathrm{d}\phi_n \, \mathcal{B} + \int \mathrm{d}\phi_n \, \mathcal{V} + \int \mathrm{d}\phi_{n+1} \, \mathcal{R}$$

Infrared singularities

$$|\mathcal{M}_J|^2 = \Bigg|$$



$$\Bigg|^2 \sim \frac{1}{(p_q - p_g)^2} = \frac{1}{E_q E_g (1 - \beta_q \cos\theta_{qg})}$$

with

$$\beta_q = \frac{|\vec{p}_q|}{E_q} \xrightarrow{m_q \to 0} 1$$

$$\int |\mathcal{M}_J|^2 F_J \mathrm{d}\Pi^{(4)} \sim \int_0^{E^{\max}} \mathrm{d}E_g \int_{-1}^{1} \mathrm{d}\cos\theta_{qg} \frac{1}{E_q E_g (1 - \beta_q \cos\theta_{qg})} \to \infty \begin{cases} \text{for } E_g \to 0 \\ \text{for } \theta_{qg} \to 0 \\ \quad \wedge \beta_q \to 1 \end{cases}$$

Infrared singularities:

- soft singularity ($E_g \to 0$)
- collinear singularity ($\theta_{qg} \to 0 \wedge \beta_q \to 1$)
- soft-collinear singularity ($E_g \to 0$ and $\theta_{qg} \to 0 \wedge \beta_q \to 1$)

Subtraction schemes

$$\lim_{d\to 4}\int |\mathcal{M}_J|^2 F_J \mathrm{d}\Pi^{(d)} \to \infty$$

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$$\lim_{d\to 4} \int |\mathcal{M}_J|^2 F_J \mathrm{d}\Pi^{(d)} \to \infty$$

Subtraction function

find a suitable subtraction function S which

- reproduces the matrix element in the unresolved limit
- is simple to integrate over the unresolved phase space

$$\lim_{\text{IR poles}} S = \lim_{\text{IR poles}} |\mathcal{M}_J|^2 F_J$$

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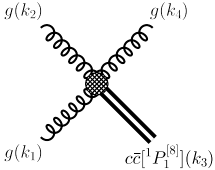
$$\lim_{\text{IR poles}} S = \lim_{\text{IR poles}} |\mathcal{M}_J|^2 F_J$$

$$\lim_{d\to 4}\int |\mathcal{M}_J|^2 F_J \mathrm{d}\Pi^{(d)} = \underbrace{\int \left(|\mathcal{M}_J|^2 F_J - S\right) \mathrm{d}\Pi^{(4)}}_{\text{free of divergences}} + \underbrace{\lim_{d\to 4}\int S\, \mathrm{d}\Pi^{(d)}}_{\text{analytic solution}}$$

FKS subtraction scheme

Frixione-Kunszt-Signer

$$\mathrm{d}\sigma^{\mathrm{R}}(r) = \mathrm{d}\sigma^{\mathrm{R}}(r)$$

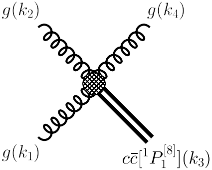


FKS subtraction scheme

Frixione-Kunszt-Signer

$$\mathrm{d}\sigma^{\mathrm{R}}(r) = (1 - \mathcal{S}_4)\, \mathrm{d}\sigma^{\mathrm{R}}(r)$$

$$+ \mathcal{S}_4\, \mathrm{d}\sigma^{\mathrm{R}}(r)$$

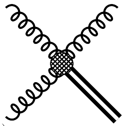


1. regulate soft singularities: $\mathcal{S}_4 = \lim_{E_4 \to 0}$
   - soft gluon can be emitted from both initial state gluons **and** from the $Q\bar{Q}'$ pair

FKS subtraction scheme

Frixione-Kunszt-Signer

$$\begin{aligned}\mathrm{d}\sigma^{\mathrm{R}}(r) &= (1-\mathcal{S}_4)\,(1-(\mathcal{C}_{41}+\mathcal{C}_{42}))\;\mathrm{d}\sigma^{\mathrm{R}}(r)\\ &\quad+(1-\mathcal{S}_4)(\mathcal{C}_{41}+\mathcal{C}_{42})\,\mathrm{d}\sigma^{\mathrm{R}}(r)\\ &\quad+\mathcal{S}_4\,\mathrm{d}\sigma^{\mathrm{R}}(r)\end{aligned}$$



