HP2 - 2024, Turin

Light by Light Scattering at NLO in QCD+QED

Ajjath Abdul Hameed

10 Sep 2024

with **Ekta Chaubey**, **Mathijs Fraaije**, **Valentin Hirschi**, and **Hua-Sheng Shao**

arXiv: 2312.16966 [hep-ph] & arXiv:2312.16956 [hep-ph]

LPTHE
LABORATOIRE DE PHYSIQUE THEORIQUE ET HAUTES ENERGIES

SORBONNE UNIVERSITÉ
CRÉATEURS DE FUTURS DEPUIS 1257

Motivation

- One of the earliest predictions in Dirac's theory : in 30s by Heisenberg & Euler

    Euler-Heisenberg Lagrangian for low energy limit

- First complete LO : [Karplus & Neuman PR'51]

- A fundamental process for many interesting questions…

Motivation

- One of the earliest predictions in Dirac's theory : in 30s by Heisenberg & Euler

  Euler-Heisenberg Lagrangian for low energy limit

- First complete LO : [Karplus & Neuman PR'51]

- A fundamental process for many interesting questions…

Consequences of Dirac's Theory of the Positron

W. Heisenberg and H. Euler in Leipzig[1]

22. December 1935

Abstract

According to Dirac's theory of the positron, an electromagnetic field tends to create pairs of particles which leads to a change of Maxwell's equations in the vacuum. These changes are calculated in the special case that no real electrons or positrons are present and the field varies little over a Compton wavelength. The resulting effective Lagrangian of the field reads:

$$\mathfrak{L}=\frac{1}{2}(\mathfrak{E}^2-\mathfrak{B}^2)+\frac{e^2}{\hbar c}\int_0^\infty e^{-\eta}\frac{d\eta}{\eta^3}\left\{i\eta^2(\mathfrak{E}\mathfrak{B})\cdot\frac{\cos\left(\frac{\eta}{|\mathfrak{E}_k|}\sqrt{\mathfrak{E}^2-\mathfrak{B}^2+2i(\mathfrak{E}\mathfrak{B})}\right)+\text{conj.}}{\cos\left(\frac{\eta}{|\mathfrak{E}_k|}\sqrt{\mathfrak{E}^2-\mathfrak{B}^2+2i(\mathfrak{E}\mathfrak{B})}\right)-\text{conj.}}+|\mathfrak{E}_k|^2+\frac{\eta^2}{3}(\mathfrak{B}^2-\mathfrak{E}^2)\right\}$$

$\mathfrak{E},\mathfrak{B}$ field strengths

$|\mathfrak{E}_k|=\frac{m^2c^3}{e\hbar}=\frac{1}{137}\frac{e}{(e^2/mc^2)^2}=$ critical field strengths

The expansion terms in small fields (compared to $\mathfrak{E}$) describe light-light scattering. The simplest term is already known from perturbation theory. For large fields, the equations derived here differ strongly from Maxwell's equations. Our equations will be compared to those proposed by Born.

German title: "Folgerungen aus der Diracschen Theorie des Positrons" Zeitschr. Phys. *98*, 714 (1936).

