

Two-loop amplitude reduction in the HELAC framework

Based on work with Costas Papadopoulos, Giuseppe Bevilacqua,
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Research & Innovation

- HELAC amplitude construction
- OPP Amplitude reduction at 1-loop
- 2-loop reduction: 4 vs D-dimensions
- Outlook

Amplitude Construction

- Qgraph \rightarrow symbolic manipulation, dimensionally regularized amplitudes \rightarrow IBP: FIRE Kira or numerical pySecDec
- Numerical unitarity \rightarrow dimensionally regularized amplitudes by gluing tree amplitudes in different integer dimensions $\rightarrow D_s$ (Abreu, Cordero, Ita, Page and Sotnikov, 2021a)
- OpenLoops \rightarrow Feynman graph \rightarrow opening the loops \rightarrow amplitudes in $d = 4 \rightarrow$ coefficients of tensor integrals (Pozzorini, Schär and Zoller, 2022)

Costas Papadopoulos and Aggeliki Kanaki introduced HELAC (Kanaki and Papadopoulos, 2000)

This was an automated way to compute amplitudes using the Dyson-Schwinger Equations i.e. recursively expressing n-point Green's Functions in terms of lesser point Green's Functions.

2-loop amplitude is required for NNLO precision.

$$\hat{\sigma}_{NNLO} \sim |A_{tree}|^2 + \alpha_s \left(2\text{Re}[A_{tree}A_{1loop}^*] + |A_{+1p}|^2 \right) + \alpha_s^2 \left(|A_{1loop}|^2 + 2\text{Re}[A_{tree}A_{1loop+1p}^* + A_{tree}A_{2loop}^*] + |A_{+2p}|^2 \right)$$

Various results:

- $pp \rightarrow W + \gamma + j$ (Badger, Hartanto, Kryś and Zoia, 2022)
- $pp \rightarrow H + b\bar{b}$ (Badger, Hartanto, Kryś and Zoia, 2021b)
- $u\bar{d} \rightarrow W^+ + b\bar{b}$ (Badger, Hartanto, Kryś and Zoia, 2022)
- $pp \rightarrow b\bar{b}$ (Hartanto, Poncelet, Popescu and Zoia, 2022)
- $pp \rightarrow Zb\bar{b}$ (Mazzitelli, Sotnikov and Wiesemann, 2024)
- $q\bar{q} \rightarrow gg\gamma$ and $q\bar{q} \rightarrow Q\bar{Q}\gamma$ (Badger, Czakon, Hartanto, Moodie, Peraro, Poncelet and Zoia, 2023)
- $pp \rightarrow \gamma\gamma + j$ (Chawdhry, Czakon, Mitov and Poncelet, 2021a)
- $pp \rightarrow W + 4\text{partons}$ (Abreu, Cordero, Ita, Klinkert, Page and Sotnikov, 2022; Hartanto, Badger, Brønnum-Hansen and Peraro, 2019)
- $gg \rightarrow g\gamma\gamma$ (Badger, Brønnum-Hansen, Chicherin, Gehrmann, Hartanto, Henn, Marcoli, Moodie, Peraro and Zoia, 2021a)
- $qg \rightarrow q\gamma\gamma$ and $q\bar{q} \rightarrow q\gamma\gamma$ (Agarwal, Buccioni, von Manteuffel and Tancredi, 2021)
- $q\bar{q} \rightarrow \gamma\gamma\gamma$ (Abreu, Page, Pascual and Sotnikov, 2021b; Chawdhry, Czakon, Mitov and Poncelet, 2021b)
- $pp \rightarrow 3j$ (Abreu, Cordero, Ita, Page and Sotnikov, 2021a; Czakon, Mitov and Poncelet, 2021)
- 5 partons (Abreu, Dormans, Febres Cordero, Ita, Page and Sotnikov, 2019b; Abreu, Febres Cordero, Ita, Page and Sotnikov, 2018)
- $gg \rightarrow ggg$ (Badger, Frellesvig and Zhang, 2013; Badger, Brønnum-Hansen, Hartanto and Peraro, 2019a; Abreu, Dormans, Cordero, Ita and Page, 2019a; Badger, Chicherin, Gehrmann, Heinrich, Henn, Peraro, Wasser, Zhang and Zoia, 2019b)
- $pp \rightarrow \gamma\gamma\gamma$ (Kallweit, Sotnikov and Wiesemann, 2021)