1. regulate soft singularities: $\mathcal{S}_4 = \lim_{E_4\to 0}$
   - soft gluon can be emitted from both initial state gluons **and** from the $Q\bar{Q}'$ pair
2. regulate collinear singularities: $\mathcal{C}_{4i} = \lim_{\theta_{i4}\to 0}$
   - collinear gluon can be emitted from both initial state gluons, but **not** from the $Q\bar{Q}'$ pair

FKS subtraction scheme

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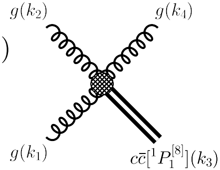
$$\mathrm{d}\sigma^{\mathrm{R}}(r) = (1-\mathcal{S}_4)\,(1-(\mathcal{C}_{41}+\mathcal{C}_{42}))\;\mathrm{d}\sigma^{\mathrm{R}}(r) + (1-\mathcal{S}_4)(\mathcal{C}_{41}+\mathcal{C}_{42})\,\mathrm{d}\sigma^{\mathrm{R}}(r) + \mathcal{S}_4\,\mathrm{d}\sigma^{\mathrm{R}}(r)$$



1. regulate soft singularities: $\mathcal{S}_4 = \lim_{E_4 \to 0}$
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2. regulate collinear singularities: $\mathcal{C}_{4i} = \lim_{\theta_{i4} \to 0}$
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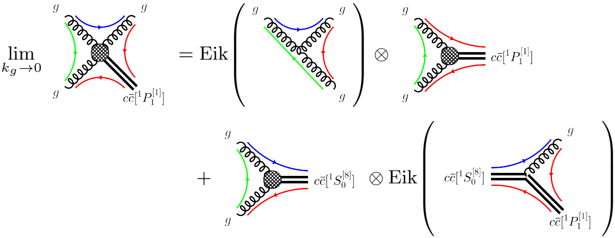
FKS for quarkonium production

- **new soft subtraction terms** w.r.t. production of elementary particles
- **collinear subtraction terms** are **unchanged**

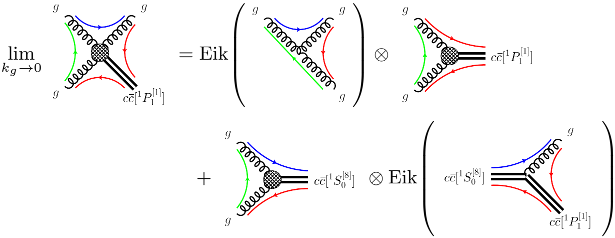
Colour-singlet spin-singlet P-wave

Soft limit of the amplitude



Colour-singlet spin-singlet P-wave

Soft limit of the amplitude



$$\lim_{k_i \to 0} \mathcal{A}^{(n+1)}_{\{[1],0,1,1\}}(r) = \sum_{j=n_I}^{n_L^{(R)}+n_H} g_s \frac{k_j \cdot \varepsilon^*_{\lambda_i}(k_i)}{k_j \cdot k_i} \vec{Q}(\mathcal{I}_j) \mathcal{A}^{(n)}_{\{[1],0,1,1\}}(r^{\not{i}})$$

$$+ g_s \left[\frac{\varepsilon^*_{\lambda_l}(K) \cdot \varepsilon^*_{\lambda_i}(k_i)}{K \cdot k_i} - \frac{K \cdot \varepsilon^*_{\lambda_i}(k_i) k_i \cdot \varepsilon^*_{\lambda_l}(K)}{(K \cdot k_i)^2} \right] \vec{Q}_{\text{eff}}(Q\bar{Q}'_{[18]}) \mathcal{A}^{(n)}_{\{[8],0,0,0\}}(r^{\not{i}})$$

Colour-singlet spin-singlet P-wave

Integrated soft counterterms

Reminder: subtraction schemes

$$\lim_{d\to 4}\int |\mathcal{M}_J|^2 F_J \mathrm{d}\Pi^{(d)} = \underbrace{\int \left(|\mathcal{M}_J|^2 F_J - S\right) \mathrm{d}\Pi^{(4)}}_{\text{free of divergences}} + \underbrace{\lim_{d\to 4}\int S\, \mathrm{d}\Pi^{(d)}}_{\text{analytic solution}}$$

