



Three-loop ladder diagrams with two off-shell legs

High Precision for Hard Processes 2024, Turin, 10 - 13 September 2024

Ming-Ming Long | to appear soon 2409.xxxx



www.kit.edu



Outline

1. Introduction

State-of-the-art

2. Integral family

- Topology
- Kinematics

3. Master integrals

- Canonical differential equation
- Solution and checks

4. Symbols

5. Summary

Introduction o	Integral family	Master integrals	Symbols 00000	Summary 00

State-of-the-art







• 2 \rightarrow 2 under good control at NNLO QCD



.





- 2 \rightarrow 2 under good control at NNLO QCD
- Progress towards massive 2 \rightarrow 3 at two-loop

Introduction	Integral family	Master integrals	Symbols	Summary
•	000	000000	00000	00





- 2 \rightarrow 2 under good control at NNLO QCD
- Progress towards massive 2 \rightarrow 3 at two-loop

Introduction	Integral family	Master integrals	Symbols	Summary
•	000	000000	00000	00





- 2 \rightarrow 2 under good control at NNLO QCD
- Progress towards massive 2 \rightarrow 3 at two-loop
- 2 \rightarrow 2 at N3LO QCD
 - massless $(\gamma \gamma, jj, \gamma j)$

Introduction	Integral family	Master integrals	Symbols	Summary
•	000	000000	00000	00





- 2 \rightarrow 2 under good control at NNLO QCD
- Progress towards massive 2 \rightarrow 3 at two-loop
- 2 \rightarrow 2 at N3LO QCD
 - massless (γγ, jj, γj)
 - one massive leg (Vγ, Vj, Hj) [talks of Jungwon, Petr]

Integral family Master integrals Symbols Summary oco oco	Introduction
--	--------------





- 2 \rightarrow 2 under good control at NNLO QCD
- Progress towards massive 2 \rightarrow 3 at two-loop
- 2 \rightarrow 2 at N3LO QCD
 - massless (γγ, jj, γj)
 - one massive leg (Vγ, Vj, Hj) [talks of Jungwon, Petr]
 - This work \rightarrow one additional massive leg (VV)

Introduction Integral family Master integrals Symbols Sum occo occo occo occo occo occo occo oc	mmary
---	-------





- 2 \rightarrow 2 under good control at NNLO QCD
- Progress towards massive 2 \rightarrow 3 at two-loop
- 2 \rightarrow 2 at N3LO QCD
 - massless (γγ, jj, γj)
 - one massive leg (Vγ, Vj, Hj) [talks of Jungwon, Petr]
 - This work \rightarrow one additional massive leg (VV)
- Two-loop six-point & three-loop five-point [Yang Zhang's talk at GGI]

Introduction	Integral family	Master integrals	Symbols	Summary
•	000	000000	00000	00





- 2 \rightarrow 2 under good control at NNLO QCD
- Progress towards massive 2 \rightarrow 3 at two-loop
- 2 \rightarrow 2 at N3LO QCD
 - massless (γγ, jj, γj)
 - one massive leg (Vγ, Vj, Hj) [talks of Jungwon, Petr]
 - This work \rightarrow one additional massive leg (VV)
- Two-loop six-point & three-loop five-point [Yang Zhang's talk at GGI]
- Expansion in limits (large mass, high energy, forward, etc.) and (semi-)numeric approach [many talks at this workshop]

Introduction	Integral family	Master integrals	Symbols	Summary
•	000	000000	00000	00



Ladder diagrams





• Family Ladder A, T₁

$$\begin{array}{ll} D_1 = l_1^2 \,, & D_2 = (l_1 + p_1)^2 \,, & D_3 = (l_1 + p_{12})^2 \,, \\ D_4 = (l_2 + p_{12})^2 \,, & D_5 = (l_3 + p_{12})^2 \,, & D_6 = (l_3 - p_4)^2 \,, \\ D_7 = l_3^2 \,, & D_8 = l_2^2 \,, & D_9 = (l_1 - l_2)^2 \,, \\ D_{10} = (l_2 - l_3)^2 \,, & D_{11} = (l_1 - p_4)^2 \,, & D_{12} = (l_2 - p_4)^2 \,, \\ D_{13} = (l_2 + p_1)^2 \,, & D_{14} = (l_3 + p_1)^2 \,, & D_{15} = (l_1 - l_3)^2 \end{array}$$

• Family Ladder B, T₂

$$\begin{array}{ll} D_1 = l_1^2 \,, & D_2 = (l_1 + p_1)^2 \,, & D_3 = (l_2 + p_1)^2 \,, \\ D_4 = (l_3 + p_1)^2 \,, & D_5 = (l_3 + p_{12})^2 \,, & D_6 = (l_3 - p_4)^2 \,, \\ D_7 = (l_2 - p_4)^2 \,, & D_8 = (l_1 - p_4)^2 \,, & D_9 = (l_1 - l_2)^2 \,, \\ D_{10} = (l_2 - l_3)^2 \,, & D_{11} = (l_1 + p_{12})^2 \,, & D_{12} = (l_2 + p_{12})^2 \,, \\ D_{13} = l_3^2 \,, & D_{14} = l_2^2 \,, & D_{15} = (l_1 - l_3)^2 \end{array}$$



•
$$\sum p_i = 0$$
, $p_{1,2}^2 = 0$, $p_{3,4}^2 = m^2$
• $s = (p_1 + p_2)^2$, $t = (p_1 + p_4)^2$, $u = (p_2 + p_4)^2$
• $t, u = m^2 - \frac{s}{2} \pm \frac{s}{2} \sqrt{1 - \frac{4m^2}{s}} \cos \theta$





•
$$\sum p_i = 0$$
, $p_{1,2}^2 = 0$, $p_{3,4}^2 = m^2$
• $s = (p_1 + p_2)^2$, $t = (p_1 + p_4)^2$, $u = (p_2 + p_4)^2$
• $t, u = m^2 - \frac{s}{2} \pm \frac{s}{2} \sqrt{1 - \frac{4m^2}{s}} \cos \theta$
• Euclidean region $s < 0$, $t < 0$, $m^2 < 0$
• Physical region $s > 4m^2 > 0$, $t < 0$

Introd o	uction	Integral family ○○●	Master integrals	Symbols 00000	Summary 00
6/19	11.09.2024	Ming-Ming Long: HP ² 2024		Three-loop ladder diagram	ns with two off-shell legs



•
$$\sum p_i = 0$$
, $p_{1,2}^2 = 0$, $p_{3,4}^2 = m^2$
• $s = (p_1 + p_2)^2$, $t = (p_1 + p_4)^2$, $u = (p_2 + p_4)^2$
• $t, u = m^2 - \frac{s}{2} \pm \frac{s}{2} \sqrt{1 - \frac{4m^2}{s}} \cos \theta$
• Euclidean region $s < 0$, $t < 0$, $m^2 < 0$
• Physical region $s > 4m^2 > 0$, $t < 0$
• $v = \frac{s}{m^2} = \frac{(1 + x)^2}{x}$, $y = \frac{t}{m^2}$