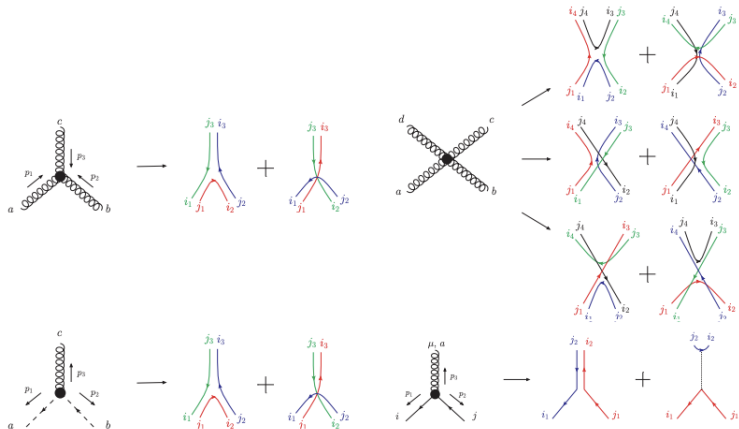
What do we need for HELAC 2-loop?

3 steps:

- 1. Amplitude Construction
- **2. Amplitude Reduction at 2-loops (My talk today)**
- 3. Do the integrals

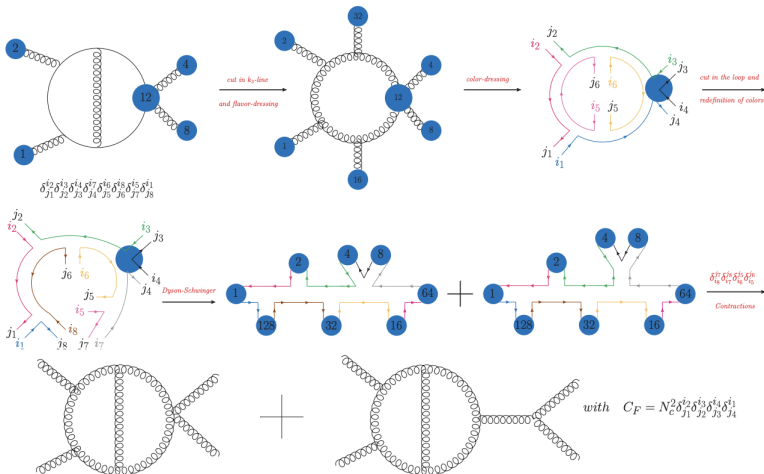
Color Flow representation

How do we deal with QCD?



Each colored line corresponds to a Kronecker delta.

Amplitude Construction at 2-Loops



Amplitude Construction at 2-Loops: Skeleton

The information for each amplitude is incorporated in the Skeleton

```
INFO =====
INFO COLOR          9 out of          24
INFO number of nums          0
INFO =====
INFO COLOR          10 out of         24
INFO number of nums        332
INFO NUM            1 of          332      8
INFO NUM            110 of          332      7
INFO =====
INFO  4  80  35  9  1  1  16  35  5  64  35  7  0  0  0  0  1  2
INFO  4  12  35 10  1  1  4  35  3  8  35  4  0  0  0  0  1  1
INFO  4  92  35 11  1  2  12  35 10 80  35  9  0  0  0  0  1  1
INFO  5  92  35 11  2  2  4  35  3  8  35  4 80 35  9  0  1  5
INFO  4 124  35 12  1  1 32  35  6 92  35 11  0  0  0  0  1  2
INFO  4 126  35 13  1  1  2  35  2 124 35 12  0  0  0  0  1  1
INFO  4 254  35 14  1  1 128 35  8 126 35 13  0  0  0  0  1  2
INFO  6  1  12  1  2 12  35 35 35 35 35 35  0  0  0  0  5  9
```

From the first line to the second-to-last line, there is a sequence of sub-amplitudes accompanied by instructions for their computation.

Amplitude Construction

So far, everything is done in 4 dimensions. But, this is not necessary! The skeleton doesn't know about the number of dimensions!

