Antenna subtraction for processes with identified particles at hadron colliders

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Introduction

- Identified hadrons at colliders described by fragmentation functions (FFs)
  - Semi-inclusive observables (SIA, SIDIS ...) $\leftarrow$ coefficient functions
  - Exclusive observables (W+D, H-in-jet(s) ...) $\leftarrow$ subtraction method
- Both light (π,K,p ...) and heavy (D,B ...) hadrons
- Exclusive observables:
  - NNLO required (FFs @ (a)NNLO)
  - Antenna subtraction $\rightarrow$ extension to identified hadrons
  - Fragmentation antenna functions

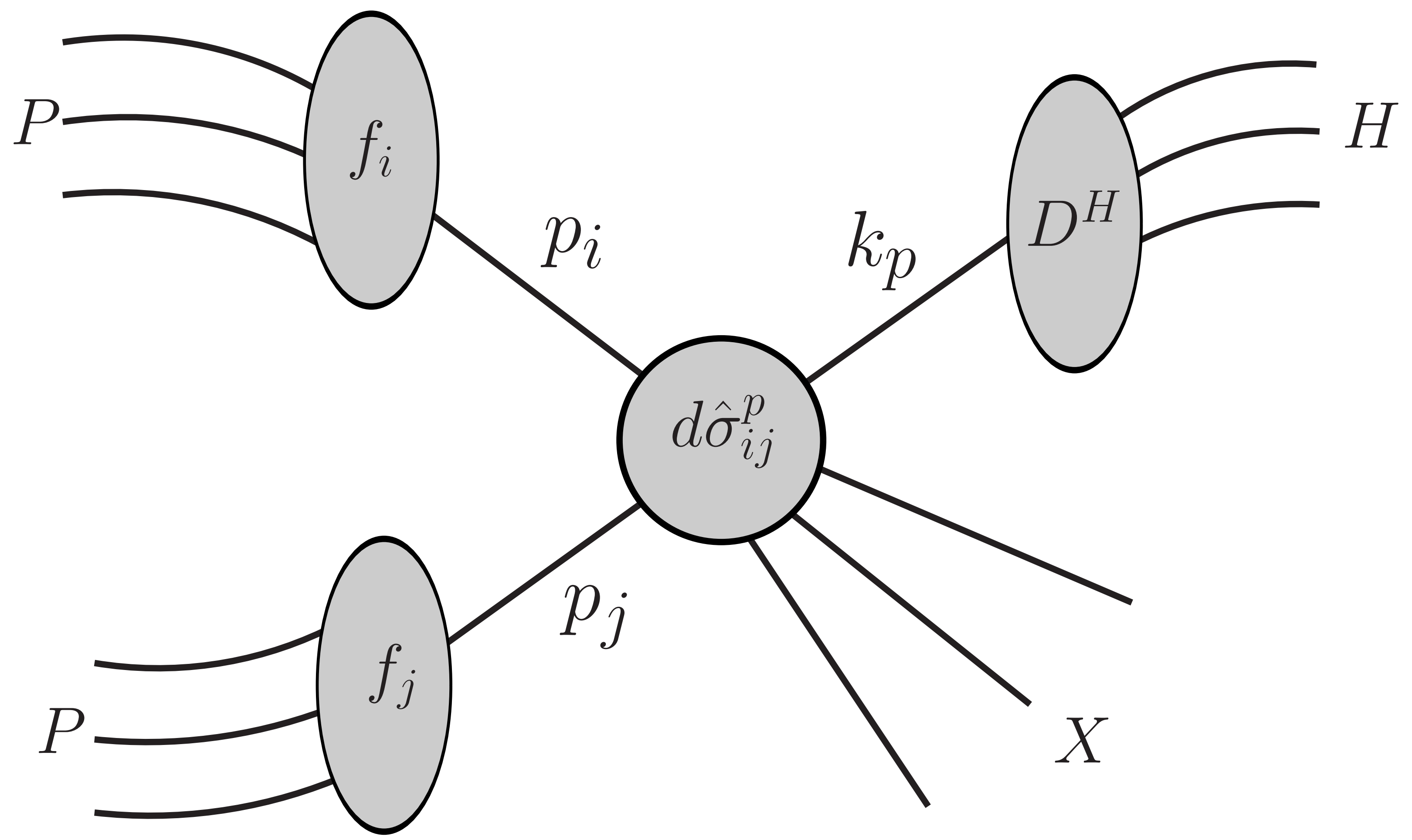
Work based on 2406.09925 with T. Gehrmann, M. Marcoli, R. Schürmann, and G. Stagnitto

Introduction

- Cross-section for identified final-state hadron at hadron colliders ($\mu_{f/a}$ factorisation scales)

$$\mathrm{d}\sigma^H = \sum_{i,j,p} \mathrm{d}\hat{\sigma}^p_{ij}(\xi_1, \xi_2, \eta, \mu_r, \mu_f, \mu_a) \otimes f_i(\xi_1, \mu_f) \otimes f_j(\xi_2, \mu_f) \otimes D^H_p(\eta, \mu_a)$$

incoming (p_i/ξ_1,p_j/ξ_2) and outgoing (ηk_p) hadron momenta



- Parton-level cross section $i + j \to p + X$

$$\mathrm{d}\hat{\sigma} = \mathrm{d}\hat{\sigma}_{\mathrm{LO}} + \left(\frac{\alpha_s}{2\pi}\right) \mathrm{d}\hat{\sigma}_{\mathrm{NLO}} + \left(\frac{\alpha_s}{2\pi}\right)^2 \mathrm{d}\hat{\sigma}_{\mathrm{NNLO}} + \ldots$$

- Leading order: parton-level process $\{k_1, \ldots, k_p, \ldots, k_n; p_i, p_j\}$

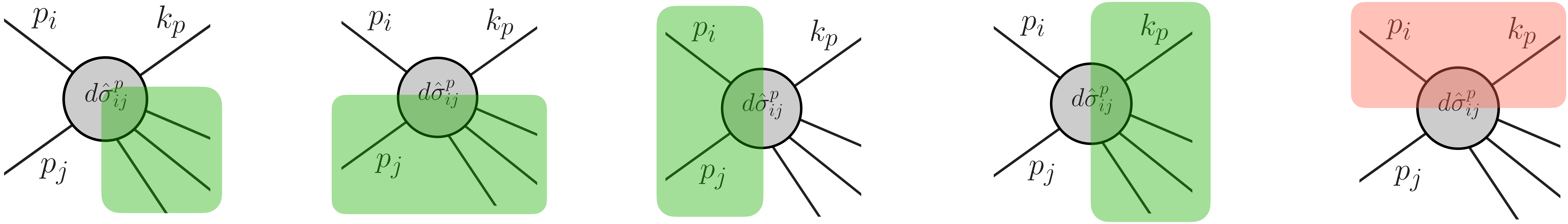
$$\mathrm{d}\hat{\sigma}_{LO} = \mathcal{N}_B \int \mathrm{d}\Phi_n(k_1, \ldots, k_p, \ldots, k_n; p_i, p_j) \frac{1}{S_n} M^0_n(k_1, \ldots, k_p, \ldots, k_n; p_i, p_j) J(k_1, \ldots, k_p, \ldots, k_n; p_i, p_j; \xi_1, \xi_2, \eta)$$

With J observable definition (e.g. hadron p_T or rapidity)

Antenna subtraction

$$d\hat{\sigma} = d\hat{\sigma}_{\mathrm{LO}} + \left(\frac{\alpha_s}{2\pi}\right) d\hat{\sigma}_{\mathrm{NLO}} + \left(\frac{\alpha_s}{2\pi}\right)^2 d\hat{\sigma}_{\mathrm{NNLO}} + \ldots$$

- Higher order corrections divergent → add subtraction terms to make them finite
- Antenna subtraction method: systematic procedure to construct the subtraction terms
  - $(n+1)$ and $(n+2)$ particle phase spaces factorise into reduced n-particle and antenna phase spaces
  - antenna functions: all unresolved radiation between two colour-ordered hard radiator partons (extracted from squared matrix elements)
- Type of antenna function based on kinematics of hard emitters
  - Without fragmentation: final-final (FF) **[Gehrmann-De Ridder et al. '05]**, initial-final (IF) **[Daleo et al. '07, '09]** and Initial-initial (II) **[Gehrmann, Monni '11] [Gehrmann-De Ridder et al. '12]**
  - With fragmentation: final-final (FFh) **[Gehrmann, Stagnitto '20]** and initial-final (IFh)



FF ✓ IF ✓ II ✓ FFh ✓ IFh This work

Antenna subtraction at NLO

- At NLO real (R) and virtual (V) corrections

$$\mathrm{d}\hat{\sigma}_{\mathrm{NLO}} = \int_{n+1} \left(\mathrm{d}\hat{\sigma}^{R}_{\mathrm{NLO}} - \mathrm{d}\hat{\sigma}^{S}_{\mathrm{NLO}}\right) + \int_{n} \left(\mathrm{d}\hat{\sigma}^{V}_{\mathrm{NLO}} - \mathrm{d}\hat{\sigma}^{T}_{\mathrm{NLO}}\right)$$

$\mathrm{d}\hat{\sigma}^{S,T}_{NLO}$ subtraction terms make integrals finite with mass factorisation (MF)

$\mathrm{d}\hat{\sigma}^{T}_{\mathrm{NLO}} = -\,\mathrm{d}\hat{\sigma}^{MF}_{\mathrm{NLO}} - \int_{1} \mathrm{d}\hat{\sigma}^{S}_{\mathrm{NLO}}$ with explicit singularities in $d = 4 - 2\epsilon$

