Generalised Antenna Functions for Higher Order Calculations

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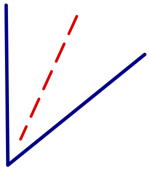
Recap of Antenna Subtraction

- At NLO, we want to construct a subtraction term

$$d\sigma_{NLO} = \int_{d\Phi_{m+1}} \left[d\sigma_{NLO}^{R} - d\sigma_{NLO}^{S} \right] + \left[\int_{d\Phi_{m+1}} d\sigma_{NLO}^{S} + \int_{d\Phi_{m}} d\sigma_{NLO}^{V} \right]$$

- The only emission topology we encounter is a single unresolved emission between a pair of hard radiator partons
- We can reproduce this singular behaviour with terms like

$$X_3^0(i^h, j, k^h) M_m^0(..., \widetilde{ij}, \widetilde{jk}, ...) J(m)_m(..., \widetilde{ij}, \widetilde{jk}, ...)$$



Recap of Antenna Subtraction

- At NNLO, we need to construct subtraction terms such that

$$d\sigma_{NNLO} = \int_{d\Phi_{m+2}} \left[d\sigma^{RR}_{NNLO} - d\sigma^{S,RR}_{NNLO} \right] \int_{d\Phi_{m+1}} \left[d\sigma^{RV}_{NNLO} - d\sigma^{S,RV}_{NNLO} \right] + \int_{d\Phi_{m}} \left[d\sigma^{VV}_{NNLO} - d\sigma^{S,VV}_{NNLO} \right]$$

where

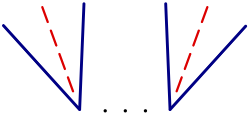
$$\int_{d\Phi_{m+2}} d\sigma^{S,RR}_{NNLO} + \int_{d\Phi_{m+1}} d\sigma^{S,RV}_{NNLO} + \int_{d\Phi_{m}} d\sigma^{S,VV}_{NNLO} = 0$$

- At the RR level we can have both single- and double- unresolved emissions
- At the RV level we have one-loop single-unresolved emissions, and bulk singularities
- This leads to more complex emission topologies

Double-Unresolved Emission Topologies

- **Colour Unconnected**
  ☞ Both emissions are well separated in the colour string and share no common hard radiator

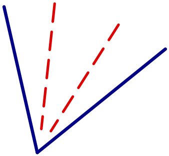
$$X_3^0(i^h, j, k^h) X_3^0(l^h, m, n^h) M_n^0(\ldots, \widetilde{ij}, \widetilde{jk}, \ldots, \widetilde{lm}, \widetilde{mn} \ldots)$$



Double-Unresolved Emission Topologies

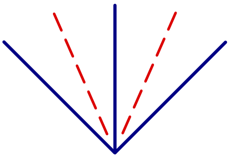
- **Colour Connected**
  ☛ Both emissions are colour connected to each other and one of the hard radiators

$$
\begin{aligned}
&X_4^0(i^h, j, k, l^h) M_n^0(\ldots, \widetilde{ijk}, \widetilde{jkl}, \ldots) \\
&- X_3^0(i^h, j, k^h) X_3^0(\widetilde{ij}^h, \widetilde{jk}, l^h) M_n^0(\ldots, \widetilde{\widetilde{ij}\widetilde{jk}}, \widetilde{\widetilde{jk}l}, \ldots) \\
&- X_3^0(l^h, k, j^h) X_3^0(\widetilde{lk}^h, \widetilde{kj}, i^h) M_n^0(\ldots, \widetilde{i\widetilde{jk}}, \widetilde{\widetilde{jk}\widetilde{kl}}, \ldots)
\end{aligned}
$$



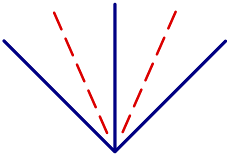
Double-Unresolved Emission Topologies

- **Almost Colour Unconnected**
  ☞ The two unresolved partons are colour unconnected, but share a common hard radiator



- Antenna functions have 2 hard radiators, so cannot cleanly describe this kind of singularity
- Instead, they are described by a non-trivial combination of $\widetilde{X}_4^0$ and Large Angle Soft Terms
- For high multiplicity processes, this is the bottleneck for the size and complexity of the subtraction terms

- **Almost Colour Unconnected**



- Instead, we would like to be able to to describe the ACU singularities like

$$
\begin{aligned}
&X_{5,3}^0(i^h, j, k^h, l, m^h) M_n^0(\dots, \widetilde{ijk}, \widetilde{ijklm}, \widetilde{jkl}, \dots) \\
&- X_3^0(i^h, j, k^h) X_3^0(\widetilde{jk}^h, l, m^h) M_n^0(\dots, \widetilde{ij}, \widetilde{\widetilde{jk}l}, \widetilde{lm}, \dots) \\
&- X_3^0(m^h, l, k^h) X_3^0(\widetilde{lk}^h, j, i^h) M_n^0(\dots, \widetilde{ij}, \widetilde{j\widetilde{kl}}, \widetilde{lm}, \dots)
\end{aligned}
$$