Amplitude reduction OPP @ 1-loop

Reduction is done at the *integrand* level. OPP (Ossola, Papadopoulos and Pittau, 2007) master formula for the 4-d numerator at 1 loop:

$$\begin{aligned} N(q) = & \sum_{i_0 < i_1 < i_2 < i_3}^{m-1} d(i_0 i_1 i_2 i_3) + \tilde{d}(q; i_0 i_1 i_2 i_3) \prod_{i \neq i_0, i_1, i_2, i_3}^{m-1} D_i \\ & + \sum_{i_0 < i_1 < i_2}^{m-1} c(i_0 i_1 i_2) + \tilde{c}(q; i_0 i_1 i_2) \prod_{i \neq i_0, i_1, i_2}^{m-1} D_i \\ & + \sum_{i_0 < i_1 < i_2}^{m-1} b(i_0 i_1) + \tilde{b}(q; i_0 i_1) \prod_{i \neq i_0, i_1}^{m-1} D_i \\ & + \sum_{i_0}^{m-1} a(i_0) + \tilde{a}(q; i_0) \prod_{i \neq i_0}^{m-1} D_i \end{aligned}$$

where \tilde{d} , \tilde{c} , \tilde{b} , \tilde{a} are terms which vanish upon integration.

The system is solved by iteratively:

- Evaluate the numerator on values of k for which $D_i(k) = 0$, starting from the first line of the previous equation, where 4 propagators are put on shell
- This condition by default sets to zero all the rest of the terms, allowing us to calculate d and \tilde{d}
- Repeat for c , \tilde{c} and so on until we fit all the coefficients

We need to account for the mismatch between 4 and $D = 4 - 2\epsilon \longrightarrow$
Rational Terms.

Categorized into two terms (Ossola, Papadopoulos and Pittau, 2008b):

- R1, arising from the mismatch between the of D- and 4- dimensional propagators.
- R2, arising from the extra dimensional parts of the loop momentum, metric tensor and gamma matrices.

1 Loop Example

Example of a 6 point amplitude

$$\mathcal{A}(q) = \sum \underbrace{\frac{N_i^{(6)}(q)}{\bar{D}_{i_0} \bar{D}_{i_1} \cdots \bar{D}_{i_5}}}_{\text{Hexagon}} + \underbrace{\frac{N_i^{(5)}(q)}{\bar{D}_{i_0} \bar{D}_{i_1} \cdots \bar{D}_{i_4}}}_{\text{Pentagon}} + \underbrace{\frac{N_i^{(4)}(q)}{\bar{D}_{i_0} \bar{D}_{i_1} \cdots \bar{D}_{i_3}}}_{\text{Square}} + \underbrace{\frac{N_i^{(3)}(q)}{\bar{D}_{i_0} \bar{D}_{i_1} \bar{D}_{i_2}}}_{\text{Triangle}} + \dots$$

- HELAC numerically computes the numerators taking into account all flavors and colors that are consistent with the given process
- The value of the loop momentum q is given by CutTools (Ossola, Papadopoulos and Pittau, 2008a)
- Do the reduction, Integral calculation, compute the rational terms
This has been implemented! → **HELAC One Loop!** (Bevilacqua, Czakon, Garzelli, van Hameren, Kardos, Papadopoulos, Pittau and Worek, 2013)

2-Loop Amplitude: Reduction

Reduction is done at the *integrand* level. **The fit will be more complex!**
A generic 2-loop integrand can be written using the following scalar product set:

$$\{p_i \cdot p_j, k_i \cdot k_j, k_i \cdot p_j, k_i \cdot \eta_j\}$$

as well as any masses inside the loops. For the rest of the discussion we will ignore the case of massive loops.

The integrand can be written in the general form

$$\mathcal{R} = \frac{\mathcal{N}}{\mathcal{D}} = \frac{\sum_a c_a (z_1^{(a)})^{\beta_1} \dots (z_{n_a}^{(a)})^{\beta_{N_a}}}{D_1 \dots D_{N_p}} \quad (1)$$

where the z_i are any of the scalar products in the set.

