# Unresolved limits of polarized matrix elements

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# Introduction

## What are polarized matrix elements? What are they good for?

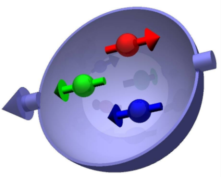
- take the "difference" over spin states instead of averaging for one (or more) of the external partons
- tree-level, $d = 4$: same as modulus square of helicity amplitudes with polarized parton taken difference over and others summed
  *Caveat:* naive treatment of $\gamma_5$ leads to wrong results, want to use a consistent $\gamma_5$ scheme
- describe spin asymmetries
  $\rightarrow$ provide direct access to proton spin structure
- here: double-longitudinal spin asymmetry

$$A_{LL} = \frac{\sigma^{++} + \sigma^{--} - \sigma^{+-} - \sigma^{-+}}{\sigma^{++} + \sigma^{--} + \sigma^{+-} + \sigma^{-+}} = \frac{g_1}{2F_1}$$

probes helicity PDF $\Delta f = f^+ - f^-$ through $g_1$ DIS coefficient function. (notation: $\sigma^{h_p h_\ell}$)

# The proton spin

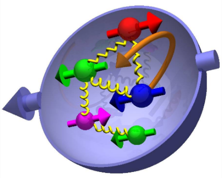
Before EMC spin measurement



$$\frac{1}{2} = \frac{1}{2}\Delta\Sigma = \frac{1}{2}\int_0^1 \mathrm{d}x\,\Delta q(x)$$

EMC '89: $\Delta\Sigma \sim 0.25$ !!!

Complete picture



$$\frac{1}{2} = \underbrace{\frac{1}{2}\Delta\Sigma}_{\sim 0.12} + \underbrace{\Delta G}_{\sim 0.18?} + \underbrace{L_q + L_g}_{?}$$

Composition of proton spin remains underconstrained, observables largely at NLO precision.

**From 2030's:** Electron-ion collider (EIC) at Brookhaven National Lab: LHC-like luminosity for polarized *ep* collisions at $\sqrt{s} = 45\ldots140\ \mathrm{GeV}$
$\rightarrow$ NNLO calculations for spin asymmetry observables

# Anatomy of jet observables up to NNLO

$$
\begin{aligned}
\sigma = & \int_m |\mathcal{M}_m^{(0)}|^2 J_m \,\mathrm{d}\phi_m \\
& + \Bigg( \underbrace{\int_{m+1} |\mathcal{M}_{m+1}^{(0)}|^2 J_m \,\mathrm{d}\phi_{m+1}}_{\mathrm{R}} + \underbrace{\int_m (\mathcal{M}_m^{(1)} \mathcal{M}_m^{(0)*} + \mathrm{c.c.}) J_m \,\mathrm{d}\phi_m}_{\mathrm{V}} \Bigg) \\
& + \Bigg( \underbrace{\int_{m+2} |\mathcal{M}_{m+2}^{(0)}|^2 J_m \,\mathrm{d}\phi_{m+2}}_{\mathrm{RR}} + \underbrace{\int_{m+1} (\mathcal{M}_{m+1}^{(1)} \mathcal{M}_{m+1}^{(0)*} + \mathrm{c.c.}) J_m \,\mathrm{d}\phi_{m+1}}_{\mathrm{RV}} \\
& \qquad + \underbrace{\int_m |\mathcal{M}_m^{(1)}|^2 J_m \,\mathrm{d}\phi_m + \int_m (\mathcal{M}_m^{(2)} \mathcal{M}_m^{(0)*} + \mathrm{c.c.}) J_m \,\mathrm{d}\phi_m}_{\mathrm{VV}} \Bigg) + \mathcal{O}(\mathrm{N^3LO})
\end{aligned}
$$

with $\mathcal{M}_m^{(l)}$ with the amplitude with $m$ final-state partons and $l$ loops, $J_m$ the jet functions selecting $m$ jets out of the final state momenta, and $\mathrm{d}\phi_m$ the $m$ particle phase space

# Goal: Subtraction in polarized processes

**Idea of Subtraction**: [e.g. talks of M. Marcoli, A. Kardos yesterday]

- To each term with divergent phase space regions add a counter term locally cancelling the divergent behaviour.
- The counter term should be sufficiently simple to integrate analytically over the unresolved phase space.