- Subtraction terms made of (un)integrated antennae

$\mathrm{d}\hat{\sigma}^{S}_{NLO} \supseteq X_3^0 M_n^0 J \leftarrow$ un-integrated 3-parton tree level antenna function X_3^0

$\mathrm{d}\hat{\sigma}^{T}_{NLO} \supseteq J_2^{(1)} M_n^0 J \leftarrow$ integral of X_3^0 ($\mathcal{X}_3^0$) and MF in $J_2^{(1)}(p, a) = c_1 \mathcal{X}_3^0 + c_2 \Gamma^{(1)}$

- Convenient formulation with integrated dipoles

Antenna subtraction at NNLO

- At NNLO real-real (RR), real-virtual (RV) and double virtual (VV) corrections

$$\mathrm{d}\hat{\sigma}_{\mathrm{NNLO}} = \int_{n+2} \left(\mathrm{d}\hat{\sigma}^{RR}_{\mathrm{NNLO}} - \mathrm{d}\hat{\sigma}^{S}_{\mathrm{NNLO}}\right) + \int_{n+1} \left(\mathrm{d}\hat{\sigma}^{RV}_{\mathrm{NNLO}} - \mathrm{d}\hat{\sigma}^{T}_{\mathrm{NNLO}}\right) + \int_{n} \left(\mathrm{d}\hat{\sigma}^{VV}_{\mathrm{NNLO}} - \mathrm{d}\hat{\sigma}^{U}_{\mathrm{NNLO}}\right)$$

each integration is numerically well defined → Monte Carlo (***NNLOJET***)

- Three types of subtraction terms related as

$$\mathrm{d}\hat{\sigma}^{S}_{\mathrm{NNLO}} = \mathrm{d}\hat{\sigma}^{S,1}_{\mathrm{NNLO}} + \mathrm{d}\hat{\sigma}^{S,2}_{\mathrm{NNLO}} \supseteq X_4^0 M_n^0 J \leftarrow \text{4-parton tree-level antennae } X_4^0$$

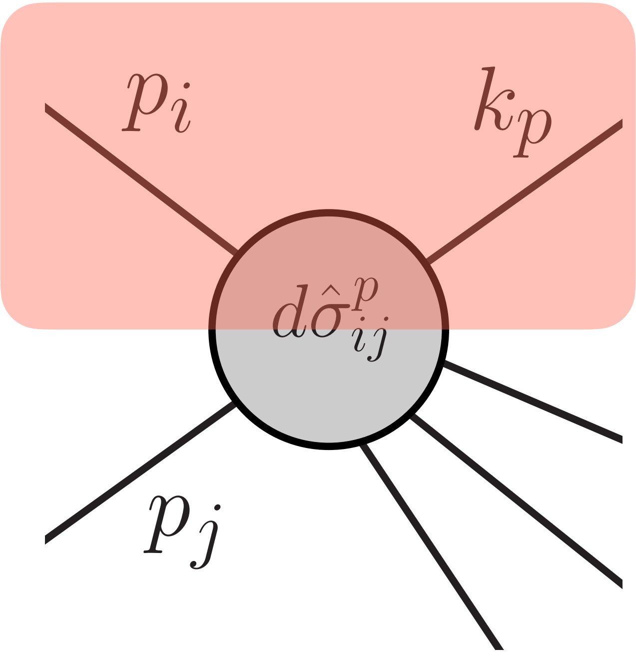
$$\mathrm{d}\hat{\sigma}^{T}_{\mathrm{NNLO}} = \mathrm{d}\hat{\sigma}^{VS,1}_{\mathrm{NNLO}} - \mathrm{d}\hat{\sigma}^{MF,1}_{\mathrm{NNLO}} - \int_1 \mathrm{d}\hat{\sigma}^{S,1}_{\mathrm{NNLO}} \supseteq X_3^1 M_n^0 J \leftarrow \text{3-parton 1-loop antennae } X_3^1$$

$$\mathrm{d}\hat{\sigma}^{U}_{\mathrm{NNLO}} = -\,\mathrm{d}\hat{\sigma}^{MF,2}_{\mathrm{NNLO}} - \int_1 \mathrm{d}\hat{\sigma}^{VS,1}_{\mathrm{NNLO}} - \int_2 \mathrm{d}\hat{\sigma}^{S,2}_{\mathrm{NNLO}} \supseteq J_2^{(2)} M_n^0 J \leftarrow \text{integrated antennae and MF}$$

counterterms in integrated dipoles

$$J_2^{(2)}(p,a) = c_1 \mathcal{X}_4^0 + c_2 \mathcal{X}_3^1 + c_3 \mathcal{X}_3^0 \otimes \mathcal{X}_3^0 + c_4 \frac{\beta_0}{\epsilon} \mathcal{X}_3^0 + c_5 \overline{\Gamma}^{(2)} + c_6 \Gamma^{(1)} \otimes \Gamma^{(1)} + c_7 \Gamma^{(1)} \otimes \mathcal{X}_3^0$$

Initial-final with fragmentation



- Same structure for initial-final fragmentation subtraction terms
- New integrated antenna functions needed (initial-final kinematics)
- $\mathcal{X}_{3,i}^{0,\mathrm{id}.p}$ and $\mathcal{X}_{3,i}^{1,\mathrm{id}.p}$ already computed in photon fragmentation **[Gehrmann, Schürmann '22]**

$$\mathcal{X}_{3,i}^{0,\mathrm{id}.p}(z,x) \propto \int \mathrm{d}\Phi_2 \frac{Q^2}{2\pi} z^{1-2\epsilon} X_{3,i}^{0,\mathrm{id}.p} \delta(z-z_3)$$

$$\mathcal{X}_{3,i}^{1,\mathrm{id}.p}(z,x) \propto \int \mathrm{d}\Phi_2 \frac{Q^2}{2\pi} z^{1-2\epsilon} X_{3,i}^{1,\mathrm{id}.p} \delta(z-z_3)$$

$$z = \frac{s_{ip}}{s_{ip}+s_{ik}} = z_3 \text{ and } x = \frac{s_{ip}+s_{ik}-s_{pk}}{s_{ip}+s_{ik}}$$

$$\mathcal{X}_{4,i}^{0,\mathrm{id}.p}(z,x) \propto \int \mathrm{d}\Phi_3 \frac{Q^2}{2\pi} z^{1-2\epsilon} X_{4,i}^{0,\mathrm{id}.p} \delta(z-z_4)$$

$$z = \frac{s_{ip}}{s_{ip}+s_{ik}+s_{il}} = z_4 \text{ and } x = \frac{s_{ip}+s_{ik}+s_{il}-s_{pk}-s_{pl}-s_{kl}}{s_{ip}+s_{ik}+s_{il}}$$

Integration of initial-final 3-parton antennae

- Initial-final phase space for 3-parton fragmentation antenna functions fully constrained

$$\mathcal{X}_{3,i}^{0,\mathrm{id}.p}(x,z) = \frac{1}{C(\epsilon)} \int \mathrm{d}\Phi_2(k_p, k_k; q, p_i) X_{3,i}^{0,\mathrm{id}.p} \frac{Q^2}{2\pi} \delta\left(z - \frac{s_{ip}}{s_{ip}+s_{ik}}\right) \propto \mathcal{J}(x,z) X_{3,i}^{0,\mathrm{id}.p}(x,z)$$

- Jacobian factor $\mathcal{J}(x,z) = (1-x)^{-\epsilon} x^{\epsilon} z^{-\epsilon} (1-z)^{-\epsilon}$
- Same for 1-loop $\mathcal{X}_{3,i}^{1,\mathrm{id}.p}$
  - Avoid ambiguities associated with the analytic continuation of box integrals: segment the (x,z)-plane into four sectors → expressions real and continuous across boundaries **[Gehrmann, Schürmann '22]**
- Full set of antennae: A, D, E, F and G types (partonic content) ✓

Integration of initial-final 4-parton antennae

- Kinematics $q + p_i \to k_p + k_l + k_k$ with $x = Q^2/(2p \cdot q)$ and $Q^2 = -q^2$

- Differential in momentum fraction z

$$\mathcal{X}^{0,\mathrm{id}.p}_{4,i}(z,x) = \frac{1}{C(\epsilon)^2} \int \mathrm{d}\Phi_3(k_p, k_k, k_l; p_i, q)\,\delta\left(z - x\frac{(p_i+k_p)^2}{Q^2}\right)\frac{Q^2}{2\pi} X^{0,\mathrm{id}.p}_{4,i}$$

- 12 denominators (4 cut propagators) → 21 MI in 12 families

- MI solved with differential equations, boundary conditions from z -integration and comparing to inclusive result

- $$I[-3,7] = \frac{Q^2(2\pi)^{-2d+3}}{x}\int \mathrm{d}^d k_p \mathrm{d}^d k_l\, \delta(D_9)\delta(D_{10})\delta(D_{11})\delta(D_{12})\frac{D_3}{D_7}$$