Double-Unresolved Emission Topologies

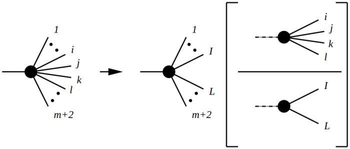
- Instead, we would like to be able to to describe the ACU singularities like

$$
\begin{aligned}
&X^0_{5,3}(i^h, j, k^h, l, m^h) M^0_n(\ldots, \widetilde{ijk}, \widetilde{ijklm}, \widetilde{jkl}, \ldots) \\
&- X^0_3(i^h, j, k^h) X^0_3(\widetilde{jk}^h, l, m^h) M^0_n(\ldots, \widetilde{ij}, \widetilde{\widetilde{jk}l}, \widetilde{lm}, \ldots) \\
&- X^0_3(m^h, l, k^h) X^0_3(\widetilde{lk}^h, j, i^h) M^0_n(\ldots, \widetilde{ij}, \widetilde{j\widetilde{kl}}, \widetilde{lm}, \ldots)
\end{aligned}
$$

- **Advantages**
  ✓A more algorithmic construction of subtraction terms
  ✓A reduction in the size of subtraction terms
  ✓Works for a single colour ordering
  ✓Reduced computational time ($\sim n!$ for n gluons)
  ✓No wide-angle soft terms

Recap of Designer Antennas

- In the original antenna subtraction scheme, antenna functions were extracted from matrix elements with the required infrared limits [Gehrmann-De Ridder, Gehrmann, Glover '05]



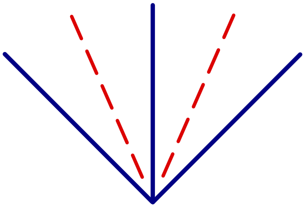
- In the designer antenna scheme, we build antennas directly from the set of infrared limits we want the antenna to have

✓We have a full set of unintegrated and integrated FF antennae [Braun-White,Glover '23]

$$X_3^0(i^h, j, k^h)\ X_4^0(i^h, j, k, l^h)\ X_3^1(i^h, j, k^h)$$

Tree level five particle three-hard radiator antenna functions

- We can construct a $X^0_{5,3}(i^h_a, j_b, k^h_c, l_d, m^h_e)$ antenna from its desired infrared limits
- We have a momentum-conserving on-shell 5to3 mapping which behaves correctly in all unresolved limits
- Hence we can use these antennas in RR subtraction terms



$$
\begin{aligned}
L_1(i^h,j,k^h,l,m^h) &= S_b(i^h,j,k^h)S_d(k^h,l,m^h)\\
L_2(i^h,j,k^h,l,m^h) &= P_{ab}(i^h,j)P_{cd}(k^h,l)\\
L_3(i^h,j,k^h,l,m^h) &= P_{ab}(i^h,j)P_{ed}(m^h,l)\\
L_4(i^h,j,k^h,l,m^h) &= P_{cb}(k^h,j)P_{ed}(m^h,l)\\
L_5(i^h,j,k^h,l,m^h) &= S_b(i^h,j,k^h)X_3^0(k^h,l,m^h)\\
L_6(i^h,j,k^h,l,m^h) &= S_d(k^h,l,m^h)X_3^0(i^h,j,k^h)\\
L_7(i^h,j,k^h,l,m^h) &= P_{ab}(i^h,j)X_3^0(k^h,l,m^h)\\
L_8(i^h,j,k^h,l,m^h) &= P_{ed}(m^h,l)X_3^0(i^h,j,k^h)\\
L_9(i^h,j,k^h,l,m^h) &= P_{bcd}(j,k^h,l)\\
L_{10}(i^h,j,k^h,l,m^h) &= P_{bc}(k^h,j)X_3^0([j+k]^h,l,m^h)\\
L_{11}(i^h,j,k^h,l,m^h) &= P_{cd}(k^h,l)X_3^0(i^h,j,[k+l]^h).
\end{aligned}
$$

$\texttt{B3g0ZepemS}(\texttt{a},\texttt{i},\texttt{j},\texttt{k},\texttt{b},\texttt{1},\texttt{2}) =$

1 $+D_3^0(a,i,j)\,B_2^{Z,0}((\widetilde{ai}),(\widetilde{ij}),k,b,1,2)\,J_3^{(4)}(\{p\}_4)$

2 $+F_3^0(i,j,k)\,B_2^{Z,0}(a,(\widetilde{ij}),(\widetilde{kj}),b,1,2)\,J_3^{(4)}(\{p\}_4)$

3 $+D_3^0(b,k,j)\,B_2^{Z,0}(a,i,(\widetilde{jk}),(\widetilde{kb}),1,2)\,J_3^{(4)}(\{p\}_4)$

Single Unresolved

4 $+D_4^0(a,i,j,k)\,B_1^{Z,0}((\widetilde{aij}),(\widetilde{kji}),b,1,2)\,J_3^{(3)}(\{p\}_3)$

5 $-D_3^0(a,i,j)\,D_3^0((\widetilde{ai}),(\widetilde{ji}),k)\,B_1^{Z,0}((\widetilde{(\widetilde{ai})(\widetilde{ij})}),(\widetilde{k(\widetilde{ji})}),b,1,2)\,J_3^{(3)}(\{p\}_3)$

6 $-F_3^0(i,j,k)\,D_3^0(a,(\widetilde{ij}),(\widetilde{kj}))\,B_1^{Z,0}((\widetilde{a(\widetilde{ij})}),(\widetilde{(\widetilde{kj})(\widetilde{ji})}),b,1,2)\,J_3^{(3)}(\{p\}_3)$

7 $+D_4^0(b,k,j,i)\,B_1^{Z,0}(a,(\widetilde{ijk}),(\widetilde{bkj}),1,2)\,J_3^{(3)}(\{p\}_3)$

8 $-F_3^0(i,j,k)\,D_3^0(b,(\widetilde{jk}),(\widetilde{ij}))\,B_1^{Z,0}(a,(\widetilde{(\widetilde{ij})(\widetilde{jk})}),(\widetilde{b(\widetilde{kj})}),1,2)\,J_3^{(3)}(\{p\}_3)$

