Karlsruhe Institute of Technology

Institute for Theoretical Particle Physics

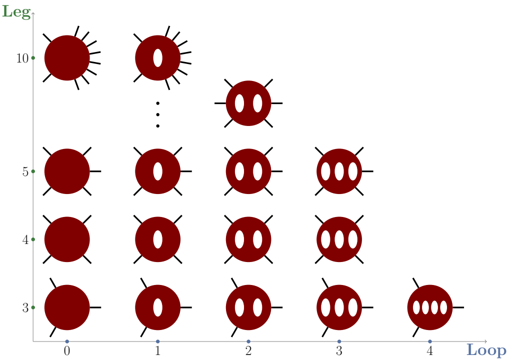
Three-loop ladder diagrams with two off-shell legs

High Precision for Hard Processes 2024, Turin, 10 - 13 September 2024

Ming-Ming Long | to appear soon 2409.xxxx

Outline

State-of-the-art



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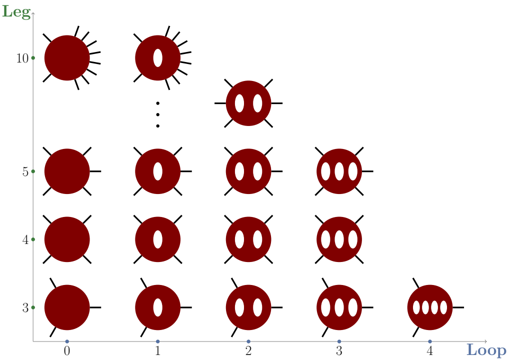

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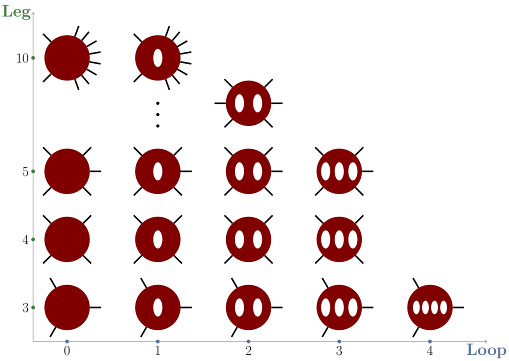

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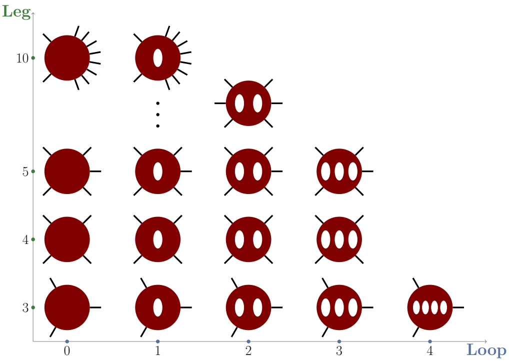

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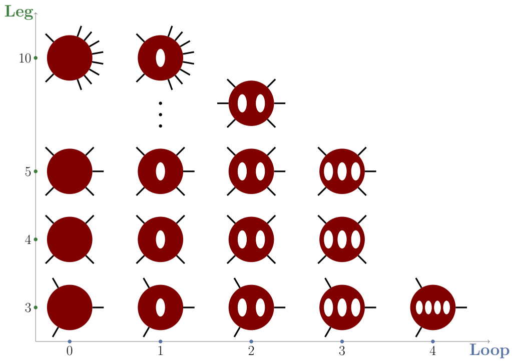

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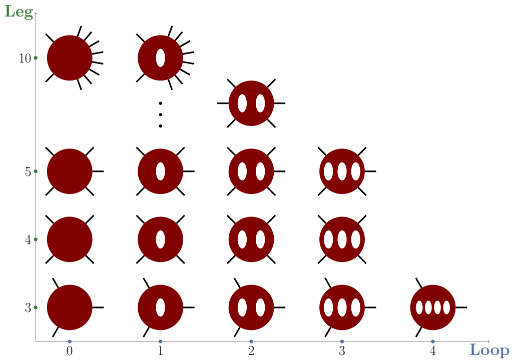

