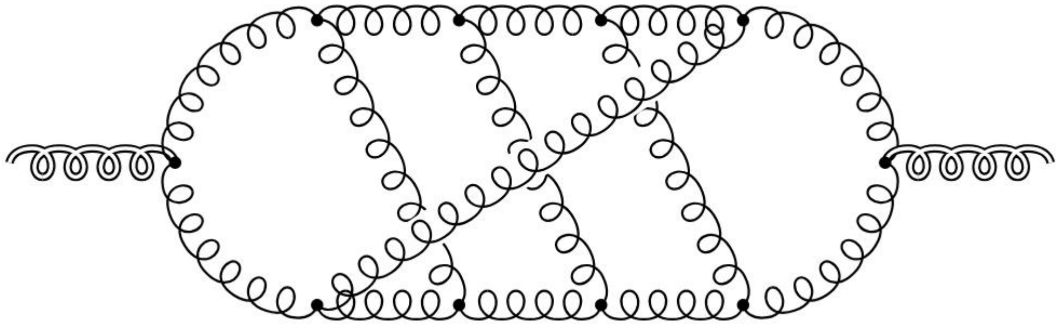
Tensor reduction for high-rank multi-loop integrals

High Precision for Hard Processes 2024

Sam Teale

In collaboration with F. Herzog, J. Goode, A. Kennedy, J. Vermaseren

Motivation

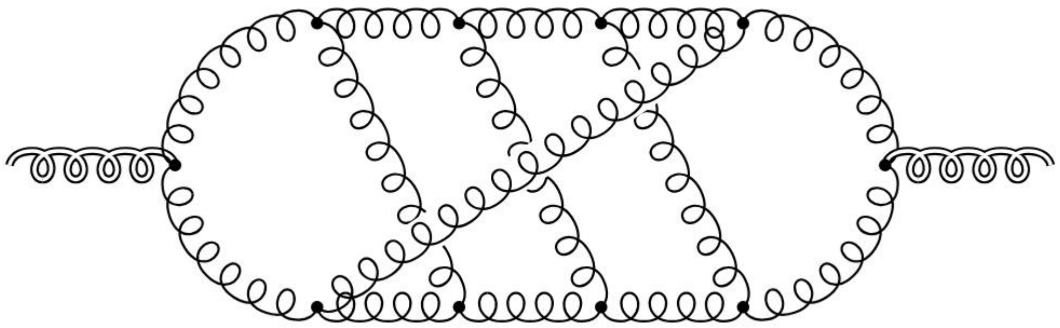
$R*$ 

[Herzog, Ruijl, Ueda, Vermaseren, Vogt 2017]

- R* renormalization
- Dimensional regularization
- High rank!

Motivation

$R*$



[Herzog, Ruijl, Ueda, Vermaseren, Vogt 2017]

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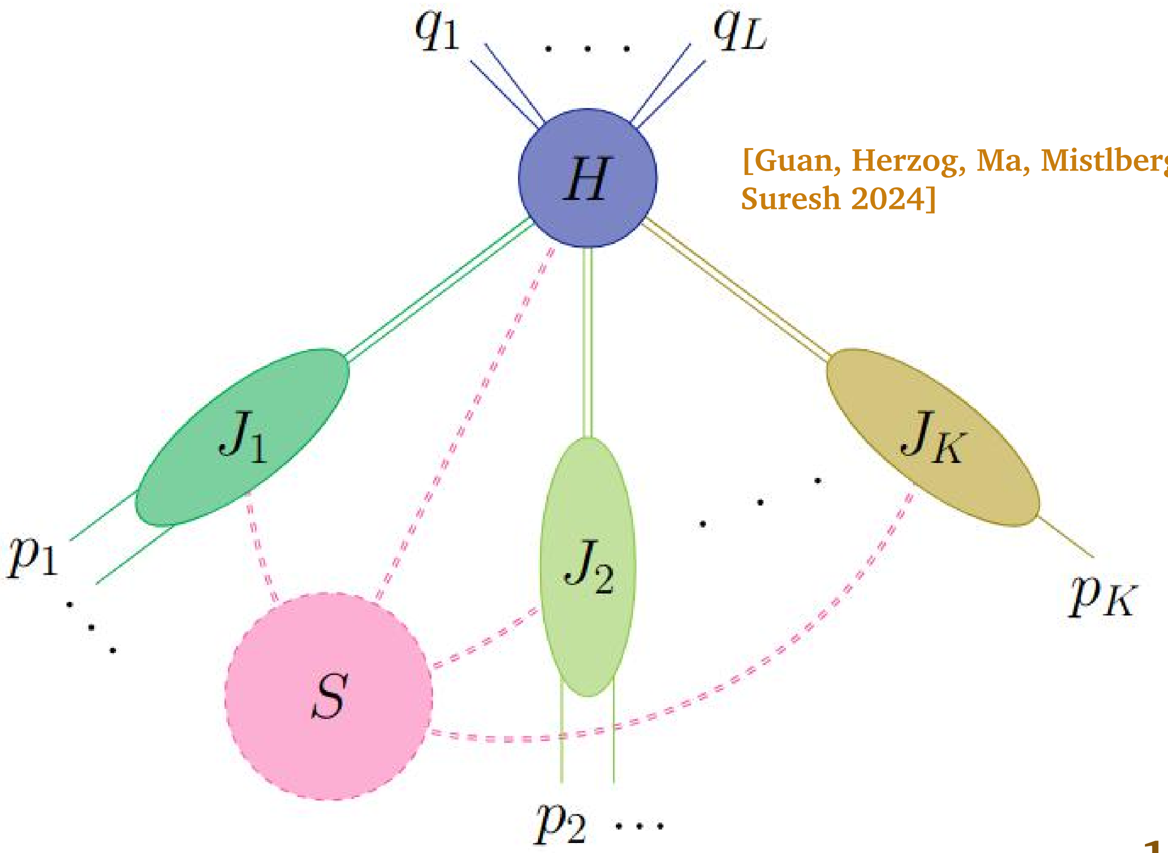
- Dimensional regularization
- High rank!



[Guan, Herzog, Ma, Mistlberger, Suresh 2024]

- Momentum space expansions
  - Method of regions
  - Small / large momentum expansions
  - Collinear expansions

Tensor reduction - Approaches

- Passarino-Veltmann [Passarino, Veltmann 1974]
  - Very general
  - High-rank gets expensive
- Parametric rep. [Tarasov, 1996] [Anastasiou, Glover, Oleari 2000]
  - Work in FP space
  - Required dimensional shift identities
  - Raises propagator powers
  - FaRe code - very general [Re Fiorentin 2015]
- Physical projectors [Peraro, Tancredi 2019]
  - 't Hooft-Veltman scheme
  - Effective for amplitude calculations
  - Physical constraints reduce basis
- Auxiliary vector [Feng, Li, Li; Hu, Li, Li; Feng, Li, Wang, Zhang; Feng, Hu, Li, Song]
  - Full analytic solution [Feng, Li, 2022]
  - So far only 2-loop for some topologies
- Unitarity [Ossola, Papadopoulos, Pittau; Forde; Giele, Kunszt, Melnikov; Berger, Bern, Dixon, Febres Cordero, Forde, Ita, Kosower, Maitre]
  - Circumvents the tensor reduction
  - So far only 2-loop [Ita; Abreu, Febres Cordero, Ita, Page, Zeng; Badger, Brønnum-Hansen, Hartanto, Peraro]
- Orbit-partition [Herzog, Ruijl, Ueda, Vermaseren, Vogt 2017]
  - In D dimensions [Goode, Herzog, Kennedy, ST, Vermaseren, 2024]
  - solved to rank 32 and arbitrary loop
  - Automated implementation - OPITeR

Motivating example

short hand

$$I^{\mu\nu\rho\sigma} = g^{\mu\nu} g^{\rho\sigma} A_1 + g^{\mu\rho} g^{\nu\sigma} A_2 + g^{\mu\sigma} g^{\nu\rho} A_3$$

$$I = \sum_i t_i A_i$$

D dimensional, Lorentz invariant + vacuum

scalar integrals

Motivating example

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D dimensional, Lorentz invariant + vacuum

scalar integrals

- Contract integral with each tensor in turn

$$\begin{bmatrix} I^{\mu}{}_{\mu}{}^{\rho}{}_{\rho} \\ I^{\mu}{}_{\rho\mu}{}^{\rho} \\ I^{\mu}{}_{\rho}{}^{\rho}{}_{\mu} \end{bmatrix} = \begin{bmatrix} D^2 & D & D \\ D & D^2 & D \\ D & D & D^2 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix}$$