9 $-D_3^0(b,k,j)\,D_3^0((\widetilde{bk}),(\widetilde{jk}),i)\,B_1^{Z,0}(a,(\widetilde{i(\widetilde{jk})}),(\widetilde{(\widetilde{bk}),(\widetilde{jk})}),1,2)\,J_3^{(3)}(\{p\}_3)$

Colour Connected Double Unresolved

10 $+A_{5,3}^0(a,i,j,k,b)\,B_1^{Z,0}(\{aij\},\{aijkb\},\{bjk\},1,2)\,J_3^{(3)}(\{p\}_3)$

11 $-D_3^0(a,i,j)\,D_3^0(b,k,(\widetilde{ij}))\,B_1^{Z,0}((\widetilde{ai}),[(\dot{\widetilde{ji}})k],[kb],1,2)\,J_3^{(3)}(\{p\}_3)$

12 $-D_3^0(b,k,j)\,D_3^0(a,i,(\widetilde{kj}))\,B_1^{Z,0}([ai],[(\dot{\widetilde{kj}})i],(\widetilde{bk}),1,2)\,J_3^{(3)}(\{p\}_3)$

Almost Colour Unconnected Double Unresolved

- 12 terms compared to 42 in the traditional scheme
- Works for a single colour ordering
- Clear algorithmic construction

Analytic Integration: Mapping dependence

- We consider an unintegrated subtraction term

$$X(\{p\})\,M(\{\widetilde{p}\},\{\widetilde{q}\})\,J^{(n_{\widetilde{p}}+n_q)}_{n_p+n_q}(\{\widetilde{p}\},\{\widetilde{q}\})\,dPS_{n_p+n_q}(p,q)$$

- We choose a mapping such that the phase space integral factorises

$$dPS_{n_p+n_q}(p,q)\to dPS_X(p/\{\widetilde{p}\})\,dPS_{n_{\widetilde{p}}+n_q}(\{\widetilde{p}\},\{\widetilde{q}\})$$

- The integrated subtraction term is then given by

$$\mathcal{X}(\{\widetilde{p}\})\,M(\{\widetilde{p}\},\{\widetilde{q}\})\,J^{(n_{\widetilde{p}}+n_q)}_{n_p+n_q}(\{\widetilde{p}\},\{\widetilde{q}\})\,dPS_m(\{\widetilde{p}\},\{\widetilde{q}\})$$

where

$$\mathcal{X}(\{\widetilde{p}\})=\int X(\{p\})\,dPS_X(p/\{\widetilde{p}\}),$$

Analytic Integration: Mapping dependence

Antennas with 2 hard radiators

- A generic n$\to$2 mapping has the form

$$\{p_1, \ldots, p_{n_p}\} \to \{\widetilde{p}_I, \widetilde{p}_J\}$$

- The only available scale after integration is s_{IJ} so on dimensional grounds

$$\mathcal{X}(\{\widetilde{p}_I, \widetilde{p}_J\}) = c(\epsilon)\,(s_{IJ})^d$$

- It is straightforward to check that all mappings give the same $c(\epsilon)$ and d

Analytic Integration: Mapping dependence

Antennas with 3 hard radiators

- A generic n→3 mapping has the form

$$\{p_1, \dots, p_{n_p}\} \to \{\widetilde{p}_I, \widetilde{p}_J, \widetilde{p}_K\}$$

- Now we have have multiple available scales

$$s_{IJ}, \quad s_{IK}, \quad s_{JK}, \quad s_{IJK}, \quad s_{IJ}+s_{JK}, \quad s_{IJ}+s_{IK}, \quad s_{JK}+s_{IK}$$

- As before the overall dimensionality of is fixed, but the dependence on each individual scale is not

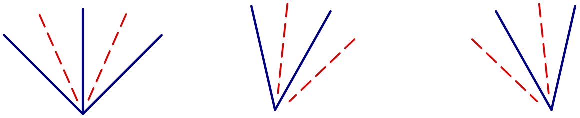
$$\mathcal{X}(\{\widetilde{p}_I, \widetilde{p}_J, \widetilde{p}_K\}) = \sum_i c_i\,(s_{IJ})^{\alpha_i}(s_{JK})^{\beta_i}(s_{JK})^{\gamma_i}(s_{IJK})^{\delta_i} + \dots$$

- Hence we have to use the same mapping for analytic integration as we do in the numerical implementation

Analytic Integration: $X_{5,3}^0$

- We split $X_{5,3}^0$ into 3 parts to make the integration more straightforward

$$X_{5,3}^0(i^h, j, k^h, l, m^h) = X_{5,3;M}^0(i^h, j, k^h, l, m^h) + X_{5,3;L}^0(i^h, j, k^h, l|m^h) + X_{5,3;R}^0(i^h|j, k^h, l, m^h)$$



- $X_{5,3;L}^0$ and $X_{5,3;R}^0$ are related to $\widetilde{X}_4^0$ so can be integrated in the same way

Analytic Integration: $X^0_{5,3}$

$$X^0_{5,3;M} \equiv X^0_3(i^h_a, j_b, k^h_c) X^0_3(k^h_c, l_d, m^h_e) - \mathbf{C}^{\downarrow}_{jk}((1-\mathbf{S}^{\downarrow}_j) X^0_3(i^h_a, j_b, k^h_c)) \mathbf{C}^{\downarrow}_{kl}((1-\mathbf{S}^{\downarrow}_l) X^0_3(k^h_c, l_d, m^h_e))$$

