# Decomposing Feynman integrals with intersection numbers

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Based on joined work with:
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+ Andrzej Pokraka

[2401.01897] [2408.16668] + ongoing

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# Pictures at an Exhibition

# Introduction

# Feynman twisted period integrals

- Amplitude at fixed loop order = sum of families. Each family = sum of master integrals

$$\mathcal{A} = \sum_i c_i \, \mathcal{I}_i$$

- Master decomposition formula (abusing notation) [Mastrolia Mizera '18] [FGLMMMM '19]

$$\langle \mathcal{A} | = \sum_{ij} \langle \mathcal{A} \, | \, \mathcal{I}_j^\vee \rangle \, (C^{-1})_{ji} \, \langle \mathcal{I}_i |$$

of a vector $\langle \mathcal{A} |$ into a basis $\langle \mathcal{I}_i |$ using metric

$$C_{ij} := \langle \mathcal{I}_i \, | \, \mathcal{I}_j^\vee \rangle$$

- The scalar product $\langle \bullet \, | \, \bullet \rangle$ is called the **intersection number** [Cho '95 Matsumoto]
- Decomposition agrees with traditional IBP-based methods! [Tkachov '81] [Chetyrkin Tkachov '81] [Laporta '00]

# Baikov representation

Parametric representation of Feynman integrals: [Baikov '96] [Frellesvig '17 Papadopoulos]

$$\mathcal{I} = \int \mathrm{d}^d k \, \frac{1}{\mathcal{D}_1^{a_1} \dots \mathcal{D}_n^{a_n}} \Longrightarrow \mathcal{I} = \int_\Gamma (\mathcal{B}(z;\, x))^\gamma \, \frac{\mathrm{d}^n z}{z_1^{a_1} \dots z_n^{a_n}} = \int_\Gamma u \, \varphi$$

Features:

1. Holomorphic measure $\mathrm{d}^n z := \mathrm{d}z_1 \wedge \ldots \wedge \mathrm{d}z_n$
2. Baikov polynomial $\mathcal{B}(z;\, x) \in \mathbb{C}[z, x]$ with $x = \{p_i \cdot p_j, m^2, \ldots\}$
3. $\gamma = (d - \texttt{\#loops} - \texttt{\#legs} - 1)/2 \in \mathbb{C} \Longrightarrow \mathcal{B}^\gamma$ is multivalued!
4. Integration contour $\Gamma \subset \mathbb{C}^n$ such that $\mathcal{B}(\partial\Gamma) = 0$
5. Soon will find out how to compute it! [Hjalte's talk]

# Twisted cohomology

- Twist, connection, and covariant derivative [Mastrolia Mizera '18]

$$u := \prod_i (\mathcal{B}_i(z))^{\gamma_i}\,,\ \omega := \mathrm{d}\log(u)\,,\ \text{and } \nabla\psi := \mathrm{d}\psi + \omega\,\psi$$

- Intersection number integral

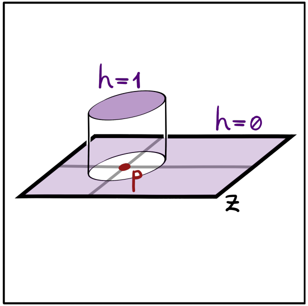
$$\langle \varphi \,|\, \varphi^\vee \rangle = \int_{\mathbb{C}} \mathrm{reg}(\varphi) \wedge \varphi^\vee$$

- Regularization in subtraction scheme . . . [Cho '95 Matsumoto]

$$\mathrm{reg}(\varphi) = \varphi - \sum_p \nabla(h_p\,\psi_p)\,, \quad \nabla\psi_p = \varphi \quad \text{near } z = p$$

- . . . localizes on singularities of $\omega$ = *zeroes* of $\mathcal{B}(z)$

$$\langle \varphi \,|\, \varphi^\vee \rangle = \sum_p \mathrm{Res}_{z=p}\big(\psi_p\,\varphi^\vee\big)$$