Motivating example

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D dimensional, Lorentz invariant + vacuum

scalar integrals

- Contract integral with each tensor in turn
- And solve

$$\begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} = \frac{1}{D(D+2)(D-1)} \begin{bmatrix} D+1 & -1 & -1 \\ -1 & D+1 & -1 \\ -1 & -1 & D+1 \end{bmatrix} \begin{bmatrix} I^{\mu}{}_{\mu}{}^{\rho}{}_{\rho} \\ I^{\mu}{}_{\rho\mu}{}^{\rho} \\ I^{\mu}{}_{\rho}{}^{\rho}{}_{\mu} \end{bmatrix}$$

Projectors

short hand

$$I^{\mu\nu\rho\sigma} = g^{\mu\nu}g^{\rho\sigma}\,A_1 + g^{\mu\rho}g^{\nu\sigma}\,A_2 + g^{\mu\sigma}g^{\nu\rho}\,A_3$$

$$I = \sum_i t_i\,A_i$$

- Define projectors such that:

$$P_i \cdot t_j = \delta_{ij}, \qquad P_i \cdot I = A_i$$

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- With an ansatz

$$P_1^{\mu\nu\rho\sigma} = g^{\mu\nu}g^{\rho\sigma}\,B_1 + g^{\mu\rho}g^{\nu\sigma}\,B_2 + g^{\mu\sigma}g^{\nu\rho}\,B_3$$

Projectors

short hand

$$I^{\mu\nu\rho\sigma} = g^{\mu\nu}g^{\rho\sigma}\, A_1 + g^{\mu\rho}g^{\nu\sigma}\, A_2 + g^{\mu\sigma}g^{\nu\rho}\, A_3 \qquad I = \sum_i t_i\, A_i$$

- Define projectors such that:

$$P_i \cdot t_j = \delta_{ij}, \qquad P_i \cdot I = A_i$$

- With an ansatz
- BUT should have the same **index symmetry** as the term $g^{\mu\nu}\, g^{\rho\sigma}$ so ....

$$P_1^{\mu\nu\rho\sigma} = g^{\mu\nu}g^{\rho\sigma}\, B_1 + \left(g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}\right) B_2$$

orbit under the action of the stabilizer group

Solving the projector

$$I^{\mu\nu\rho\sigma} = g^{\mu\nu}g^{\rho\sigma}\,A_1 + g^{\mu\rho}g^{\nu\sigma}\,A_2 + g^{\mu\sigma}g^{\nu\rho}\,A_3$$

short hand

$$I = \sum_i t_i\,A_i$$

- Define projectors such that:

$$P_i \cdot t_j = \delta_{ij}, \qquad P_i \cdot I = A_i$$

- Set up a **smaller** system to solve

$$\begin{bmatrix} P_1 \cdot t_1 \\ P_1 \cdot t_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} D^2 & 2D \\ D & D(D+1) \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$$

$$B_1 = \frac{D+1}{D(D+2)(D-1)}$$

$$B_2 = \frac{1}{D(D+2)(D-1)}$$

Applying the projector

short hand

$$I^{\mu\nu\rho\sigma} = g^{\mu\nu} g^{\rho\sigma} A_1 + g^{\mu\rho} g^{\nu\sigma} A_2 + g^{\mu\sigma} g^{\nu\rho} A_3 \qquad I = \sum_i t_i A_i$$

$$A_1 = P_1 \cdot I = B_1 I^{\mu}{}_{\mu}{}^{\rho}{}_{\rho} + B_2 (I^{\mu}{}_{\rho\mu}{}^{\rho} + I^{\mu}{}_{\rho}{}^{\rho}{}_{\mu})$$

Applying the projector

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$$I^{\mu\nu\rho\sigma} = g^{\mu\nu} g^{\rho\sigma} A_1 + g^{\mu\rho} g^{\nu\sigma} A_2 + g^{\mu\sigma} g^{\nu\rho} A_3 \qquad I = \sum_i t_i A_i$$

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- Other projectors related by index symmetry

$$A_2 = P_2 \cdot I = B_1 I^{\mu}{}_{\rho\mu}{}^{\rho} + B_2 (I^{\mu}{}_{\mu}{}^{\rho}{}_{\rho} + I^{\mu}{}_{\rho}{}^{\rho}{}_{\mu})$$

$$A_3 = P_3 \cdot I = B_1 I^{\mu}{}_{\rho}{}^{\rho}{}_{\mu} + B_2 (I^{\mu}{}_{\rho\mu}{}^{\rho} + I^{\mu}{}_{\mu}{}^{\rho}{}_{\rho})$$

Scaling of the problem

$$I^{\mu_1 \cdots \mu_N} = \sum_{\sigma \in S_2^N} g^{\mu_{\sigma(1)}\mu_{\sigma(2)}} \dots g^{\mu_{\sigma(N-1)}\mu_{\sigma(N)}} I_{\sigma(1)\dots\sigma(N)} = \sum_{\sigma \in S_2^N} g(\sigma) I(\sigma)$$

Set of independent permutations

- # tensor structures given by:

$$|S_2^N| = (N-1)!!$$

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Set of independent permutations

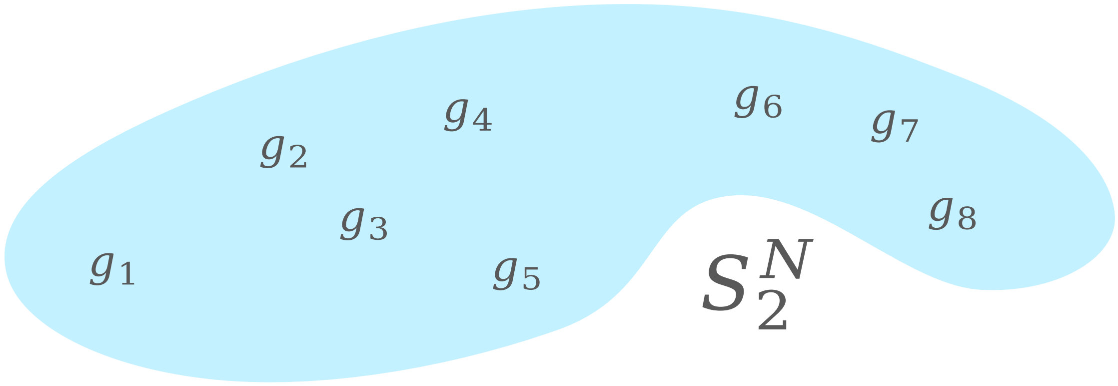
- # tensor structures given by:

$$|S_2^N| = (N-1)!!$$

N	2	4	6	8	10	12	14	16	18	20
$\lvert S_2^N\rvert$	1	3	15	105	945	10,395	135,135	2,027,025	34,459,425	654,729,075

The orbit-partition formula

[Goode, Herzog, Kennedy, ST, Vermaseren, 2024]
2408.05137



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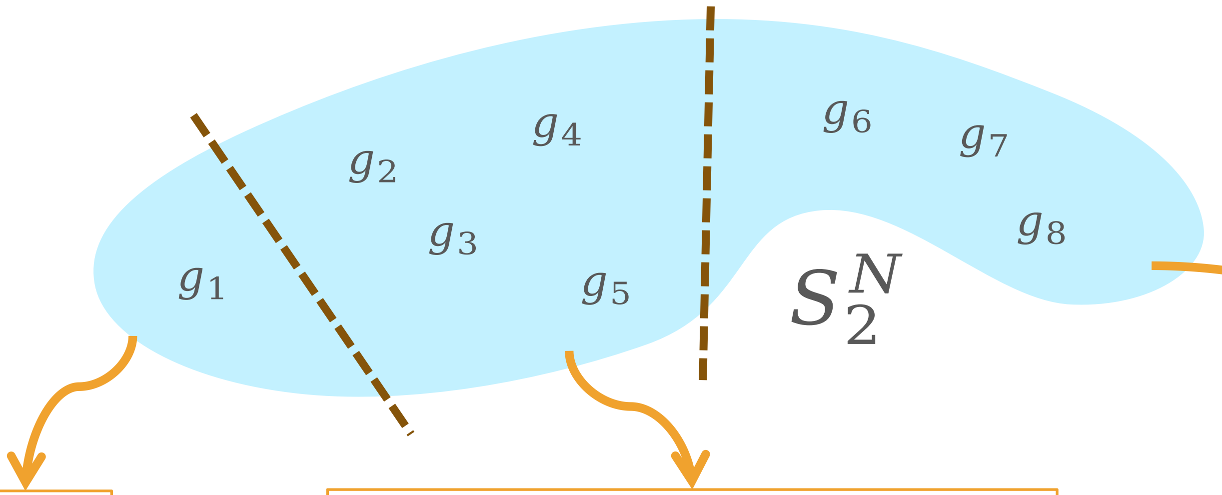