- This satisfies fully 8 out of the 11 limits, and only include terms with invariants built from momenta in one half of the $X^0_{5,3}$
- We can integrate this analytically over the mapping

$$:\quad \begin{aligned} p_I &= p_i + p_j - \frac{s_{ij}}{s_{ik}+s_{jk}} p_k, \\ p_K &= \left(1 + \frac{s_{ij}}{s_{ik}+s_{jk}} + \frac{s_{lm}}{s_{lk}+s_{mk}}\right) p_k, \\ p_M &= p_l + p_m - \frac{s_{lm}}{s_{lk}+s_{mk}} p_k, \end{aligned}$$

Real-Virtual Subtraction Term

- First we include the terms from the RR subtraction term that naturally land at the RV level

R R

1 $+D_3^0(a,i,j)\, B_2^{Z,0}((\widetilde{ai}),(\widetilde{ij}),k,b,1,2)\, J_3^{(4)}(\{p\}_4)$

2 $+F_3^0(i,j,k)\, B_2^{Z,0}(a,(\widetilde{ij}),(\widetilde{kj}),b,1,2)\, J_3^{(4)}(\{p\}_4)$

3 $+D_3^0(b,k,j)\, B_2^{Z,0}(a,i,(\widetilde{jk}),(\widetilde{kb}),1,2)\, J_3^{(4)}(\{p\}_4)$

4 $+D_4^0(a,i,j,k)\, B_1^{Z,0}((\widetilde{aij}),(\widetilde{kji}),b,1,2)\, J_3^{(3)}(\{p\}_3)$

5 $-D_3^0(a,i,j)\, D_3^0((\widetilde{ai}),(\widetilde{ji}),k)\, B_1^{Z,0}(((\widetilde{\widetilde{ai})(\widetilde{ij}})),(\widetilde{k(\widetilde{ji}})),b,1,2)\, J_3^{(3)}(\{p\}_3)$

6 $-F_3^0(i,j,k)\, D_3^0(a,(\widetilde{ij}),(\widetilde{kj}))\, B_1^{Z,0}((\widetilde{a(\widetilde{ij}})),((\widetilde{\widetilde{kj})(\widetilde{ji}})),b,1,2)\, J_3^{(3)}(\{p\}_3)$

7 $+D_4^0(b,k,j,i)\, B_1^{Z,0}(a,(\widetilde{ijk}),(\widetilde{bkj}),1,2)\, J_3^{(3)}(\{p\}_3)$

8 $-F_3^0(i,j,k)\, D_3^0(b,(\widetilde{jk}),(\widetilde{ij}))\, B_1^{Z,0}(a,((\widetilde{\widetilde{ij})(\widetilde{jk}})),(\widetilde{b(\widetilde{kj}})),1,2)\, J_3^{(3)}(\{p\}_3)$

9 $-D_3^0(b,k,j)\, D_3^0((\widetilde{bk}),(\widetilde{jk}),i)\, B_1^{Z,0}(a,(\widetilde{i(\widetilde{jk}})),((\widetilde{\widetilde{bk}),(\widetilde{jk}})),1,2)\, J_3^{(3)}(\{p\}_3)$

10 $+A_{5,3}^0(a,i,j,k,b)\, B_1^{Z,0}(\{aij\},\{aijkb\},\{bjk\},1,2)\, J_3^{(3)}(\{p\}_3)$

11 $-D_3^0(a,i,j)\, D_3^0(b,k,(\widetilde{ij}))\, B_1^{Z,0}((\widetilde{ai}),[(\dot{\widetilde{ji}})k],[kb],1,2)\, J_3^{(3)}(\{p\}_3)$

12 $-D_3^0(b,k,j)\, D_3^0(a,i,(\widetilde{kj}))\, B_1^{Z,0}([ai],[(\dot{\widetilde{kj}})i],(\widetilde{bk}),1,2)\, J_3^{(3)}(\{p\}_3)$

R V

1 $-J_{2,QG}^{1,FF}(s_{ai})\, B_2^{Z,0}(a,i,j,b,1,2)\, J_3^{(4)}(\{p\}_4)$

2 $-J_{2,GG}^{1,FF}(s_{ij})\, B_2^{Z,0}(a,i,j,b,1,2)\, J_3^{(4)}(\{p\}_4)$

3 $-J_{2,QG}^{1,FF}(s_{jb})\, B_2^{Z,0}(a,i,j,b,1,2)\, J_3^{(4)}(\{p\}_4)$

4 $+J_{2,QG}^{1,FF}(s_{ai}) D_3^0(a,i,j)\, B_1^{Z,0}((\widetilde{ai}),(\widetilde{ij}),b,1,2)\, J_3^{(3)}(\{p\}_3)$

5 $+J_{2,GG}^{1,FF}(s_{ij}) D_3^0(a,i,j)\, B_1^{Z,0}((\widetilde{ai}),(\widetilde{ij}),b,1,2)\, J_3^{(3)}(\{p\}_3)$

6 $+J_{2,GG}^{1,FF}(s_{ij}) D_3^0(b,j,i)\, B_1^{Z,0}(a,(\widetilde{ij}),(\widetilde{jb}),1,2)\, J_3^{(3)}(\{p\}_3)$

7 $+J_{2,QG}^{1,FF}(s_{jb}) D_3^0(b,j,i)\, B_1^{Z,0}(a,(\widetilde{ij}),(\widetilde{jb}),1,2)\, J_3^{(3)}(\{p\}_3)$

8 $+J_{2,QG}^{1,FF}(s_{jb}) D_3^0(a,i,j)\, B_1^{Z,0}([ai],[ij],b,1,2)\, J_3^{(3)}(\{p\}_3)$