# Polynomial reduction and $p(z)$-adic expansion

- Ansatz for $p(z)$-adic expansion and the $\beta$-deformation [Fontana '22] [Fontana Peraro '23]

$$\psi = \sum_{n\in\mathbb{Z}} \sum_{i=0}^{\deg\mathcal{B}-1} \left(\mathcal{B}(z)\right)^n z^i\, \psi_{ni} \quad \rightsquigarrow \quad \psi = \sum_{n\in\mathbb{Z}} \sum_{i=0}^{\deg\mathcal{B}-1} \beta^n\, z^i\, \psi_{ni}$$

- Differential equation with polynomial reduction

$$\Big\lfloor \left(\mathcal{B}'(z)\,\partial_\beta + \partial_z + \omega\right)\psi - \varphi \Big\rfloor_{\langle\mathcal{B}(z)-\beta\rangle} = 0$$

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- Sum over poles via the global residue $\operatorname{Res}_{\langle\mathcal{B}\rangle}$ [Weinzierl '20]

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$$\langle \varphi \,|\, \varphi^\vee \rangle = \mathrm{Res}_{\langle\mathcal{B}\rangle}\big(\psi\,\varphi^\vee\big) = \tilde{c}_{-1,\deg\mathcal{B}-1}\,, \quad \psi\,\varphi^\vee = \sum_{ni} \big(\mathcal{B}(z)\big)^n \, z^i \, \tilde{c}_{ni}$$

- Ansatz issues: many variables and expressions $\Rightarrow$ replace everything with matrices!

# Companion matrices

# Promenade: Polynomials, ideals, and quotient rings

- Polynomial ideals are constraint equations: $\mathcal{B}(z) = 0$
- Reaccuring theme: roots of Baikov polynomial, changes of variables, expansions near singularities of DEQs ...
- Running example:

$$\langle \mathcal{B}(z) \rangle = \langle z^2 + b_1\, z + b_0 \rangle := \{\text{poly } p(z) \,|\, p(z) = \mathcal{B}(z) \times \text{other poly}\}$$

- Remainders of polynomial division modulo $\mathcal{B}(z)$ form a linear space

$$\mathcal{Q} = {}^{\{\text{polys in } z\}}\!\big/_{\langle \mathcal{B}(z) \rangle} = \mathrm{Span}(\{1,\, z\})$$

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$$z\, \mathcal{B}(z) = z^3 + b_1\, z^2 + b_0\, z = 0 \quad \Longrightarrow \quad \lfloor z^3 \rfloor_{\langle \mathcal{B}(z) \rangle} = \lfloor -b_1\, z^2 - b_0\, z \rfloor_{\langle \mathcal{B}(z) \rangle}$$

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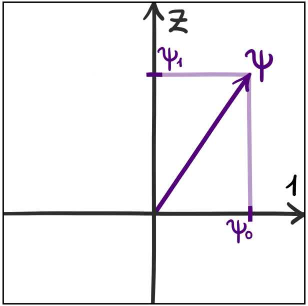
# Companion matrices

- Space of remainders $\mathcal{Q} = \mathrm{Span}(\{1,\ z\})$ is populated with vectors

$$\psi = \psi_0 + z\,\psi_1 = \left[\begin{array}{cc} 1 & z \end{array}\right] \cdot \left[\begin{array}{c} \psi_0 \\ \psi_1 \end{array}\right]$$

- **Companion matrices** $Q_f$ = operators $f$ acting on $\psi$



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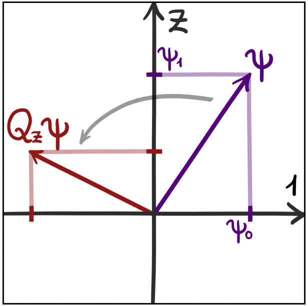

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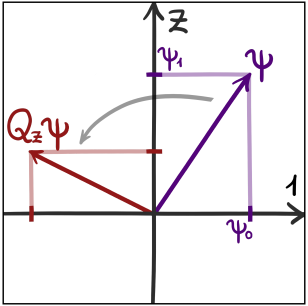


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- Example: using the reduction rule $z^2 \mapsto -b_1\,z - b_0$

$$\lfloor z\,\psi \rfloor_{\langle \mathcal{B}(z)\rangle} = \lfloor z\,\psi_0 + z^2\,\psi_1 \rfloor_{\langle \mathcal{B}(z)\rangle}$$