**What is required?**

✓ Phase space factorization
same in polarized and unpolarized processes

- How to build a subtraction term?
  $\rightarrow$ Full understanding of factorization of matrix elements:
  - Do polarized MEs factorize ($\gamma_5$) in infrared limits?
  - What do they factorize into?

- From now on consider only color-ordered matrix elements
  $\rightarrow$ singular limits only between color-adjacent partons
- Particle $\gamma$ will refer to an unordered, abelian gluon.

# Approach

Calculate matrix elements contributing to $g_1$ coefficient function for

- photon DIS [Zijlstra, van Neerven '93]
- graviton DIS [Lam, Li '81] [Stirling, Vryonidou '11] [Moch, Vermaseren, Vogt '14]

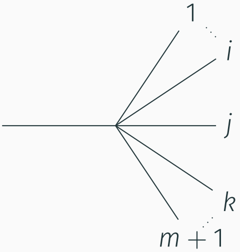
at

- tree-level with up to 5 external partons
- 1-loop with up to 3 external partons

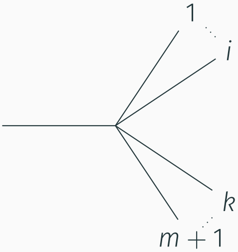
in Larin $\gamma_5$ scheme, and investigate the leading-power behavior in the unresolved limits.

## Single-soft

parametrize $p_j \to \lambda p_j$ with $\lambda \to 0$




$$\xrightarrow{j \text{ soft}} 4\pi\alpha_s \frac{1}{\lambda^2}\frac{2s_{ik}}{s_{ij}s_{jk}} \times$$




$$|(\Delta)\mathcal{M}^{(0)}_{m+1}(p_1,\dots,p_i,p_j,p_k,\dots p_{m+1})|^2$$
$$\xrightarrow{j \text{ soft}} 4\pi\alpha_s \frac{1}{\lambda^2}\frac{2s_{ik}}{s_{ij}s_{jk}}|(\Delta)\mathcal{M}^{(0)}_{m}(p_1,\dots,p_i,p_k,\dots p_{m+1})|^2 + \mathcal{O}(\lambda^0)$$

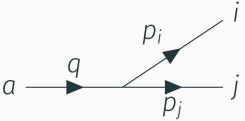
if $j$ is an unpolarized gluon, else 0.

# IR limits: the single-unresolved case (2)

**Single-collinear:** $i \parallel j$, final-final

$$\begin{cases} p_i^\mu \to z p^\mu + k_\perp^\mu - \frac{k_\perp^2}{2z}\frac{n^\mu}{p\cdot n} \\ p_j^\mu \to (1-z) p^\mu - k_\perp^\mu - \frac{k_\perp^2}{2(1-z)}\frac{n^\mu}{p\cdot n} \end{cases}$$

(after azimuth. avg. $\int \mathrm{d}\varphi\, k_T^\mu(\varphi) k_T^\nu(\varphi) \propto -g^{\mu\nu} + (p^\mu n^\nu + n^\mu p^\nu)/(p\cdot n)$)




$$\xrightarrow{i\|j} 4\pi\alpha_s \frac{1}{s_{ij}} \mathcal{P}_{ij\leftarrow a} \times$$


$$\begin{aligned} &|(\Delta)\mathcal{M}_{m+1}^{(0)}(p_1,\ldots,p_i,p_j,\ldots p_{m+1})|^2 \\ &\quad \xrightarrow{i\|j} 4\pi\alpha_s \frac{1}{s_{ij}} \mathcal{P}_{ij\leftarrow a}(z)\, |(\Delta)\mathcal{M}_m^{(0)}(p_1,\ldots,p,\ldots,p_{m+1})|^2 + \mathcal{O}(k_T^0) \end{aligned}$$

with $\mathcal{P}_{ij\leftarrow a}(z) = \begin{cases} P_{ij\leftarrow a}(z) & i,j \text{ unpolarized} \\ P_{\Delta i\, j\leftarrow a}(z) & i \text{ polarized} \end{cases}$

