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New tools for N-jettiness computations

High Precision for Hard Processes (HP2 2024)

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Talk plan

This talk is based in our work presented in hep-ph/2403.03078 and ongoing research on power corrections

1. **Introduction**
2. **Soft function calculation**
3. **Numerical checks**
4. **Power Corrections**
5. **Conclusions**

Introduction

Higher-order QCD corrections (at NNLO)

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Subtraction methods
Analytically removes $1/\epsilon^n$ poles by constructing integrable counterterms

- Antenna subtraction
  Gehrmann-De Ridder, Gehrmann, Glover - hep-ph/0505111
- CoLoRFul subtraction
  Somogyi, Trócsányi, Del Duca - hep-ph/0502226
- Sector subtraction
  Czakon - hep-ph/1005.0274, Boughezal et al. - hep-ph/1111.7041
- Projection-to-Born
  Cacciari et al. - hep-ph/1506.02660
- Nested soft-collinear subtraction
  Caola, Melnikov, Röntsch - hep-ph/1702.01352
- Local analytic sector subtraction
  Magnea et al. - hep-ph/1806.09570

Slicing methods
Imposes cuts in some variable to split the phase space. Below the cut a soft-collinear approximation is used

- q_t-subtraction
  Catani, Grazzini - hep-ph/0703012
- N-jettiness subtraction
  Boughezal et al. - hep-ph/1504.02131, Gaunt et al. - hep-ph/1505.04794

N-jettiness subtraction

The N-jettiness variable is defined by

$$\mathcal{T}(\mathcal{R},\mathcal{U}) = \sum_{x\in\mathcal{U}} \min\left\{\frac{2p_x p_{h_1}}{P_{h_1}}, \frac{2p_x p_{h_2}}{P_{h_2}}, \frac{2p_x p_{h_3}}{P_{h_3}}, \ldots\right\}$$

Can be used to perform a slicing of the phase space (like in q_T subtraction)

$$\sigma = \int^{\mathcal{T}_0} d\mathcal{T}\frac{d\sigma}{d\mathcal{T}} + \int_{\mathcal{T}_0} d\mathcal{T}\frac{d\sigma}{d\mathcal{T}}$$

and, thanks to the factorization theorem from SCET, we can calculate

$$\int^{\mathcal{T}_0} d\mathcal{T}\frac{d\sigma}{d\mathcal{T}} = \int B\otimes B\otimes S\otimes H\otimes \prod_i^N J_i + \mathcal{O}(\mathcal{T}_0)$$

N-jettiness subtraction

$$\int^{\mathcal{T}_0} d\mathcal{T} \frac{d\sigma}{d\mathcal{T}} = \int B \otimes B \otimes S \otimes H \otimes \prod_i^N J_i + \mathcal{O}(\mathcal{T}_0)$$

- The **Beam** and **Jet functions** (B, J_i) describe initial- and final-state collinear radiation, the **Soft function** S the soft radiation, and the (process dependent) **Hard function** H encodes the virtual corrections
- Small cutoff $\mathcal{T}_0$ required so that power corrections in $\mathcal{T}_0/Q$ are under control
- At NNLO, **all ingredients are known**. S was available for 0-, 1- and 2-jettiness, but only recently for generic N-jettiness (hep-ph/2312.11626, hep-ph/2403.03078)
- At N3LO, only S is missing. The N3LO 0-jettiness soft function is **almost here** (see Pikelner's and Chen's talks)