- A, B and C families in **[Gehrmann, Schürmann '22]**

- MIs in ancillary file

$$\begin{aligned}
D_1 &= (q-k_p)^2\,,\\
D_2 &= (p_i+q-k_p)^2\,,\\
D_3 &= (p_i-k_l)^2\,,\\
D_4 &= (q-k_l)^2\,,\\
D_5 &= (p_i+q-k_l)^2\,,\\
D_6 &= (q-k_p-k_l)^2\,,\\
D_7 &= (p_i-k_p-k_l)^2\,,\\
D_8 &= (k_p+k_l)^2\,,\\
D_9 &= k_p^2\,,\\
D_{10} &= k_l^2\,,\\
D_{11} &= (q+p_i-k_p-k_l)^2\,,\\
D_{12} &= (p_i-k_p)^2 + Q^2\frac{z}{x}\,,
\end{aligned}$$

family	master	deepest pole	at $x=1$	at $z=1$
	$I[0]$	ϵ^0	$(1-x)^{1-2\epsilon}$	$(1-z)^{1-2\epsilon}$
A	$I[5]$	ϵ^{-1}	$(1-x)^{-2\epsilon}$	$(1-z)^{1-2\epsilon}$
	$I[2,3,5]$	ϵ^{-2}	$(1-x)^{-1-2\epsilon}$	$(1-z)^{-1-2\epsilon}$
B	$I[7]$	ϵ^0	$(1-x)^{1-2\epsilon}$	$(1-z)^{1-2\epsilon}$
	$I[-2,7]$	ϵ^0	$(1-x)^{1-2\epsilon}$	$(1-z)^{1-2\epsilon}$
	$I[-3,7]$	ϵ^0	$(1-x)^{1-2\epsilon}$	$(1-z)^{1-2\epsilon}$
	$I[2,3,7]$	ϵ^{-2}	$(1-x)^{-2\epsilon}$	$(1-z)^{-1-2\epsilon}$
C	$I[5,7]$	ϵ^{-1}	$(1-x)^{-2\epsilon}$	$(1-z)^{1-2\epsilon}$
	$I[3,5,7]$	ϵ^{-2}	$(1-x)^{-2\epsilon}$	$(1-z)^{-2\epsilon}$
D	$I[1]$	ϵ^0	$(1-x)^{-2\epsilon}$	$(1-z)^{-2\epsilon}$
	$I[1,4]$	ϵ^0	$(1-x)^{-2\epsilon}$	$(1-z)^{-2\epsilon}$
	$I[1,3,4]$	ϵ^{-1}	$(1-x)^{-2\epsilon}$	$(1-z)^{-1-2\epsilon}$
E	$I[1,3,5]$	ϵ^{-2}	$(1-x)^{-2\epsilon}$	$(1-z)^{-1-2\epsilon}$
G	$I[1,3,8]$	ϵ^{-2}	$(1-x)^{-2\epsilon}$	$(1-z)^{-1-2\epsilon}$
H	$I[1,4,5]$	ϵ^{-1}	$(1-x)^{-1-2\epsilon}$	$(1-z)^{-2\epsilon}$
I	$I[2,4,5]$	ϵ^{-2}	$(1-x)^{-1-2\epsilon}$	$(1-z)^{-2\epsilon}$
J	$I[4,7]$	ϵ^0	$(1-x)^{-2\epsilon}$	$(1-z)^{-2\epsilon}$
	$I[3,4,7]$	ϵ^{-1}	$(1-x)^{-2\epsilon}$	$(1-z)^{-2\epsilon}$
K	$I[3,5,8]$	ϵ^{-2}	$(1-x)^{-1-2\epsilon}$	$(1-z)^{-2\epsilon}$
L	$I[4,5,7]$	ϵ^{-1}	$(1-x)^{-1-2\epsilon}$	$(1-z)^{-2\epsilon}$
M	$I[4,5,8]$	ϵ^{-1}	$(1-x)^{-1-2\epsilon}$	$(1-z)^{-2\epsilon}$

Integration of initial-final antennae (RR)

- Full set of $\mathscr{X}_{4,i}^{0,\text{id}.p}$ now computed
- Different hadronic species in initial and final-state → larger set of antennae wrt initial-initial and final-final kinematics
- E.g. $\mathscr{A}_{4,q}^{0,\text{id}.g_3} \neq \mathscr{A}_{4,q}^{0,\text{id}.g_4}$ since $A_4^0(\hat{1}_q,3_g^{\text{id.}},4_g,2_{\bar{q}}) \neq A_4^0(\hat{1}_q,3_g,4_g^{\text{id.}},2_{\bar{q}})$
- A, B, C, D, E, F, G and H types
- Pole structure can be checked ↓

Notation	Integral of	Integrand symm.
Hard radiators: quark-quark		
$\mathcal{A}_{4,q}^{0,\text{id}.q}$	$A_4^0(1_q^{\text{id.}},3_g,4_g,\hat{2}_{\bar{q}})$	$1\leftrightarrow 2 + 3\leftrightarrow 4$
$\mathcal{A}_{4,g_3}^{0,\text{id}.q}$	$A_4^0(1_q^{\text{id.}},\hat{3}_g,4_g,2_{\bar{q}})$	
$\mathcal{A}_{4,g_4}^{0,\text{id}.q}$	$A_4^0(1_q^{\text{id.}},3_g,\hat{4}_g,2_{\bar{q}})$	
$\mathcal{A}_{4,q}^{0,\text{id}.g_3}$	$A_4^0(1_q,3_g^{\text{id.}},4_g,\hat{2}_{\bar{q}})$	
$\mathcal{A}_{4,q}^{0,\text{id}.g_4}$	$A_4^0(1_q,3_g,4_g^{\text{id.}},\hat{2}_{\bar{q}})$	
$\mathcal{A}_{4,g}^{0,\text{id}.g}$	$A_4^0(1_q,3_g^{\text{id.}},\hat{4}_g,2_{\bar{q}})$	
$\widetilde{\mathcal{A}}_{4,q}^{0,\text{id}.q}$	$\widetilde{A}_4^0(1_q^{\text{id.}},3_g,4_g,\hat{2}_{\bar{q}})$	$1\leftrightarrow 2\ ,\ 3\leftrightarrow 4$
$\widetilde{\mathcal{A}}_{4,g}^{0,\text{id}.q}$	$\widetilde{A}_4^0(1_q^{\text{id.}},\hat{3}_g,4_g,2_{\bar{q}})$	
$\widetilde{\mathcal{A}}_{4,q}^{0,\text{id}.g}$	$\widetilde{A}_4^0(1_q,3_g^{\text{id.}},4_g,\hat{2}_{\bar{q}})$	
$\widetilde{\mathcal{A}}_{4,g}^{0,\text{id}.g}$	$\widetilde{A}_4^0(1_q,3_g^{\text{id.}},\hat{4}_g,2_{\bar{q}})$	
$\mathcal{B}_{4,q}^{0,\text{id}.q}$	$B_4^0(1_q^{\text{id.}},3_{q'},4_{\bar{q}'},\hat{2}_{\bar{q}})$	
$\mathcal{B}_{4,q'}^{0,\text{id}.q}$	$B_4^0(1_q^{\text{id.}},3_{q'},\hat{4}_{\bar{q}'},2_{\bar{q}})$	
$\mathcal{B}_{4,q}^{0,\text{id}.q'}$	$B_4^0(1_q,3_{q'}^{\text{id.}},4_{\bar{q}'},\hat{2}_{\bar{q}})$	
$\mathcal{C}_{4,q_2}^{0,\text{id}.q_1}$	$C_4^0(1_q^{\text{id.}},3_q,4_{\bar{q}},\hat{2}_{\bar{q}})$	
$\mathcal{C}_{4,q_3}^{0,\text{id}.q_1}$	$C_4^0(1_q^{\text{id.}},\hat{3}_q,4_{\bar{q}},2_{\bar{q}})$	
$\mathcal{C}_{4,\bar{q}_1}^{0,\text{id}.\bar{q}_2}$	$C_4^0(\hat{1}_q,3_q,4_{\bar{q}},2_{\bar{q}}^{\text{id.}})$	
$\mathcal{C}_{4,\bar{q}_1}^{0,\text{id}.q_3}$	$C_4^0(\hat{1}_q,3_q^{\text{id.}},4_{\bar{q}},2_{\bar{q}})$	

Notation	Integral of	Integrand symm.
Hard radiators: gluon-gluon		
$\mathcal{F}_{4,g_2}^{0,\text{id}.g_1}$	$F_4^0(1_g^{\text{id.}},\hat{2}_g,3_g,4_g)$	
$\mathcal{F}_{4,g_3}^{0,\text{id}.g_1}$	$F_4^0(1_g^{\text{id.}},2_g,\hat{3}_g,4_g)$	
$\mathcal{G}_{4,g}^{0,\text{id}.g}$	$G_4^0(1_g^{\text{id.}},3_q,4_{\bar{q}},\hat{2}_g)$	
$\mathcal{G}_{4,q}^{0,\text{id}.g_1}$	$G_4^0(1_g^{\text{id.}},3_q,\hat{4}_{\bar{q}},2_g)$	
$\mathcal{G}_{4,q}^{0,\text{id}.g_2}$	$G_4^0(1_g,3_q,\hat{4}_{\bar{q}},2_g^{\text{id.}})$	
$\mathcal{G}_{4,g_1}^{0,\text{id}.q}$	$G_4^0(\hat{1}_g,3_q^{\text{id.}},4_{\bar{q}},2_g)$	
$\mathcal{G}_{4,q}^{0,\text{id}.q}$	$G_4^0(1_g,3_q^{\text{id.}},\hat{4}_{\bar{q}},2_g)$	
$\mathcal{G}_{4,g_2}^{0,\text{id}.q}$	$G_4^0(1_g,3_q^{\text{id.}},4_{\bar{q}},\hat{2}_g)$	
$\widetilde{\mathcal{G}}_{4,q}^{0,\text{id}.g}$	$\widetilde{G}_4^0(1_g^{\text{id.}},3_q,\hat{4}_{\bar{q}},2_g)$	
$\widetilde{\mathcal{G}}_{4,g}^{0,\text{id}.g}$	$\widetilde{G}_4^0(\hat{1}_g,3_q,4_{\bar{q}},2_g^{\text{id.}})$	
$\widetilde{\mathcal{G}}_{4,g}^{0,\text{id}.q}$	$\widetilde{G}_4^0(\hat{1}_g,3_q^{\text{id.}},4_{\bar{q}},2_g)$	
$\widetilde{\mathcal{G}}_{4,q}^{0,\text{id}.q}$	$\widetilde{G}_4^0(1_g,3_q^{\text{id.}},\hat{4}_{\bar{q}},2_g)$	
$\mathcal{H}_{4,q}^{0,\text{id}.q}$	$H_4^0(1_q^{\text{id.}},\hat{2}_{\bar{q}},3_{q'},4_{\bar{q}'})$	$1\leftrightarrow 2\ ,\ 3\leftrightarrow 4\ ,\ 1\leftrightarrow 3 + 2\leftrightarrow 4$
$\mathcal{H}_{4,q'}^{0,\text{id}.q}$	$H_4^0(1_q^{\text{id.}},2_{\bar{q}},3_{q'},\hat{4}_{\bar{q}'})$	