Colour-singlet spin-singlet P-wave

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Analytic solution:

$$d\hat{\sigma}^{(S)}(r) = \frac{\alpha_s}{2\pi}\phi_{n-1}(r^{\not i})\frac{J^{n_L^{(B)}}}{\mathcal{N}(r^{\not i})}\mathcal{G}(r^{\not i})\Bigg[\sum_{k=n_I}^{n_L^{(B)}+n_H}\sum_{l=k}^{n_L^{(B)}+n_H}\bar{\mathcal{E}}(\{1,1\},\{k_k,k_l\})\mathcal{M}_{kl}(r^{\not i})$$
$$+\sum_{k=n_I}^{n_L^{(B)}+n_H}\left(\bar{\mathcal{E}}(\{1,1\},\{k_k,K\})\frac{k_{k,\mu}}{K\cdot k_k} - \bar{\mathcal{E}}_\mu(\{1,2\},\{k_k,K\})\right)\mathcal{M}^{\mu}_{k[18]}(r^{\not i},r_1^{\not i})$$
$$+\left(\frac{1}{2N_c}\frac{8}{m_Q m_{\bar{Q}'}}\left(\frac{1}{\bar{\epsilon}}+\log\frac{\mu^2}{\mu^2_{\mathrm{NRQCD}}}\right) - \frac{2\epsilon-2}{K^2}\bar{\mathcal{E}}(\{1,1\},\{K,K\})C_{\mathrm{eff}}(Q\bar{Q}'_{[18]})\right)\mathcal{M}(r_1^{\not i})\Bigg]$$

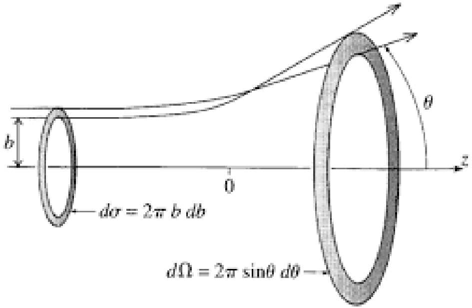
where the eikonal integrals

$$\bar{\mathcal{E}}^{\alpha_1\ldots\alpha_{n_1-1}\beta_1\ldots\beta_{n_2-1}}(\{n_1,n_2\},\{k_k,k_l\}) = 8\pi^2\mu^{2\epsilon}k_k\cdot k_l\int\frac{d^{3-2\epsilon}\mathbf{k}_i}{(2\pi)^{3-2\epsilon}2k_i^0}\frac{k_i^{\alpha_1}\ldots k_i^{\alpha_{n_1-1}}k_i^{\beta_1}\ldots k_i^{\beta_{n_2-1}}}{(k_k\cdot k_i)^{n_1}(k_l\cdot k_i)^{n_2}}\theta(\xi_{\mathrm{cut}}-\xi_i)$$

can be solved analytically.

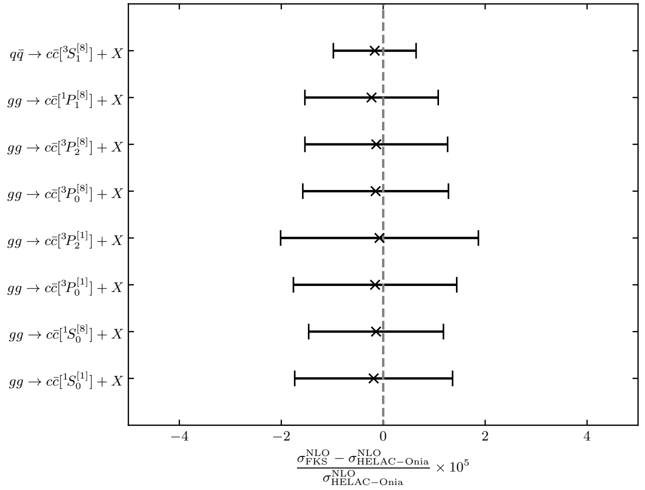
- singularities appear explicitly as poles in ϵ

Cross Sections



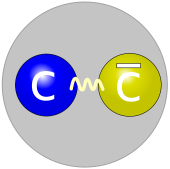
Inclusive NLO cross sections

Validation of $2 \to 1$ processes

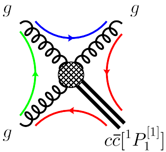


- inclusive cross sections of FKS approach (**fully numerical**) can be compared to `HELAC-Onia` [Shao, 2016] (**analytic phase-space integration**)