Introduction o	Integral family ○○●	Master integrals	Symbols 00000	Summary



$$\sum_{i=1}^{n} p_{i} = 0, \quad p_{1,2}^{2} = 0, \quad p_{3,4}^{2} = m^{2}$$

$$s = (p_{1} + p_{2})^{2}, \quad t = (p_{1} + p_{4})^{2}, \quad u = (p_{2} + p_{4})^{2}$$

$$t, u = m^{2} - \frac{s}{2} \pm \frac{s}{2} \sqrt{1 - \frac{4m^{2}}{s}} \cos \theta$$

$$Euclidean region s < 0, \quad t < 0, \quad m^{2} < 0$$

$$Physical region s > 4m^{2} > 0, \quad t < 0$$

$$v = \frac{s}{m^{2}} = \frac{(1 + x)^{2}}{x}, \quad y = \frac{t}{m^{2}}$$





Master integrals

Definition of integrals

$$F_{\vec{a}} = \int \mathcal{D}^{d} l_{1} \mathcal{D}^{d} l_{2} \mathcal{D}^{d} l_{3} \frac{\prod_{j=11}^{15} D_{j}^{-a_{j}}}{\prod_{j=1}^{10} D_{j}^{a_{j}}}, \quad \begin{cases} a_{j} \in \mathbb{Z} & j \leq 10 \\ a_{j} \in \mathbb{Z}^{\leq 0} & j > 10 \end{cases},$$

$$\mathcal{D}^{d}I_{i} = C_{\epsilon} \frac{(-m^{2})^{\epsilon}}{i\pi^{d/2}} d^{d}I_{i}, \quad C_{\epsilon} = \frac{\Gamma(1-2\epsilon)}{\Gamma(1-\epsilon)^{2}\Gamma(1+\epsilon)}$$

Master integrals

Торо	# of MI	# of Sector	Max # of MI
<i>T</i> ₁	94	50	6^{\dagger}
<i>T</i> ₂	84	50	10 ^{††}

Introduction	Integral family	Master integrals	Symbols	Summary
0	000	00000	00000	00

Master integrals



• Definition of integrals $F_{\vec{a}} = \int \mathcal{D}^{d} l_{1} \mathcal{D}^{d} l_{2} \mathcal{D}^{d} l_{3} \frac{\prod_{j=11}^{15} \mathcal{D}_{j}^{-a_{j}}}{\prod_{j=1}^{10} \mathcal{D}_{j}^{a_{j}}}, \quad \begin{cases} a_{j} \in \mathbb{Z} \quad j \leq 10 \\ a_{j} \in \mathbb{Z}^{\leq 0} \quad j > 10 \end{cases},$ $\mathcal{D}^{d} l_{i} = C_{\epsilon} \frac{(-m^{2})^{\epsilon}}{j\pi^{d/2}} d^{d} l_{i}, \quad C_{\epsilon} = \frac{\Gamma(1-2\epsilon)}{\Gamma(1-\epsilon)^{2}\Gamma(1+\epsilon)}$ • $\dagger \dagger$

•00000

Master integrals

Introduction

Торо	# of MI	# of Sector	Max # of MI
T_1	94	50	6†
<i>T</i> ₂	84	50	10 ^{††}
	Integr	al family	Master inte



7/19 11.09.2024 Ming-Ming Long: HP² 2024

Canonical differential equation

Canonical basis

Introduction o	Integral family	Master integrals	Symbols 00000	Summary 00

8/19 11.09.2024 Ming-Ming Long: HP² 2024

Canonical differential equation

Canonical basis

Choose a starting basis based on educated-guess and leading-singularity analysis

Introduction	Integral family	Master integrals	Symbols	Summary
o		○●○○○○	00000	00



Canonical differential equation

Canonical basis

Choose a starting basis based on educated-guess and leading-singularity analysis

Iower sectors: learn from 2-loop calculations

Introduction	Integral family	Master integrals	Symbols	Summary
o		○●○○○○	00000	00

Canonical differential equation

Canonical basis

Choose a starting basis based on educated-guess and leading-singularity analysis

- Iower sectors: learn from 2-loop calculations
- higher sectors: leading-singularity analysis in 4-dimensions

Introduction o	Integral family	Master integrals ●●○○○○	Symbols	Summary 00
-				



Canonical differential equation

Canonical basis

Choose a starting basis based on educated-guess and leading-singularity analysis

- Iower sectors: learn from 2-loop calculations
- higher sectors: leading-singularity analysis in 4-dimensions
- hard when a sector has too many MIs

Introduction o	Integral family	Master integrals	Symbols 00000	Summary 00



Canonical differential equation

Canonical basis

Choose a starting basis based on educated-guess and leading-singularity analysis

- Iower sectors: learn from 2-loop calculations
- higher sectors: leading-singularity analysis in 4-dimensions
- hard when a sector has too many MIs

2 apply a transformation depending on ϵ only

Introduction o	Integral family	Master integrals ○●○○○○	Symbols ooooo	Summary

Canonical differential equation

Canonical basis

Choose a starting basis based on educated-guess and leading-singularity analysis

- Iower sectors: learn from 2-loop calculations
- higher sectors: leading-singularity analysis in 4-dimensions
- hard when a sector has too many MIs
- 2 apply a transformation depending on ϵ only
 - \hfill obtaining a differential system which is linear in ϵ

 $\partial_{\mathbf{x}}\mathbf{G} = (\mathbf{A}_0 + \epsilon \mathbf{A}_1)\mathbf{G}$

Introduction o	Integral family	Master integrals	Symbols 00000	Summary 00

Canonical differential equation

Canonical basis

Choose a starting basis based on educated-guess and leading-singularity analysis

- Iower sectors: learn from 2-loop calculations
- higher sectors: leading-singularity analysis in 4-dimensions
- hard when a sector has too many MIs
- 2 apply a transformation depending on ϵ only
 - \hfill obtaining a differential system which is linear in ϵ

 $\partial_{\mathbf{x}}\mathbf{G} = (\mathbf{A}_0 + \epsilon \mathbf{A}_1)\mathbf{G}$

normalization adjusts weight structure

Introduction	Integral family	Master integrals	Symbols	Summary
o		○●○○○○	00000	00

Canonical differential equation

Canonical basis

Choose a starting basis based on educated-guess and leading-singularity analysis

- Iower sectors: learn from 2-loop calculations
- higher sectors: leading-singularity analysis in 4-dimensions
- hard when a sector has too many MIs
- 2 apply a transformation depending on ϵ only
 - \hfill obtaining a differential system which is linear in ϵ

$$\partial_{\mathbf{x}}\mathbf{G} = (\mathbf{A}_0 + \epsilon \mathbf{A}_1)\mathbf{G}$$

- normalization adjusts weight structure
- use Magnus series expansion
 - obtaining canonical basis F = TG with

$$\partial_x T^{-1} = A_0 T^{-1}$$

Introduction	Integral family	Master integrals	Symbols	Summary
0	000	00000	00000	00

8/19 11.09.2024 Ming-Ming Long: HP² 2024

Canonical differential equation

$$\partial_x \mathbf{F} = \epsilon \mathbf{A}_x \mathbf{F}, \quad \partial_y \mathbf{F} = \epsilon \mathbf{A}_y \mathbf{F}$$

which can be recast into the *d*log form

$$\mathrm{d}\mathbf{F} = \epsilon \mathrm{d}\mathbb{A}\mathbf{F}, \qquad ext{with } \mathbb{A} = \sum_{i=1}^{9} \mathbb{C}_i \log(\omega_i)$$