# Unpolarized and Polarized single-collinear splitting amplitudes

Unpolarized and polarized splitting amplitudes differ:

$$P_{qg\leftarrow q}(z) = C_F\left[\frac{1+z^2}{1-z} - (1-z)\varepsilon\right], \qquad P_{\Delta q\, g\leftarrow q}(z) = C_F\left[\frac{1+z^2}{1-z} + (1-z)\frac{\varepsilon(3+\varepsilon)}{1-\varepsilon}\right]$$

$$P_{gq\leftarrow q}(z) = C_F\left[\frac{1+(1-z)^2}{z} - z\varepsilon\right], \qquad P_{\Delta g\, q\leftarrow q}(z) = C_F\left[\frac{2-z(1+\varepsilon)}{1-\varepsilon}\right]$$

$$P_{q\bar{q}\leftarrow g}(z) = 2n_f T_R\left[1 - 2\frac{z(1-z)}{1-\varepsilon}\right], \qquad P_{\Delta q\, \bar{q}\leftarrow g}(z) = 2n_f T_R\left[1 - 2\frac{1-z}{1-\varepsilon}\right]$$

$$P_{gg\leftarrow g}(z) = C_A\left[2z(1-z) + \frac{2}{1-z} + \frac{2}{z} - 4\right], \quad P_{\Delta g\, g\leftarrow g}(z) = 2C_A\left[\frac{1}{1-z} - 1 + 2\frac{1-z}{1-\varepsilon}\right]$$

$\rightarrow$ different infrared structure!

To achieve proper factorization of matrix element need to retain *entire* $d$-dimensional structure.

# The single-unresolved case at 1-loop

Using method of [Bern, Del Duca, Kilgore '99]

$$|(\Delta)\mathcal{M}^{(1)}_{m+1}|^2 \xrightarrow{i||j} \frac{1}{s_{ij}}\mathcal{P}^{(0)}_{ij\leftarrow a}|(\Delta)\mathcal{M}^{(1)}_{m}|^2 + (s_{ij})^{-1-\varepsilon}\mathcal{P}^{(1)}_{ij\leftarrow a}|(\Delta)\mathcal{M}^{(0)}_{m}|^2$$

E.g.

$$\begin{aligned} P^{(1)}_{\Delta q\, g\leftarrow q} =& \frac{1}{\varepsilon^2}\sum_{k=1}^{\infty}\varepsilon^k \,\mathrm{Li}_k\left(\frac{z}{z-1}\right) P^{(0)}_{\Delta q\, g\leftarrow q}(z) \\ &+ \frac{1}{\varepsilon^2}\frac{1+z^2}{1-z} + \frac{3}{\varepsilon}(1-z) + \frac{7z-9}{4(1-2\varepsilon)} - \left(\frac{1}{4} - \frac{6}{1-\varepsilon}\right)(1-z) \end{aligned}$$

in the final-final limit (preliminary).

# IR limits: the double-unresolved case (1)

**Unconnected**
singularities factorize individually like single-unresolved case

**Double soft** Like single-soft described by eikonal factors, initiated by gluon

- color-ordered double-soft eikonal factors $S_{gg}$, $S_{\gamma\gamma}$, and $S_{q\bar{q}}$
- $S_{\gamma\gamma}$ and $S_{gg}$ contain iterated soft limits
- $S_{gg}$ and $S_{q\bar{q}}$ contain genuine double-soft limit

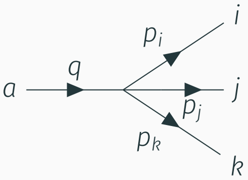
in the case of unpolarised soft partons $i$, $j$; otherwise zero

**Soft & collinear**
Overlap between subsequent triple-collinear & soft, and soft & single-collinear, depending on position of soft parton in cluster