N-jettiness subtraction - power corrections

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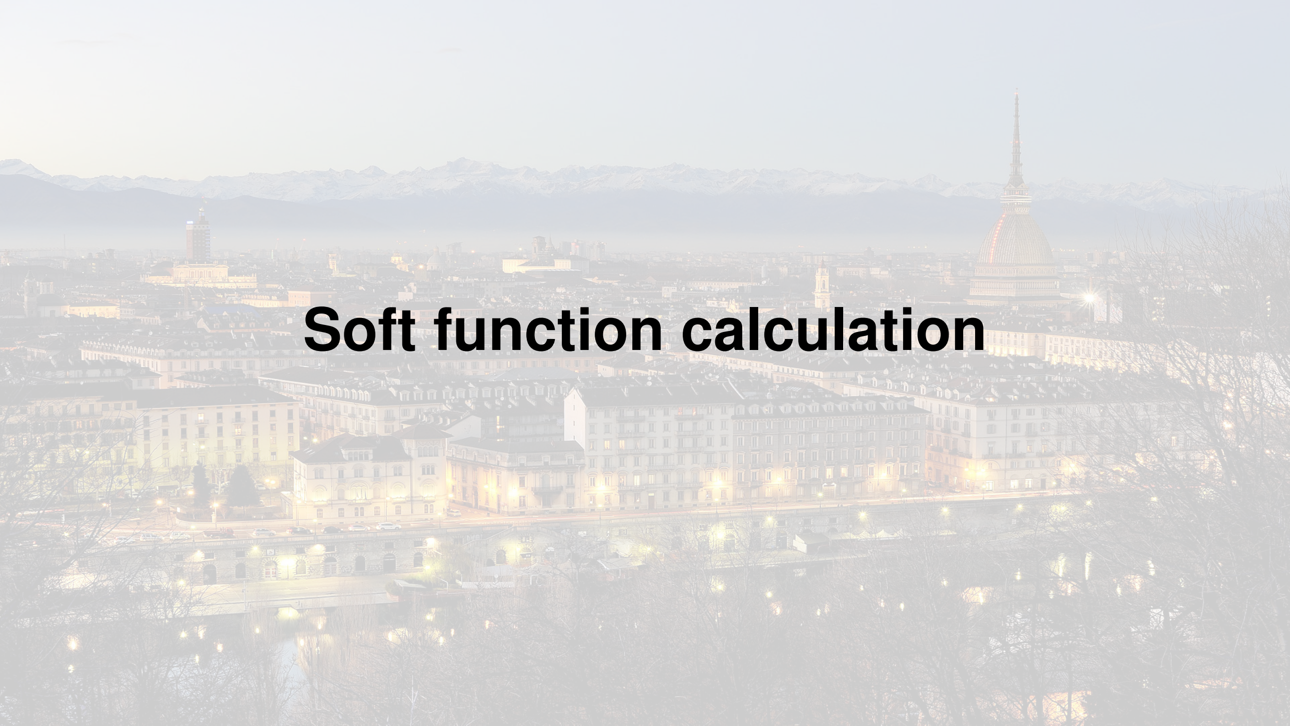
$$\int^{\mathcal{T}_0} d\mathcal{T} \frac{d\sigma}{d\mathcal{T}} = \int B \otimes B \otimes S \otimes H \otimes \prod_i^N J_i + \mathcal{O}(\mathcal{T}_0)$$

- Slicing methods suffer from **instabilities** due to large cancellations between contributions if the slicing parameter (cutoff) is not sufficiently small
- To improve this we need to include more terms in the approximate computation of the singular contribution
- Power-suppressed terms, particularly *subleading* ones, were studied in recent years, mostly at NLO (hep-ph/1802.00456, hep-ph/1807.10764, hep-ph/1907.12213, hep-ph/1905.08741)



Soft function calculation

In our soft function calculation

- Previous NNLO calculations mainly based on mapping the available phase space of soft-gluon emissions onto hemispheres and computing numerically.
  (Boughezal et al. - hep-ph/1504.02540, Campbell et al. - hep-ph/1711.09984, Bell et al. - hep-ph/2312.11626)

- We use **subtraction methods** to calculate this ingredient of a slicing method, showing the *explicit analytical cancellation of divergences*. Also, *N is treated genuinely as a parameter*.

- We show that borrowing ideas from NNLO QCD subtraction schemes is beneficial for computing ingredients of modern slicing calculations.