Introduction	Integral family	Master integrals	Symbols	Summary
o		○○●○○○	00000	00

Canonical differential equation

$$\partial_x \mathbf{F} = \epsilon \mathbf{A}_x \mathbf{F}, \quad \partial_y \mathbf{F} = \epsilon \mathbf{A}_y \mathbf{F}$$

which can be recast into the *d*log form

$$\mathrm{d}\mathbf{F} = \epsilon \mathrm{d}\mathbb{A}\mathbf{F}, \qquad ext{with } \mathbb{A} = \sum_{i=1}^{9} \mathbb{C}_i \log(\omega_i)$$

Letters in family T₁

$$\begin{split} & \omega_1 = x \,, \qquad \omega_4 = 1 + x \,, \quad \omega_7 = 1 + xy \,, \\ & \omega_2 = y \,, \qquad \omega_5 = 1 - y \,, \quad \omega_8 = 1 + xy + x^2 \,, \\ & \omega_3 = 1 - x \,, \quad \omega_6 = x + y \,, \quad \omega_9 = 1 + x + xy + x^2 \end{split}$$

Introduction	Integral family	Master integrals	Symbols	Summary
o		○○●○○○	00000	00

Canonical differential equation



$$\partial_x \mathbf{F} = \epsilon \mathbf{A}_x \mathbf{F}, \quad \partial_y \mathbf{F} = \epsilon \mathbf{A}_y \mathbf{F}$$

which can be recast into the *d*log form

$$d\mathbf{F} = \epsilon d\mathbb{A}\mathbf{F}, \qquad \text{with } \mathbb{A} = \sum_{i=1}^{9} \mathbb{C}_i \log(\omega_i)$$

Letters in family T₁

Letters in family T2

$$\begin{array}{ll} \omega_1 = x \,, & \omega_4 = 1 + x \,, & \omega_7 = 1 + xy \,, & \omega_1 = x \,, & \omega_4 = 1 + x \,, & \omega_7 = 1 + xy \,, \\ \omega_2 = y \,, & \omega_5 = 1 - y \,, & \omega_8 = 1 + xy + x^2 \,, & \omega_2 = y \,, & \omega_5 = 1 - y \,, & \omega_8 = 1 + x + xy + x^2 \,, \\ \omega_3 = 1 - x \,, & \omega_6 = x + y \,, & \omega_9 = 1 + x + xy + x^2 \,, & \omega_3 = 1 - x \,, & \omega_6 = x + y \,, & \omega_9 = x + y(1 + x + x^2) \end{array}$$

Introduction	Integral family	Master integrals	Symbols	Summary
o		○○●○○○	00000	00

9/19 11.09.2024 Ming-Ming Long: HP² 2024



Solution and boundary

$$\mathbf{F}(x,y) = \sum_{n=0}^{\infty} \mathbf{F}^{(n)}(x,y) \ \epsilon^n$$

 $\mathbf{F}^{(n)}(x, y)$ are obtained recursively,

$$\mathbf{F}^{(n)}(x,y) = \mathbf{F}^{(n)}_{y}(x,y) + \mathbf{F}^{(n)}_{x}(x) + \mathbf{c}^{(n)}$$

they evaluate to MPLs,

$$G(w_n, ..., w_1; z) = \int_0^z \frac{1}{t - w_n} G(w_{n-1}, ..., w_1; t) dt,$$

$$G(\underbrace{0, ..., 0}_{n \text{ times}}; z) = \frac{\log^n(z)}{n!}$$
Introduction
Integral family
Master integrals
Symbols
OOOO

10/19 11.09.2024 Ming-Ming Long: HP² 2024

Three-loop ladder diagrams with two off-shell legs

Summary



Solution and boundary

$$\mathbf{F}(x,y) = \sum_{n=0}^{\infty} \mathbf{F}^{(n)}(x,y) \,\epsilon^n$$

 $\mathbf{F}^{(n)}(x, y)$ are obtained recursively,

$$\mathbf{F}^{(n)}(x,y) = \mathbf{F}^{(n)}_{y}(x,y) + \mathbf{F}^{(n)}_{x}(x) + \mathbf{c}^{(n)}$$

they evaluate to MPLs.



Argument

y

x

Indices

 $\left\{ 0, 1, -x, -\frac{1}{x}, -\frac{x^2+1}{x}, -\frac{x}{x^2+x+1} \right\}$ $\left\{ -1, 0, 1, -i, i, e^{-2i\pi/3}, e^{2i\pi/3} \right\}$



Solution and boundary

$$\mathbf{F}(x,y) = \sum_{n=0}^{\infty} \mathbf{F}^{(n)}(x,y) \, \epsilon^n$$

 $\mathbf{F}^{(n)}(x, y)$ are obtained recursively,

$$\mathbf{F}^{(n)}(x,y) = \mathbf{F}^{(n)}_{y}(x,y) + \mathbf{F}^{(n)}_{x}(x) + \mathbf{c}^{(n)}$$

they evaluate to MPLs,

$$G(w_n,...,w_1;z) = \int_0^z \frac{1}{t-w_n} G(w_{n-1},...,w_1;t) \,\mathrm{d}t,$$

 Argument
 Indices

 y
 $\left\{0, 1, -x, -\frac{1}{x}, -\frac{x^2+1}{x}, -\frac{x}{x^2+x+1}\right\}$

 x
 $\left\{-1, 0, 1, -i, i, e^{-2i\pi/3}, e^{2i\pi/3}\right\}$

 $\mathbf{c}^{(n)}$ are fixed by the regularities in the limits,

$$t \rightarrow m^2, \quad s \rightarrow -\frac{(t-m^2)^2}{t}, \quad s \rightarrow 2m^2 - t, \quad s \rightarrow m^2 - t$$

together with simple input MIs (HyperInt), e.g.,



10/19 11.09.2024 Ming-Ming Long: HP² 2024

Continuation to physical region

In physical region,

$$0 < x < 1, -1/x < y < -x$$

y should carry a positive infinitesimal imaginary

 $y \rightarrow y + i0^+$

	Introduction o	Integral family 000	Master integrals	Symbols 00000	Summary 00
--	-------------------	------------------------	------------------	------------------	---------------

Continuation to physical region

In physical region,

0 < x < 1, -1/x < y < -x

y should carry a positive infinitesimal imaginary

 $y \rightarrow y + i0^+$

- practically let 0⁺ be a small number
 - precision loss

implicit imaginary part

1/10 11 00 0004			Three-loon ladder diagrams with two off-shall logs	
Introduction	Integral family	Master integrals ○○○○●○	Symbols	Summary
Continuation to physical region

In physical region,

0 < x < 1, -1/x < y < -x

y should carry a positive infinitesimal imaginary

 $y \rightarrow y + i0^+$

- practically let 0⁺ be a small number
 - precision loss
 - implicit imaginary part
- spurious indices

• {
$$\pm i, e^{\pm 2i\pi/3}$$
}
• G[$-i, x$] + G[i, x] = log(1 + x^2)

Introduction	Integral family	Master integrals	Symbols	Summary
0	000	000000	00000	00

Continuation to physical region

In physical region,

0 < x < 1, -1/x < y < -x

y should carry a positive infinitesimal imaginary

 $y \rightarrow y + i0^+$

- practically let 0⁺ be a small number
 - precision loss
 - implicit imaginary part
- spurious indices

• {
$$\pm i, e^{\pm 2i\pi/3}$$
}
• G[$-i, x$] + G[i, x] = log(1 + x^2)

Perform a change of variables,

$$y = -z - x$$

it follows that

 $z \rightarrow z - i0^+$



Continuation to physical region

In physical region,

0 < x < 1, -1/x < y < -x

y should carry a positive infinitesimal imaginary

 $y \rightarrow y + i0^+$

- practically let 0⁺ be a small number
 - precision loss
 - implicit imaginary part
- spurious indices