**Triple-collinear:** $i \parallel j \parallel k$ final-final

$$\begin{cases} p_i^\mu = z_i p^\mu + k_{T,i}^\mu + \frac{k_T^2}{2z_i}\frac{n^\mu}{p\cdot n} \\ p_j^\mu = z_j p^\mu + k_{T,j}^\mu + \frac{k_T^2}{2z_j}\frac{n^\mu}{p\cdot n} \\ p_k^\mu = z_k p^\mu + k_{T,k}^\mu + \frac{k_T^2}{2z_k}\frac{n^\mu}{p\cdot n} \end{cases}$$

with $k_{T,i}^\mu + k_{T,j}^\mu + k_{T,k}^\mu = 0, \quad z_i + z_j + z_k = 1$




$$\xrightarrow{i\parallel j\parallel k} (4\pi\alpha_s)^2 \frac{1}{s_{ijk}^2}\mathcal{P}_{ijk}(z_i, z_j, z_k, s_{ij}, s_{ik}, s_{jk}) \times$$


after azimuthal averaging

# Triple-collinear splitting amplitudes

- contain eikonal factors, iterated splitting, and "genuine" triple-collinear splitting
- unpolarized case: 7 independent splitting amplitudes
  $P_{ggg}$, $P_{qgg}$, $P_{q\gamma\gamma}$, $P_{gq\bar{q}}$, $P_{\gamma q\bar{q}}$, $P_{q\bar{q}'q'}$, $P_{q\bar{q}q}$ [Campbell, Glover '97]
- polarized case: 7 unpolarized + 16 polarized = 23
  $\Delta P_{\Delta q\, g_1 g_2}$, $\Delta P_{\Delta q\, \gamma\gamma}$, $\Delta P_{\Delta g\, gq}$, $\Delta P_{g\, \Delta g\, q}$, $\Delta P_{\Delta\gamma\, \gamma q}$, $\Delta P_{\Delta q\, q'\bar{q}'}$,
  $\Delta P_{\Delta q'\, \bar{q}' q}$, $\Delta P_{\Delta q\, \bar{q} q}$, $\Delta P_{\Delta\bar{q}\, qq}$, $\Delta P^{\text{unconn.}}_{\Delta g\, gg}$, $\Delta P^{\text{conn.}}_{\Delta g\, gg}$,$\Delta P^{\text{conn.}}_{\Delta q\, \bar{q} g}$, $\Delta P^{\text{unconn.}}_{\Delta q\, \bar{q} g}$,
  $\Delta P_{\Delta q\, \bar{q}\gamma}$, $\Delta P_{\Delta g\, q\bar{q}}$, $\Delta P_{\Delta\gamma\, q\bar{q}}$
- complicated structure in terms of momentum fractions
- physically understood and strongly constrained structure in terms of Mandelstam variables [Braun-White, Glover '22]

# How to obtain splitting amplitudes?

- unpolarized: proof of factorization with constructive method

  [Catani, Grazzini '99]

- polarized:

  same proof does not work for $g_1$

  instead extraction with above parametrizations from DIS coefficient functions à la [Glover, Campbell '97]

  - $g_1$ coefficient function in Photon-DIS [e.g. Zijlstra, van Neerven '93]
  - $g_1$ coefficient functions in Graviton-DIS [Moch et al. '14]

$\rightarrow$ only with averaged "azimuthal correlation", suffer from coefficient function features in soft limit ($1/z_{\text{pol}}$)