$$P(g_1) = c_1 (g_1) + c_2 (g_2 + g_3 + g_4 + g_5) + c_3 (g_6 + g_7 + g_8)$$

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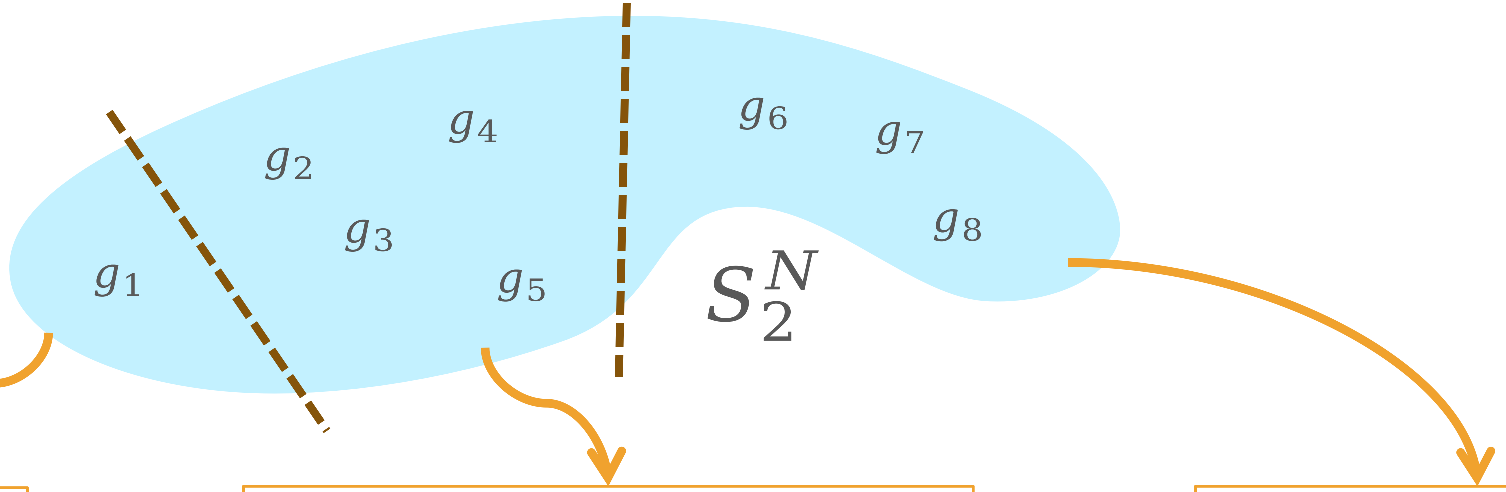


$$P(g_1) = c_1(g_1) + c_2(g_2 + g_3 + g_4 + g_5) + c_3(g_6 + g_7 + g_8)$$

$$P(\sigma) = \sum_k c_k \sum_{\sigma' \in C_k(\sigma)} g(\sigma')$$

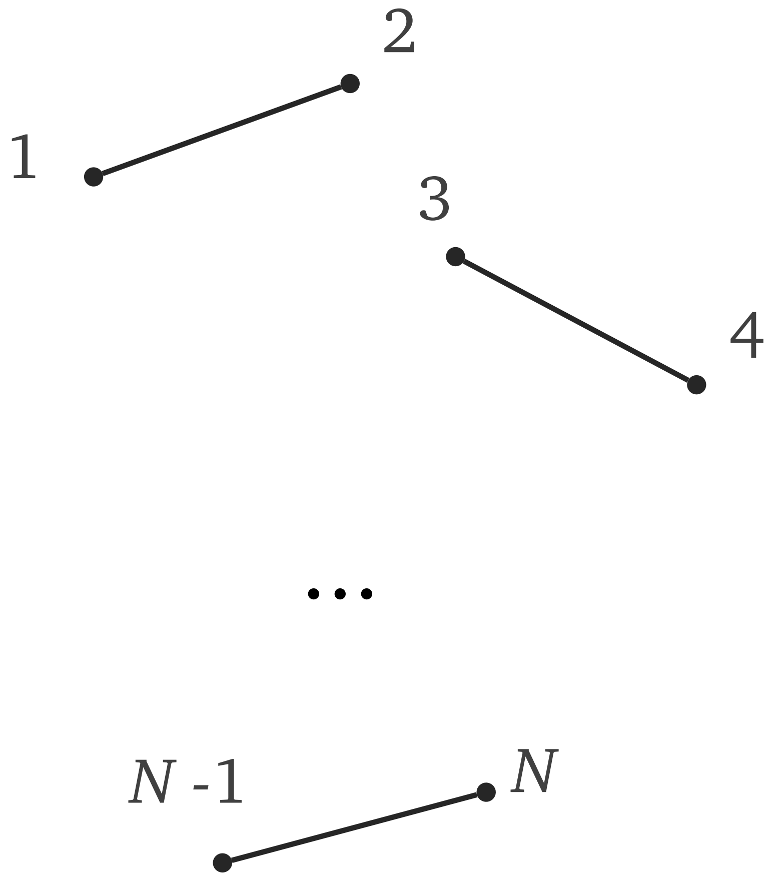
- k labels orbit
- $C_k(\sigma)$ a set that builds the orbits

Finding orbits

[Goode, Herzog, Kennedy, ST, Vermaseren, 2024]
2408.05137

$$g^{\mu_1\mu_2} g^{\mu_3\mu_4} \ldots g^{\mu_{N-1}\mu_N}$$

$$\underbrace{S_2 \times \cdots \times S_2}_{N/2} \times S_{N/2}$$



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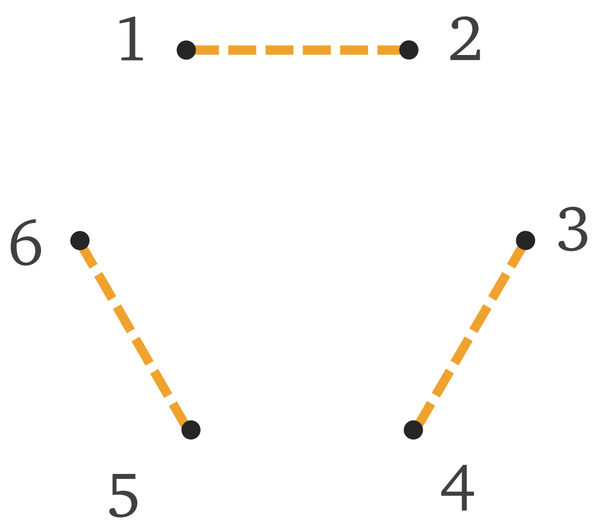
Finding orbits

- Projector for $g^{\mu_1\mu_2}g^{\mu_3\mu_4}g^{\mu_5\mu_6}$



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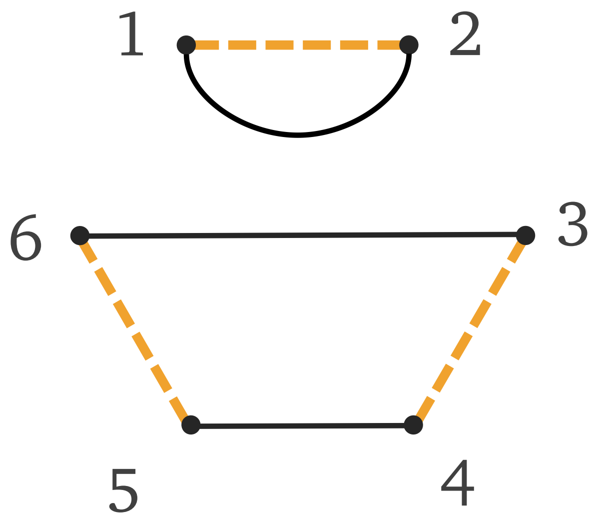