Soft function renormalization

Loop-corrections are not present, so IR divergences turn into UV ones that require renormalization
It is convenient to work in Laplace space

$$S(u) = \int_0^\infty d\mathcal{T}\ S_{\mathcal{T}}(\mathcal{T}) e^{-u\mathcal{T}}$$

Since the renormalization is multiplicative (with matrices in color space)

$$S = Z\tilde{S}Z^\dagger$$

If we write the expansion in powers of α_s

$$Z = 1 + Z_1 + Z_2,$$
$$S = 1 + S_1 + S_2,$$
$$\tilde{S} = 1 + \tilde{S}_1 + \tilde{S}_2,$$

$$\tilde{S}_1 = S_1 - Z_1 - Z_1^\dagger,$$

$$\left(Z_2 = \tfrac{1}{2} Z_1 Z_1 + Z_{2,r}\right)$$

$$\tilde{S}_2 = S_2 - Z_2 - Z_2^\dagger + Z_1 Z_1 + Z_1^\dagger Z_1^\dagger - Z_1 S_1 - S_1 Z_1^\dagger + Z_1 Z_1^\dagger$$
$$= \frac{1}{2}\tilde{S}_1\tilde{S}_1 + \frac{1}{2}[Z_1, Z_1^\dagger] + \frac{1}{2}\left[S_1, Z_1 - Z_1^\dagger\right] + S_{2,r} - Z_{2,r} - Z_{2,r}^\dagger.$$

Soft function at NLO

If we take $P_{h_i} = 2E_i$ with an unresolved gluon m, the N-jettiness is given by

$$\mathcal{T}(m) = E_m \psi_m = E_m \min\{\rho_{1m}, \rho_{2m}, \rho_{3m}, ..., \rho_{Nm}\},$$

where $\rho_{ij} = 1 - \vec{n}_i \cdot \vec{n}_j$. Then, the soft function is given by

$$S(\tau) = -\sum_{(ij)} \mathbf{T}_i \cdot \mathbf{T}_j \, g_s^2 \int \frac{d\Omega_m^{(d-1)}}{2(2\pi)^{d-1}} \frac{dE_m}{E_m^{1+2\epsilon}} E_m^2 \, \delta(\tau - E_m\psi_m) \, \langle S_{ij}(m) \rangle_m , \qquad S_{ij}(m) = \frac{1}{E_m^2} \frac{\rho_{ij}}{\rho_{im}\rho_{jm}}$$

We integrate over energy and use that we know the limit $\lim_{m||i} \psi_m = \rho_{im}$., so we can rewrite

$$\psi_m^{2\epsilon} \frac{\rho_{ij}}{\rho_{im}\rho_{jm}} = \left(\frac{\psi_m \rho_{ij}}{\rho_{im}\rho_{jm}}\right)^{2\epsilon} \frac{\rho_{ij}^{1-2\epsilon}}{\rho_{im}^{1-2\epsilon}\rho_{jm}^{1-2\epsilon}} = \left(1 + 2\epsilon g_{ij,m}^{(2)}\right) \frac{\rho_{ij}^{1-2\epsilon}}{\rho_{im}^{1-2\epsilon}\rho_{jm}^{1-2\epsilon}}$$

Soft function at NLO

Knowing that ($\eta_{ij} = \rho_{ij}/2$)

$$\left\langle \frac{\rho_{ij}^{1-2\epsilon}}{\rho_{im}^{1-2\epsilon}\rho_{jm}^{1-2\epsilon}} \right\rangle_m = \frac{2\eta_{ij}^{\epsilon}}{\epsilon} K_{ij}^{(2)} = \frac{2\eta_{ij}^{\epsilon}}{\epsilon} \frac{\Gamma(1+\epsilon)^2}{\Gamma(1+2\epsilon)} \, {}_2F_1\left(\epsilon, \epsilon, 1-\epsilon, 1-\eta_{ij}\right),$$

in Laplace space we arrive to the following bare soft function

$$S_1 = a_s\,(\mu\bar{u})^{2\epsilon} \frac{\Gamma(1-2\epsilon)}{\Gamma(1-\epsilon)e^{\epsilon\gamma_E}} \sum_{(ij)} \mathbf{T}_i \cdot \mathbf{T}_j \left[\frac{\eta_{ij}^{\epsilon}}{\epsilon^2} K_{ij}^{(2)} + \left\langle g_{ij,m}^{(2)} \frac{\rho_{ij}^{1-2\epsilon}}{\rho_{im}^{1-2\epsilon}\rho_{jm}^{1-2\epsilon}} \right\rangle_m \right].$$