Notation	Integral of	Integrand symm.
Hard radiators : quark-gluon		
$\mathcal{D}_{4,g_2}^{0,\text{id}.q}$	$D_4^0(1_q^{\text{id.}},\hat{2}_g,3_g,4_g)$	$2\leftrightarrow 4$
$\mathcal{D}_{4,g_3}^{0,\text{id}.q}$	$D_4^0(1_q^{\text{id.}},2_g,\hat{3}_g,4_g)$	
$\mathcal{D}_{4,q}^{0,\text{id}.g_2}$	$D_4^0(\hat{1}_{\bar{q}},2_g^{\text{id.}},3_g,4_g)$	
$\mathcal{D}_{4,g_3}^{0,\text{id}.g_2}$	$D_4^0(1_q,2_g^{\text{id.}},\hat{3}_g,4_g)$	
$\mathcal{D}_{4,g_4}^{0,\text{id}.g_2}$	$D_4^0(1_q,2_g^{\text{id.}},3_g,\hat{4}_g)$	
$\mathcal{D}_{4,q}^{0,\text{id}.g_3}$	$D_4^0(\hat{1}_{\bar{q}},2_g,3_g^{\text{id.}},4_g)$	
$\mathcal{D}_{4,g_2}^{0,\text{id}.g_3}$	$D_4^0(1_q,\hat{2}_g,3_g^{\text{id.}},4_g)$	
$\mathcal{E}_{4,\bar{q}'}^{0,\text{id}.q}$	$E_4^0(1_q^{\text{id.}},\hat{2}_{q'},3_{\bar{q}'},4_g)$	No symm.
$\mathcal{E}_{4,q'}^{0,\text{id}.q}$	$E_4^0(1_q^{\text{id.}},2_{q'},\hat{3}_{\bar{q}'},4_g)$	
$\mathcal{E}_{4,g}^{0,\text{id}.q}$	$E_4^0(1_q^{\text{id.}},2_{q'},3_{\bar{q}'},\hat{4}_g)$	
$\mathcal{E}_{4,q}^{0,\text{id}.q'}$	$E_4^0(\hat{1}_{\bar{q}},2_{q'}^{\text{id.}},3_{\bar{q}'},4_g)$	
$\mathcal{E}_{4,q'}^{0,\text{id}.q'}$	$E_4^0(1_q,2_{q'}^{\text{id.}},\hat{3}_{\bar{q}'},4_g)$	
$\mathcal{E}_{4,g}^{0,\text{id}.q'}$	$E_4^0(1_q,2_{q'}^{\text{id.}},3_{\bar{q}'},\hat{4}_g)$	
$\mathcal{E}_{4,q}^{0,\text{id}.\bar{q}'}$	$E_4^0(\hat{1}_{\bar{q}},2_{q'},3_{\bar{q}'}^{\text{id.}},4_g)$	
$\mathcal{E}_{4,\bar{q}'}^{0,\text{id}.\bar{q}'}$	$E_4^0(1_q,\hat{2}_{q'},3_{\bar{q}'}^{\text{id.}},4_g)$	
$\mathcal{E}_{4,g}^{0,\text{id}.\bar{q}'}$	$E_4^0(1_q,2_{q'},3_{\bar{q}'}^{\text{id.}},\hat{4}_g)$	
$\mathcal{E}_{4,q}^{0,\text{id}.g}$	$E_4^0(\hat{1}_{\bar{q}},2_{q'},3_{\bar{q}'},4_g^{\text{id.}})$	
$\mathcal{E}_{4,\bar{q}'}^{0,\text{id}.g}$	$E_4^0(1_q,\hat{2}_{q'},3_{\bar{q}'},4_g^{\text{id.}})$	
$\mathcal{E}_{4,q'}^{0,\text{id}.g}$	$E_4^0(1_q,2_{q'},\hat{3}_{\bar{q}'},4_g^{\text{id.}})$	
$\widetilde{\mathcal{E}}_{4,q'}^{0,\text{id}.q}$	$\widetilde{E}_4^0(1_q^{\text{id.}},2_{q'},\hat{3}_{\bar{q}'},4_g)$	$2\leftrightarrow 3$
$\widetilde{\mathcal{E}}_{4,g}^{0,\text{id}.q}$	$\widetilde{E}_4^0(1_q^{\text{id.}},2_{q'},3_{\bar{q}'},\hat{4}_g)$	
$\widetilde{\mathcal{E}}_{4,q}^{0,\text{id}.q'}$	$\widetilde{E}_4^0(\hat{1}_{\bar{q}},2_{q'}^{\text{id.}},3_{\bar{q}'},4_g)$	
$\widetilde{\mathcal{E}}_{4,q'}^{0,\text{id}.q'}$	$\widetilde{E}_4^0(1_q,2_{q'}^{\text{id.}},\hat{3}_{\bar{q}'},4_g)$	
$\widetilde{\mathcal{E}}_{4,g}^{0,\text{id}.q'}$	$\widetilde{E}_4^0(1_q,2_{q'}^{\text{id.}},3_{\bar{q}'},\hat{4}_g)$	
$\widetilde{\mathcal{E}}_{4,q}^{0,\text{id}.g}$	$\widetilde{E}_4^0(\hat{1}_{\bar{q}},2_{q'},3_{\bar{q}'},4_g^{\text{id.}})$	
$\widetilde{\mathcal{E}}_{4,q'}^{0,\text{id}.g}$	$\widetilde{E}_4^0(1_q,2_{q'},\hat{3}_{\bar{q}'},4_g^{\text{id.}})$	

Mass factorisation with integrated dipoles

- Collect antennae into integrated dipoles **[Currie et al. '13] [Gehrmann, Marcoli, Glover '22]** $\rightarrow$ natural organisation of IR singularities
- Same assembly as for non-frag processes
- Identity preserving (IP) and identity changing (IC) initial-final and final-final integrated dipoles
  - MF kernels absorb poles from PDFs and FFs (collinear)
  - Residual infrared poles reproduce the ones of 1 and 2 loop virtual corrections (IP)
- Strong check against Catani's operators **[Catani '98]** $\mathcal{I}^{(1,2)}_{pp'}$

$$\mathcal{P}oles\left[\mathcal{J}_2^{(1)}(i,j)\right] = \mathcal{P}oles\left[\mathrm{Re}\left(\mathcal{I}_{ij}^{(1)}\left(\epsilon,\mu_r^2\right)\right)\right],$$

1-loop

$$\mathcal{P}oles\left[N_c\,\mathcal{J}_2^{(2)}(q,\bar{q}) - \frac{\beta_0}{\epsilon}\mathcal{J}_2^{(1)}(q,\bar{q})\right] = \mathcal{P}oles\left[\mathrm{Re}\left(\mathcal{I}_{q\bar{q}}^{(2)}\left(\epsilon,\mu_r^2\right) - \frac{\beta_0}{\epsilon}\mathcal{I}_{q\bar{q}}^{(1)}\left(\epsilon,\mu_r^2\right)\right)\right],$$

$$\mathcal{P}oles\left[N_c\,\mathcal{J}_2^{(2)}(g,g) - \frac{\beta_0}{\epsilon}\mathcal{J}_2^{(1)}(g,g)\right] = \mathcal{P}oles\left[\mathrm{Re}\left(\mathcal{I}_{gg}^{(2)}\left(\epsilon,\mu_r^2\right) - \frac{\beta_0}{\epsilon}\mathcal{I}_{gg}^{(1)}\left(\epsilon,\mu_r^2\right)\right)\right],$$

$$\mathcal{P}oles\Bigg[N_c\left(\mathcal{J}_2^{(2)}(q,g) + \mathcal{J}_2^{(2)}(g,\bar{q}) - 2\overline{\mathcal{J}}_2^{(2)}(q,\bar{q})\right) - \frac{\beta_0}{\epsilon}\left(\mathcal{J}_2^{(1)}(q,g) + \mathcal{J}_2^{(1)}(g,\bar{q})\right)\Bigg] = \mathcal{P}oles\left[\mathrm{Re}\left(\mathcal{I}_{qg}^{(2)}\left(\epsilon,\mu_r^2\right) + \mathcal{I}_{g\bar{q}}^{(2)}\left(\epsilon,\mu_r^2\right) - \frac{\beta_0}{\epsilon}\left(\mathcal{I}_{qg}^{(1)}\left(\epsilon,\mu_r^2\right) + \mathcal{I}_{g\bar{q}}^{(1)}\left(\epsilon,\mu_r^2\right)\right)\right)\right],$$