2-Loop Amplitude reduction

The above can be written in a more reduced form:

$$\mathcal{R} = \frac{\mathcal{N}}{\mathcal{D}} = \sum_{m=0}^{N_p} \sum_{\sigma} \frac{\sum_a \bar{c}_a (\bar{z}_1^{(a)})^{\alpha_1} \dots (\bar{z}_{N_a}^{(a)})^{\alpha_N}}{D_{\sigma_1} \dots D_{\sigma_m}} \quad (2)$$

where now the \bar{z}_i are only the scalar products which cannot be eliminated by being written as linear combinations of D_i , known as irreducible scalar products (ISPs) or the transverse $k_i \cdot \eta_j$ and σ is any subset of $\{1, \dots, N_p\}$ with m elements.

Write the numerator of the integrand level amplitude as follows:

Numerator Formula

$$\mathcal{N} = \sum_{m=0}^{N_p} \sum_{\sigma} \sum_a \bar{c}_a \prod_i^{N_{T+ISP}} (\bar{z}_i)^{\alpha_i} \prod_j D_j \quad (3)$$

where N_{T+ISP} is the number of ISP and transverse scalar products and $j \neq \sigma_i$ for all i (Bevilacqua, Canko and Papadopoulos, 2024)

2-Loop Amplitude reduction

Our goal now is to calculate the coefficients \bar{c}_a which generically depend on the set of scalar products. For each topology, we

- Identify the maximal set of loop propagators we can set to zero, i.e. the Maximal Cut and solve the equations that put all of them on shell simultaneously, AKA *cut solutions*
- Write a linear system for the coefficients $\mathbf{M} \cdot \vec{c} = \vec{\mathcal{N}}$ where \mathbf{M} is a matrix of all monomials \bar{z}_i evaluated on different values of cut-solutions, \vec{c} is all the \bar{c}_a and $\vec{\mathcal{N}}$ is a vector of equal length with values of the numerator evaluated on the cut-solutions
- Solve the system of equations
- Subtract the result from the numerator

2-Loop Amplitude reduction

- Move to the next cut, where one less propagator is put on shell, AKA a *subtopology*.
- Do this till this for all subtopologies (up to a 2-cut for massless legs)
- Algebraic test that the reduced amplitude is equal to the original amplitude, known as the $N=N$ test.
- The reduction is complete!

2-Loop Amplitude: Final Form

$$\mathcal{N} = P_{maxcut} + \sum_i P_{maxcut-1} D_i + \sum_{ij} P_{maxcut-2} D_i D_j + \dots \quad (4)$$

where the Ps are polynomials in the transverse elements and ISPs, with the values of \bar{c}_a such that this equation is satisfied.

Final Form:

$$\mathcal{A} = \sum_i \bar{c}_i F_i \quad (5)$$

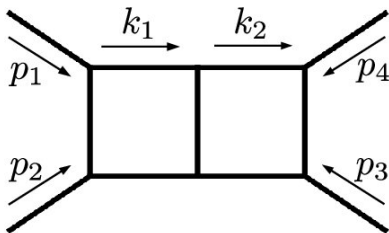
where:

$$F_i \equiv F_{a_1 \dots a_N} = \int d^d k \frac{\overbrace{(D_{m+1})^{a_{m+1}} \dots (D_N)^{a_N}}^{ISP}}{\underbrace{(D_1)^{a_1} \dots (D_m)^{a_m}}_{RSP}}$$

Use Integration by Parts (IBP) and/or other methods to further simplify the integral basis.

2-Loop reduction example

Let's look at a specific $2 \rightarrow 2$ gluon-topology example, part of current work:



Maximal cut: 7 propagators on shell. Question arises: Can/should we fit in 4 dimensions or $D = 4 - 2\epsilon$ dimension?

2-Loop reduction example: 4-dimensions

Begin with 8 free parameters which construct solutions to the cut equations. After imposing the on-shell conditions in the maximal cut only 1 parameter is left independent.

Can use relations between the various monomials to reduce the number of coefficients we need to fit, implemented through BasisDet (Zhang, 2012).

Recent result

Completed a Mathematica simulation for all subtopologies of this example, including the **N=N test**.

2-Loop reduction example: 4-dimensions

More specific issues in 4d:

- Some monomials which form the matrix \mathbf{M} have the same values for different cut solution sets. Danger that we could get $\text{Rank}[\mathbf{M}] < \text{Length}[\vec{c}]$
→ **Need to use full set of solutions, treating each of them independently.**
- Computationally difficult to reduce the terms in the numerator using Gram determinant relations
→ **Write directly in terms of 4d k_i components**

2-Loop reduction example: D-dimensions

Begin with 11 free parameters: 8 from the components of 2 4-momenta, and 3 $\mu_{11}, \mu_{22}, \mu_{12}$ the ϵ components of k_1^2 , k_2^2 and $k_1 \cdot k_2$ respectively. The maximal cut has 7 cut equations, we have a remainder of 4 free parameters.