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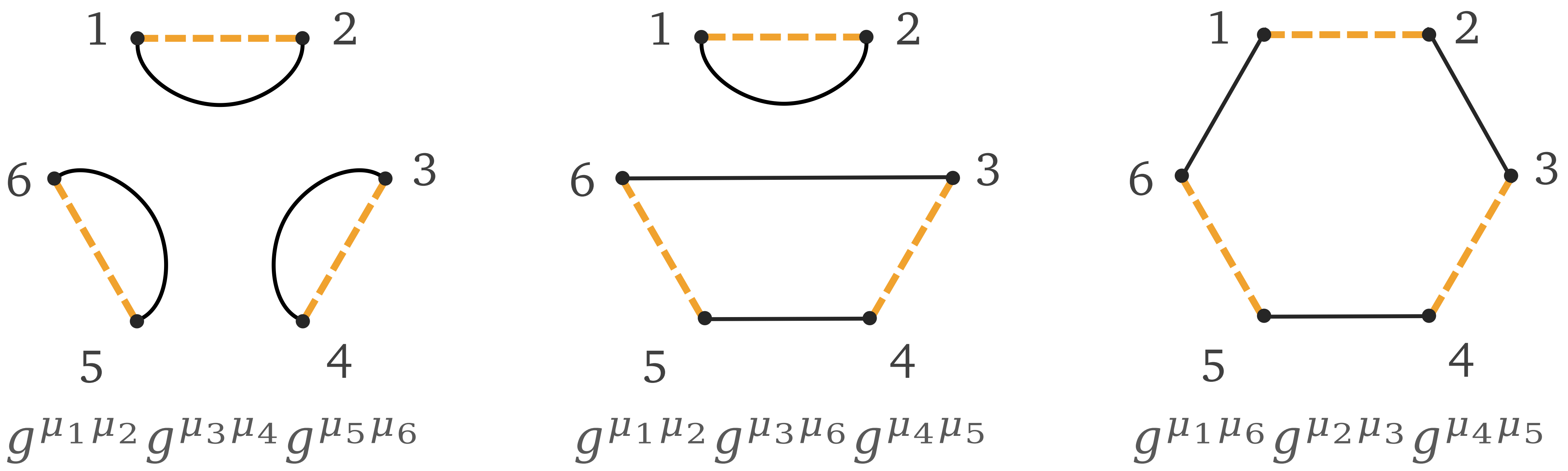

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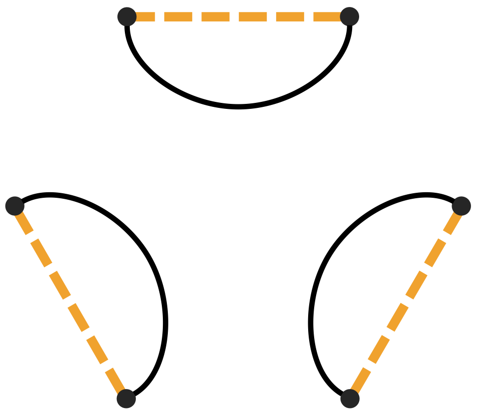

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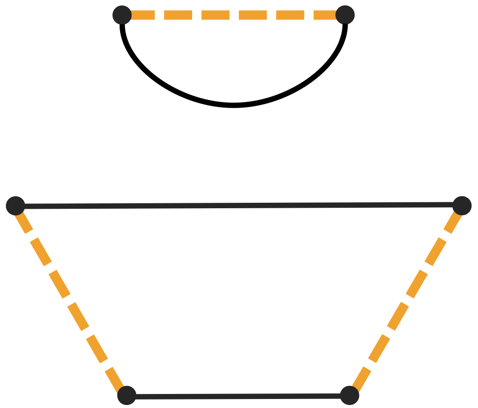
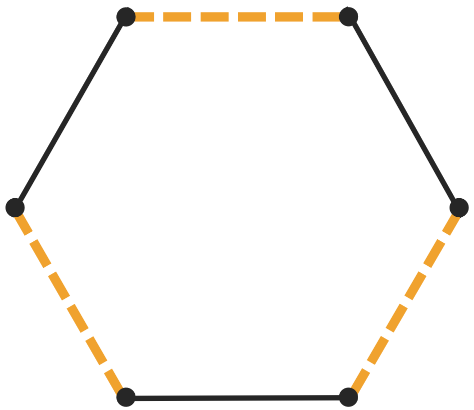


$g^{\mu_1\mu_2}g^{\mu_3\mu_6}g^{\mu_4\mu_5}$

+ 5 terms



$g^{\mu_1\mu_6}g^{\mu_2\mu_3}g^{\mu_4\mu_5}$

+ 7 terms

$$15 = 1 + 6 + 8$$

Finding orbits

[Goode, Herzog, Kennedy, ST, Vermaseren, 2024]
2408.05137

- Orbits $\leftrightarrow$ Diagrams

- Diagrams characterised by *cycle structure*

$$P(\sigma) = \sum_{\lambda \vdash N/2} c_\lambda T_\lambda(\sigma)$$

diagrams = integer partitions of $N/2$

Finding orbits

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N	**2**	**4**	**6**	**8**	**10**	**12**	**14**	**16**	**18**	**20**
$\lvert S_2^N \rvert$	1	3	15	105	945	10,395	135,135	2,027,025	34,459,425	654,729,075
$p\left(N/2\right)$	1	2	3	5	7	11	15	22	30	42

Finding orbits

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- We calculated the coefficients **rank 32**

N	Graph	c_λ
$N=2$		$c_1 = \frac{1}{D}$
$N=4$		$c_2 = \frac{1}{D(D+2)(D-1)}$,
		$c_{11} = \frac{D+1}{D(D+2)(D-1)}$
$N=6$		$c_3 = \frac{2}{D(D+2)(D-1)(D+4)(D-2)}$,
		$c_{21} = -\frac{1}{D(D-1)(D-2)(D+4)}$,
		$c_{111} = \frac{D^2+3D-2}{D(D+2)(D-1)(D+4)(D-2)}$
$N=8$		$c_4 = \frac{-(5D+6)}{D(D-1)(D-2)(D-3)(D+6)(D+4)(D+2)(D+1)}$,
		$c_{31} = \frac{2}{(D-1)(D-2)(D+2)(D-3)(D+6)(D+1)}$,
		$c_{22} = \frac{D^2+5D+18}{D(D-1)(D-2)(D-3)(D+6)(D+4)(D+2)(D+1)}$
		$c_{211} = \frac{-(D^3+6D^2+3D-6)}{D(D-1)(D-2)(D-3)(D+6)(D+4)(D+2)(D+1)}$,
		$c_{1111} = \frac{(D+3)(D^2+6D+1)}{D(D+4)(D+2)(D-1)(D-3)(D+6)(D+1)}$

- Ready to apply to integrals

Applying to Integrands

[Goode, Herzog, Kennedy, ST, Vermaseren, 2024]
2408.05137

- ☑ Vacuum projectors
- ☑ Coefficients to rank 32

Applying to Integrands

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- ☑ Vacuum projectors
- ☑ Coefficients to rank 32
- ☐ Symmetry of the integrand?

Fully symmetric example

[Goode, Herzog, Kennedy, ST, Vermaseren, 2024]
2408.05137

$$I_1^{\mu_1\dots\mu_6} = \int d^Dk\ k^{\mu_1}\, k^{\mu_2}\, k^{\mu_3}\, k^{\mu_4}\, k^{\mu_5} k^{\mu_6} (\dots)$$

- Naively **15 terms!**

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- Naively **15 terms!**

e.g. $\delta^{\mu_1\mu_2\mu_3\mu_4} = g^{\mu_1\mu_2}g^{\mu_3\mu_4} + g^{\mu_1\mu_3}g^{\mu_2\mu_4} + g^{\mu_1\mu_4}g^{\mu_2\mu_3}$

- Only one invariant tensor

$$I_1^{\mu_1\dots\mu_N} = \int d^D k\, k^{\mu_1}\dots k^{\mu_N}(\dots) = \frac{\delta^{\mu_1\dots\mu_N}}{C(N)}\int d^D k (k^2)^{N/2}(\dots)$$

- More loops?

$$C(N) = 2^{N/2}\frac{\Gamma((D+N)/2)}{\Gamma(D/2)}$$

Integrand symmetry

[Goode, Herzog, Kennedy, ST, Vermaseren, 2024]
[Goode, Herzog, ST, in preparation]

$$I_2^{\mu_1\dots\mu_{N_1};\nu_1\dots\nu_{N_2}} = \int d^D k_1\, d^D k_2\, k_1^{\mu_1}\, k_1^{\mu_2}\, k_2^{\nu_1}\, k_2^{\nu_2}\, k_2^{\nu_3}\, k_2^{\nu_4} (\dots)$$

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- Contract integrand with symmetric tensor $\delta^{\mu_1\mu_2\nu_1\nu_2\nu_3\nu_4}$

$$\delta^{k_1k_1k_2k_2k_2k_2} = \underbrace{3(k_1\cdot k_1)(k_2\cdot k_2)^2}_{\delta_1^{k_1k_1k_2k_2k_2k_2}} + \underbrace{12(k_1\cdot k_2)^2(k_2\cdot k_2)}_{\delta_2^{k_1k_1k_2k_2k_2k_2}}$$

Integrand symmetry

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- Contract integrand with symmetric tensor $\delta^{\mu_1\mu_2\nu_1\nu_2\nu_3\nu_4}$

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- recover δ_1 and δ_2 by :
  - Replace dot product with metric
  - Replace k_1 with μ_i
  - Replace k_2 with ν_i