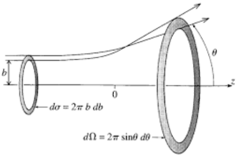
Summary



c
$\bar{c}$



g
g
g
$c\bar{c}[{}^1P_1^{[1]}]$



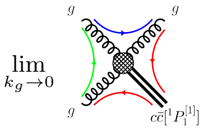
b
$d\sigma = 2\pi\, b\, db$
$d\Omega = 2\pi \sin\theta\, d\theta$
θ
0
z

Summary

- automated cross section computations for processes with bound states in a future `MadGraph5_aMC@NLO` release

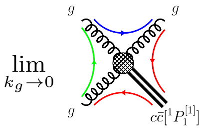
Summary

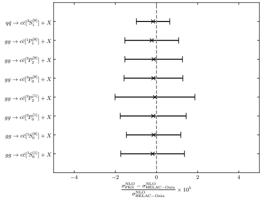
- automated cross section computations for processes with bound states in a future `MadGraph5_aMC@NLO` release
- fully differential NLO results require knowledge about soft gluon emission from quarkonia
  - S-wave quarkonia behave like “massive gluons”
  - P-wave quarkonia lead to more complex structures



Summary

- automated cross section computations for processes with bound states in a future `MadGraph5_aMC@NLO` release
- fully differential NLO results require knowledge about soft gluon emission from quarkonia
  - S-wave quarkonia behave like “massive gluons”
  - P-wave quarkonia lead to more complex structures
- local FKS subtraction term and the corresponding integrated counterterm were computed
  - eikonal limits are check numerically
  - ϵ-poles of integrated counterterms cancel analytically against virtual corrections

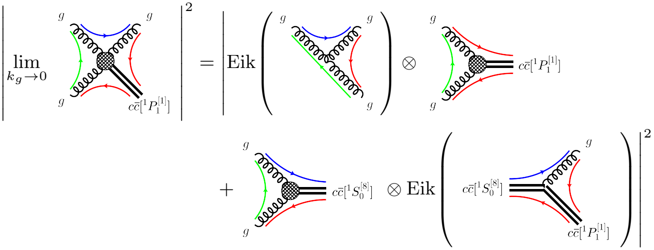




Backup

Colour-singlet spin-singlet P-wave

Soft limit of the squared amplitude



$$
\lim_{k_i \to 0} \mathcal{M}_{\{[1],0,1,1\}}(r) = g_s^2 \Bigg\{ \sum_{\substack{k,l=n_I \\ k,l \neq i, k \leq l}}^{n_L^{(R)}+n_H} \frac{k_k \cdot k_l}{k_k \cdot k_i k_l \cdot k_i} \mathcal{M}_{kl}(r^{\not{i}})
$$

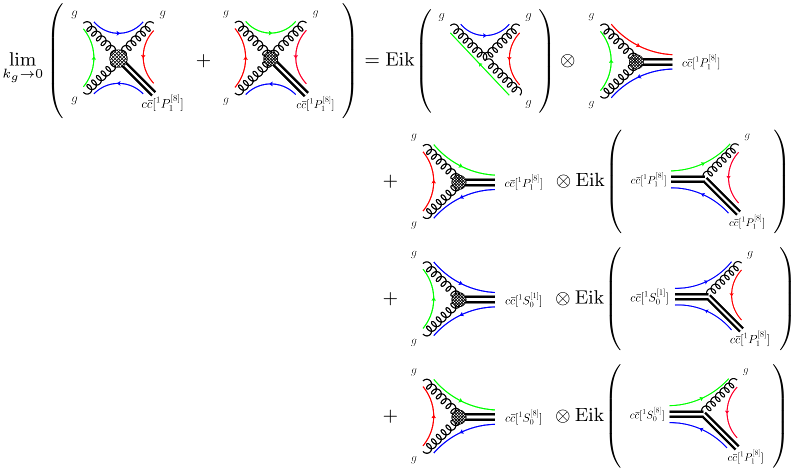
$$
+ \sum_{\substack{k=n_I \\ k \neq i}}^{n_L^{(R)}+n_H} \left[\frac{k_{k,\mu}}{k_k \cdot k_i K \cdot k_i} - \frac{K \cdot k_k k_{i,\mu}}{k_k \cdot k_i \, (K \cdot k_i)^2} \right] \mathcal{M}^{\mu}_{k[18]}(r^{\not{i}}, r_1^{\not{i}})
$$

$$
- \frac{2\epsilon - 2}{(K \cdot k_i)^2} C_{\text{eff}}(Q\bar{Q}'_{[18]}) \mathcal{M}(r_1^{\not{i}}) \Bigg\}
$$