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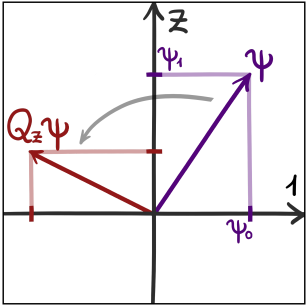


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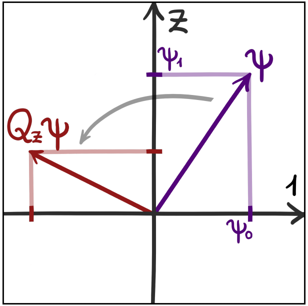


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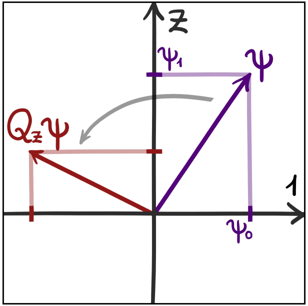


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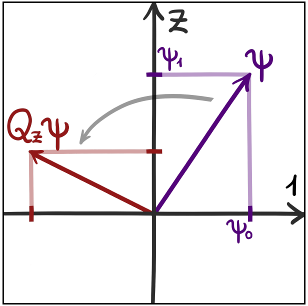


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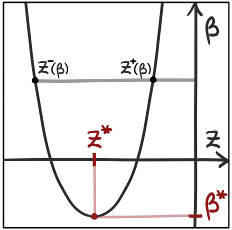
- Form commutative matrix algebra
- Polynomial reduction done with matrix multiplication!
- But how can we reach this space of remainders $\mathcal{Q}$?

[Sturmfels] [Cox, Little O'Shea]

# Riemann sheets

For $\mathcal{B}(z) - \beta = 0$ two roots $z^{\pm}(\beta) = -b_1/2 \pm \sqrt{b_1^2 - 4b_0 + 4\beta}/2$ give two Riemann sheets



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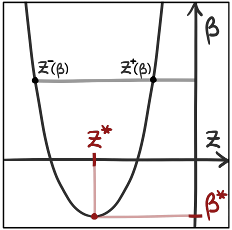


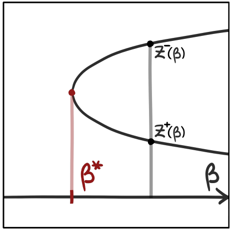

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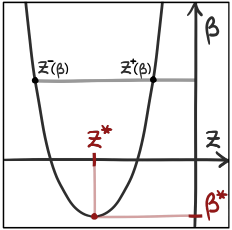


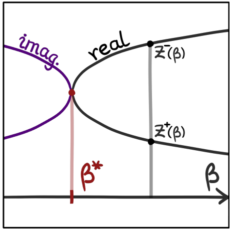

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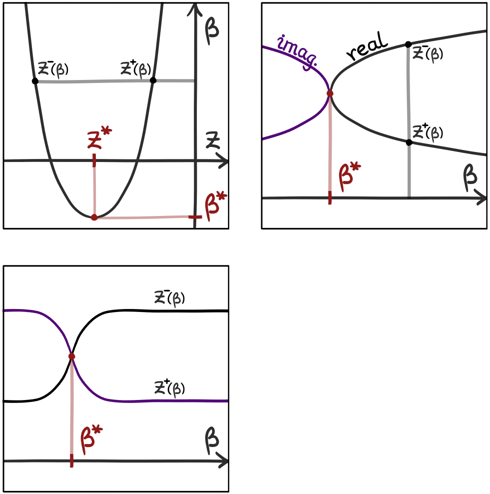

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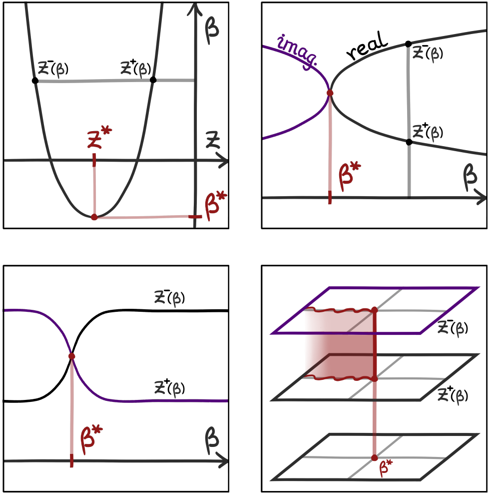