2-loops

$$J_2^{(2)}(p,a) = c_1\mathcal{X}_4^0 + c_2\mathcal{X}_3^1 + c_3\mathcal{X}_3^0\otimes\mathcal{X}_3^0 + c_4\frac{\beta_0}{\epsilon}\mathcal{X}_3^0 + c_5\overline{\Gamma}^{(2)} + c_6\Gamma^{(1)}\otimes\Gamma^{(1)} + c_7\Gamma^{(1)}\otimes\mathcal{X}_3^0$$

Mass factorisation with integrated dipoles

- Integrated dipoles are building blocks for construction of VV subtraction terms

$$\mathrm{d}\hat{\sigma}^{U}_{\mathrm{NNLO}} = -\,\mathrm{d}\hat{\sigma}^{MF,2}_{\mathrm{NNLO}} - \int_1 \mathrm{d}\hat{\sigma}^{VS,1}_{\mathrm{NNLO}} - \int_2 \mathrm{d}\hat{\sigma}^{S,2}_{\mathrm{NNLO}}$$

- Different types: qq, qg, gg …

One dipole used for time-like crossing thresholds: see Christian Biello's talk

qq IP

	Integrated dipoles
FF	$J_2^{(2)}(3_q,i_{\bar{q}}) = \mathcal{A}_4^{0,\mathrm{id}.q} + \mathcal{A}_3^{1,\mathrm{id}.q} + \frac{b_0}{\epsilon}\left(\frac{\lvert s_{3i}\rvert}{\mu_r^2}\right)^{-\epsilon}\mathcal{A}_3^{0,\mathrm{id}.q} - \frac{1}{2}\left[\mathcal{A}_3^{0,\mathrm{id}.q}\otimes\mathcal{A}_3^{0,\mathrm{id}.q}\right] - \overline{\Gamma}_{qq}^{(2)\mathrm{id}.}(x_3)\,\delta_1$
	$\tilde{J}_2^{(2)}(3_q,i_{\bar{q}}) = \frac{1}{2}\widetilde{\mathcal{A}}_4^{0,\mathrm{id}.q} + 2\mathcal{C}_4^{0,\mathrm{id}.\bar{q}} + \mathcal{C}_4^{0,\mathrm{id}.q_1} + \widetilde{\mathcal{A}}_3^{1,\mathrm{id}.q} - \frac{1}{2}\left[\mathcal{A}_3^{0,\mathrm{id}.q}\otimes\mathcal{A}_3^{0,\mathrm{id}.q}\right] + \tilde{\tilde{\overline{\overline{\Gamma}}}}_{qq}^{(2)\mathrm{id}.}(x_3)\,\delta_1$
	$\hat{J}_2^{(2)}(3_q,i_{\bar{q}}) = \mathcal{B}_4^{0,\mathrm{id}.q} + \widehat{\mathcal{A}}_3^{1,\mathrm{id}.q} + \frac{b_{0,F}}{\epsilon}\left(\frac{\lvert s_{3i}\rvert}{\mu_r^2}\right)^{-\epsilon}\mathcal{A}_3^{0,\mathrm{id}.q} - \hat{\overline{\overline{\Gamma}}}_{qq}^{(2)\mathrm{id}.}(x_3)\,\delta_1$
IF	$J_2^{(2)}(1_q,3_q) = \mathcal{A}_{4,q}^{0,\mathrm{id}.q} + \mathcal{A}_{3,q}^{1,\mathrm{id}.q} + \frac{b_0}{\epsilon}\left(\frac{\lvert s_{13}\rvert}{\mu_r^2}\right)^{-\epsilon}\mathcal{A}_{3,q}^{0,\mathrm{id}.q} - \frac{1}{2}\left[\mathcal{A}_{3,q}^{0,\mathrm{id}.q}\otimes\mathcal{A}_{3,q}^{0,\mathrm{id}.q}\right] - \overline{\Gamma}_{qq}^{(2)}(x_1)\,\delta_3 - \overline{\Gamma}_{qq}^{(2)\mathrm{id}.}(x_3)\,\delta_1$
	$\tilde{J}_2^{(2)}(1_q,3_q) = \frac{1}{2}\widetilde{\mathcal{A}}_{4,q}^{0,\mathrm{id}.q} + 2\mathcal{C}_{4,\bar{q}_1}^{0,\mathrm{id}.\bar{q}_2} + 2\mathcal{C}_{4,q_4}^{0,\mathrm{id}.q_1} + \widetilde{\mathcal{A}}_{3,q}^{1,\mathrm{id}.q} - \frac{1}{2}\left[\mathcal{A}_{3,q}^{0,\mathrm{id}.q}\otimes\mathcal{A}_{3,q}^{0,\mathrm{id}.q}\right] + \tilde{\tilde{\overline{\overline{\Gamma}}}}_{qq}^{(2)}(x_1)\,\delta_3 + \tilde{\tilde{\overline{\overline{\Gamma}}}}_{qq}^{(2)\mathrm{id}.}(x_3)\,\delta_1$
	$\hat{J}_2^{(2)}(1_q,3_q) = \mathcal{B}_{4,q}^{0,\mathrm{id}.q} + \widehat{\mathcal{A}}_{3,q}^{1,\mathrm{id}.q} + \frac{b_{0,F}}{\epsilon}\left(\frac{\lvert s_{13}\rvert}{\mu_r^2}\right)^{-\epsilon}\mathcal{A}_{3,q}^{0,\mathrm{id}.q} - \hat{\overline{\overline{\Gamma}}}_{qq}^{(2)}(x_1)\,\delta_3 - \hat{\overline{\overline{\Gamma}}}_{qq}^{(2)\mathrm{id}.}(x_3)\,\delta_1$

gg IP

	Integrated dipoles
FF	$J_2^{(2)}(3_g,i_g) = \frac{1}{2}\mathcal{F}_4^{0,\mathrm{id}.g} + \frac{1}{2}\mathcal{F}_3^{1,\mathrm{id}.g} + \frac{1}{2}\frac{b_0}{\epsilon}\left(\frac{\lvert s_{3i}\rvert}{\mu_r^2}\right)^{-\epsilon}\mathcal{F}_3^{0,\mathrm{id}.g} - \frac{1}{4}\left[\mathcal{F}_3^{0,\mathrm{id}.g}\otimes\mathcal{F}_3^{0,\mathrm{id}.g}\right] - \frac{1}{2}\overline{\Gamma}_{gg}^{(2)\mathrm{id}.}(x_3)\,\delta_1$
	$\hat{J}_2^{(2)}(3_g,i_g) = \mathcal{G}_4^{0,\mathrm{id}.g} + \frac{1}{2}\widehat{\mathcal{F}}_3^{1,\mathrm{id}.g} + \frac{1}{2}\mathcal{G}_3^{1,\mathrm{id}.g} + \frac{1}{2}\frac{b_{0,F}}{\epsilon}\left(\frac{\lvert s_{3i}\rvert}{\mu_r^2}\right)^{-\epsilon}\mathcal{F}_3^{0,\mathrm{id}.g} + \frac{1}{2}\frac{b_0}{\epsilon}\left(\frac{\lvert s_{3i}\rvert}{\mu_r^2}\right)^{-\epsilon}\mathcal{G}_3^{0,\mathrm{id}.g} - \frac{1}{2}\left[\mathcal{G}_3^{0,\mathrm{id}.g}\otimes\mathcal{F}_3^{0,\mathrm{id}.g}\right] - \frac{1}{2}\hat{\overline{\overline{\Gamma}}}_{gg}^{(2)\mathrm{id}.}(x_3)\,\delta_1 + \hat{J}_{2,\mathrm{f/f}}^{(2)}(3_g,i_g)$
	$\hat{\tilde{J}}_2^{(2)}(3_g,i_g) = \frac{1}{2}\widetilde{\mathcal{G}}_4^{0,\mathrm{id}.g} + \frac{1}{2}\widetilde{\mathcal{G}}_3^{1,\mathrm{id}.g} + \frac{1}{2}\hat{\tilde{\overline{\overline{\Gamma}}}}_{gg}^{(2)\mathrm{id}.}(x_3)\,\delta_1 + \hat{\tilde{J}}_{2,\mathrm{f/f}}^{(2)}(3_g,i_g)$
	$\hat{\hat{J}}_2^{(2)}(3_g,i_g) = \frac{1}{2}\widehat{\mathcal{G}}_3^{1,\mathrm{id}.g} + \frac{1}{2}\frac{b_{0,F}}{\epsilon}\left(\frac{\lvert s_{1i}\rvert}{\mu_r^2}\right)^{-\epsilon}\mathcal{G}_3^{0,\mathrm{id}.g} - \frac{1}{4}\left[\mathcal{G}_3^{0,\mathrm{id}.g}\otimes\mathcal{G}_3^{0,\mathrm{id}.g}\right] - \frac{1}{2}\hat{\hat{\overline{\overline{\Gamma}}}}_{gg}^{(2)\mathrm{id}.}(x_3)\,\delta_1$
IF	$J_2^{(2)}(1_g,3_g) = \mathcal{F}_{4,g_2}^{0,\mathrm{id}.g_1} + \frac{1}{2}\mathcal{F}_{4,g_3}^{0,\mathrm{id}.g_1} + \mathcal{F}_{3,g}^{1,\mathrm{id}.g} + \frac{b_0}{\epsilon}\left(\frac{\lvert s_{13}\rvert}{\mu_r^2}\right)^{-\epsilon}\mathcal{F}_{3,g}^{0,\mathrm{id}.g} - \left[\mathcal{F}_{3,g}^{0,\mathrm{id}.g}\otimes\mathcal{F}_{3,g}^{0,\mathrm{id}.g}\right] - \frac{1}{2}\overline{\Gamma}_{gg}^{(2)}(x_1)\,\delta_3 - \frac{1}{2}\overline{\Gamma}_{gg}^{(2)\mathrm{id}.}(x_3)\,\delta_1$
	$\hat{J}_2^{(2)}(1_g,3_g) = \mathcal{G}_{4,g}^{0,\mathrm{id}.g} + \widehat{\mathcal{F}}_{3,g}^{1,\mathrm{id}.g} + \frac{b_{0,F}}{\epsilon}\left(\frac{\lvert s_{13}\rvert}{\mu_r^2}\right)^{-\epsilon}\mathcal{F}_{3,g}^{0,\mathrm{id}.g} - \frac{1}{2}\hat{\overline{\overline{\Gamma}}}_{gg}^{(2)}(x_1)\,\delta_3 - \frac{1}{2}\hat{\overline{\overline{\Gamma}}}_{gg}^{(2)\mathrm{id}.}(x_3)\,\delta_1 + \hat{J}_{2,\mathrm{f/f}}^{(2)}(1_g,3_g)$
	$\hat{\tilde{J}}_2^{(2)}(1_g,3_g) = \frac{1}{2}\mathcal{G}_{4,g}^{0,\mathrm{id}.g} + \frac{1}{2}\hat{\tilde{\overline{\overline{\Gamma}}}}_{gg}^{(2)}(x_1)\,\delta_3 + \frac{1}{2}\hat{\tilde{\overline{\overline{\Gamma}}}}_{gg}^{(2)\mathrm{id}.}(x_1)\,\delta_3 + \hat{\tilde{J}}_{2,\mathrm{f/f}}^{(2)}(1_g,3_g)$
	$\hat{\hat{J}}_2^{(2)}(1_g,3_g) = -\frac{1}{2}\hat{\hat{\overline{\overline{\Gamma}}}}_{gg}^{(2)}(x_1)\,\delta_3 - \frac{1}{2}\hat{\hat{\overline{\overline{\Gamma}}}}_{gg}^{(2)\mathrm{id}.}(x_3)\,\delta_1$