Integrand symmetry

[Goode, Herzog, Kennedy, ST, Vermaseren, 2024]
[Goode, Herzog, ST, in preparation]

$$I_2^{\mu_1\dots\mu_{N_1};\nu_1\dots\nu_{N_2}} = \int d^D k_1\, d^D k_2\, k_1^{\mu_1}\, k_1^{\mu_2}\, k_2^{\nu_1}\, k_2^{\nu_2}\, k_2^{\nu_3}\, k_2^{\nu_4}(\dots)$$

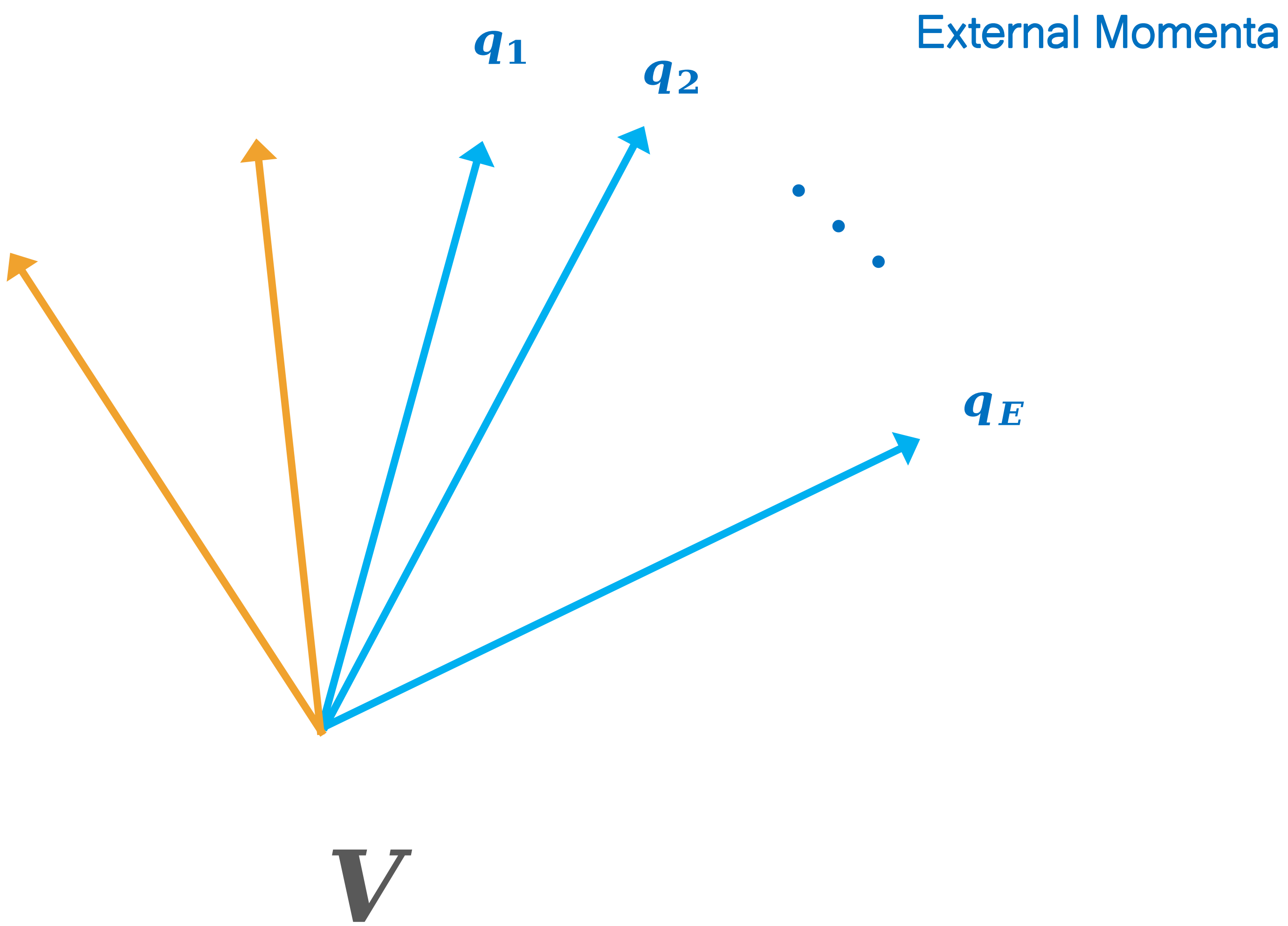
$$\delta_1^{\mu_1\mu_2\nu_1\nu_2\nu_3\nu_4} = \delta^{\mu_1\mu_2}\delta^{\nu_1\nu_2\nu_3\nu_4}$$

- Left with the very nice form

$$\begin{aligned} I_2^{\mu_1\mu_2;\nu_1\cdots\nu_4} = \quad & \delta^{\mu_1\mu_2}\delta^{\nu_1\nu_2\nu_3\nu_4}\int d^D k_1\, d^D k_2\, P^{k_1k_1k_2k_2k_2k_2}\,(\dots) \\ & + \delta_2^{\mu_1\mu_2\nu_1\nu_2\nu_3\nu_4}\int d^D k_1\, d^D k_2\, P^{k_1k_2k_1k_2k_2k_2}\,(\dots) \end{aligned}$$

- Beyond vacuum?