• {
$$\pm i, e^{\pm 2i\pi/3}$$
}
• G[$-i, x$] + G[i, x] = log(1 + x^2)

Perform a change of variables,

$$y = -z - x$$

it follows that

$$z \rightarrow z - i0^+$$

Argument	Indices
Z	$\{0, -x, -1 - x, \frac{1}{x}, \frac{1+x}{x}, \frac{1-x^2}{x}\}$
X	$\{-1, 0, 1\}$

Introduction	Integral family	Master integrals	Symbols	Summary
0	000	000000	00000	00



Continuation to physical region

In physical region,

0 < x < 1, -1/x < y < -x

y should carry a positive infinitesimal imaginary

 $y \rightarrow y + i0^+$

- practically let 0⁺ be a small number
 - precision loss
 - implicit imaginary part
- spurious indices

Introduction

• {
$$\pm i, e^{\pm 2i\pi/3}$$
}
• G[$-i, x$] + G[i, x] = log(1 + x^2)

Integral family

Perform a change of variables,

$$y = -z - x$$

it follows that

Master inte

000000

$$z \rightarrow z - i0^+$$

Argument	Indices
z x	$\{0, -x, -1 - x, \frac{1}{x}, \frac{1 + x}{x}, \frac{1 - x^2}{x}\}\$
each sin	gle MPL is real-valued
explicit in	naginary part
fewer MF	PLs
egrals	Symbols Summary





Numerical checks

a point in Euclidean region

$$(s, t, m^2) = (-5, -0.5, -1)$$

check against pySecDec





Numerical checks

a point in Euclidean region

$$(s, t, m^2) = (-5, -0.5, -1)$$

check against pySecDec



MPL distribution, (y, x)

weight	1	2	3	4	5	6
# of MPLs	14	62	305	975	836	1959
Time	-	-	-	-	2%	98%

Introduction	Integral family	Master integrals	Symbols	Summary
0	000	00000	00000	00

Numerical checks

a point in Euclidean region

$$(s, t, m^2) = (-5, -0.5, -1)$$

check against pySecDec



1 2 3

of MPLs | 14 62 305 975 836

4

-

5

2%

MPL distribution, (y, x)

weight

Time

a point in physical region

$$(s, t, m^2) = (9, -2, 1)$$

- check using DiffExp
 - numeric integration of DE from Euclidean to physical region
 - perfect agreement

Introduction	Integral family	Master integrals	Symbols	Summary
0	000	00000	00000	00

6

1959

98%



Numerical checks

a point in Euclidean region

$$(s, t, m^2) = (-5, -0.5, -1)$$

check against pySecDec



• MPL distribution, (y, x)

Introduction

0

weight	1	2	3	4	5	6
# of MPLs	14	62	305	975	836	1959
Time	-	-	-	-	2%	98%

Integral family



a point in physical region

$$(s, t, m^2) = (9, -2, 1)$$

- check using DiffExp
 - numeric integration of DE from Euclidean to physical region
 - perfect agreement
- MPL distribution, (z, x)

weight # of MPLs	1 9	2 39	3 181	4 626	5 1513	6 1647
als		Symbo	ols O		Su oo	mmary
	weight # of MPLs	weight 1 # of MPLs 9	weight 1 2 # of MPLs 9 39 als Symbol	weight 1 2 3 # of MPLs 9 39 181	weight # of MPLs 1 2 3 4 # of MPLs 9 39 181 626 als Symbols cocco Symbols Symbols Symbols	weight # of MPLs 1 2 3 4 5 # of MPLs 9 39 181 626 1513 als Symbols 00000 Sumon Sumon Sumon



- Fast and stable numeric evaluations
 - faster MPL
 - 2 parallelization
 - **isimplified expressions** (less MPLs, no spurious singularities, convergent power series)
 - re-express the MIs in terms of another set of MPLs
 - use non-trivial relations among MPLs

Introduction o	Integral family	Master integrals	Symbols ●○○○○	Summary 00



shuffle/stuffle relations

$$G(a; z)G(b; z) = G(a, b; z) + G(b, a; z),$$

Li₁(x)Li₁(y) = Li_{1,1}(x, y) + Li_{1,1}(y, x) + Li₂(xy)

Introduction o	Integral family	Master integrals	Symbols ●০০০০	Summary 00



shuffle/stuffle relations

$$G(a; z)G(b; z) = G(a, b; z) + G(b, a; z),$$

$$Li_{1}(x)Li_{1}(y) = Li_{1,1}(x, y) + Li_{1,1}(y, x) + Li_{2}(xy)$$

$$Li_{m_{1},...,m_{k}}(x_{1},...,x_{k}) = \sum_{n_{1}<...< n_{k}} \frac{x_{1}^{n_{1}}...x_{k}^{n_{k}}}{n_{1}^{m_{1}}...n_{k}^{m_{k}}}, |x_{i}| < 1,$$

$$\hookrightarrow (-1)^{k}G_{m_{k},...,m_{1}}(\frac{1}{x_{k}},...,\frac{1}{x_{1}...x_{k}}),$$

with $G_{n_{1},...,n_{k}}(z_{1},...,z_{k}) = G(\underbrace{0,...,0}_{n_{1}-1},z_{1},...,\underbrace{0,...,0}_{n_{k}-1},z_{k}; 1$

Introduction	Integral family	Master integrals	Symbols	Summary
0	000	000000	00000	00

)



functional equations

shuffle/stuffle relations

$$G(a; z)G(b; z) = G(a, b; z) + G(b, a; z),$$

$$Li_{1}(x)Li_{1}(y) = Li_{1,1}(x, y) + Li_{1,1}(y, x) + Li_{2}(xy)$$

$$Li_{m_{1},...,m_{k}}(x_{1},...,x_{k}) = \sum_{n_{1}<...

$$\hookrightarrow (-1)^{k}G_{m_{k},...,m_{1}}(\frac{1}{x_{k}},...,\frac{1}{x_{1}...x_{k}}),$$

with $G_{n_{1},...,n_{k}}(z_{1},...,z_{k}) = G(\underbrace{0,...,0}_{n_{1}-1},z_{1},...,\underbrace{0,...,0}_{n_{k}-1},z_{k}; 1)$$$

Introduction	Integral family	Master integrals	Symbols	Summary
0	000	000000	●0000	00

shuffle/stuffle relations

$$G(a; z)G(b; z) = G(a, b; z) + G(b, a; z),$$

$$Li_{1}(x)Li_{1}(y) = Li_{1,1}(x, y) + Li_{1,1}(y, x) + Li_{2}(xy)$$

$$Li_{m_{1},...,m_{k}}(x_{1},...,x_{k}) = \sum_{\substack{n_{1} < ... < n_{k} \\ n_{1}^{m_{1}}...n_{k}^{m_{k}}}} \frac{x_{1}^{n_{1}}...x_{k}^{n_{k}}}{n_{1}^{m_{1}}...n_{k}^{m_{k}}}, |x_{i}| < 1,$$

$$\hookrightarrow (-1)^{k}G_{m_{k},...,m_{1}}(\frac{1}{x_{k}},...,\frac{1}{x_{1}...x_{k}}),$$

with $G_{n_{1},...,n_{k}}(z_{1},...,z_{k}) = G(\underbrace{0,...,0}_{n_{k}-1},z_{1},...,\underbrace{0,...,0}_{n_{k}-1},z_{k};1)$