Byproducts: (un)polarized SIDIS

$$Q^2 = -q^2\,,\, x = \frac{Q^2}{2P\cdot q}\,,\, z = \frac{P\cdot P_h}{P\cdot q}\,,\, y = \frac{P\cdot q}{P\cdot k}$$

- Kinematics of initial-final fragmentation antennae same as semi-inclusive deep-inelastic scattering (SIDIS) → same set of MI
- NNLO corrections to
  - unpolarized $F_T\, F_L$ **[LB, Gehrmann, Stagnitto 2401.16281]**

$$\mathscr{F}_i^h(x,z,Q^2) = \sum_{p,p'} \int_x^1 \frac{\mathrm{d}\hat{x}}{\hat{x}} \int_z^1 \frac{\mathrm{d}\hat{z}}{\hat{z}} f_p\left(\frac{x}{\hat{x}},\mu_F^2\right) D_{p'}^h\left(\frac{z}{\hat{z}},\mu_A^2\right) \mathscr{C}^i_{p'p}\left(\hat{x},\hat{z},Q^2,\mu_R^2,\mu_F^2,\mu_A^2\right)\,, \quad i = T, L$$

  - polarized g_1 **[LB, Gehrmann, Löchner, Schönwald, Stagnitto 2404.08597]**

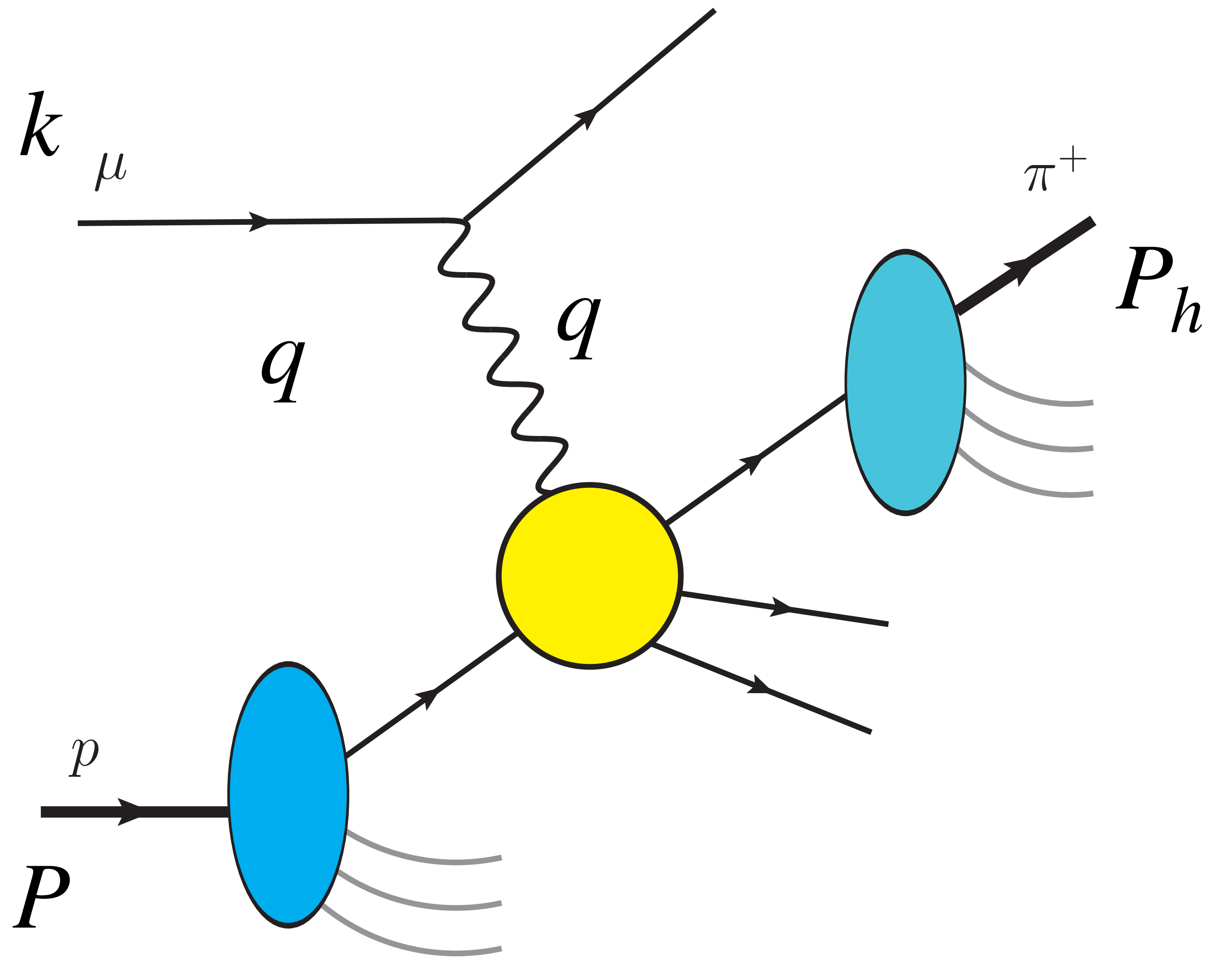
$$2g_1^h(x,z,Q^2) = \sum_{p,p'} \int_x^1 \frac{\mathrm{d}\hat{x}}{\hat{x}} \int_z^1 \frac{\mathrm{d}\hat{z}}{\hat{z}} \Delta f_p\left(\frac{x}{\hat{x}},\mu_F^2\right) D_{p'}^h\left(\frac{z}{\hat{z}},\mu_A^2\right) \Delta\mathscr{C}_{p'p}\left(\hat{x},\hat{z},Q^2,\mu_R^2,\mu_F^2,\mu_A^2\right)$$

- New channels opening at NNLO

$$(\Delta)\mathscr{C}_{p'p} = (\Delta)C^{(0)}_{p'p} + \frac{\alpha_s(\mu_R^2)}{2\pi}(\Delta)C^{(1)}_{p'p} + \left(\frac{\alpha_s(\mu_R^2)}{2\pi}\right)^2 (\Delta)C^{(2)}_{p'p} + \mathscr{O}(\alpha_s^3)$$

Agreement with 2404.09959 and 2312.17711 (See Sven Moch's talk)