9 $+J_{2,QG}^{1,FF}(s_{ai}) D_3^0(b,j,i)\, B_1^{Z,0}(a,[ij],[jb],1,2)\, J_3^{(3)}(\{p\}_3)$

Real-Virtual Subtraction Term

- First we include the terms from the RR subtraction term that naturally land at the RV level
- Then we add in the single unresolved behaviour using loop*tree and tree*loop terms

B2g1ZepemT(a,i,j,b,1,2)

1 $-\Big[+J_{2,QG}^{1,FF}(s_{ai})+J_{2,GG}^{1,FF}(s_{ij})+J_{2,QG}^{1,FF}(s_{jb})\Big]\, B_2^{Z,0}(a,i,j,b,1,2)\, J_3^{(4)}(\{p\}_4)$

2 $+\Big[+J_{2,QG}^{1,FF}(s_{ai})+J_{2,GG}^{1,FF}(s_{ij})\Big]\, D_3^0(a,i,j)\, B_1^{Z,0}((\widetilde{ai}),(\widetilde{ij}),b,1,2)\, J_3^{(3)}(\{p\}_3)$

3 $+\Big[+J_{2,GG}^{1,FF}(s_{ij})+J_{2,QG}^{1,FF}(s_{jb})\Big]\, D_3^0(b,j,i)\, B_1^{Z,0}(a,(\widetilde{ij}),(\widetilde{jb}),1,2)\, J_3^{(3)}(\{p\}_3)$

4 $+J_{2,QG}^{1,FF}(s_{jb})D_3^0(a,i,j)\, B_1^{Z,0}([ai],[ij],b,1,2)\, J_3^{(3)}(\{p\}_3)$

5 $+J_{2,QG}^{1,FF}(s_{ai})D_3^0(b,j,i)\, B_1^{Z,0}(a,[ij],[jb],1,2)\, J_3^{(3)}(\{p\}_3)$

6 $+D_3^1(a,i,j)\, B_1^{Z,0}((\widetilde{ai}),(\widetilde{ij}),b,1,2)\, J_3^{(3)}(\{p\}_3)$

7 $+D_3^1(b,j,i)\, B_1^{Z,0}(a,(\widetilde{ij}),(\widetilde{jb}),1,2)\, J_3^{(3)}(\{p\}_3)$

8 $+D_3^0(a,i,j)\, B_1^{Z,1}((\widetilde{ai}),(\widetilde{ij}),b,1,2)\, J_3^{(3)}(\{p\}_3)$

9 $+D_3^0(b,j,i)\, B_1^{Z,1}(a,(\widetilde{ij}),(\widetilde{jb}),1,2)\, J_3^{(3)}(\{p\}_3)$

From the R R (lines 1–5)

Added in at R V (lines 6–9)

Real-Virtual Subtraction Term

B2g1ZepemT(a,i,j,b,1,2)

Correct Poles

1 $-\left[+J_{2,QG}^{1,FF}(s_{ai})+J_{2,GG}^{1,FF}(s_{ij})+J_{2,QG}^{1,FF}(s_{jb})\right] B_2^{Z,0}(a,i,j,b,1,2)\, J_3^{(4)}(\{p\}_4)$

Extra Poles

2 $+\left[+J_{2,QG}^{1,FF}(s_{ai})+J_{2,GG}^{1,FF}(s_{ij})\right] D_3^0(a,i,j)\, B_1^{Z,0}((\widetilde{ai}),(\widetilde{ij}),b,1,2)\, J_3^{(3)}(\{p\}_3)$

3 $+\left[+J_{2,GG}^{1,FF}(s_{ij})+J_{2,QG}^{1,FF}(s_{jb})\right] D_3^0(b,j,i)\, B_1^{Z,0}(a,(\widetilde{ij}),(\widetilde{jb}),1,2)\, J_3^{(3)}(\{p\}_3)$

4 $+J_{2,QG}^{1,FF}(s_{jb}) D_3^0(a,i,j)\, B_1^{Z,0}([ai],[\dot{ij}],b,1,2)\, J_3^{(3)}(\{p\}_3)$

5 $+J_{2,QG}^{1,FF}(s_{ai}) D_3^0(b,j,i)\, B_1^{Z,0}(a,[\dot{ij}],[jb],1,2)\, J_3^{(3)}(\{p\}_3)$

Non-Divergent

6 $+D_3^1(a,i,j)\, B_1^{Z,0}((\widetilde{ai}),(\widetilde{ij}),b,1,2)\, J_3^{(3)}(\{p\}_3)$

7 $+D_3^1(b,j,i)\, B_1^{Z,0}(a,(\widetilde{ij}),(\widetilde{jb}),1,2)\, J_3^{(3)}(\{p\}_3)$

8 $+D_3^0(a,i,j)\, B_1^{Z,1}((\widetilde{ai}),(\widetilde{ij}),b,1,2)\, J_3^{(3)}(\{p\}_3)$

9 $+D_3^0(b,j,i)\, B_1^{Z,1}(a,(\widetilde{ij}),(\widetilde{jb}),1,2)\, J_3^{(3)}(\{p\}_3)$

Correct Divergence

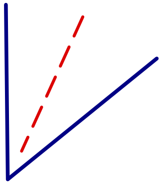
✓Gives the correct ϵ poles in all unresolved limits

✓Gives the correct implicit singularity behaviour in all unresolved limits

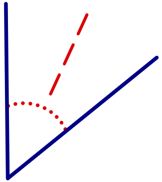
$\otimes$ Gives incorrect $\frac{1}{\epsilon^2}$ and $\frac{1}{\epsilon}$ poles in the bulk