- functional equations
 - simple examples

$$log(xy) = log(x) + log(y),$$

Li₂(1 - x) + Li₂(x) + log(1 - x) log(x) = ζ_2

Introduction	Integral family	Master integrals	Symbols	Summary
0	000	000000	00000	00

shuffle/stuffle relations

$$G(a; z)G(b; z) = G(a, b; z) + G(b, a; z),$$

$$\text{Li}_{1}(x)\text{Li}_{1}(y) = \text{Li}_{1,1}(x, y) + \text{Li}_{1,1}(y, x) + \text{Li}_{2}(xy)$$

$$\text{Li}_{m_{1},...,m_{k}}(x_{1},...,x_{k}) = \sum_{n_{1}<...< n_{k}} \frac{x_{1}^{n_{1}}...x_{k}^{n_{k}}}{n_{1}^{m_{1}}...n_{k}^{m_{k}}}, |x_{i}| < 1,$$

$$\hookrightarrow (-1)^{k}G_{m_{k},...,m_{1}}(\frac{1}{x_{k}},...,\frac{1}{x_{1}...x_{k}}),$$

with $G_{n_{1},...,n_{k}}(z_{1},...,z_{k}) = G(\underbrace{0,...,0}_{n_{1}-1},z_{1},...,\underbrace{0,...,0}_{n_{k}-1},z_{k};1)$



- functional equations
 - simple examples

log(xy) = log(x) + log(y),Li₂(1 - x) + Li₂(x) + log(1 - x) log(x) = ζ_2

- Q: general MPLs
- A: symbol map

Introduction	Integral family	Master integrals	Symbols	Summary
0	000	000000	00000	00

0

shuffle/stuffle relations

$$\begin{split} G(a; z)G(b; z) &= G(a, b; z) + G(b, a; z), \\ \text{Li}_1(x)\text{Li}_1(y) &= \text{Li}_{1,1}(x, y) + \text{Li}_{1,1}(y, x) + \text{Li}_2(xy) \\ \text{Li}_{m_1,...,m_k}(x_1,...,x_k) &= \sum_{n_1 < ... < n_k} \frac{x_1^{n_1} \dots x_k^{n_k}}{n_1^{m_1} \dots n_k^{m_k}}, |x_i| < 1, \\ &\hookrightarrow (-1)^k G_{m_k,...,m_1}(\frac{1}{x_k},...,\frac{1}{x_1...x_k}), \\ \text{with } G_{n_1,...,n_k}(z_1,...,z_k) &= G(\underbrace{0,...,0}_{n_1-1},z_1,...,\underbrace{0,...,0}_{n_k-1},z_k;1) \\ \text{Introduction} \qquad \text{Integral family} \qquad \text{Master integrals} \end{split}$$



- functional equations
 - simple examples



Symbol calculus

• definition [1006.5703, 1110.0458]

Introduction o	Integral family	Master integrals	Symbols ⊙●⊙⊙⊙	Summary 00

14/19 11.09.2024 Ming-Ming Long: HP² 2024



- definition [1006.5703, 1110.0458]
 - recursive derivative

$$\mathrm{d}F_{\mathsf{w}} = \sum_{i} F_{i,\mathsf{w}-1} d \log R_{i} \to \mathcal{S}[F_{\mathsf{w}}] = \sum_{i} \mathcal{S}[F_{i,\mathsf{w}-1}] \otimes R_{i}$$

Introduction o	Integral family	Master integrals	Symbols ⊙●⊙⊙⊙	Summary 00



- **definition** [1006.5703, 1110.0458]
 - recursive derivative

$$dF_w = \sum_i F_{i,w-1} d \log R_i \to S[F_w] = \sum_i S[F_{i,w-1}] \otimes R_i$$

maximal coproduct

$$\mathcal{S}[f] = \Delta_{1,\ldots,1}(f) = \sum_{i_1,\ldots,i_n} c_{i_1,\ldots,i_n} t_{i_1} \otimes \ldots \otimes t_{i_n}$$

Introduction o	Integral family	Master integrals	Symbols oooo	Summary 00



- definition [1006.5703, 1110.0458]
 - recursive derivative

$$dF_{w} = \sum_{i} F_{i,w-1} d \log R_{i} \rightarrow S[F_{w}] = \sum_{i} S[F_{i,w-1}] \otimes R_{i}$$

maximal coproduct

$$\mathcal{S}[f] = \Delta_{1,\ldots,1}(f) = \sum_{i_1,\ldots,i_n} c_{i_1,\ldots,i_n} t_{i_1} \otimes \ldots \otimes t_{i_n}$$

 (rooted & decorated) polygon attached to a MPL (PolyLogTools)

Introduction Integral family Master integra o oo ooo ooooo	als Symbols Summary



- **definition** [1006.5703, 1110.0458]
 - recursive derivative

$$dF_{w} = \sum_{i} F_{i,w-1} d \log R_{i} \rightarrow S[F_{w}] = \sum_{i} S[F_{i,w-1}] \otimes R_{i}$$

maximal convoluct

maximal coproduct

$$\mathcal{S}[f] = \Delta_{1,\ldots,1}(f) = \sum_{i_1,\ldots,i_n} c_{i_1,\ldots,i_n} t_{i_1} \otimes \ldots \otimes t_{i_n}$$

- (rooted & decorated) polygon attached to a MPL (PolyLogTools)
- examples

$$\mathcal{S}[\log x] = \otimes x,$$

$$\mathcal{S}[-\operatorname{Li}_n x] = (1 - x) \otimes \underbrace{x \otimes \ldots \otimes x}_{n-1}$$

Introduction	Integral family	Master integrals	Symbols	Summary
0	000	000000	0000	00

Symbol calculus

- definition [1006.5703, 1110.0458]
 - recursive derivative

distributivity

$$dF_{w} = \sum_{i} F_{i,w-1} d \log R_{i} \to \mathcal{S}[F_{w}] = \sum_{i} \mathcal{S}[F_{i,w-1}] \otimes R_{i} \qquad A \otimes (a \ b) \otimes B = A \otimes a \otimes B + A \otimes b \otimes B$$

maximal coproduct
$$\hookrightarrow \begin{cases} A \otimes a^{n} \otimes B = n (A \otimes a \otimes B) \\ A \otimes r \otimes B = 0, \quad r^{n} = 1 \end{cases}, n \in \mathbb{Z}$$

$$\mathcal{S}[f] = \Delta_{1,...,1}(f) = \sum_{i_1,...,i_n} c_{i_1,...,i_n} t_{i_1} \otimes ... \otimes t_{i_n}$$

 (rooted & decorated) polygon attached to a MPL (PolyLogTools)

examples

$$\mathcal{S}[\log x] = \otimes x,$$

$$\mathcal{S}[-\text{Li}_n x] = (1 - x) \otimes \underbrace{x \otimes \dots \otimes x}_{n-1}$$

Introduction Integral family Master integrals Symbols	Summary 00
---	---------------