γ-exchange only



$$C^{i,(2)}_{qq} = C^{i,(2)}_{\bar{q}\bar{q}} = e_q^2 C^{i,\mathrm{NS}}_{qq} + \left(\sum_j e_{q_j}^2\right) C^{i,\mathrm{PS}}_{qq}\,,$$

$$C^{i,(2)}_{\bar{q}q} = C^{i,(2)}_{q\bar{q}} = e_q^2 C^{i}_{\bar{q}q}\,,$$

$$C^{i,(2)}_{q'q} = C^{i,(2)}_{\bar{q}'\bar{q}} = e_q^2 C^{i,1}_{q'q} + e_{q'}^2 C^{i,2}_{q'q} + e_q e_{q'} C^{i,3}_{q'q}\,,$$

$$C^{i,(2)}_{\bar{q}'q} = C^{i,(2)}_{q'\bar{q}} = e_q^2 C^{i,1}_{q'q} + e_{q'}^2 C^{i,2}_{q'q} - e_q e_{q'} C^{i,3}_{q'q}\,,$$

$$C^{i,(2)}_{gq} = C^{i,(2)}_{g\bar{q}} = e_q^2 C^{i}_{gq}\,,$$

$$C^{i,(2)}_{qg} = C^{i,(2)}_{\bar{q}g} = e_q^2 C^{i}_{qg}\,,$$

$$C^{i,(2)}_{gg} = \left(\sum_j e_{q_j}^2\right) C^{i}_{gg}\,,$$

Same for $\Delta C^i_{p'p}$

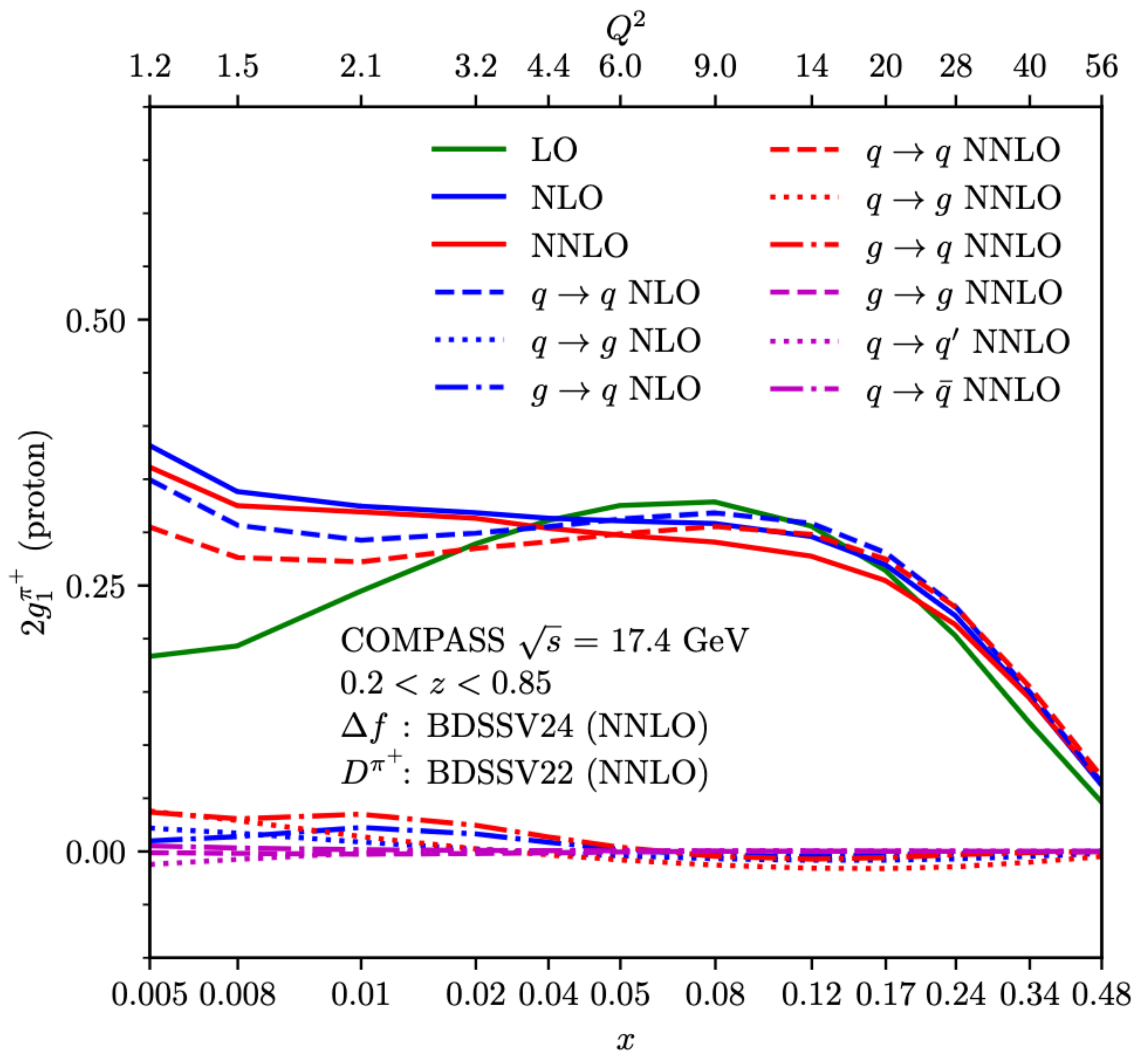
Byproducts: (un)polarized SIDIS

- Phenomenological studies
  - Unpolarized scattering cross section

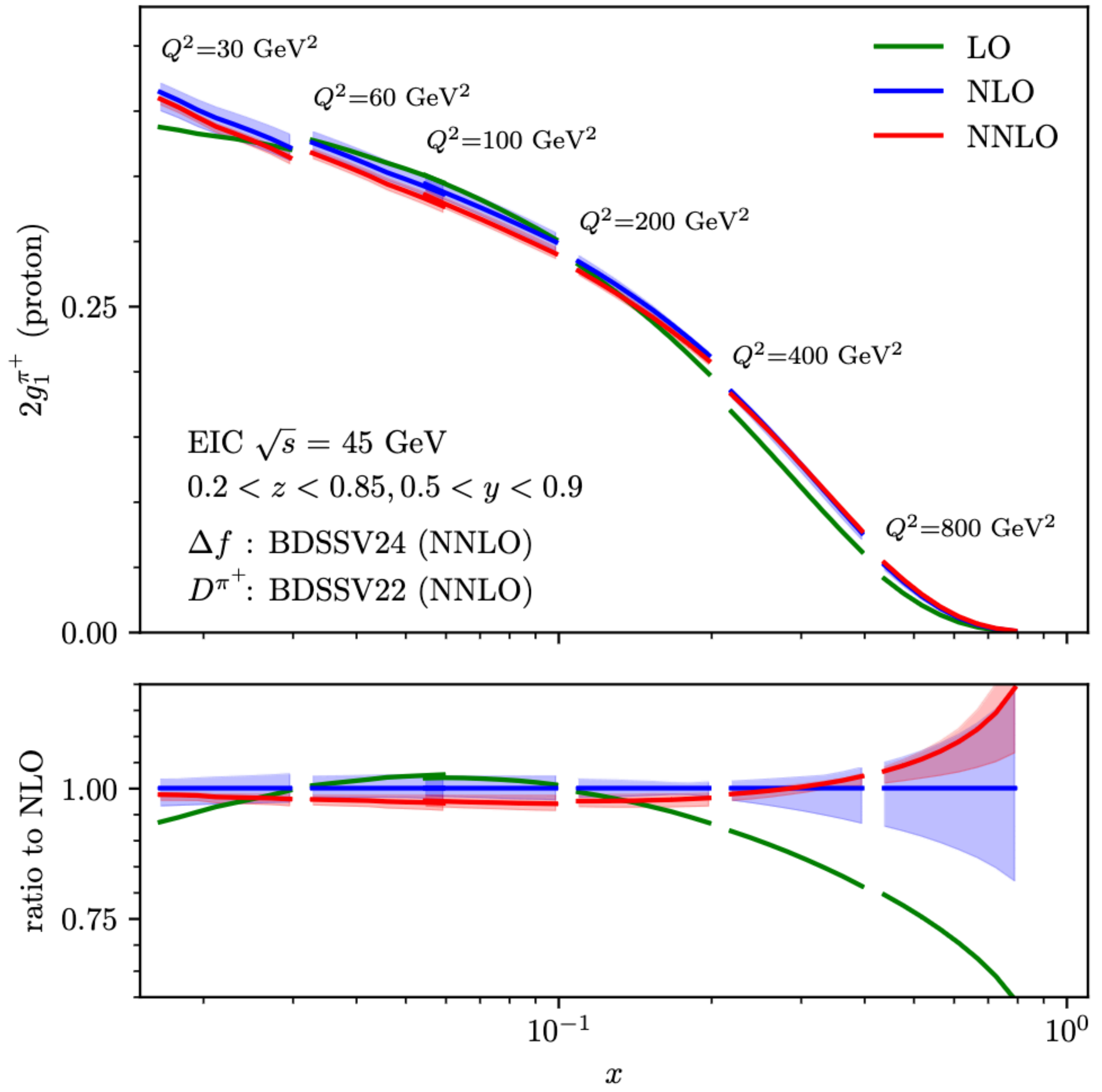
$$\frac{d^3\sigma^h}{dx dy dz} = \frac{4\pi\alpha^2}{Q^2}\left[\frac{1+(1-y)^2}{2y}\mathcal{F}_T^h(x,z,Q^2) + \frac{1-y}{y}\mathcal{F}_L^h(x,z,Q^2)\right]$$

  - Double spin asymmetry $A_1^h(x,z,Q^2) = \dfrac{g_1^h(x,z,Q^2)}{F_1^h(x,z,Q^2)}$ with $2F_1^h = \mathcal{F}_T^h$

- New channels, gluonic channels large at small-z
- Reduction of scale dependence
- Relevant for FFs and polPDFs global fits @ NNLO