By combining S_1 with the renormalization matrices Z_1 and $Z_1^{\dagger}$, we finally obtain ($L_{ij} = \ln\left(\mu\bar{u}\sqrt{\eta_{ij}}\right)$)

$$\tilde{S}_1 = a_s \sum_{(ij)} \mathbf{T}_i \cdot \mathbf{T}_j \left[2L_{ij}^2 + \mathrm{Li}_2(1-\eta_{ij}) + \frac{\pi^2}{12} + \left\langle \ln\left(\frac{\psi_m \rho_{ij}}{\rho_{im}\rho_{jm}}\right) \frac{\rho_{ij}}{\rho_{im}\rho_{jm}} \right\rangle_m + \mathcal{O}(\epsilon) \right]$$

Soft function at NNLO

The NNLO contribution to the **bare soft function** is

$$S_2 = S_{2,RR} + S_{2,RV} - a_s \frac{\beta_0}{\epsilon} S_1$$

We further split the double-real contribution into correlated and uncorrelated pieces

$$S_{2,RR,\tau} = S_{2,RR,T^4} + S_{2,RR,T^2} = \frac{1}{2} \sum_{(ij),(k,l)} \{\mathbf{T}_i \cdot \mathbf{T}_j, \mathbf{T}_k \cdot \mathbf{T}_l\} \; I_{T^4,ij,kl} - \frac{C_A}{2} \sum_{(ij)} \mathbf{T}_i \cdot \mathbf{T}_j \; I_{T^2,ij}$$

The real-virtual contribution reads

$$S_{2,RV,\tau} = S_{RV,T^2} + S_{RV,\text{tc}} = \frac{[\alpha_s]\, 2^{-\epsilon}}{\epsilon^2} C_A A_K(\epsilon) \sum_{(ij)} \mathbf{T}_i \cdot \mathbf{T}_j \; I_{RV,ij} + [\alpha_s] \frac{4\pi N_\epsilon}{\epsilon} \sum_{(kij)} \kappa_{ij} F^{kij} I_{kij}$$

where $\kappa_{ij} = \lambda_{ij} - \lambda_{im} - \lambda_{jm}$, with $\lambda_{ij} = 1$ if both i and j refer to incoming/outgoing partons and zero otherwise. We have defined $F^{kij} = f_{abc} T_k^a T_i^b T_j^c$, while $A_K(\epsilon)$ and N_ϵ are normalization factors.

Soft function at NNLO

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The calculation of the **renormalized soft function** is organized as follows

$$\tilde{S}_2 = \tilde{S}_2^{\text{uncorr}} + \tilde{S}_2^{\text{corr}} + \tilde{S}_2^{\text{tc}}$$

Where the pieces are given by the following contributions

Uncorrelated emission

$$\tilde{S}_2^{\text{uncorr}} = \frac{1}{2}\tilde{S}_1\tilde{S}_1$$

Correlated emission

$$\tilde{S}_2^{\text{corr}} = S_{2,RR,T^2} + S_{RV,T^2} - Z_{2,r} - Z_{2,r}^\dagger - \frac{a_s\beta_0}{\epsilon}S_1\,,$$

Triple color terms

$$\tilde{S}_2^{\text{tc}} = \frac{1}{2}\left[Z_1, Z_1^\dagger\right] + \frac{1}{2}\left[S_1, Z_1 - Z_1^\dagger\right] + S_{RV,\text{tc}}$$

Soft function at NNLO

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Uncorrelated emission

$$\tilde{S}_2^{\text{uncorr}} = \frac{1}{2}\tilde{S}_1 \tilde{S}_1$$

→ Easy to relate to iterations of NLO

Correlated emission

$$\tilde{S}_2^{\text{corr}} = S_{2,RR,\mathcal{T}^2} + S_{RV,\mathcal{T}^2} - Z_{2,r} - Z_{2,r}^\dagger - \frac{a_s \beta_0}{\epsilon} S_1 ,$$