# Gauging away the roots

- Variable change $z \mapsto \beta = \mathcal{B}(z)$ leads to roots $z^{\pm}(\beta)$
- Turning a function $\psi(z)$ into a *vector*

$$\begin{bmatrix} \psi^+(\beta) \\ \psi^-(\beta) \end{bmatrix} := \begin{bmatrix} \psi\big(z^+(\beta)\big) \\ \psi\big(z^-(\beta)\big) \end{bmatrix}$$

- Vandermonde gauge transformation to the quotient space $\mathcal{Q}$

$$\begin{bmatrix} \psi^+(\beta) \\ \psi^-(\beta) \end{bmatrix} = \begin{bmatrix} 1 & z^+(\beta) \\ 1 & z^-(\beta) \end{bmatrix} \cdot \begin{bmatrix} \psi_0(\beta) \\ \psi_1(\beta) \end{bmatrix}$$

- In case of linear DEQs

$$\Big(\partial_z - \omega\Big)\,\psi(z) = \varphi$$

this implies

- $\psi^+$ and $\psi^-$ obey non-rational diagonal system of DEQs
- $\psi_0$ and $\psi_1$ obey rational non-diagonal system of DEQs

# Rational DEQs from companion matrices

- Consider an example: $u = z^{\rho}\,(\mathcal{B}(z))^{\gamma}$ and the DEQ

$$\left(\partial_z + \frac{\rho}{z} + \gamma\frac{\mathcal{B}'(z)}{\mathcal{B}(z)}\right)\psi(z) = \varphi(z)$$

- Change of variables $z \mapsto \beta = \mathcal{B}(z)$ introduces roots $z^{\pm}$ into a diagonal DEQ

$$\left(\mathcal{B}'(z^{\pm})\,\partial_\beta + \frac{\rho}{z^{\pm}} + \gamma\frac{\mathcal{B}'(z^{\pm})}{\mathcal{B}(z^{\pm})}\right)\psi^{\pm}(\beta) = \varphi^{\pm}(\beta)$$

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$$\left(\begin{bmatrix} \mathcal{B}'(z^+) & 0 \\ 0 & \mathcal{B}'(z^-) \end{bmatrix}\partial_\beta + \begin{bmatrix} \rho/z^+ + \gamma\mathcal{B}'/\mathcal{B}^+ & 0 \\ 0 & \rho/z^- + \gamma\mathcal{B}'/\mathcal{B}^- \end{bmatrix}\right)\cdot\begin{bmatrix} \psi^+ \\ \psi^- \end{bmatrix} = \begin{bmatrix} \varphi^+ \\ \varphi^- \end{bmatrix}$$

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- Rational DEQ system after gauging

$$\left(Q_{\mathcal{B}'}\,\partial_\beta + Q_{\partial_z} + \rho\,Q_{1/z} + \gamma\,Q_{\mathcal{B}'}/\beta\right)\begin{bmatrix} \psi_0 \\ \psi_1 \end{bmatrix} = \begin{bmatrix} \varphi^+ \\ \varphi^- \end{bmatrix}$$

with companion matrices

$$Q_{\mathcal{B}'} = \begin{bmatrix} b_1 & 2(-b_0+\beta) \\ 2 & -b_1 \end{bmatrix},\; Q_{1/z} = \frac{1}{-b_0+\beta}\begin{bmatrix} b_1 & -b_0+\beta \\ 1 & 0 \end{bmatrix},\; Q_{\partial_z} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

# Intersection numbers

# Expanding and taking the residue

- To finish the story, let's compute the intersection $\langle \varphi \,|\, \varphi^\vee \rangle = \langle 1/\mathcal{B}(z)^2 \,|\, 1/\mathcal{B}(z) \rangle$
- Multiply with the inverse $Q_{\mathcal{B}'}^{-1}$, series expand $\beta \to 0$

$$\left(\partial_\beta + \frac{\gamma}{\beta} + \mathcal{O}(1)\right) \cdot \left( \begin{bmatrix} \psi_{00} \\ \psi_{10} \end{bmatrix} + \beta \begin{bmatrix} \psi_{01} \\ \psi_{11} \end{bmatrix} \right) - \left( \frac{1}{\beta(b_1^2 - 4b_0)} + \mathcal{O}(1) \right) \cdot \begin{bmatrix} b_1 \\ 2 \end{bmatrix} = 0$$