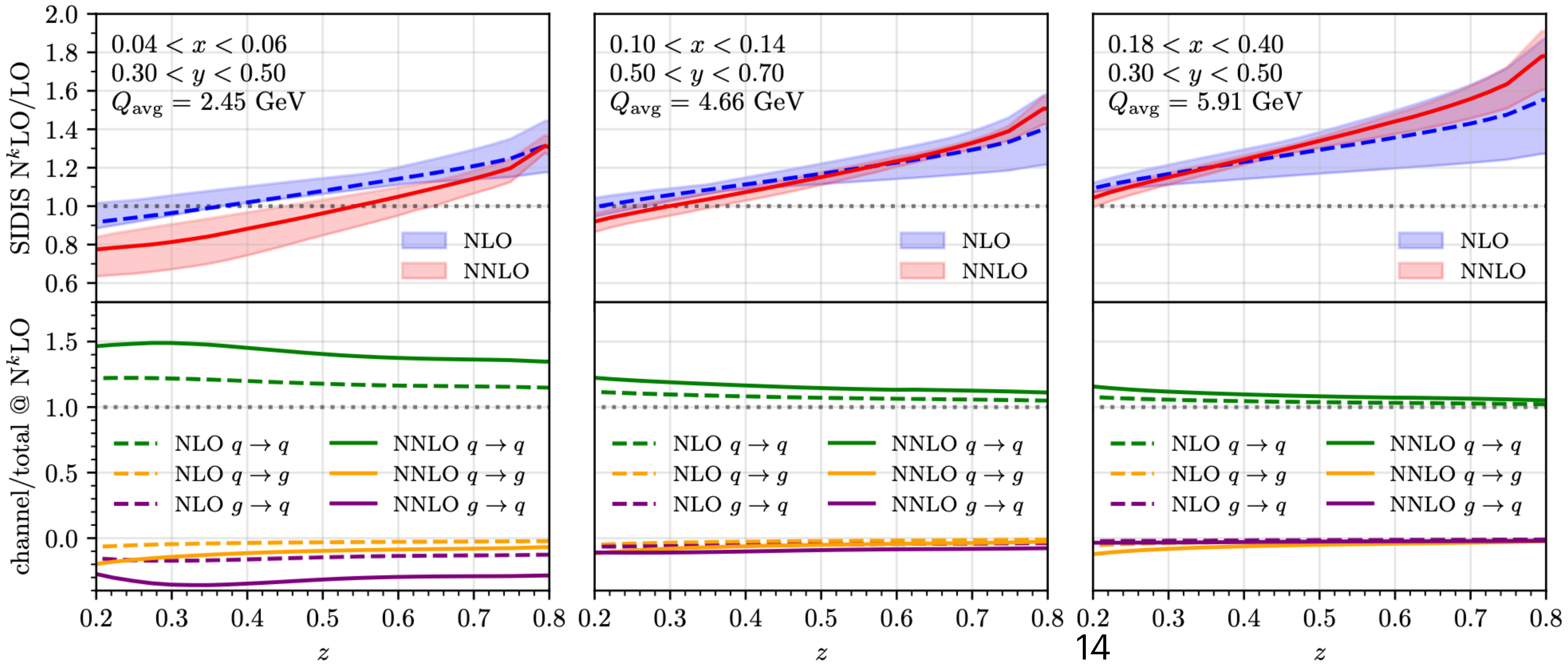
COMPASS



EIC

PDFs: NNPDF3.1

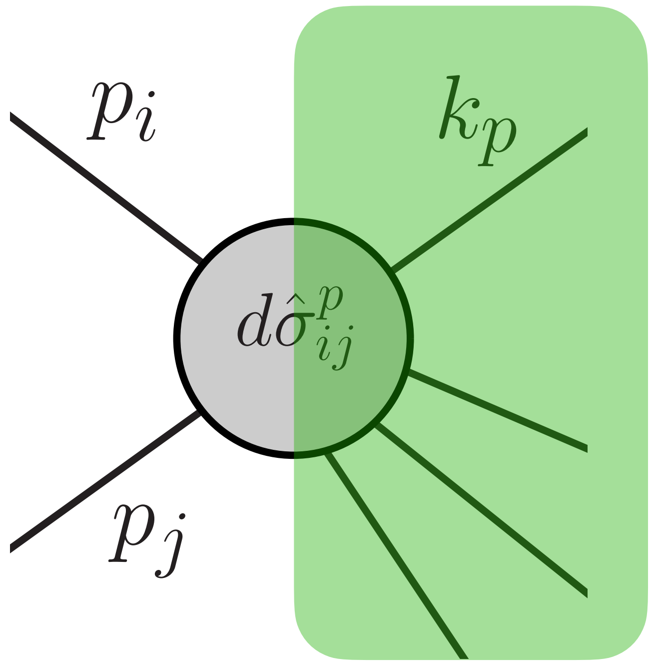
FFs: BSDDV22



Conclusion

- Antenna subtraction method extended to observables with identified hadrons in hadronic collisions at NNLO
  - Computed last ingredient: initial-final integrated antenna functions
  - Mass factorisation counterterms included in integrated dipoles
- Fully general framework: versatile for different observables and processes
- Byproduct: NNLO corrections to (pol)SIDIS coefficient functions
- Looking forward to phenomenological results for LHC

Final-final with fragmentation



- Subtraction terms for final-final configurations

$\mathrm{d}\hat{\sigma}^{S,\mathrm{id}.p}_{NNLO} \supseteq X_4^{0,\mathrm{id}.p} M_n^0 J \leftarrow$ 4-partons tree-level antennae X_4^0

$\mathrm{d}\hat{\sigma}^{T,\mathrm{id}.p}_{NNLO} \supseteq X_3^{1,\mathrm{id}.p} M_n^0 J \leftarrow$ 3-partons 1-loop antennae X_3^1

$\mathrm{d}\hat{\sigma}^{U,\mathrm{id}.p}_{NNLO} \supseteq J_2^{(2)} M_n^0 J \leftarrow$ all integrated antennae and MF counterterms in integrated dipoles

$$J_2^{(2)}(p,a) = c_1 \mathcal{X}_4^{0,\mathrm{id}.p} + c_2 \mathcal{X}_3^{1,\mathrm{id}.p} + c_3 \mathcal{X}_3^{0,\mathrm{id}.p} \otimes \mathcal{X}_3^{0,\mathrm{id}.p} + c_4 \frac{\beta_0}{\epsilon} \mathcal{X}_3^{0,\mathrm{id}.p} + c_5 \overline{\Gamma}^{(2)} + c_6 \Gamma^{(1)} \otimes \Gamma^{(1)} + c_7 \Gamma^{(1)} \otimes \mathcal{X}_3^0$$

- Final-final integrated antenna functions **[Gehrmann, Stagnitto '20]**

$$\mathcal{X}_3^{0,\mathrm{id}.p}(z) \propto \int \mathrm{d}\Phi_2 \frac{q^2}{2\pi} z^{1-2\epsilon} X_3^0$$

$$\mathcal{X}_3^{1,\mathrm{id}.p}(z) \propto \int \mathrm{d}\Phi_2 \frac{q^2}{2\pi} z^{1-2\epsilon} X_3^1$$

$$\mathcal{X}_4^{0,\mathrm{id}.p}(z) \propto \int \mathrm{d}\Phi_3 \frac{q^2}{2\pi} z^{1-2\epsilon} X_4^0$$

$$z = \frac{s_{pj} + s_{pk}}{s_{pj} + s_{pk} + s_{jk}}$$

$$z = \frac{s_{pj} + s_{pk} + s_{pl}}{s_{pj} + s_{pk} + s_{jk} + s_{pl} + s_{jl} + s_{kl}}$$

Example of master integral: I[3,5,8]

- $D_5 = (k_p + k_k)^2$ and $D_8 = (k_p + k_l)^2$ singular in $z \to 0$ limit: $I[358]$ accounts for this behaviour
  - Naive approach (solving DEQ in x and z as Laurent expansion in ϵ) constraints at most ϵ^{-1}
  - z-integrated result diverges as ϵ^{-3}
- DEQs for $I[358]$ homogeneous in x and $Q^2 \to I[358](Q^2, x, z) \propto \left(\frac{1-2\epsilon}{\epsilon}\right)^2 \left(Q^2\right)^{-2-2\epsilon}(1-x)^{-1-2\epsilon}x^{2+2\epsilon}z^{-1-2\epsilon}I'[358](z)$
- DEQ $\partial I'[358](z)/\partial z$ solved in closed form up to integration constant C'
- Integration constant determined by comparing to inclusive MI ($I_{\text{inc}}[0](Q^2, x)$ inclusive PS)

$$I_{\text{inc}}[358](Q^2, x) = \frac{3(1-2\epsilon)(4-2\epsilon)(2-6\epsilon)}{\epsilon^3}\frac{x^3}{\left(Q^2\right)^3(1-x)^2}I_{\text{inc}}[0](Q^2, x)$$

- Closed form expression

$$I[358](Q^2, x, z) = N_\Gamma \left(\frac{1-2\epsilon}{\epsilon}\right)^2 \left(Q^2\right)^{-2-2\epsilon}(1-x)^{-1-2\epsilon}x^{2+2\epsilon}z^{-1-2\epsilon}\left(-2(1-z)^{-2\epsilon}z^{\epsilon} + 2z^{\epsilon}{}_3F_2(\epsilon,\epsilon,2\epsilon,1+\epsilon,1+\epsilon,z) - \frac{2\epsilon\Gamma(1-2\epsilon)\Gamma(1+\epsilon)}{\Gamma(1-\epsilon)}(\pi\cot(\pi\epsilon) + \ln(z))\right)$$