→ Use nested soft-collinear subtraction, reuse results from calculation without jettiness-constraint

Triple color terms

$$\tilde{S}_2^{\text{tc}} = \frac{1}{2}\left[Z_1, Z_1^\dagger\right] + \frac{1}{2}\left[S_1, Z_1 - Z_1^\dagger\right] + S_{RV,\text{tc}}$$

→ Similar to NLO, reuse results from calculation without jettiness-constraint

The final result

- The NLO contribution was

$$\tilde{S}_1 = a_s \sum_{(ij)} \mathbf{T}_i \cdot \mathbf{T}_j \left[2L_{ij}^2 + \mathrm{Li}_2(1-\eta_{ij}) + \frac{\pi^2}{12} + \left\langle L^{\psi}_{ij,m} \frac{\rho_{ij}}{\rho_{im}\rho_{jm}} \right\rangle_m \right],$$

where $L_{ij} = \ln(\bar{u}\sqrt{\eta_{ij}}\mu)$ and $L^{\psi}_{ij,m} = \ln\left(\frac{\psi_m \rho_{ij}}{\rho_{im}\rho_{jm}}\right)$.

- The NNLO one is

$$\tilde{S}_2 = \frac{1}{2}\,\tilde{S}_1^2 + a_s^2 C_A \sum_{(ij)} \mathbf{T}_i \cdot \mathbf{T}_j \; G_{ij} + a_s^2 \; n_f \;\; T_R \sum_{(ij)} \mathbf{T}_i \cdot \mathbf{T}_j \; Q_{ij} + a_s^2 \pi \sum_{(kij)} F^{kij} \;\; \kappa_{kj} G_{kij}^{\text{triple}},$$

where G_{ij}, Q_{ij} and G_{kij}^{triple} are finite functions with analytical terms along with a *low number numerical four-dimensional integrations* over one- and two-particle phase space

Numerical checks

Numerical checks

- We compared our results with (see Rahn's talk):

The NNLO soft function for N-jettiness in hadronic collisions

Bell, Dehnadi, Mohrmann, Rahn, arXiv hep-ph/2312.11626.

- We focus in the "new" 3-jettiness case, with two back-to-back beams. The five directions are

$$n_1 = (0,0,1), \quad n_2 = (0,0,-1), \quad n_3 = (\sin\theta_{13}, 0, \cos\theta_{13}),$$
$$n_4 = (\sin\theta_{14}\cos\phi_4, \sin\theta_{14}\sin\phi_4, \cos\theta_{14}), \quad n_5 = (\sin\theta_{15}\cos\phi_5, \sin\theta_{15}\sin\phi_5, \cos\theta_{15}),$$

in the following phase-space point

$$\theta_{13} = \frac{3\pi}{10}, \quad \theta_{14} = \frac{6\pi}{10}, \quad \theta_{15} = \frac{9\pi}{10}, \quad \phi_4 = \frac{3\pi}{5}, \quad \phi_5 = \frac{6\pi}{5}$$

Numerical checks

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Dipole configurations

Dipoles	Gluons		Quarks	
	G_{ij}^{nl}	Bell et al.	Q_{ij}^{nl}	Bell et al.
12	116.20 ± 0.01	116.20 ± 0.16	-36.249 ± 0.001	-36.244 ± 0.009
13	38.13 ± 0.03	37.63 ± 0.03	-21.717 ± 0.007	-21.732 ± 0.005
14	63.63 ± 0.01	63.66 ± 0.06	-25.189 ± 0.003	-25.192 ± 0.006
15	107.17 ± 0.01	106.99 ± 0.12	-35.268 ± 0.001	-35.256 ± 0.009
23	97.11 ± 0.01	96.97 ± 0.10	-32.875 ± 0.002	-32.872 ± 0.008
24	67.36 ± 0.02	67.51 ± 0.08	-26.821 ± 0.003	-26.815 ± 0.007
25	30.87 ± 0.03	30.73 ± 0.04	-21.561 ± 0.009	-21.561 ± 0.005
34	69.43 ± 0.01	69.24 ± 0.07	-25.854 ± 0.002	-25.861 ± 0.006
35	106.13 ± 0.02	105.97 ± 0.13	-34.799 ± 0.002	-34.796 ± 0.008
45	74.45 ± 0.02	74.36 ± 0.09	-28.247 ± 0.004	-28.251 ± 0.007