Beyond vacuum



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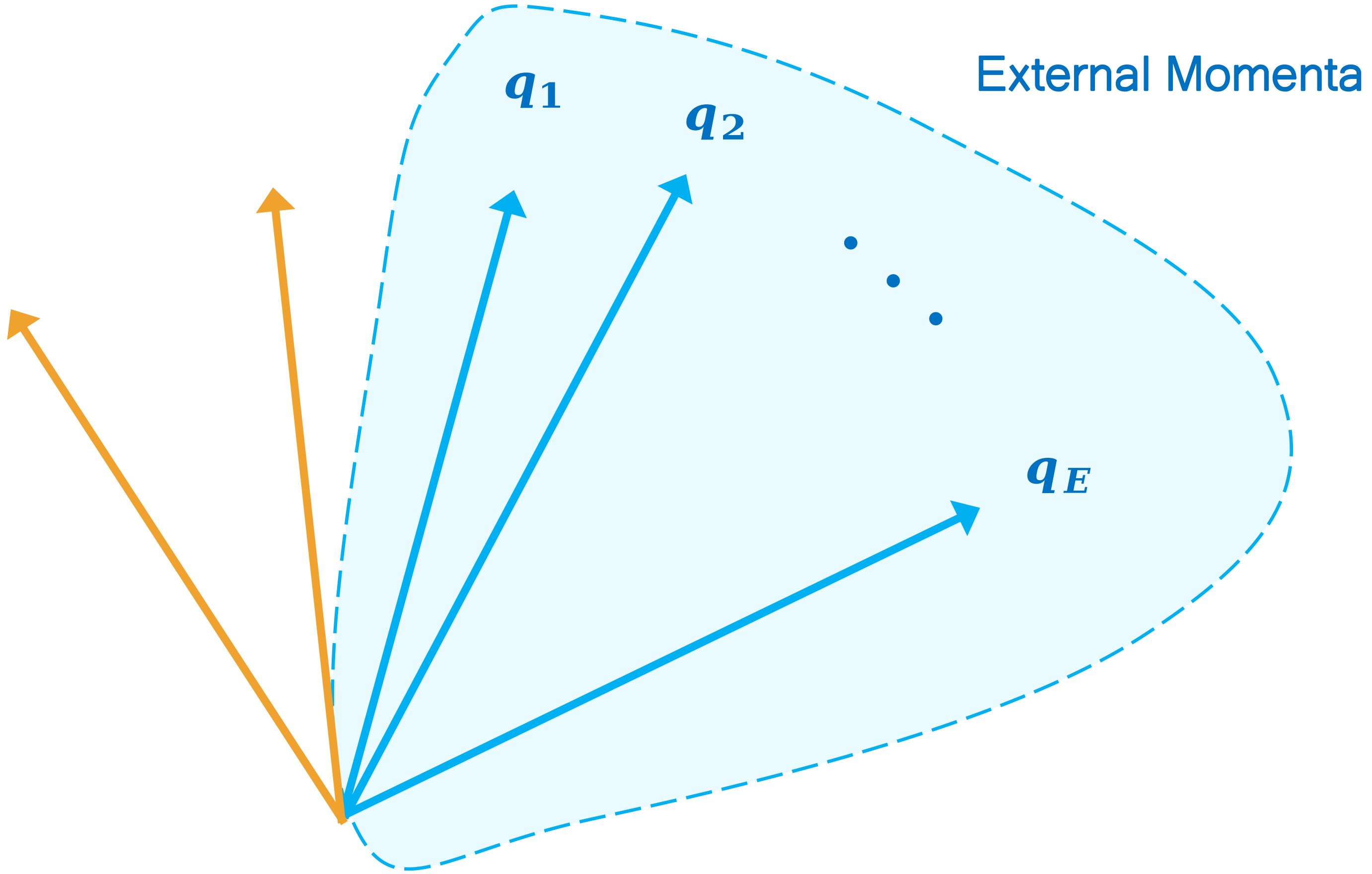
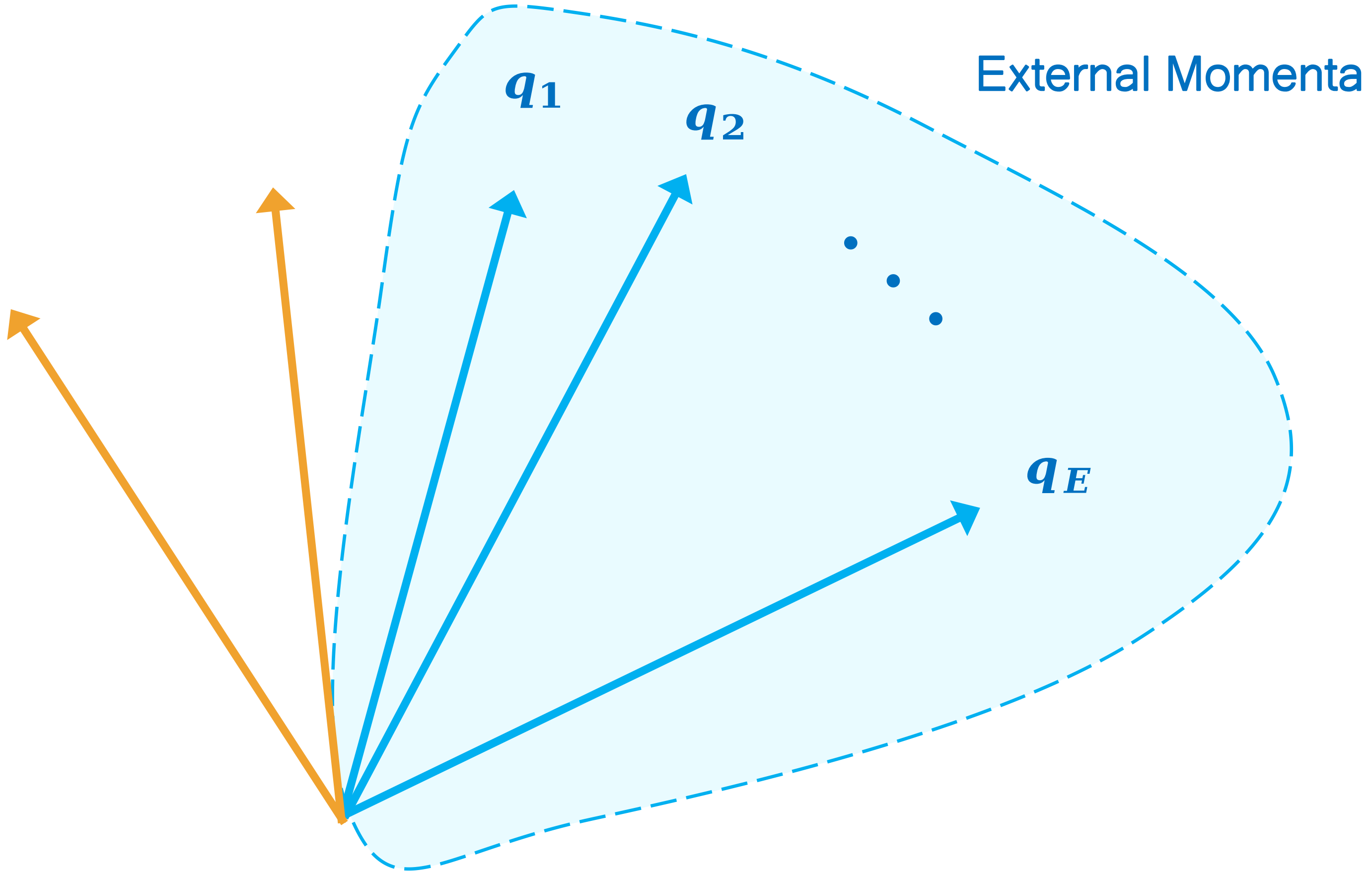


$$V = V_{\perp} \oplus V_{\parallel}$$

[Goode, Herzog, ST, in preparation]

Beyond vacuum



$$\boldsymbol{V} = \boldsymbol{V}_{\perp} \oplus \boldsymbol{V}_{\parallel}$$

Van Neerven-Vermaseren basis

[van Neerven, Vermaseren 1984]
[Ellis, Kunszt, Melnikov, Zanderighi 2012]

$$g^{\mu\nu} = g_{\perp}^{\mu\nu} + g_{\parallel}^{\mu\nu}$$

$$g_{\perp}^{\mu\nu} = g^{\mu\nu} - \sum_{i,j} p_i^{\mu} H_{ij} p_j^{\nu},$$

$$q_i \cdot q_j = G_{ij}\,, \quad H_{ij} = (G^{-1})_{ij}$$

Beyond vacuum

[Goode, Herzog, ST, in preparation]

- Decompose momenta

$$p_i^\mu = (p_i)_\perp^\mu + (p_i)_\parallel^\mu$$

Beyond vacuum

[Goode, Herzog, ST, in preparation]

- Decompose momenta

$$p_i^\mu = (p_i)_\perp^\mu + (p_i)_\parallel^\mu$$

- Introduce dual vectors r_i^μ $\quad q_i \cdot r_j = \delta_{ij}$

- Decompose all the loop momenta p_i^μ

$$\boxed{p_i^\mu = (p_i)_\perp^\mu + \sum_{j=1}^{E} p_i \cdot q_j \, r_j^\mu}$$

Beyond vacuum

- Use our vacuum projectors on transverse integrals with:

$$D \to D - E, \qquad g^{\mu\nu} \to g_{\perp}^{\mu\nu}$$

Beyond vacuum

[Goode, Herzog, ST, in preparation]

- Use our vacuum projectors on transverse integrals with:

$$D \to D - E, \qquad g^{\mu\nu} \to g_{\perp}^{\mu\nu}$$

- Convert back after integration

$$r_i^{\mu} = \sum_{j=1}^{E} H_{ij}\, q_j^{\mu}$$

$$H_{ij} = \frac{\delta_{\|q_1 \dots q_{j-1}\, q_{j+1} \dots q_E}^{\; q_1 \dots q_{i-1}\, q_{i+1} \dots q_E}}{\Delta(q_1 \dots q_E)} (-1)^{i+j}$$

Generalised Kronecker delta

new !

$$\Delta(q_1 \dots q_E) = \delta_{q_1 \dots q_E}^{q_1 \dots q_E}$$

Fermion lines

[Goode, Herzog, Kennedy, ST, Vermaseren, 2024]
2408.05137

☑ Pure Lorentz

☑ Beyond vacuum

☐ fermions????

Fermion lines

[Goode, Herzog, Kennedy, ST, Vermaseren, 2024]
2408.05137

☑ Pure Lorentz

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☐ fermions????

Fermionic projectors

- construct Projectors including γ matrices

γ-factorisation

- Factorise the γ structure out of the integral

$$\int d^D k \, \ldots \not{k} \ldots = \gamma_\mu \int d^D k \, \ldots k^\mu \ldots$$

OPITeR

Orbit Partition Improved Tensor Reduction

OPITeR

Orbit Partition Improved Tensor Reduction

- Tensor reduction up to **rank 20**
- Arbitrary number of spinor indices
- Arbitrary loop
- Up to **8 external momenta**
- Options for exploiting integrand symmetry
- Packaged as a **`Form`** procedure

OPITeR examples

[Goode, Herzog, ST, in preparation]

- Define the integral

$$\int d^D p_1\, d^D p_2\, p_1^{\mu_1}\, p_1^{\mu_2}\, p_2^{\nu_1}\, p_2^{\nu_2}\, p_2^{\nu_3}\, p_2^{\nu_4} (\dots)$$

```
*-------------------------------------
#include opiter.frm
*-------------------------------------
Local integral=ext()*loop(p1,p2)
                *p1(mu1)*p1(mu2)*p2(nu1)
                *p2(nu2)*p2(nu3)*p2(nu4);
```

OPITeR examples

[Goode, Herzog, ST, in preparation]