Symbol calculus

- definition [1006.5703, 1110.0458]
 - recursive derivative

distributivity

$$dF_{w} = \sum_{i} F_{i,w-1} d \log R_{i} \to \mathcal{S}[F_{w}] = \sum_{i} \mathcal{S}[F_{i,w-1}] \otimes R_{i} \qquad A \otimes (a \ b) \otimes B = A \otimes a \otimes B + A \otimes b \otimes B$$

maximal coproduct
$$\mathcal{S}[f] = \Delta_{1} \qquad _{1}(f) = \sum_{i} c_{i} \qquad _{i} t_{i} \otimes \dots \otimes t_{i}$$
$$\hookrightarrow \begin{cases} A \otimes a^{n} \otimes B = n (A \otimes a \otimes B) \\ A \otimes r \otimes B = 0, \quad r^{n} = 1 \end{cases}, n \in \mathbb{Z}$$

$$\mathcal{S}[f] = \Delta_{1,...,1}(f) = \sum_{i_1,...,i_n} c_{i_1,...,i_n} t_{i_1} \otimes ... \otimes t_{i_n}$$

 (rooted & decorated) polygon attached to a MPL (PolyLogTools)

examples

$$\mathcal{S}[\log x] = \otimes x,$$

$$\mathcal{S}[-\operatorname{Li}_n x] = (1 - x) \otimes \underbrace{x \otimes \ldots \otimes x}_{n-1}$$

shuffle product

$$\mathcal{S}[f \ g] = \mathcal{S}[f] \sqcup \mathcal{I}\mathcal{S}[g]$$

Introduction	Integral family	Master integrals	Symbols	Summary
0	000	000000	0000	00

14/19 11.09.2024 Ming-Ming Long: HP² 2024

definition [1006.5703, 1110.0458]

recursive derivative

Symbol calculus

distributivity

$$dF_{w} = \sum_{i} F_{i,w-1} d \log R_{i} \rightarrow S[F_{w}] = \sum_{i} S[F_{i,w-1}] \otimes R_{i} \qquad A \otimes (a \ b) \otimes B = A \otimes a \otimes B + A \otimes b \otimes B$$

maximal coproduct
$$S[d] = A \otimes a^{n} \otimes B = n (A \otimes a \otimes B), n \in \mathbb{Z}$$

$$\mathcal{S}[f] = \Delta_{1,...,1}(f) = \sum_{i_1,...,i_n} c_{i_1,...,i_n} t_{i_1} \otimes ... \otimes t_{i_n}$$

 (rooted & decorated) polygon attached to a MPL (PolyLogTools)

examples

$$\mathcal{S}[\log x] = \otimes x,$$

$$\mathcal{S}[-\text{Li}_n x] = (1 - x) \otimes \underbrace{x \otimes \dots \otimes x}_{n-1}$$

shuffle product

 $\mathcal{S}[f \ g] = \mathcal{S}[f] \sqcup \! \sqcup \! \mathcal{S}[g]$

• symbol does not see π and (multiple) ζ value

- restored via non-maximal coproduct operators
- pure constants (*primitive elements*) are reconstructed, PSLQ algorithm

Introduction	Integral family	Master integrals	Symbols	Summary
0	000	000000	0000	00





Input: $F_i^{(n)}(x, y)$, Output: $F_i^{(n)} = \sum_j c_{ij} f_j^{(n)}$

IntroductionIntegral familyMaster integralsSymbolsSummoo	15/19 11.09.2024	Mina-Mina Lona: HP ² 2024		Three-loop ladder diagram	ns with two off-shell leas
	Introduction o	Integral family	Master integrals	Symbols ⊙⊙●⊙⊙	Summary 00



Input:
$$F_i^{(n)}(x, y)$$
, Output: $F_i^{(n)} = \sum_j c_{ij} f_j^{(n)}$

(1) prime factors in $S[F_i^{(n)}]$

Introduction o	Integral family	Master integrals	Symbols ooeoo	Summary 00
15/19 11.09.2024	Mina-Mina Lona: HP ² 2024		Three-loop ladder diagram	ns with two off-shell legs

15/19 11.09.2024 Ming-Ming Long: HP² 2024



Input:
$$F_i^{(n)}(x, y)$$
, Output: $F_i^{(n)} = \sum_j c_{ij} f_j^{(n)}$

(1) prime factors in $\mathcal{S}[F_i^{(n)}]$

 $\mathbf{\Omega} = \{\omega_i\}$

Introduction o	Integral family	Master integrals	Symbols ○○●○○	Summary 00



Input:
$$F_i^{(n)}(x, y)$$
, Output: $F_i^{(n)} = \sum_j c_{ij} f_j^{(n)}$
(1) prime factors in $\mathcal{S}[F_i^{(n)}]$

 $\mathbf{\Omega} = \{\omega_i\} \cup \{\mathbf{2}\}$

Introduction o	Integral family	Master integrals	Symbols ○○●○○	Summary 00



Input:
$$F_i^{(n)}(x, y)$$
, Output: $F_i^{(n)} = \sum_j c_{ij} f_j^{(n)}$
(1) prime factors in $\mathcal{S}[F_i^{(n)}]$

$$\mathbf{\Omega} = \{\omega_i\} \cup \{\mathbf{2}\}$$

(2) choose functions

	Weight	Functions	
_	1	$\boldsymbol{\Psi}^{(1)} = \{ log \}$	
	2	$\mathbf{\Psi}^{(2)} = \{\mathrm{Li}_2\}$	
	3	$\mathbf{\Psi}^{(3)} = \{\mathrm{Li}_3\}$	
	4	$\boldsymbol{\Psi}^{(4)} = \{\mathrm{Li}_4, \mathrm{Li}_{2,2}\}$	
	5	$\boldsymbol{\Psi}^{(5)} = \{\mathrm{Li}_5, \mathrm{Li}_{2,3}\}$	
	6	$\mathbf{\Psi}^{(6)} = \{ \mathrm{Li}_{6}, \mathrm{Li}_{2,4}, \mathrm{Li}_{3,3}, \mathrm{Li}_{2,2,2} \}$	
o N	ction	Integral family	Master integrals

Summary

00

Symbols

Algorithms [1110.0458]

Input:
$$F_i^{(n)}(x, y)$$
, Output: $F_i^{(n)} = \sum_j c_{ij} f_j^{(n)}$
(1) prime factors in $\mathcal{S}[F_i^{(n)}]$

$$\mathbf{\Omega} = \{\omega_i\} \cup \{\mathbf{2}\}$$

(3) construct arguments of $\Psi^{(i)}$

$$\mathbf{\Phi} = \left\{ \phi_{n_1,\dots,n_l}^{\pm} = \pm \prod_i \omega_i^{n_i} \middle| \omega_i \in \mathbf{\Omega}, n_i \in \mathbb{Z} \right\} \cup \{\mathbf{0}\}$$

Symbols

00000

(2) choose functions

Weight	Functions	
1	$\boldsymbol{\Psi}^{(1)} = \{ log \}$	
2	$\mathbf{\Psi}^{(2)} = \{\mathrm{Li}_2\}$	
3	$\mathbf{\Psi}^{(3)} = \{\mathrm{Li}_3\}$	
4	$\boldsymbol{\Psi}^{(4)} = \{\mathrm{Li}_4, \mathrm{Li}_{2,2}\}$	
5	$\mathbf{\Psi}^{(5)} = \{ \mathrm{Li}_5, \mathrm{Li}_{2,3} \}$	
6	$\Psi^{(6)} = \{ \text{Li}_6, \text{Li}_{2,4}, \text{Li}_{3,3}, \text{Li}_{2,2,2} \}$	
o Introduction	Integral family	Master integrals

Three-loop ladder diagrams with two off-shell legs

Summary

Algorithms [1110.0458]