Tripole sums

	$\tilde{c}_{\text{tripoles}}$	Bell et al.
$\tilde{c}_{\text{tripoles}}^{(2,124)}$	-683.25 ± 0.01	-683.23 ± 0.04
$\tilde{c}_{\text{tripoles}}^{(2,125)}$	-2203.3 ± 0.2	-2203.5 ± 0.1
$\tilde{c}_{\text{tripoles}}^{(2,145)}$	-6.324 ± 0.004	-6.325 ± 0.04
$\tilde{c}_{\text{tripoles}}^{(2,245)}$	-0.837 ± 0.008	-0.830 ± 0.039

The tripole sums correspond to the four independent color structures as specified in hep-ph/2312.11626

Power Corrections

Power corrections to color-singlet production

In the process with LO $f_a(p_a) + f_b(p_b) \to X(\tilde{P}_X)$, at NLO we have

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\tau} = \mathcal{N} \int [\mathrm{d}\tilde{P}_X]_m [\mathrm{d}k] \delta(p_a + p_b - k - \tilde{P}_X) \delta(\mathcal{T} - \mathcal{T}_0(p_a, p_b, k))\, \mathcal{O}(\tilde{P}_X) \sum_{\mathrm{col,pol}} |\mathcal{M}|^2 (p_a, p_b, k, \tilde{P}_X)$$

Power corrections require the expansion in $\mathcal{T}$ of two building blocks:

Phase space

Dependence on $\mathcal{T}$ can be expressed in a process independent manner

Matrix element

Process independent Next-to-soft corrections from **LBK** theorem
No analogous theorem for for collinear radiation

The soft and collinear contributions

Expansion in $\mathcal{T}$ controlled by gluon emission angle θ_k

- If θ_k is $\mathcal{O}(1)$, the gluon energy is $\mathcal{O}(\tau)$ and so the expansion is the **soft expansion**
- If θ_k is $\mathcal{O}(\tau/m_V)$, then the angle is small and we expand in θ_k (**collinear expansion**)

The two distinct integration regions – soft and collinear – are associated with two “branches” of the cross section with respect to τ:

$$\frac{d\Sigma}{\mathrm{d}\tau} \sim \tau^{-1-2\epsilon} f_s(\tau) + \tau^{-1-\epsilon} f_c(\tau)$$

The soft contribution

- For **Phase space**: use mapping that absorbs the gluon k into the colorless final state (hep-ph/1910.01024)

$$P_{ab}^{\mu} = \lambda^{-1}\Lambda^{\mu}_{\nu}(P_{ab}^{\nu} - k^{\nu}), \qquad \lambda = \sqrt{1 - \frac{2P_{ab}\cdot k}{P_{ab}^2}} \approx 1 - \frac{P_{ab}\cdot k}{P_{ab}^2} + \mathcal{O}(k^2).$$

- For **Matrix element**: we can use the **LBK** theorem to get the subleading terms

We arrive to the general expression:

$$\frac{\mathrm{d}\sigma^{(s)}}{\mathrm{d}\tau} = \mathcal{N}\int [\mathrm{d}\Phi_m(p_a, p_b, P_X)] \Bigg\{ \mathcal{O}(P_X) \Bigg[I_1 - \kappa_m I_2 - I_2 \sum_{i\in L_f} p_i^{\mu} \frac{\partial}{\partial p_i^{\mu}} \Bigg] |\mathcal{M}|^2(p_a, p_b, P_X) - I_2 |\mathcal{M}|^2(p_a, p_b, P_X) \sum_{i=1}^{m} p_i^{\mu} \frac{\partial}{\partial p_i^{\mu}} \mathcal{O}(P_X) \Bigg\}$$