Use the 4 free parameters to get sets of solutions in order to solve the system. Sufficient rank is less of a concern.

Result

We have completed a Mathematica simulation of this fit analytically. Agreement with known results: Caravel (Abreu, Cordero, Ita, Page and Sotnikov, 2021a)

Upcoming

Complete cut+fit for all subtologies, and then implement numerically.

D-dimensional amplitude construction

Currently underway in our group, including a method of numerically evaluating the amplitude in D-dimensions by keeping track of ϵ and μ terms.

→ Is an option since **the skeleton is dimensionally agnostic!**

Note that in principle construction and reduction of the amplitude are completely modular, i.e **independent of each other!**

2 loop reduction interesting questions

- What goes into constructing the polynomial Ansatz? One option is BasisDet (Zhang, 2012). Is there some apriori way of determining it for each topology/subtopology apriori? What is the "correct" power for each monomial at each subtopology?
- Especially for 4-d: Can we build the polynomial in some other way, without relying on monomials?
- Structure of the solution space for the cut solutions (Algebraic Geometry question) (Frellesvig, 2014)

Immediate steps

- Numerically implement the 4-d reduction within the HELAC framework @ 2-Loops for all $2 \rightarrow 2$ topologies
- Implement the fully D-dimensional fit @ 1 and 2 loops

Outlook-Near Future

- D-dimensional fit at 2-loops for all $2 \rightarrow 2$ topologies
- $2 \rightarrow 3$ topology reduction

Outlook-Further Future Generalize for further topologies!

Thank you!

Thank you for listening!

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Dyson Schwinger Equations

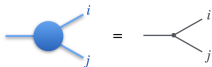


Figura: First Step of the recursion

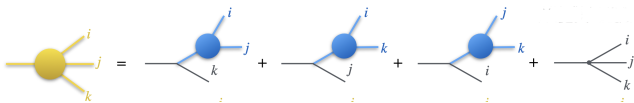


Figura: Second Step of the recursion

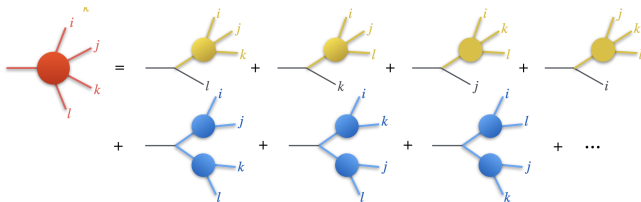


Figura: Third Step of the recursion

Simple Example $f\bar{f} \rightarrow f\bar{f}b$

Binary ID assigned to each particle in powers of 2, so we can write our building blocks as $\psi(1), \bar{\psi}(2), \bar{\psi}(4), \psi(8), A(16)$.

Use this to build all the level 2 sub-amplitudes

- $A_\mu(12) = (ig)\Pi_\mu^\nu \bar{\psi}(4)\gamma_\nu \psi(8)$
- $\bar{\psi}(18) = (ig)\bar{\psi}(2)A(16)\mathcal{P}$
- $\bar{\psi}(20) = (ig)\bar{\psi}(4)A(16)\mathcal{P}$
- $\psi(24) = (ig)\mathcal{P}A(16)\psi(8)$

where $\Pi_{\mu\nu}$ is the boson propagator and \mathcal{P} is the fermion propagator.

Simple Example $f\bar{f} \rightarrow f\bar{f}b$

Continue by building the 3-level subamplitudes

- $A_\mu(12) = \Pi_\mu^\nu \bar{\psi}(4) \gamma_\nu \psi(8)$
- $A_\mu(28) = (ig) \Pi_\mu^\nu (\bar{\psi}(20) \gamma_\nu \psi(8) + \bar{\psi}(4) \gamma_\nu \psi(24))$

4-level subamplitudes

- $\bar{\psi}_0(30) = (ig) (\bar{\psi}(2)A(28) + \bar{\psi}(14)A(16) + \bar{\psi}(18)\psi(12))$

And so the amplitude is finally given by

$$\mathcal{A} = \bar{\psi}_0(30)\psi(1)$$

The recursion is made to always end and the Binary ID 1.