- Check: factorization still true with one extra radiated particle

  $\rightarrow$ non-trivial factorization

# Example of a triple-collinear splitting function

$$
\begin{aligned}
P_{\Delta q\,g\,g\leftarrow q} = &+\frac{1}{s_{23}s_{123}}\left[P_{\Delta q\,g\leftarrow q}(z_1)\,P_{gg\leftarrow g}\left(\frac{z_2}{1-z_1}\right)\right] + \frac{1}{s_{12}s_{123}}\left[P_{\Delta q\,g\leftarrow q}(1-z_3)\,P_{\Delta q\,g\leftarrow q}\left(\frac{z_1}{1-z_3}\right)\right] \\
&+\frac{1}{s_{123}^2}\left[17+15z_2+12\varepsilon+2\varepsilon z_2+2\varepsilon^2-\frac{1+7\varepsilon+13\varepsilon^2+9\varepsilon^3+2\varepsilon^4}{(1-\varepsilon)(1-2\varepsilon)}\frac{1+z_2}{z_2+z_1}-\frac{17+6\varepsilon}{2}z_1\right. \\
&+\frac{2\varepsilon+4\varepsilon^2+2\varepsilon^3}{(1-\varepsilon)(1-2\varepsilon)}\frac{1-2z_2+z_2^2}{z_1}-\frac{16\varepsilon}{1-2\varepsilon}\frac{z_2-z_2^2}{z_3+z_2}-\frac{8}{1-\varepsilon}[6+2z_2-3z_1]+\frac{4-4\varepsilon}{z_3+z_2} \\
&\left.+\frac{62+2z_2-31z_1}{2(1-2\varepsilon)}-2\frac{2z_2-z_2^2}{(z_2+z_3)^2}+2\frac{2z_2-z_2^2}{(z_2+z_3)^2}\varepsilon\right]+\frac{\mathrm{Tr}(231l)}{s_{23}s_{123}^2}\left[\frac{4\varepsilon}{1-2\varepsilon}\frac{1}{z_1}-\frac{12\varepsilon}{1-2\varepsilon}\right. \\
&\left.+\frac{16\varepsilon}{1-2\varepsilon}\frac{z_2}{z_3+z_2}-\frac{4-4\varepsilon}{z_3+z_2}+4\frac{z_2}{(z_2+z_3)^2}-4\frac{z_2}{(z_2+z_3)^2}\varepsilon\right]+\frac{\mathrm{Tr}(123l)}{s_{12}s_{123}^2}\left[9\frac{z_2}{z_1}+2\frac{z_2}{z_1}\varepsilon\right. \\
&+\frac{1+7\varepsilon+13\varepsilon^2+9\varepsilon^3+2\varepsilon^4}{(1-\varepsilon)(1-2\varepsilon)}\frac{1}{z_2+z_1}-\frac{8\varepsilon+4\varepsilon^2+4\varepsilon^3}{(1-\varepsilon)(1-2\varepsilon)}-\frac{1}{1-2\varepsilon}\frac{9-7z_2}{z_1}+\frac{16}{1-\varepsilon}\frac{1-z_2}{z_1} \\
&\left.-\frac{2\varepsilon+7}{z_1}\right]+\frac{\mathrm{Tr}(123l)}{s_{12}s_{23}s_{123}}\left[\frac{1+2\varepsilon+\varepsilon^2}{1-\varepsilon}-\frac{1+2\varepsilon+\varepsilon^2}{1-\varepsilon}\frac{z_2}{z_3}+\frac{4\varepsilon}{1-2\varepsilon}\frac{z_2}{z_1}-\frac{8}{1-\varepsilon}\frac{1-z_1}{z_2}\right. \\
&+\frac{10+2\varepsilon}{z_2}-6\frac{z_1}{z_2}-2\frac{z_1}{z_2}\varepsilon-\frac{z_2}{z_3+z_2}+\frac{z_2}{z_3+z_2}\varepsilon-\frac{2}{z_3+z_2}-\frac{2}{z_3(z_2+z_3)}+\frac{2}{z_3} \\
&\left.-\frac{4}{z_2(z_2+z_3)}\right]+\frac{W_{23}}{s_{23}^2s_{123}^2}\left[-\frac{4\varepsilon}{1-2\varepsilon}\frac{1}{z_1}+\frac{2-2\varepsilon}{(z_2+z_3)^2}\right]+\frac{W_{12}}{s_{12}^2s_{123}^2}\left[-\frac{2\varepsilon+4\varepsilon^2+2\varepsilon^3}{(1-\varepsilon)(1-2\varepsilon)}\frac{1}{z_1}\right]
\end{aligned}
$$

with $\mathrm{Tr}(ijkl) = z_i s_{jk} - z_j s_{ik} + z_k s_{ij}$ and
$W_{ij} = (z_i s_{jk} - z_j s_{ik})^2 - \frac{2}{1-\varepsilon}\frac{z_i z_j z_k}{1-z_k} s_{ij} s_{ijk}$

# Conclusion

- derived all universal objects appearing in single- and double-unresolved limits up to tree-level NNLO
- simple factorization
- 1-loop single-collinear (NNLO RV): first limits identified
- close to complete picture of IR structure up to NNLO
- next step: generate polarized antenna functions (i.e. building blocks of antenna subtraction formalism)
- goal: antenna subtraction for polarized observables

Thanks for your attention!