- Tensor $Q_f$ with matrix "representation" of Weyl algebra $\Rightarrow$ final block-triangular system

$$\left[\begin{array}{c|cccc} 1 & 0 & 0 & 0 & -1 \\ \hline 0 & \gamma & 0 & 0 & 0 \\ 0 & 0 & \gamma & 0 & 0 \\ 0 & * & * & \gamma+1 & 0 \\ 0 & * & * & 0 & \gamma+1 \end{array}\right] \cdot \left[\begin{array}{c} \langle \varphi \,|\, \varphi^\vee \rangle \\ \hline \psi_{00} \\ \psi_{10} \\ \psi_{01} \\ \psi_{11} \end{array}\right] = \begin{bmatrix} 0 \\ \frac{b_1}{b_1^2 - 4b_0} \\ \frac{2}{b_1^2 - 4b_0} \\ * \\ * \end{bmatrix}$$

- The solution [Crisanti Smith '24]

$$\langle \varphi \,|\, \varphi^\vee \rangle = \psi_{11} = \frac{4(2\gamma - 1)}{(b_1^2 - 4b_0)^2 \gamma(\gamma - 1)}$$

- Companion matrices and tensor avoid the auxiliary variables $z$ and $\beta$

# Promenade: The multivariate case

- Use fibration method: integrate one variable at a time

- Each layer of this recursion will have its own set of masters

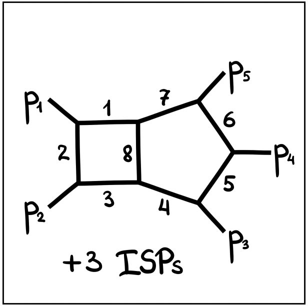
- Leads to a similar univariate DEQ system for vector-valued $\psi(z)$

- Similar to transition from QED to QCD: one new non-abelian tensor index

# Application to 2-loop 5-point reduction

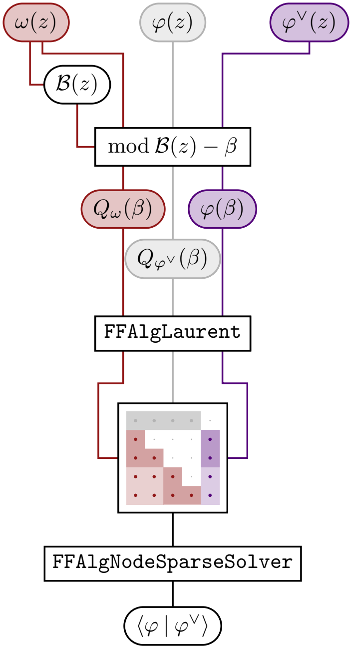
- Result of recent efforts [2401.01897] [2408.16668] [repo @]
- Decomposition into 62 masters (no symmetry relations)

$$\mathcal{I} = \int u \, \frac{z_9^2}{z_1 z_2 z_3 z_4 z_5 z_6 z_7 z_8} \, \mathrm{d}z \equiv \mathcal{I}_{11111111-200}$$




- Computed intersections of 11-variate integrals
- Used spanning cuts strategy
- Intermediate dimensions: $\{1,1,2,2,4,12,28,31\}$, $\{1,1,2,3,6,18,26,27\}$ and easier
- Automatic intermediate (candidate) bases generation `GetBasis.m`
- Extensive usage of tensors in FiniteFlow [Peraro '19 FiniteFlow]
- For now only numerical: $s_{ij}$ set to $\mathbb{Q}$-numbers

# The FiniteFlow data graph



# Conclusion

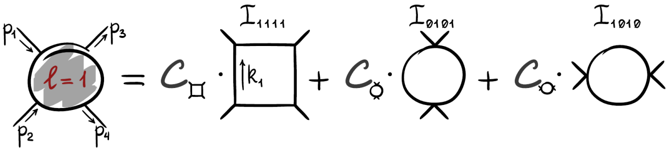
# Conclusion

- **Twisted cohomology** describes multivariate integrals of multivalued functions, which are commonly encountered in perturbative analysis.
- **Intersection number** is a scalar product on the space of twisted periods, allowing for the direct decomposition of Feynman integrals via the master decomposition formula.
- Today we explored **tensor structures** of intersection numbers that arise from the companion matrix representation.
- This new formulation enabled the full reduction to master integrals of the massless 5-point 2-loop Feynman integral family.