This is how HELAC calculates amplitudes!

Color Flow representation

Therefore, the color factor can in total be written as

$$\mathcal{F} = \delta_{1\sigma_I(1)}\delta_{2\sigma_I(2)} \cdots \delta_{n\sigma_I(n)}$$

where σ is some permutation of $1 \dots n$ and $I = 1, \dots, n!$

We can therefore write a general amplitude in the form

$$\mathcal{M} = \sum_{\sigma} \mathcal{F} \mathcal{A}_{\sigma}$$


where the color striped amplitude is properly calculated using appropriate Feynman rules.

Benefits:

- Computational complexity is polynomial
- Amplitude squared form is simply a product of Kroenecker deltas leading to a simple form.

1 Loop Example

Example of a 6 point amplitude

$$\mathcal{A}(q) = \sum \underbrace{\frac{N_i^{(6)}(q)}{\bar{D}_{i_0} \bar{D}_{i_1} \cdots \bar{D}_{i_5}}}_{\text{Hexagon}} + \underbrace{\frac{N_i^{(5)}(q)}{\bar{D}_{i_0} \bar{D}_{i_1} \cdots \bar{D}_{i_4}}}_{\text{Pentagon}} + \underbrace{\frac{N_i^{(4)}(q)}{\bar{D}_{i_0} \bar{D}_{i_1} \cdots \bar{D}_{i_3}}}_{\text{Square}} + \underbrace{\frac{N_i^{(3)}(q)}{\bar{D}_{i_0} \bar{D}_{i_1} \bar{D}_{i_2}}}_{\text{Triangle}} + \dots$$


Now we can do the reduction:

- Calculate the coefficients of the OPP formula for each numerator
- Use Integration by Parts relations to get a Master Integral Basis
- Compute the rational terms

Has been done! → **HELAC One Loop!** (Bevilacqua, Czakon, Garzelli, van Hameren, Kardos, Papadopoulos, Pittau and Worek, 2013)

Amplitude Construction in $D = 4 - 2\epsilon$ dimensions

Clarification: Going from 4 to D dimensions is simple to do analytically for example in Mathematica.

Structure in $d = 4$		Extra term
$q^2 X$	\rightarrow	μX
$\sum_{\lambda} (\epsilon_{\ell_1, \lambda} \cdot \epsilon_{\ell_2, \lambda}) X$	\rightarrow	$(d - 4) X$
$\sum_{\lambda} (q \cdot \epsilon_{\ell_1, \lambda}) (q \cdot \epsilon_{\ell_2, \lambda}) X$	\rightarrow	$(q^2 + \mu) X$

How can we do this numerically?

Work is ongoing, led by Giuseppe B., optimistic progress so far.

Amplitude Construction: Performance

Performance of amplitude generation for various processes

<i>Process</i>	<i>Loops</i>	<i>Loop-Flavors</i>	<i>Color</i>	<i>Skeleton Size</i>	<i>Timing</i>	<i>Numerators</i>
$gg \rightarrow gg$	2	$\{g, c, \bar{c}\}$	Leading	8.9 MB	15.017s	4560
$gg \rightarrow gg$	2	$\{g, q, \bar{q}, c, \bar{c}\}$	Full	110.6 MB	6m 54.574s	89392
$gg \rightarrow q\bar{q}$	2	$\{g, q, \bar{q}, c, \bar{c}\}$	Full	16.1 MB	3m 14.509s	13856
$gg \rightarrow ggg$	2	$\{g, c, \bar{c}\}$	Leading	300.0 MB	21m 42.609s	81480
$gg \rightarrow gg$	1	$\{g, q, \bar{q}, c, \bar{c}\}$	Full	537.8 kB	2.386s	768
$gg \rightarrow ggg$	1	$\{g, q, \bar{q}, c, \bar{c}\}$	Full	15.1 MB	8m 53.349s	11496
$gg \rightarrow gggg$	1	$\{g, c, \bar{c}\}$	Leading	394.0 MB	104m 14.95s	19680

Results obtained running 1-core in a personal laptop (i7 processor, 8-core, 25GB RAM).