Input:
$$F_i^{(n)}(x, y)$$
, Output: $F_i^{(n)} = \sum_j c_{ij} f_j^{(n)}$
(1) prime factors in $\mathcal{S}[F_i^{(n)}]$

$$\mathbf{\Omega} = \{\omega_i\} \cup \{\mathbf{2}\}$$

(2) choose functions

	Weight	Functions
-	1	$\boldsymbol{\Psi}^{(1)} = \{ log \}$
	2	$\boldsymbol{\Psi}^{(2)} = \{\mathrm{Li}_2\}$
	3	$\mathbf{\Psi}^{(3)} = \{\mathrm{Li}_3\}$
	4	$\boldsymbol{\Psi}^{(4)} = \{\mathrm{Li}_4, \mathrm{Li}_{2,2}\}$
	5	$\boldsymbol{\Psi}^{(5)} = \{\mathrm{Li}_5,\mathrm{Li}_{2,3}\}$
	6	$\mathbf{\Psi}^{(6)} = \{ \mathrm{Li}_{6}, \mathrm{Li}_{2,4}, \mathrm{Li}_{3,3}, \mathrm{Li}_{2,2,2} \}$
ntrodu	ction	Integral family

(3) construct arguments of $\Psi^{(i)}$

Master integrals

000000

$$\mathbf{\Phi} = \left\{ \phi_{n_1,\dots,n_l}^{\pm} = \pm \prod_i \omega_i^{n_i} \bigg| \omega_i \in \mathbf{\Omega}, n_i \in \mathbb{Z} \right\} \cup \{\mathbf{0}\}$$
$$\mathbf{\Phi}^{(1)} = \left\{ \phi \in \mathbf{\Phi} \bigg| 1 - \phi \in \mathbf{\Phi}, \ \phi \neq \mathbf{0} \right\},$$

Symbols

00000

15/19 11.09.2024 Ming-Ming Long: HP² 2024

Three-loop ladder diagrams with two off-shell legs

Summary

Algorithms [1110.0458]

Input:
$$F_i^{(n)}(x, y)$$
, Output: $F_i^{(n)} = \sum_j c_{ij} f_j^{(n)}$
(1) prime factors in $\mathcal{S}[F_i^{(n)}]$

$$\mathbf{\Omega} = \{\omega_i\} \cup \{\mathbf{2}\}$$

(2) choose functions

	Weight	Functions	$\mathbf{\Phi}^{(1)} = \left\{ \cdot \right\}$
	1	$\boldsymbol{\Psi}^{(1)} = \{ log \}$	ſ
	2	$\boldsymbol{\Psi}^{(2)} = \{\mathrm{Li}_2\}$	$\mathbf{\Phi}^{(n)} = \left\{ \right.$
	3	$\boldsymbol{\Psi}^{(3)} = \{\mathrm{Li}_3\}$	l
	4	$\boldsymbol{\Psi}^{(4)} = \{\mathrm{Li}_4,\mathrm{Li}_{2,2}\}$	
	5	$\boldsymbol{\Psi}^{(5)} = \{\mathrm{Li}_5,\mathrm{Li}_{2,3}\}$	
	6	$\boldsymbol{\Psi}^{(6)} = \{\mathrm{Li}_{6}, \mathrm{Li}_{2,4}, \mathrm{Li}_{3,3}, \mathrm{Li}_{2,2,2}\}$	
Introd o	uction	Integral family	Master integrals

(3) construct arguments of $\Psi^{(i)}$

00000

Three-loop ladder diagrams with two off-shell legs

Algorithms [1110.0458]

Input:
$$F_i^{(n)}(x, y)$$
, Output: $F_i^{(n)} = \sum_j c_{ij} f_j^{(n)}$
(1) prime factors in $S[F_i^{(n)}]$

$$\mathbf{\Omega} = \{\omega_i\} \cup \{\mathbf{2}\}$$

(2) choose functions

Weight	Functions
1	$\boldsymbol{\Psi}^{(1)} = \{ log \}$
2	$\mathbf{\Psi}^{(2)} = \{\mathrm{Li}_2\}$
3	$\mathbf{\Psi}^{(3)} = \{ ext{Li}_3\}$
4	$\mathbf{\Psi}^{(4)} = \{\mathrm{Li}_4,\mathrm{Li}_{2,2}\}$
5	$\boldsymbol{\Psi}^{(5)} = \{\mathrm{Li}_5,\mathrm{Li}_{2,3}\}$
6	$\boldsymbol{\Psi}^{(6)} = \{ \mathrm{Li}_{6}, \mathrm{Li}_{2,4}, \mathrm{Li}_{3,3}, \mathrm{Li}_{2,2,2} \}$
ntroduction	Integral family

(3) construct arguments of $\Psi^{(i)}$

$$\mathbf{\Phi} = \left\{ \phi_{n_1,...,n_l}^{\pm} = \pm \prod_i \omega_i^{n_i} \Big| \omega_i \in \mathbf{\Omega}, n_i \in \mathbb{Z} \right\} \cup \{\mathbf{0}\}$$

$$\mathbf{\Phi}^{(1)} = \left\{ \phi \in \mathbf{\Phi} \Big| 1 - \phi \in \mathbf{\Phi}, \ \phi \neq \mathbf{0} \right\},$$

$$\mathbf{\Phi}^{(n)} = \left\{ (\phi_1,...,\phi_n) \in \underbrace{\mathbf{\Phi}^{(1)} \times ... \times \mathbf{\Phi}^{(1)}}_{n \ \mathbf{\Phi}^{(1)}} \Big| \phi_i - \phi_j \in \mathbf{\Phi}, \ \phi_i \neq \mathbf{0} \right\}$$

e.g.,
$$\phi = \frac{x(x+y)(1+xy)}{(1+x+xy+x^2)^2}, (\phi_1, \phi_2) = \left(\frac{1}{1-x}, -\frac{1+x}{x(x+y)}\right)$$

Master integrals	Symbols	Summary
000000	00000	00



(4) decomposition

$$F_{i}^{(w)} = \sum_{j} c_{j} \psi_{j}^{(w)}$$

$$+ \sum_{j_{1}, j_{2}; w_{1} \ge w_{2} > 0} c_{j_{1}, j_{2}}^{(w_{1}, w_{2})} \psi_{j_{1}}^{(w_{1})} \psi_{j_{2}}^{(w_{2})} \delta\left(n - \sum_{k}^{2} w_{k}\right)$$

$$+ \dots$$

$$+ \sum_{j_{1}, \dots, j_{n}; w_{1} \ge \dots \ge w_{n} > 0} c_{j_{1}, \dots, j_{n}}^{(w_{1}, \dots, w_{n})} \psi_{j_{1}}^{(w_{1})} \dots \psi_{j_{n}}^{(w_{n})} \delta\left(n - \sum_{k}^{n} w_{k}\right)$$

Apply S and solve linear equations of $c_j, c_{j_1, j_2}, ...$

Introduction o	Integral family	Master integrals	Symbols 00000	Summary 00

16/19 11.09.2024 Ming-Ming Long: HP² 2024



(4) decomposition

$$\begin{aligned} F_{i}^{(w)} &= \sum_{j} c_{j} \psi_{j}^{(w)} \\ &+ \sum_{j_{1}, j_{2}; w_{1} \geq w_{2} > 0} c_{j_{1}, j_{2}}^{(w_{1}, w_{2})} \psi_{j_{1}}^{(w_{1})} \psi_{j_{2}}^{(w_{2})} \delta\left(n - \sum_{k}^{2} w_{k}\right) \\ &+ \dots \\ &+ \sum_{j_{1}, \dots, j_{n}; w_{1} \geq \dots \geq w_{n} > 0} c_{j_{1}, \dots, j_{n}}^{(w_{1}, \dots, w_{n})} \psi_{j_{1}}^{(w_{1})} \dots \psi_{j_{n}}^{(w_{n})} \delta\left(n - \sum_{k}^{n} w_{k}\right) \end{aligned}$$