- Define the integral

$$\int d^D p_1\, d^D p_2\, p_1^{\mu_1}\, p_1^{\mu_2}\, p_2^{\nu_1}\, p_2^{\nu_2}\, p_2^{\nu_3}\, p_2^{\nu_4} (\dots)$$

```
*-------------------------------------
#include opiter.frm
*-------------------------------------
Local integral=ext()*loop(p1,p2)
                *p1(mu1)*p1(mu2)*p2(nu1)
                *p2(nu2)*p2(nu3)*p2(nu4);
```

- Call the `opiter` procedure

```
#call opiter
```

OPITeR examples

[Goode, Herzog, ST, in preparation]

- `opiter/opitersettings.dat`

- Reduction mode

```
tensormode 1
```

no symmetry improvement

```
tensormode 2
```

with integrand symmetries

- Output basis

```
tensorbasis 1
```

$g_\perp$ and dual momenta

```
tensorbasis 2
```

g and external momenta

OPITeR examples

[Goode, Herzog, ST, in preparation]

- `tensormode 2`

```
integral =

   + sym(ind1(mu1,mu2)*ind2(MMu1,MMu2))*sym(ind1(nu1,nu2,nu3,nu4)*ind2(
  MMu3,MMu4,MMu5,MMu6))*ext*loop(p1,p2)*d_(MMu1,MMu2)*d_(MMu3,MMu6)*
  d_(MMu4,MMu5)  * (
       + p1.p1*p2.p2^2*rat(3*D + 9,D^4 + 5*D^3 + 2*D^2 - 8*D)
       + p1.p2^2*p2.p2*rat(-12,D^4 + 5*D^3 + 2*D^2 - 8*D)
       )

   + sym(ind1(mu1,mu2)*ind2(MMu1,MMu2))*sym(ind1(nu1,nu2,nu3,nu4)*ind2(
  MMu3,MMu4,MMu5,MMu6))*ext*loop(p1,p2)*d_(MMu1,MMu6)*d_(MMu2,MMu4)*
  d_(MMu3,MMu5)  * (
       + p1.p1*p2.p2^2*rat(-12,D^4 + 5*D^3 + 2*D^2 - 8*D)
       + p1.p2^2*p2.p2*rat(12,D^3 + 5*D^2 + 2*D - 8)
       );
```

[Goode, Herzog, ST, in preparation]

OPITeR examples

- `tensormode 2`

```
integral =

   + sym(ind1(mu1,mu2)*ind2(MMu1,MMu2))*sym(ind1(nu1,nu2,nu3,nu4)*ind2(
  MMu3,MMu4,MMu5,MMu6))*ext*loop(p1,p2)*d_(MMu1,MMu2)*d_(MMu3,MMu6)*
  d_(MMu4,MMu5)  * (
     + p1.p1*p2.p2^2*rat(3*D + 9,D^4 + 5*D^3 + 2*D^2 - 8*D)
     + p1.p2^2*p2.p2*rat(-12,D^4 + 5*D^3 + 2*D^2 - 8*D)
     )

   + sym(ind1(mu1,mu2)*ind2(MMu1,MMu2))*sym(ind1(nu1,nu2,nu3,nu4)*ind2(
  MMu3,MMu4,MMu5,MMu6))*[illegible],MMu6)*d_(MMu2,MMu4)*
  d_(MMu3,MMu5)  * (
     + p1.p1*p2.p2^2*rat(-12,D^4 + 5*D^3 + 2*D^2 - 8*D)
     + p1.p2^2*p2.p2[illegible] 2*D - 8)
     );
```

Invariant tensor

Scalar integral

OPITeR examples

[Goode, Herzog, ST, in preparation]

- `tensormode 1` or after `#call symmetrise`

```
integral =
       + ext*loop(p1,p2)*d_(mu1,mu2)*d_(nu1,nu2)*d_(nu3,nu4) * ( p1.p1*p2.p2^2
         *rat(D + 3,D^4 + 5*D^3 + 2*D^2 - 8*D) + p1.p2^2*p2.p2*rat(-4,D^4 + 5*
         D^3 + 2*D^2 - 8*D) )

       + ext*loop(p1,p2)*d_(mu1,mu2)*d_(nu1,nu3)*d_(nu2,nu4) * ( p1.p1*p2.p2^2
         *rat(D + 3,D^4 + 5*D^3 + 2*D^2 - 8*D) + p1.p2^2*p2.p2*rat(-4,D^4 + 5*
         D^3 + 2*D^2 - 8*D) )

       + ext*loop(p1,p2)*d_(mu1,mu2)*d_(nu1,nu4)*d_(nu2,nu3) * ( p1.p1*p2.p2^2
         *rat(D + 3,D^4 + 5*D^3 + 2*D^2 - 8*D) + p1.p2^2*p2.p2*rat(-4,D^4 + 5*
         D^3 + 2*D^2 - 8*D) )

       + ext*loop(p1,p2)*d_(mu1,nu1)*d_(mu2,nu2)*d_(nu3,nu4) * ( p1.p1*p2.p2^2
         *rat(-1,D^4 + 5*D^3 + 2*D^2 - 8*D) + p1.p2^2*p2.p2*rat(1,D^3 + 5*D^2
          + 2*D - 8) )

       + ext*loop(p1,p2)*d_(mu1,nu1)*d_(mu2,nu3)*d_(nu2,nu4) * ( p1.p1*p2.p2^2
         *rat(-1,D^4 + 5*D^3 + 2*D^2 - 8*D) + p1.p2^2*p2.p2*rat(1,D^3 + 5*D^2
          + 2*D - 8) )

       + ext*loop(p1,p2)*d_(mu1,nu1)*d_(mu2,nu4)*d_(nu2,nu3) * ( p1.p1*p2.p2^2
         *rat(-1,D^4 + 5*D^3 + 2*D^2 - 8*D) + p1.p2^2*p2.p2*rat(1,D^3 + 5*D^2
          + 2*D - 8) )

       + ext*loop(p1,p2)*d_(mu1,nu2)*d_(mu2,nu1)*d_(nu3,nu4) * ( p1.p1*p2.p2^2
         *rat(-1,D^4 + 5*D^3 + 2*D^2 - 8*D) + p1.p2^2*p2.p2*rat(1,D^3 + 5*D^2
          + 2*D - 8) )

       + ext*loop(p1,p2)*d_(mu1,nu2)*d_(mu2,nu3)*d_(nu1,nu4) * ( p1.p1*p2.p2^2
         *rat(-1,D^4 + 5*D^3 + 2*D^2 - 8*D) + p1.p2^2*p2.p2*rat(1,D^3 + 5*D^2
          + 2*D - 8) )

       + ext*loop(p1,p2)*d_(mu1,nu2)*d_(mu2,nu4)*d_(nu1,nu3) * ( p1.p1*p2.p2^2
         *rat(-1,D^4 + 5*D^3 + 2*D^2 - 8*D) + p1.p2^2*p2.p2*rat(1,D^3 + 5*D^2
          + 2*D - 8) )

       + ext*loop(p1,p2)*d_(mu1,nu3)*d_(mu2,nu1)*d_(nu2,nu4) * ( p1.p1*p2.p2^2
         *rat(-1,D^4 + 5*D^3 + 2*D^2 - 8*D) + p1.p2^2*p2.p2*rat(1,D^3 + 5*D^2
          + 2*D - 8) )

       + ext*loop(p1,p2)*d_(mu1,nu3)*d_(mu2,nu2)*d_(nu1,nu4) * ( p1.p1*p2.p2^2
         *rat(-1,D^4 + 5*D^3 + 2*D^2 - 8*D) + p1.p2^2*p2.p2*rat(1,D^3 + 5*D^2
          + 2*D - 8) )

       + ext*loop(p1,p2)*d_(mu1,nu3)*d_(mu2,nu4)*d_(nu1,nu2) * ( p1.p1*p2.p2^2
         *rat(-1,D^4 + 5*D^3 + 2*D^2 - 8*D) + p1.p2^2*p2.p2*rat(1,D^3 + 5*D^2
          + 2*D - 8) )

       + ext*loop(p1,p2)*d_(mu1,nu4)*d_(mu2,nu1)*d_(nu2,nu3) * ( p1.p1*p2.p2^2
         *rat(-1,D^4 + 5*D^3 + 2*D^2 - 8*D) + p1.p2^2*p2.p2*rat(1,D^3 + 5*D^2
          + 2*D - 8) )

       + ext*loop(p1,p2)*d_(mu1,nu4)*d_(mu2,nu2)*d_(nu1,nu3) * ( p1.p1*p2.p2^2
         *rat(-1,D^4 + 5*D^3 + 2*D^2 - 8*D) + p1.p2^2*p2.p2*rat(1,D^3 + 5*D^2
          + 2*D - 8) )

       + ext*loop(p1,p2)*d_(mu1,nu4)*d_(mu2,nu3)*d_(nu1,nu2) * ( p1.p1*p2.p2^2
         *rat(-1,D^4 + 5*D^3 + 2*D^2 - 8*D) + p1.p2^2*p2.p2*rat(1,D^3 + 5*D^2
          + 2*D - 8) );
```

Summary

Projectors

- Orbit-partition ⟶ rank 32 projectors
- Exploiting integrand symmetry

OPITeR

- Fully automated reduction
- Soon to be published!