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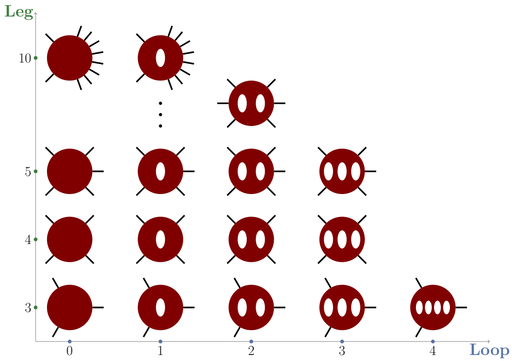

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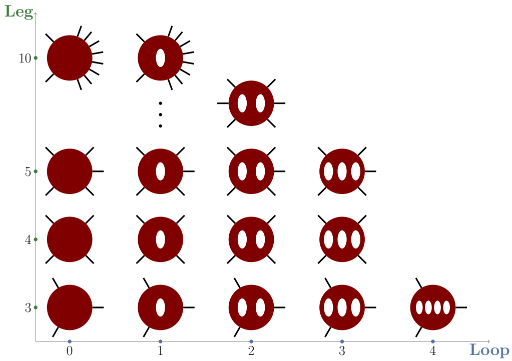

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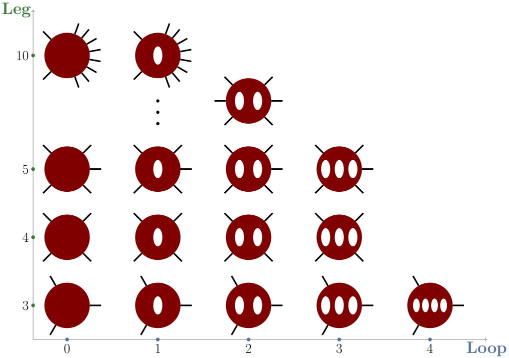

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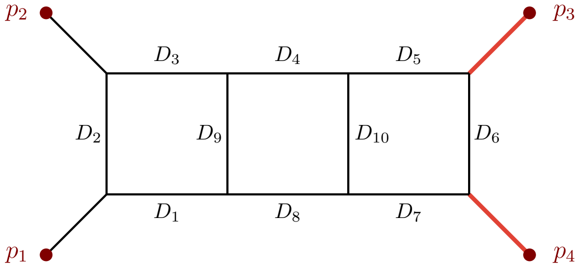
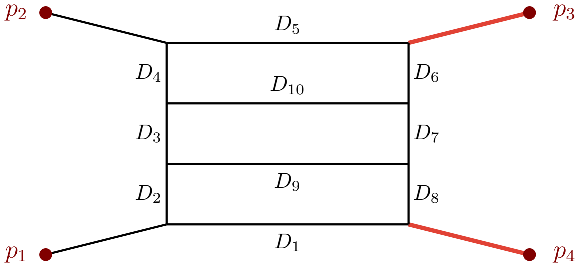
- Two-loop six-point & three-loop five-point [Yang Zhang's talk at GGI]
- Expansion in limits (large mass, high energy, forward, etc.) and (semi-)numeric approach [many talks at this workshop]

Ladder diagrams





- **Family Ladder A,** T_1

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\begin{aligned}
D_1 &= l_1^2\,, & D_2 &= (l_1+p_1)^2\,, & D_3 &= (l_1+p_{12})^2\,,\\
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D_7 &= l_3^2\,, & D_8 &= l_2^2\,, & D_9 &= (l_1-l_2)^2\,,\\
D_{10} &= (l_2-l_3)^2\,, & D_{11} &= (l_1-p_4)^2\,, & D_{12} &= (l_2-p_4)^2\,,\\
D_{13} &= (l_2+p_1)^2\,, & D_{14} &= (l_3+p_1)^2\,, & D_{15} &= (l_1-l_3)^2
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- **Family Ladder B,** T_2