BACKUP

Color-octet spin-singlet P-wave

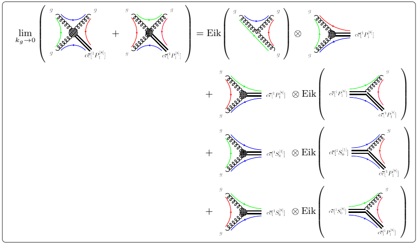
Soft limit of the amplitude



BACKUP

Color-octet spin-singlet P-wave

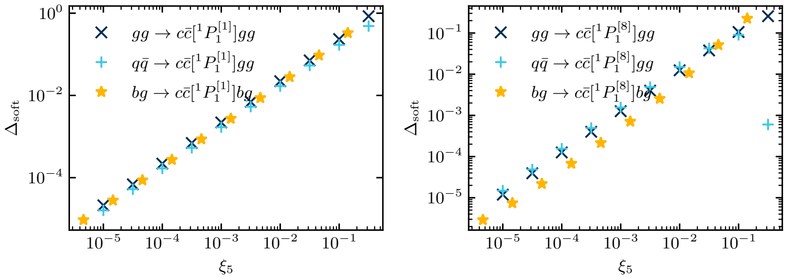
Soft limit of the amplitude



$$
\begin{aligned}
\lim_{k_i \to 0} \mathcal{A}_{\{[8],0,1,1\}}(r) = & \sum_{\substack{j=n_I \\ j\neq i}}^{n_L^{(R)}+n_H} g_s \frac{k_j \cdot \varepsilon^*_{\lambda_i}(k_i)}{k_j \cdot k_i} \vec{Q}(\mathcal{I}) \mathcal{A}_{\{[8],0,1,1\}}(r^{\not{i}}) \\
& + g_s \frac{K \cdot \varepsilon^*_{\lambda_i}(k_i)}{K \cdot k_i} \vec{Q}(Q\bar{Q}'[{}^1P_1^{[8]}]) \mathcal{A}_{\{[8],0,1,1\}}(r^{\not{i}}) \\
& + g_s \left[\frac{\varepsilon^*_{\lambda_l}(K) \cdot \varepsilon^*_{\lambda_i}(k_i)}{K \cdot k_i} - \frac{K \cdot \varepsilon^*_{\lambda_i}(k_i) k_i \cdot \varepsilon^*_{\lambda_l}(K)}{(K \cdot k_i)^2} \right] \\
& \times \left[\vec{Q}_{\text{eff}}(Q\bar{Q}'_{[81]}) \mathcal{A}_{\{[1],0,0,0\}}(r^{\not{i}}) + \vec{Q}_{\text{eff}}(Q\bar{Q}'_{[88]}) \mathcal{A}_{\{[8],0,0,0\}}(r^{\not{i}}) \right]
\end{aligned}
$$

BACKUP

Local soft subtraction terms



$$\Delta_{\mathrm{soft}} = \left| \frac{\mathcal{M}(r) - \lim_{k_5 \to 0} \mathcal{M}(r)}{\mathcal{M}(r)} \right| \propto \xi_5 + \mathcal{O}(\xi_5^2)\,, \quad \text{with} \quad E_5 = \xi_5 \frac{\sqrt{s}}{2}$$

Reminder: subtraction schemes

$$\lim_{d\to 4} \int |\mathcal{M}_J|^2 F_J \mathrm{d}\Pi^{(d)} = \underbrace{\int \left(|\mathcal{M}_J|^2 F_J - S\right) \mathrm{d}\Pi^{(4)}}_{\text{free of divergences}} + \underbrace{\lim_{d\to 4} \int S \, \mathrm{d}\Pi^{(d)}}_{\text{analytic solution}}$$

BACKUP

Subtraction schemes

Concept

$$I = \int_0^1 \mathrm{d}x \frac{1}{x^{1+\epsilon}} F(x)$$
$$= \int_0^1 \mathrm{d}x \frac{1}{x^{1+\epsilon}} \left[F(x) - F(0)\right] + F(0) \int_0^1 \mathrm{d}x \frac{1}{x^{1+\epsilon}}$$

Subtraction function $F(0)$

- $\frac{1}{x^{1+\epsilon}} \left[F(x) - F(0)\right]$ is free of divergences (numerical integration)
- $F(0) \int_0^1 \mathrm{d}x \frac{1}{x^{1+\epsilon}}$ divergent but simple to integrate analytically