# Backup

# Feynman integrals

A popular tool to compute theoretical predictions for scattering processes




Integrals over loop momenta of rational functions

$$\mathcal{I}^{(\ell)}_{a_1 \ldots a_n}(p_1, \ldots, p_{E+1}) = \int \prod_{j=1}^{\ell} \mathrm{d}^d k_j \, \frac{1}{\mathcal{D}_1^{a_1} \ldots \mathcal{D}_n^{a_n}}$$

1. $E =$ #legs $- 1 =$ independent external momenta
2. Exponents $a_i \in \mathbb{Z}$
3. Dimensional regularization: $d = 4 - 2\varepsilon \in \mathbb{C}$
4. $\mathcal{D}_i$ is either propagator or scalar product: e.g. $k^2 - m^2$ or $k \cdot p$
5. Number of factors $n = \ell(\ell + 1)/2 + E\ell$

# A glimpse of twisted cohomology

- Finite dimensional vector space of Feynman integrals with *intersection number* as metric [Mastrolia Mizera '18] [FGLMMMM '19] [FGMMMM '19] [Weinzierl '20] [FGLMMMM '20] [2209.01997] [Cacciatori Mastrolia] [Fontana Peraro '23] [2401.01897] [Crisanti Smith '24]
- Many ways to count master integrals $m$:
  1. Laporta algorithm [Laporta '00]
  2. Number of critical points $\mathrm{d}\log \mathcal{B}(z) = 0$ [Baikov '05] [Lee '13 Pomeransky]
  3. Number of independent integration contours [Bosma, Sogaard Zhang '17] [Primo '17 Tancredi] [Frellesvig '21]
  4. Number of independent integrands [Mastrolia Mizera '18]
  5. Holonomic rank of GKZ system (volume of $\Delta_A$ polytope) [de la Cruz '19] [Klausen '21] [2204.12983]
- Master decomposition formula $\Rightarrow$ no need to row reduce huge systems! [Mastrolia Mizera] [FGLMMMM]
- Benefits from finite fields techniques [Peraro '19 FiniteFlow]

Some nice reviews: [MathemAmplitudes'19] [Cacciatori '21 Conti, Trevisan]

# Companion matrices in FiniteFlow

To get $Q_p$ of a polynomial $p(z) = p_0 + z\,p_1 + \ldots + z^n\,p_n$ with the ideal $\langle b_0 - \beta + z\,b_1 + \ldots + z^{\kappa-1}\,b_{\kappa-1} + z^\kappa \rangle$ row-reduce the augmented Sylvester matrix

$$S_{p,\mathcal{B}} = \left[\begin{array}{ccc|ccccc|cccc}
1 & & & p_n & p_{n-1} & \cdots & \cdots & p_1 & p_0 & & & \\
 & \ddots & & & \ddots & & & & & \ddots & & \\
 & & 1 & & & p_n & p_{n-1} & \cdots & \cdots & p_1 & p_0 & \\
 & & & & & & p_n & p_{n-1} & \cdots & \cdots & p_1 & p_0 \\
\hline
 & & & 1 & b_{\kappa-1} & \cdots & b_1 & b_0 & & & & \\
 & & & & 1 & b_{\kappa-1} & \cdots & b_1 & b_0 & & & \\
 & & & & & \ddots & & & & \ddots & & \\
 & & & & & & 1 & b_{\kappa-1} & \cdots & b_1 & b_0 & \\
 & & & & & & & 1 & b_{\kappa-1} & \cdots & b_1 & b_0
\end{array}\right]
\begin{array}{l}
\left.\vphantom{\begin{array}{c}1\\1\\1\\1\end{array}}\right\updownarrow \kappa \\
\left.\vphantom{\begin{array}{c}1\\1\\1\\1\\1\end{array}}\right\updownarrow n
\end{array}$$

with the last $\kappa$ columns marked $\overleftrightarrow{\ \kappa\ }$.