One-loop single-unresolved emission topologies

- X_3^0 tree*loop terms are able to capture the singular behaviour from tree-level sub-graphs of a full matrix element

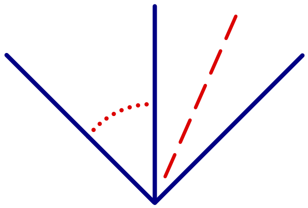


- X_3^1 loop*tree terms are able to capture the singular behaviour when the unresolved emission couples directly to a loop



One-loop single-unresolved emission topologies

- We also have a third case where the unresolved emission is neither directly connected to the loop nor fully detached from it
- In the traditional approach these explicit singularities were captured by a combination of LAST terms coming from the RR, but now we need a new structure to do this



One-loop four-particle three-hard-radiators antenna functions ($X^1_{4,3}$)

- To construct the $X^1_{4,3}$ we collect the leftover poles equipped with their mappings in the subtraction term, and assemble them into a function multiplying an X^0_3 antenna
- We find that in general we have 3 different mappings;
  - The dipole mapping with i scaled $\rightarrow X^1_{4,3;L}(i^h, j, k^h; l^h)$
  - The antenna mapping $\rightarrow X^1_{4,3;M}(i^h, j, k^h; l^h)$
  - The dipole mapping with k scaled $\rightarrow X^1_{4,3;R}(i^h, j, k^h; l^h)$
- The analytic integration is mapping dependent, but now we have specified the mapping it is straightforward

One-loop four-particle three-hard-radiators antenna functions ($X^1_{4,3}$)

In this case, we get

$$A^1_{4,3;L}(i,j,k;b) = 0$$

$$A^1_{4,3;M}(i,j,k;b) = \left(\frac{1}{\epsilon^2} + \frac{1}{\epsilon}\left(\log\left(\frac{s_{ijk}\mu^2}{(s_{ik}+s_{jk})s_{\widetilde{jk}b}}\right) + \frac{5}{3}\right)\right) D^0_3(i,j,k)$$

$$A^1_{4,3;R}(i,j,k;b) = \left(-\frac{1}{\epsilon^2} + \frac{1}{\epsilon}\left(\log\left(\frac{s_{kb}}{\mu^2}\right) - \frac{5}{3}\right)\right) D^0_3(i,j,k)$$

$$A^1_{4,3;T}(i,j,k;b) = \frac{1}{\epsilon}\log\left(\frac{s_{ijk}s_{kb}}{(s_{ik}+s_{jk})s_{\widetilde{jk}b}}\right) D^0_3(i,j,k)$$

- This contributes to poles in the bulk, but does not contribute in unresolved limits

Real-Virtual Subtraction Term

B2g1ZepemT(a,i,j,b,1,2)

Correct Poles

1 $-\left[+ J_{2,QG}^{1,FF}(s_{ai}) + J_{2,GG}^{1,FF}(s_{ij}) + J_{2,QG}^{1,FF}(s_{jb}) \right] B_2^{Z,0}(a,i,j,b,1,2)\, J_3^{(4)}(\{p\}_4)$

Pole Free

2 $+\left[+ J_{2,QG}^{1,FF}(s_{ai}) + J_{2,GG}^{1,FF}(s_{ij}) \right] D_3^0(a,i,j)\, B_1^{Z,0}((\widetilde{ai}),(\widetilde{ij}),b,1,2)\, J_3^{(3)}(\{p\}_3)$

3 $+\left[+ J_{2,GG}^{1,FF}(s_{ij}) + J_{2,QG}^{1,FF}(s_{jb}) \right] D_3^0(b,j,i)\, B_1^{Z,0}(a,(\widetilde{ij}),(\widetilde{jb}),1,2)\, J_3^{(3)}(\{p\}_3)$

4 $+J_{2,QG}^{1,FF}(s_{jb}) D_3^0(a,i,j)\, B_1^{Z,0}([ai],[\dot{i}j],b,1,2)\, J_3^{(3)}(\{p\}_3)$

5 $+J_{2,QG}^{1,FF}(s_{ai}) D_3^0(b,j,i)\, B_1^{Z,0}(a,[\dot{i}j],[jb],1,2)\, J_3^{(3)}(\{p\}_3)$

Non-Divergent (lines 1–5)

6 $+D_3^1(a,i,j)\, B_1^{Z,0}((\widetilde{ai}),(\widetilde{ij}),b,1,2)\, J_3^{(3)}(\{p\}_3)$

7 $+D_3^1(b,j,i)\, B_1^{Z,0}(a,(\widetilde{ij}),(\widetilde{jb}),1,2)\, J_3^{(3)}(\{p\}_3)$

8 $+D_3^0(a,i,j)\, B_1^{Z,1}((\widetilde{ai}),(\widetilde{ij}),b,1,2)\, J_3^{(3)}(\{p\}_3)$

9 $+D_3^0(b,j,i)\, B_1^{Z,1}(a,(\widetilde{ij}),(\widetilde{jb}),1,2)\, J_3^{(3)}(\{p\}_3)$

Correct Divergence (lines 6–9)

10 $+A_{4,3}^1(a,i,j,b)\, B_1^{Z,0}(\{ai\},\{ij\},b,1,2)\, J_3^{(3)}(\{p\}_3)$

11 $+A_{4,3}^1(b,j,i,a)\, B_1^{Z,0}(a,\{ij\},\{jb\},1,2)\, J_3^{(3)}(\{p\}_3)$

Non-Divergent (lines 10–11)