Apply S and solve linear equations of $c_j, c_{j_1, j_2}, ...$

(5) restore info forgot by symbol

Introduction	Integral family	Master integrals	Symbols	Summary
0	000	000000	00000	00

16/19 11.09.2024 Ming-Ming Long: HP² 2024



Remarks

• further constraints on the arguments

Introduction o	Integral family	Master integrals	Symbols ○○○○●	Summary 00

17/19 11.09.2024 Ming-Ming Long: HP² 2024



Remarks

- further constraints on the arguments
- projectors eliminate terms
 - e.g., modulo products
 - smaller linear system to solve

toto di strato di	Later and Grand	Martinelation	O set als	0
o	ooo	Master Integrals	Symbols 00000	oo oo


- further constraints on the arguments
- projectors eliminate terms
 - e.g., modulo products
 - smaller linear system to solve
- **s**ystem *A***c** = **b**,
 - A and **b** are rational
 - expect rational solution
 - sparse and underdetermined
 - optimal solution?
 - can be very huge

Introduction o	Integral family	Master integrals	Symbols ○○○○●	Summary 00



- further constraints on the arguments
- projectors eliminate terms
 - e.g., modulo products
 - smaller linear system to solve
- **s**ystem *A***c** = **b**,
 - A and **b** are rational
 - expect rational solution
 - sparse and underdetermined
 - optimal solution?
 - can be very huge

$\hfill\blacksquare$ extend functions in ansatz, e.g., ${\rm Li}_{1,1,3}$

Introduction o	Integral family	Master integrals	Symbols ○○○○●	Summary



- further constraints on the arguments
- projectors eliminate terms
 - e.g., modulo products
 - smaller linear system to solve
- **s**ystem *A***c** = **b**,
 - A and **b** are rational
 - expect rational solution
 - sparse and underdetermined
 - optimal solution?
 - can be very huge
- extend functions in ansatz, e.g., Li_{1,1,3}
- discover identities among ansatz [2407.12503]

Introduction	Integral family	Master integrals	Symbols	Summary
0	000	000000	00000	00



- projectors eliminate terms
 - e.g., modulo products
 - smaller linear system to solve

system *A***c** = **b**,

- A and **b** are rational
- expect rational solution
- sparse and underdetermined
 - optimal solution?
- can be very huge
- extend functions in ansatz, e.g., Li_{1,1,3}
- discover identities among ansatz [2407.12503]

Introduction Int	tegral family	Master integrals	Symbols	Summary
0 00	00	000000	00000	00



- FORM speeds up heavy manipulations of symbols
- Kira solves linear system

- further constraints on the arguments
- projectors eliminate terms
 - e.g., modulo products
 - smaller linear system to solve
- **s**ystem *A***c** = **b**,
 - A and **b** are rational
 - expect rational solution
 - sparse and underdetermined
 - optimal solution?
 - can be very huge
- extend functions in ansatz, e.g., Li_{1,1,3}
- discover identities among ansatz [2407.12503]



- FORM speeds up heavy manipulations of symbols
- Kira solves linear system
- preliminary results

- further constraints on the arguments
- projectors eliminate terms
 - e.g., modulo products
 - smaller linear system to solve
- **s**ystem *A***c** = **b**,
 - A and **b** are rational
 - expect rational solution
 - sparse and underdetermined
 - optimal solution?
 - can be very huge
- extend functions in ansatz, e.g., Li_{1,1,3}
- discover identities among ansatz [2407.12503]
- Introduction Integral family Ooo Master integrals Ooooo Symbols Symbols Summary

FORM speeds up heavy manipulations of symbols

- Kira solves linear system
- preliminary results
 - simplification

of weight 4 MPL(y, x) : \sim 1000 \rightarrow \sim 150



- further constraints on the arguments
- projectors eliminate terms
 - e.g., modulo products
 - smaller linear system to solve
- **s**ystem *A***c** = **b**,
 - A and **b** are rational
 - expect rational solution
 - sparse and underdetermined
 - optimal solution?
 - can be very huge
- extend functions in ansatz, e.g., Li_{1,1,3}
- discover identities among ansatz [2407.12503]



- FORM speeds up heavy manipulations of symbols
- Kira solves linear system
- preliminary results
 - simplification

of weight 4 MPL(y, x) : \sim 1000 \rightarrow \sim 150

complication

$$F_{12}^{(6)} = 8 \operatorname{Li}_{6} \left(\frac{1}{1+x+z} \right) + \dots,$$

$$F_{94}^{(6)} = -\frac{595599.\dots}{265168.\dots} \operatorname{Li}_{6} \left(\frac{1}{1+x+z} \right) + \dots$$

$$\hookrightarrow (80 \text{ digits})/(69 \text{ digits})$$

Introduction	Integral family	Master integrals	Symbols	Summary
0	000	000000	00000	00

Karlsruhe Institute of Technology

Summary

- Analytic results for the master integrals in two three-loop families.
- Both the solutions of canonical differential equation in Euclidean and physical regions are validated.
- The complex expressions of master integrals are (partially) simplified by applying symbol technique.

Introduction o	Integral family	Master integrals	Symbols 00000	Summary ●○

Karlsruhe Institute of Technology

Summary

- Analytic results for the master integrals in two three-loop families.
- Both the solutions of canonical differential equation in Euclidean and physical regions are validated.
- The complex expressions of master integrals are (partially) simplified by applying symbol technique.

Outlook

18/19 11.09.2024	Ming-Ming Long: HP ² 2024		Three-loop ladder diagram	ns with two off-shell legs
Introduction o	Integral family	Master integrals	Symbols 00000	Summary ●○

Karlsruhe Institute of Technology

Summary

- Analytic results for the master integrals in two three-loop families.
- Both the solutions of canonical differential equation in Euclidean and physical regions are validated.
- The complex expressions of master integrals are (partially) simplified by applying symbol technique.
- Outlook
 - reduce the complexity at weight 5 and 6

18/19 11.09.2024	Ming-Ming Long: HP ² 2024		Three-loop ladder diagram	ns with two off-shell legs
Introduction o	Integral family	Master integrals	Symbols 00000	Summary ●○



Summary

- Analytic results for the master integrals in two three-loop families.
- Both the solutions of canonical differential equation in Euclidean and physical regions are validated.
- The complex expressions of master integrals are (partially) simplified by applying symbol technique.
- Outlook
 - reduce the complexity at weight 5 and 6
 - leading-color amplitudes of diboson production at N3LO QCD

Introduction o	Integral family	Master integrals	Symbols 00000	Summary ●○



Summary

- Analytic results for the master integrals in two three-loop families.
- Both the solutions of canonical differential equation in Euclidean and physical regions are validated.
- The complex expressions of master integrals are (partially) simplified by applying symbol technique.

Outlook

0

- reduce the complexity at weight 5 and 6
- leading-color amplitudes of diboson production at N3LO QCD
- more complicated diagrams



Thank you for your attention!