The soft contribution

We arrive to the general expression in terms of the LO cross section:

$$\frac{\mathrm{d}\sigma^{(s)}}{\mathrm{d}\tau} = \mathcal{N}\int[\mathrm{d}\Phi_m(p_a,p_b,P_X)]\Bigg\{\mathcal{O}(P_X)\Bigg[I_1-\kappa_m I_2 - I_2\sum_{i\in L_f} p_i^\mu \frac{\partial}{\partial p_i^\mu}\Bigg]|\mathcal{M}|^2(p_a,p_b,P_X) - I_2|\mathcal{M}|^2(p_a,p_b,P_X)\sum_{i=1}^{m} p_i^\mu \frac{\partial}{\partial p_i^\mu}\mathcal{O}(P_X)\Bigg\}$$

Where the integrals I_i and the constant κ_m are

$$I_1 = [\alpha_s]\left(\frac{Q}{\sqrt{s}}\right)^{-2\epsilon}\frac{4}{\epsilon\,\tau^{1+2\epsilon}},\quad I_2 = [\alpha_s]\left(\frac{Q\tau}{\sqrt{s}}\right)^{-2\epsilon}\frac{4Q}{s}\left(\frac{1}{2\epsilon}-\frac{1}{2}-\frac{\epsilon}{2}+\mathcal{O}(\epsilon^2)\right),\quad \kappa_m = m(d-2)-d$$

The collinear contribution

- For **Phase space**: we use a mapping that absorbs the transverse momentum of k into the colorless final state through a Lorentz Transformation

$$k^\mu = (1-x)p_a^\mu + \tilde{k}^\mu \quad \Rightarrow \quad P_X^\mu = \Lambda^\mu_\nu \left(\tilde{P}_X^\nu + \tilde{k}^\nu\right)$$

- For **Matrix element**: we have **no analogous theorem (yet) available** to get the subleading terms

  Use the real emission matrix element, but some simplifications can be applied

This is still ongoing work...

Conclusions

Conclusions

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In our work

- Calculated the *N*-jettiness soft function, demonstrating **analytical cancellation** of poles
- Derived a **simple representation** for finite jettiness-dependent remainder, allowing for *faster implementations*. In agreement with other calculations
- Showcased the benefits of using subtraction-inspired methods to derive building blocks of slicing methods

- We aim to establish a **process-independent framework** for power corrections in a generic color-singlet case
- Next-to-soft corrections can be obtained from LBK theorem, however, next-to-collinear term is not known in a compact form

Thank you!

Correlated emission

Introducing sectors, we arrive to

$$\bar{S}_\omega[I_{ij}^{tc}] = \frac{N_u}{\epsilon}\int_0^1 \frac{d\omega}{\omega^{1+2\epsilon}} \left\langle C_{mn} \left[d\Omega_{mn}\right] \theta^{b+d} w^{tc} \psi_{mn}^{4\epsilon} \bar{S}_\omega \left[\tilde{S}_{ij}^{gg}\right]\right\rangle_{mn}$$

$$+ \frac{N_u}{\epsilon} \sum_{x\in\{i,j\}} \int_0^1 \frac{d\omega}{\omega^{1+2\epsilon}} \left\langle \left(1 - \theta^{b+d} C_{mn}\right) \left[d\Omega_{mn}\right] C_{xmn} w^{tc} \psi_{mn}^{4\epsilon} \bar{S}_\omega \left[\tilde{S}_{ij}^{gg}\right]\right\rangle_{mn}$$

$$+ \frac{N_u}{\epsilon} \sum_{x\in\{i,j\}} \int_0^1 \frac{\mathrm{d}\omega}{\omega^{1+2\epsilon}} \left\langle \left(1 - \theta^{b+d} C_{mn}\right) \left[\mathrm{d}\Omega_{mn}\right] \bar{C}_{xmn}\, w^{mx,nx} \psi_{mn}^{4\epsilon} \bar{S}_\omega \left[\tilde{S}_{ij}^{gg}\right]\right\rangle_{mn},$$

where $\bar{C}_{xmn} = I - C_{xmn}$ and $[\mathrm{d}\Omega_{mn}] = [\mathrm{d}\Omega_m][\mathrm{d}\Omega_n]$. The idea is to calculate the first two terms explicitly, and expand the last integrand in ϵ.