✓Gives the correct ϵ poles in all unresolved limits

✓Gives the correct implicit singularity behaviour in all unresolved limits

✓Gives the correct poles in the bulk

Virtual Virtual Subtraction Term

B1g2ZepemU(3,4,5,1,2)

$$\texttt{B1g2ZepemU}(3,4,5,1,2) =$$

$$1 \quad -[\mathcal{D}_3^0(s_{34}) + \mathcal{D}_3^0(s_{45})]B_1^{Z,1}(3,4,5,1,2)$$

$$2 \quad +\frac{\beta_0}{\epsilon}[\mathcal{D}_3^0(s_{34}) + \mathcal{D}_3^0(s_{45})]B_1^{Z,0}(3,4,5,1,2)$$

$$3 \quad -[\mathcal{D}_3^1(s_{34}) + \mathcal{D}_3^1(s_{45})]B_1^{Z,0}(3,4,5,1,2)$$

$$4 \quad -\frac{\beta_0}{\epsilon}\left[\left(\frac{s_{34}}{\mu_R^2}\right)^{-\epsilon}\mathcal{D}_3^0(s_{34}) + \left(\frac{s_{45}}{\mu_R^2}\right)^{-\epsilon}\mathcal{D}_3^0(s_{45})\right]B_1^{Z,0}(3,4,5,1,2)$$

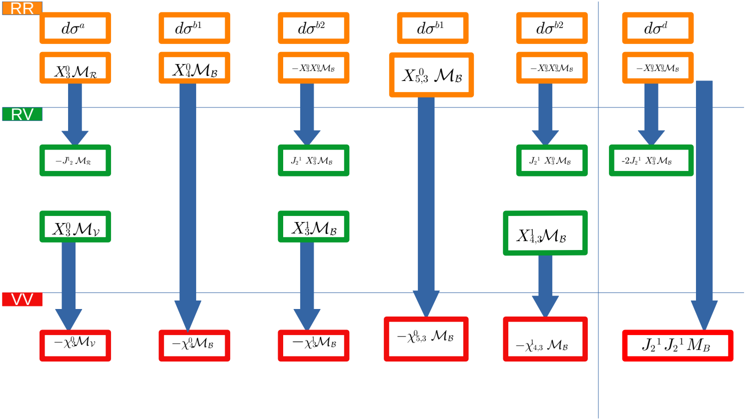
$$5 \quad -[\mathcal{D}_4^0(s_{34}) + \mathcal{D}_4^0(s_{45})]B_1^{Z,0}(3,4,5,1,2)$$

$$6 \quad -[\mathcal{A}_5^3(s_{34}, s_{45})]B_1^{Z,0}(3,4,5,1,2)$$

$$7 \quad -[\mathcal{D}_{4,3}^1(s_{34}, s_{45}) + \mathcal{D}_{4,3}^1(s_{45}, s_{34})]B_1^{Z,0}(3,4,5,1,2)$$

✓We indeed get full pole cancellation with the two loop matrix element as expected

New map between layers of the subtraction scheme



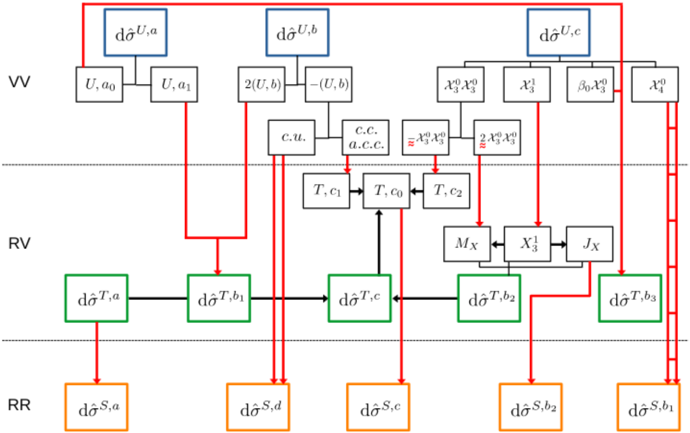
Summary (RR)

- We already have clean ways to describe singularities that are
  - Single unresolved (3 parton description)
  - Colour connected double unresolved (4 parton description)
  - Colour unconnected double unresolved (6^+ parton description)
- Almost colour unconnected double unresolved singularities would naturally have a 5 parton description
- The current treatment with $\widetilde{X}_4^0$ is very messy
- We have constructed and integrated 5 parton antennas with 3 hard radiators, $X_{5,3}^0$, that cleanly captures these almost colour unconnected double unresolved singularities
- At the RV level we need a new one-loop 4 parton three hard radiator non-divergent structure to cancel poles in the bulk
- We have constructed and integrated all $X_{4,3}^1$ relevant for $e^+e^- \to$ 3jet
- We have constructed subtraction terms for all colour layers of $e^+e^- \to$ 3jet and achieved poll cancellation

Outlook

- Finish the validation of our new method compared with the old implementation
- Quantify the speedup we can achieve
- Fit this new formalism into the colourful antenna subtraction framework, and automate the construction of subtraction terms for arbitrary processes at any multiplicity
- Extend the formalism to include intial state radiation

Map between layers of the subtraction scheme



Designer Antenna Algorithm

$$
\begin{aligned}
X^0_{n;1} &= \mathbf{P}^{\uparrow}_1 L_1 \,, \\
X^0_{n;2} &= X^0_{n;1} + \mathbf{P}^{\uparrow}_2 (L_2 - \mathbf{P}^{\downarrow}_2 X^0_{n;1}) \,, \\
&\vdots \\
X^0_{n;N} &= X^0_{n;N-1} + \mathbf{P}^{\uparrow}_N (L_N - \mathbf{P}^{\downarrow}_N X^0_{n;N-1})
\end{aligned}
$$