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Kinematics

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- $\sum p_i = 0, \quad p_{1,2}^2 = 0, \quad p_{3,4}^2 = m^2$
- $s = (p_1 + p_2)^2, \quad t = (p_1 + p_4)^2, \quad u = (p_2 + p_4)^2$
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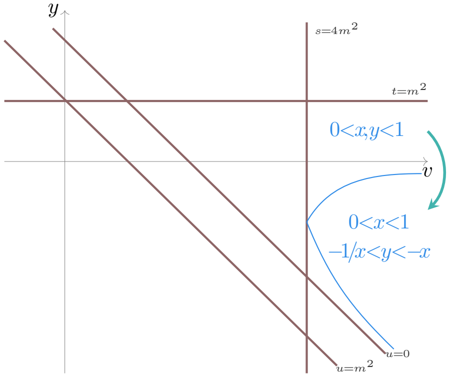

Master integrals

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- Definition of integrals

$$F_{\vec{a}} = \int \mathcal{D}^d l_1 \mathcal{D}^d l_2 \mathcal{D}^d l_3 \frac{\prod_{j=11}^{15} D_j^{-a_j}}{\prod_{j=1}^{10} D_j^{a_j}}, \quad \begin{cases} a_j \in \mathbb{Z} & j \leq 10 \\ a_j \in \mathbb{Z}^{\leq 0} & j > 10 \end{cases},$$

$$\mathcal{D}^d l_i = C_\epsilon \frac{(-m^2)^\epsilon}{i\pi^{d/2}} d^d l_i, \quad C_\epsilon = \frac{\Gamma(1-2\epsilon)}{\Gamma(1-\epsilon)^2\Gamma(1+\epsilon)}$$

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Topo	# of MI	# of Sector	Max # of MI
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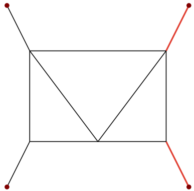

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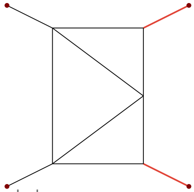

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$$\partial_x \mathbf{G} = (\mathrm{A}_0 + \epsilon \mathrm{A}_1)\mathbf{G}$$

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  3. use Magnus series expansion
     - obtaining canonical basis $\mathbf{F} = \mathrm{T}\mathbf{G}$ with

$$\partial_x \mathrm{T}^{-1} = \mathrm{A}_0 \mathrm{T}^{-1}$$

Canonical differential equation

$$\partial_x \mathbf{F} = \epsilon \mathrm{A}_x \mathbf{F}, \quad \partial_y \mathbf{F} = \epsilon \mathrm{A}_y \mathbf{F}$$

which can be recast into the *d*log form

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- Letters in family T_1

$$\omega_1 = x\,, \quad \omega_4 = 1 + x\,, \quad \omega_7 = 1 + xy\,,$$
$$\omega_2 = y\,, \quad \omega_5 = 1 - y\,, \quad \omega_8 = 1 + xy + x^2\,,$$
$$\omega_3 = 1 - x\,, \quad \omega_6 = x + y\,, \quad \omega_9 = 1 + x + xy + x^2$$

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- Letters in family T_2

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$$\omega_2 = y\,, \quad \omega_5 = 1-y\,, \quad \omega_8 = 1+x+xy+x^2\,,$$
$$\omega_3 = 1-x\,, \quad \omega_6 = x+y\,, \quad \omega_9 = x+y(1+x+x^2)$$

Solution and boundary

$$\mathbf{F}(x,y) = \sum_{n=0}^{\infty} \mathbf{F}^{(n)}(x,y)\, \epsilon^n$$

$\mathbf{F}^{(n)}(x,y)$ are obtained recursively,

$$\mathbf{F}^{(n)}(x,y) = \mathbf{F}_y^{(n)}(x,y) + \mathbf{F}_x^{(n)}(x) + \mathbf{c}^{(n)}$$

they evaluate to MPLs,

$$G(w_n, \ldots, w_1; z) = \int_0^z \frac{1}{t - w_n} G(w_{n-1}, \ldots, w_1; t)\, \mathrm{d}t,$$

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y	$\left\{0,1,-x,-\frac{1}{x},-\frac{x^2+1}{x},-\frac{x}{x^2+x+1}\right\}$
x	$\left\{-1,0,1,-i,i,e^{-2i\pi/3},e^{2i\pi/3}\right\}$