Summary

Projectors

- Orbit-partition $\longrightarrow$ rank 32 projectors
- Exploiting integrand symmetry

OPITeR

- Fully automated reduction
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Thank you for listening

Bonus slides

Fermion lines

[Goode, Herzog, Kennedy, ST, Vermaseren, 2024]

- Adding gamma matrices the tensors
- Work in anti symmetric basis $\Gamma^{\mu_1 \dots \mu_p} = \gamma^{[\mu_1} \dots \gamma^{\mu_p]}$

Fermion lines

[Goode, Herzog, Kennedy, ST, Vermaseren, 2024]

- Adding gamma matrices the tensors
- Work in anti symmetric basis $\Gamma^{\mu_1 \ldots \mu_p} = \gamma^{[\mu_1} \ldots \gamma^{\mu_p]}$

$$\mathrm{tr}\left(\Gamma_{\mu_1 \ldots \mu_a} \Gamma^{\nu_1 \ldots \nu_b}\right) = \delta_{ab}\, \mathrm{tr}(\mathbb{I}) \delta^{\nu_1 \ldots \nu_b}_{\mu_1 \ldots \mu_a} (-1)^{(a(a-1)/2)}$$

[Goode, Herzog, Kennedy, ST, Vermaseren, 2024]

Fermion lines

- Adding gamma matrices the tensors
- Work in anti symmetric basis $\Gamma^{\mu_1\ldots\mu_p} = \gamma^{[\mu_1}\ldots\gamma^{\mu_p]}$

$$\mathrm{tr}\left(\Gamma_{\mu_1\ldots\mu_a}\Gamma^{\nu_1\ldots\nu_b}\right) = \delta_{ab}\,\mathrm{tr}(\mathbb{I})\delta^{\nu_1\ldots\nu_b}_{\mu_1\ldots\mu_a}(-1)^{(a(a-1)/2)}$$

- Efficiently transform into basis

$$\gamma^{\mu_1}\ldots\gamma^{\mu_n} = \sum_{k=0}^{n}\sum_{\pi\in\Sigma_n^k}\mathrm{sgn}(\pi)\Gamma^{\mu_{\pi(1)}\cdots\mu_{\pi(k)}}\,\mathrm{tr}\left(\gamma^{\mu_{\pi(k+1)}}\ldots\gamma^{\mu_{\pi(n)}}\right)$$

new !

Σ_n^k shuffles the first k indices with the remaining $n-k$ indices over the two tensors

Fermion lines

[Goode, Herzog, Kennedy, ST, Vermaseren, 2024]

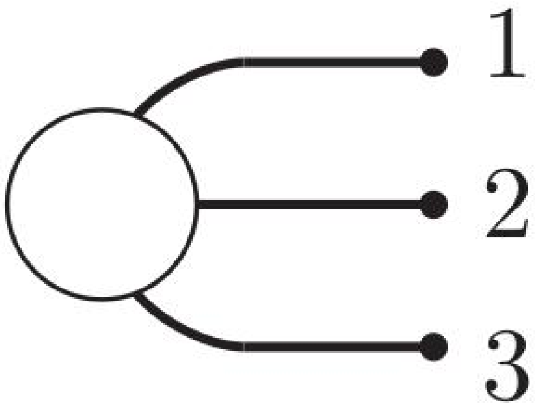
- Add new tensors
- Constrain the basis
- Create projector ansatz as before

[Goode, Herzog, Kennedy, ST, Vermaseren, 2024]

Fermion lines

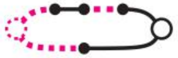
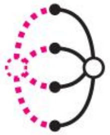
- Add new tensors
- Constrain the basis
- Create projector ansatz as before
- Need extra diagram elements

$$\Gamma^{\mu_1\mu_2\mu_3}\dots \quad \rightarrow$$

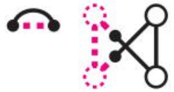
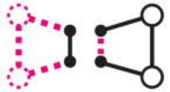
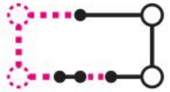
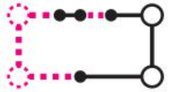
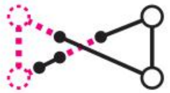
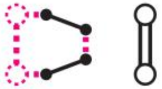
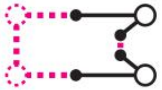
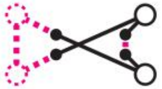


Fermion lines

[Goode, Herzog, Kennedy, ST, Vermaseren, 2024]

N	n_γ	Graph	$c_{\lambda_g,\lambda_\gamma}$
$N=1$	1		$c_{(),()} = \frac{1}{D\,\mathrm{tr}\,(\mathbb{1})}$
$N=2$	0		$c_{(1),()} = \frac{1}{D\,\mathrm{tr}\,(\mathbb{1})}$
	2		$c_{(),()} = \frac{1}{2D(D-1)\,\mathrm{tr}\,(\mathbb{1})}$
$N=3$	1		$c_{(1),()} = \frac{-1}{D(D+2)(D-1)\,\mathrm{tr}\,(\mathbb{1})}$,
			$c_{(),(1)} = \frac{D+1}{D(D+2)(D-1)\,\mathrm{tr}\,(\mathbb{1})}$
	3		$c_{(),()} = \frac{1}{D(D-2)(D-1)\,\mathrm{tr}\,(\mathbb{1})}$
$N=4$	0		$c_{(1,1),()} = \frac{D+1}{D(D+2)(D-1)\,\mathrm{tr}\,(\mathbb{1})}$,
			$c_{(2),()} = \frac{-1}{D(D+2)(D-1)\,\mathrm{tr}\,(\mathbb{1})}$
	2		$c_{(1),()} = \frac{1}{D(D-1)(D-2)(D+2)\,\mathrm{tr}\,(\mathbb{1})}$,
			$c_{(),(1)} = \frac{1}{(D-1)(D-2)(D+2)\,\mathrm{tr}\,(\mathbb{1})}$
	4		$c_{(),()} = \frac{1}{D(D-1)(D-2)(D-3)\,\mathrm{tr}\,(\mathbb{1})}$

Fermion lines

$\vec{n}$	k	Graph	c_k
$(1,1,1,1)$	1		$\frac{D^3+3D^2-3D-3}{D(D-1)(D-2)(D-3)(D+2)(D+1)(D+4)\,\mathrm{tr}\,(\mathbb{1})^2}$
	2		$\frac{-D^2-2D+1}{D(D-1)(D-2)(D-3)(D+2)(D+1)(D+4)\,\mathrm{tr}\,(\mathbb{1})^2}$
	3		$\frac{3D+5}{D(D-1)(D-2)(D-3)(D+2)(D+1)(D+4)\,\mathrm{tr}\,(\mathbb{1})^2}$
	4		$\frac{-D^2-3D-3}{D(D-1)(D-2)(D-3)(D+2)(D+1)(D+4)\,\mathrm{tr}\,(\mathbb{1})^2}$
	5		$\frac{-D^2-3D-3}{D(D-1)(D-2)(D-3)(D+2)(D+1)(D+4)\,\mathrm{tr}\,(\mathbb{1})^2}$
	6		$\frac{2D+1}{D(D-1)(D-2)(D-3)(D+2)(D+1)(D+4)\,\mathrm{tr}\,(\mathbb{1})^2}$
	7		$\frac{2D+1}{D(D-1)(D-2)(D-3)(D+2)(D+1)(D+4)\,\mathrm{tr}\,(\mathbb{1})^2}$
$(2,0,0,2)$	8		$\frac{-D^2-2D+1}{D(D-1)(D-2)(D-3)(D+2)(D+1)(D+4)\,\mathrm{tr}\,(\mathbb{1})^2}$
	9		$\frac{2D+1}{D(D-1)(D-2)(D-3)(D+2)(D+1)(D+4)\,\mathrm{tr}\,(\mathbb{1})^2}$
$(0,2,2,0)$	10		$\frac{-1}{(D-1)(D-2)(D-3)(D+2)(D+1)\,\mathrm{tr}\,(\mathbb{1})^2}$
	11		$\frac{1}{D(D-1)(D-2)(D-3)(D+2)(D+1)\,\mathrm{tr}\,(\mathbb{1})^2}$

Workflow

[Goode, Herzog, ST, in preparation]