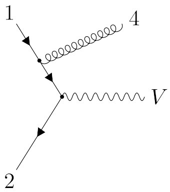
$$I = \underbrace{\int_0^1 \mathrm{d}x \frac{1}{x} \left[F(x) - F(0)\right]}_{\text{free of divergences}} - \underbrace{\frac{1}{\epsilon} F(0)}_{\text{analytic solution}} + \mathcal{O}(\epsilon)$$

BACKUP

FKS subtraction

Local soft subtraction function

$$\langle F_{\rm LM}(1,2\,|\,4)\rangle = \langle (1-\mathcal{S}_4)F_{\rm LM}(1,2\,|\,4)\rangle + \langle \mathcal{S}_4 F_{\rm LM}(1,2\,|\,4)\rangle$$



Step 1: Find suitable subtraction functions.

- $\mathcal{S}_4$ extracts the leading soft singularity
- Eikonal approximation

$$\mathcal{S}_4 F_{\rm LM}(1,2\,|\,4) = 2C_F g_s^2 \frac{p_1\cdot p_2}{(p_1\cdot p_4)(p_2\cdot p_4)}\, F_{\rm LM}(1,2)$$

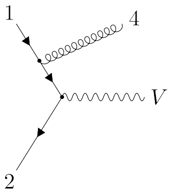
Free of soft divergences

$$F_{\rm LM}(1,2\,|\,4) - 2C_F g_s^2 \frac{p_1\cdot p_2}{(p_1\cdot p_4)(p_2\cdot p_4)}\, F_{\rm LM}(1,2)$$

FKS subtraction

Integrated soft subtraction function

$$\langle F_{\mathrm{LM}}(1,2\,|\,4)\rangle = \langle (1-\mathcal{S}_4) F_{\mathrm{LM}}(1,2\,|\,4)\rangle + \langle \mathcal{S}_4 F_{\mathrm{LM}}(1,2\,|\,4)\rangle$$



Step 2: Analytic solution of the subtraction functions.

$$\langle \mathcal{S}_4 F_{\mathrm{LM}}(1,2\,|\,4)\rangle = \int [\mathrm{d}g_4]\mathcal{S}_4 F_{\mathrm{LM}}(1,2\,|\,4)$$
$$= \int \frac{\mathrm{d}^{d-1}p_4}{(2\pi)^d 2E_4} \boxed{2C_F g_s^2 \frac{p_1\cdot p_2}{(p_1\cdot p_4)(p_2\cdot p_4)}} F_{\mathrm{LM}}(1,2)$$
$$= \mathcal{F}(1,2;\epsilon) F_{\mathrm{LM}}(1,2)$$

Analytic solution

- singularities are manifest in $\mathcal{F}(1,2;\epsilon)$

BACKUP

FKS subtraction
Finite remainder

$$\langle F_{\mathrm{LM}}(1,2\,|\,4)\rangle = \langle(1-\mathcal{S}_4)(1-(\mathcal{C}_{41}+\mathcal{C}_{42}))F_{\mathrm{LM}}(1,2\,|\,4)\rangle$$
$$+\langle(1-\mathcal{S}_4)(\mathcal{C}_{41}+\mathcal{C}_{42})F_{\mathrm{LM}}(1,2\,|\,4)\rangle$$
$$+\langle\mathcal{S}_4 F_{\mathrm{LM}}(1,2\,|\,4)\rangle$$

Step 3: Combination of real and virtual contributions.

$$\mathrm{d}\sigma^{\mathrm{NLO}} = F_{\mathrm{LV}}^{\mathrm{fin}}(1,2) + \big\langle(1-\mathcal{S}_4)(1-(\mathcal{C}_{41}+\mathcal{C}_{42}))F_{\mathrm{LM}}(1,2\,|\,4)\big\rangle$$
$$+\frac{\alpha_s}{2\pi}\int \mathrm{d}z\,\mathcal{P}_{qq}(z)\frac{F_{\mathrm{LM}}(z\cdot 1,2)+F_{\mathrm{LM}}(1,z\cdot 2)}{z}$$
$$+\frac{\alpha_s}{2\pi}\frac{2\pi^2}{3}C_F F_{\mathrm{LM}}(1,2)$$

Finite remainder

- can be integrated (numerically) in four dimension

BACKUP