Designer Antenna Algorithm

$$\mathbf{S}_j^{\downarrow}: \begin{cases} s_{ij} \mapsto \lambda s_{ij}, s_{jk} \mapsto \lambda s_{jk}, \\ s_{ijk} \mapsto s_{ik}, \end{cases}$$

$$\mathbf{C}_{ij}^{\downarrow}: \begin{cases} s_{ij} \mapsto \lambda s_{ij}, \\ s_{ik} \mapsto (1-x_j)(s_{ik}+s_{jk}), s_{jk} \mapsto x_j(s_{ik}+s_{jk}), s_{ijk} \mapsto s_{ik}+s_{jk}. \end{cases}$$

$$\mathbf{C}_{ij}^{\uparrow}: \begin{cases} x_j \mapsto s_{jk}/s_{ijk}, (1-x_j) \mapsto s_{ik}/s_{ijk} \\ s_{ik}+s_{jk} \mapsto s_{ijk} \end{cases}$$

5to3 mapping limits

$$
\begin{aligned}
\{A, B, C\} &\xrightarrow{\text{DS 12}} \{a, b, c\} \\
&\xrightarrow{\text{TC 1||b||2}} \{a, 1 + b + 2, c\} \\
&\xrightarrow{\text{DC a||1 b||2}} \{a + 1, b + 2, c\} \\
&\xrightarrow{\text{DC a||1 2||c}} \{a + 1, b, 2 + c\} \\
&\xrightarrow{\text{SC 1 b||2}} \{a, b + 2, c\} \\
&\xrightarrow{\text{SC 1 2||c}} \{a, b, 2 + c\} \\
&\xrightarrow{\text{Soft 1}} \{a, \widetilde{b2}, \widetilde{2c}\} \\
&\xrightarrow{\text{Collinear a||1}} \{a + 1, \widetilde{b2}, \widetilde{2c}\} \\
&\xrightarrow{\text{Collinear 1||b}} \{a, \widetilde{(1 + b)2}, \widetilde{2c}\}
\end{aligned}
$$

Dipole and Tripole Mappings

The **dipole** mapping (with k rescaled) is an onshell momentum conserving map

$$\{p_i, p_j, p_k\} \to \{p_I, p_K\}$$
$$p_I = p_i + p_j - \frac{s_{ij}}{s_{ik} + s_{jk}} p_k,$$
$$p_K = \frac{s_{ijk}}{s_{ik} + s_{jk}} p_k.$$

The **tripole** mapping (with k rescaled) is an onshell momentum conserving map

$$\{p_i, p_j, p_k, p_l\} \to \{p_I, p_K\}$$
$$p_I = p_i + p_j + p_l - \frac{s_{ijl}}{s_{ik} + s_{jk} + s_{kl}} p_k$$
$$p_K = \frac{s_{ijkl}}{s_{ik} + s_{jk} + s_{kl}} p_k$$

Antenna 3 to 2 mapping

For the 3 to 2 mapping $\{i, j, k\} \to \{I, K\}$

$$I = xp_i + rp_j + yp_k$$
$$K = (1-x)p_i + (1-r)p_j + (1-y)p_k$$

$$x = \frac{1}{2(s_{ik}+s_{ij})}((1+\rho_1)s_{ijk} - 2rs_{jk})$$
$$r = \frac{s_{jk}}{s_{ij}+s_{jk}}$$
$$y = \frac{1}{2(s_{ik}+s_{jk})}((1-\rho_1)s_{ijk} - 2rs_{ij})$$
$$\rho_1^2 = 1 + 4r(1-r)\frac{s_{ij}s_{jk}}{s_{ijk}s_{ik}}$$

RR Spike Plots

Figure: gg Double Soft

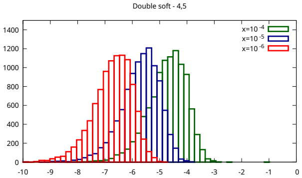


Figure: g Soft

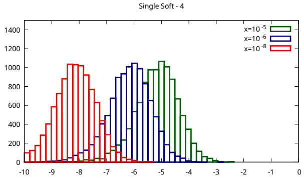


Figure: ggg Triple Collinear

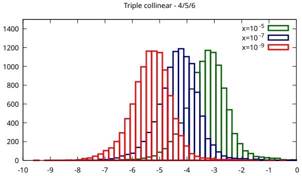
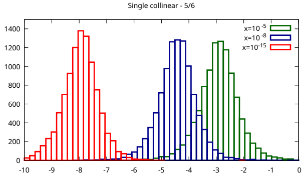


Figure: gg Collinear



RR Spike Plots

Figure: qgg Triple Collinear

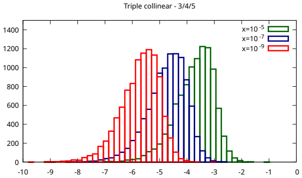


Figure: qg qg Double Collinear

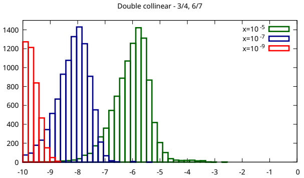


Figure: qg gg Double Collinear

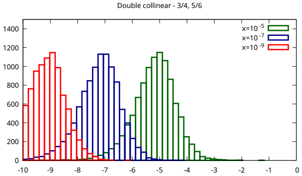


Figure: qg Collinear