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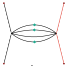
$\mathbf{c}^{(n)}$ are fixed by the regularities in the limits,

$$t \to m^2, \quad s \to -\frac{(t-m^2)^2}{t}, \quad s \to 2m^2 - t, \quad s \to m^2 - t$$

together with simple input MIs (`HyperInt`), e.g.,

 $= v^{-3\epsilon}\left[-1 + 22\zeta_3\epsilon^3 + 33\zeta_4\epsilon^4 + 234\zeta_5\epsilon^5 + O\left(\epsilon^6\right)\right]$

Continuation to physical region

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In physical region,

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$$y \to y + i0^+$$

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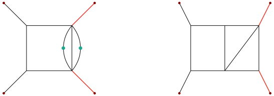
- each single MPL is real-valued
- explicit imaginary part
- fewer MPLs

Numerical checks

- a point in Euclidean region

$$(s, t, m^2) = (-5, -0.5, -1)$$

- check against `pySecDec`



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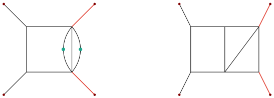

- MPL distribution, (y, x)

weight	1	2	3	4	5	6
# of MPLs	14	62	305	975	836	1959
Time	-	-	-	-	2%	98%

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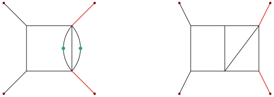

- MPL distribution, (y, x)

weight	1	2	3	4	5	6
# of MPLs	14	62	305	975	836	1959
Time	-	-	-	-	2%	98%

- a point in physical region

$$(s, t, m^2) = (9, -2, 1)$$

- check using `DiffExp`
  - numeric integration of DE from Euclidean to physical region
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Numerical checks

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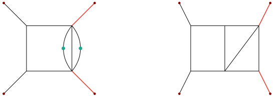

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weight	1	2	3	4	5	6
# of MPLs	9	39	181	626	1513	1647

Symbols

- Fast and stable numeric evaluations
  1. faster MPL
  2. parallelization
  3. **simplified expressions** (less MPLs, no spurious singularities, convergent power series)
     - re-express the MIs in terms of another set of MPLs
     - use non-trivial relations among MPLs

Symbols

- shuffle/stuffle relations

$$G(a;z)G(b;z) = G(a,b;z) + G(b,a;z),$$
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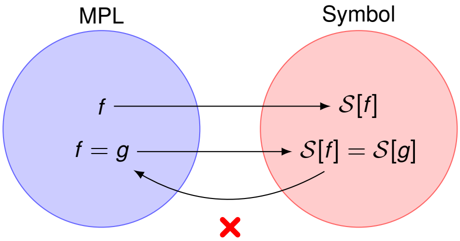

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- symbol does not see π and (multiple) ζ value
  - restored via non-maximal coproduct operators
  - pure constants (*primitive elements*) are reconstructed, PSLQ algorithm

Algorithms [1110.0458]

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e.g., $\phi = \frac{x(x+y)(1+xy)}{(1+x+xy+x^2)^2}, (\phi_1, \phi_2) = \left(\frac{1}{1-x}, -\frac{1+x}{x(x+y)}\right)$

Algorithms [1110.0458]

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  - simplification

$$\text{\# of weight 4 MPL}(y, x): \quad \sim 1000 \to \sim 150$$

Remarks

- further constraints on the arguments
- projectors eliminate terms
  - e.g., modulo products
  - smaller linear system to solve
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  - complication

$$F_{12}^{(6)} = 8\,\mathrm{Li}_6\left(\frac{1}{1+x+z}\right) + \ldots,$$

$$F_{94}^{(6)} = -\frac{595599\ldots}{265168\ldots}\,\mathrm{Li}_6\left(\frac{1}{1+x+z}\right) + \ldots$$

$$\hookrightarrow \text{(80 digits)}/\text{(69 digits)}$$

Summary

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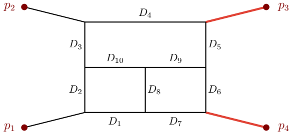
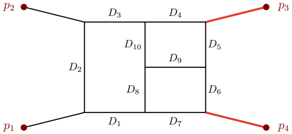
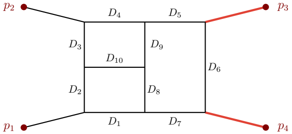


196



189



109

Thank you for your attention!