Motivation

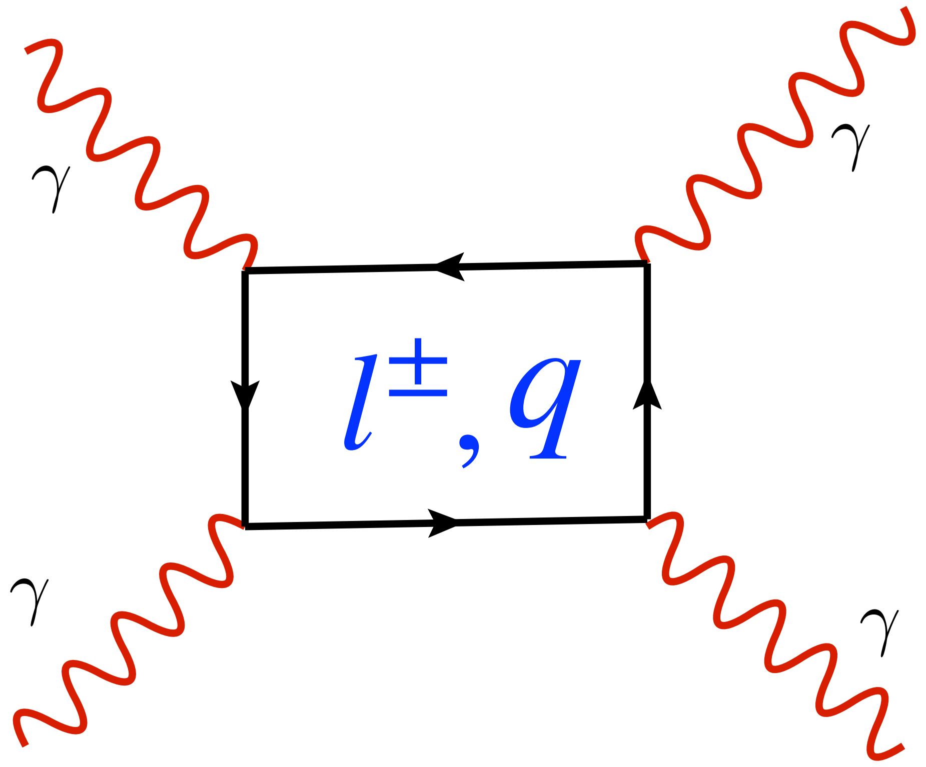
- One of the earliest predictions in Dirac's theory : in 30s by Heisenberg & Euler

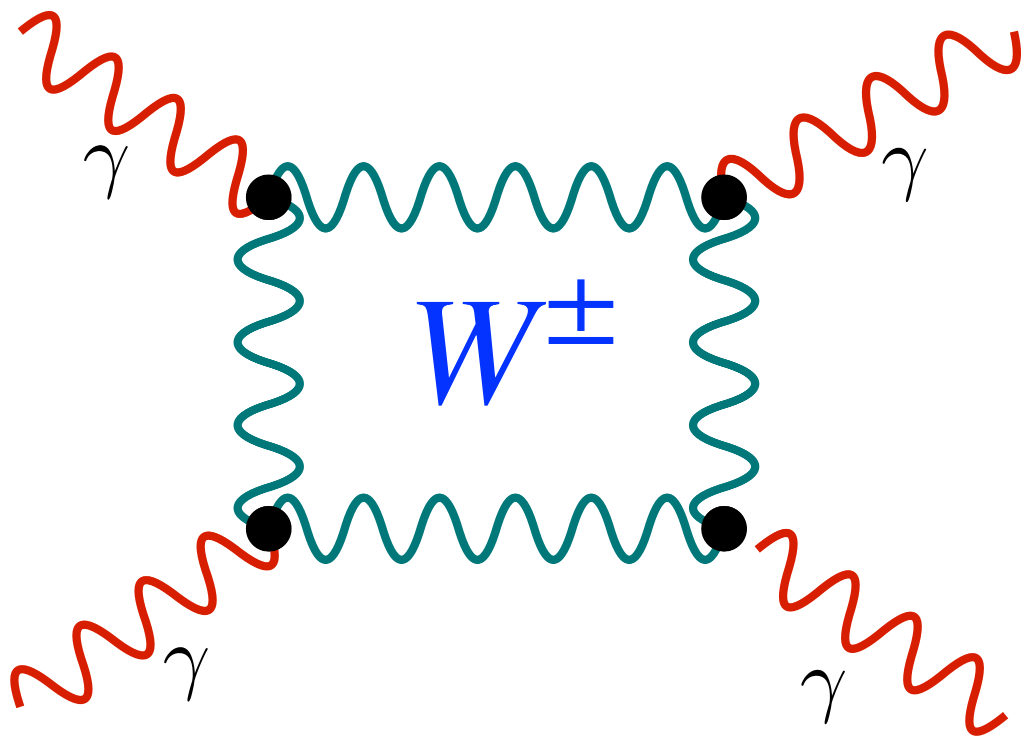
  Euler-Heisenberg Lagrangian for low energy limit

- First complete LO : [Karplus & Neuman PR'51]

- A fundamental process for many interesting questions…

- How light interacts with itself?





SM

$\sim \mathcal{O}(\alpha^4)$ at LO

Motivation

- One of the earliest predictions in Dirac's theory : in 30s by Heisenberg & Euler

  Euler-Heisenberg Lagrangian for low energy limit

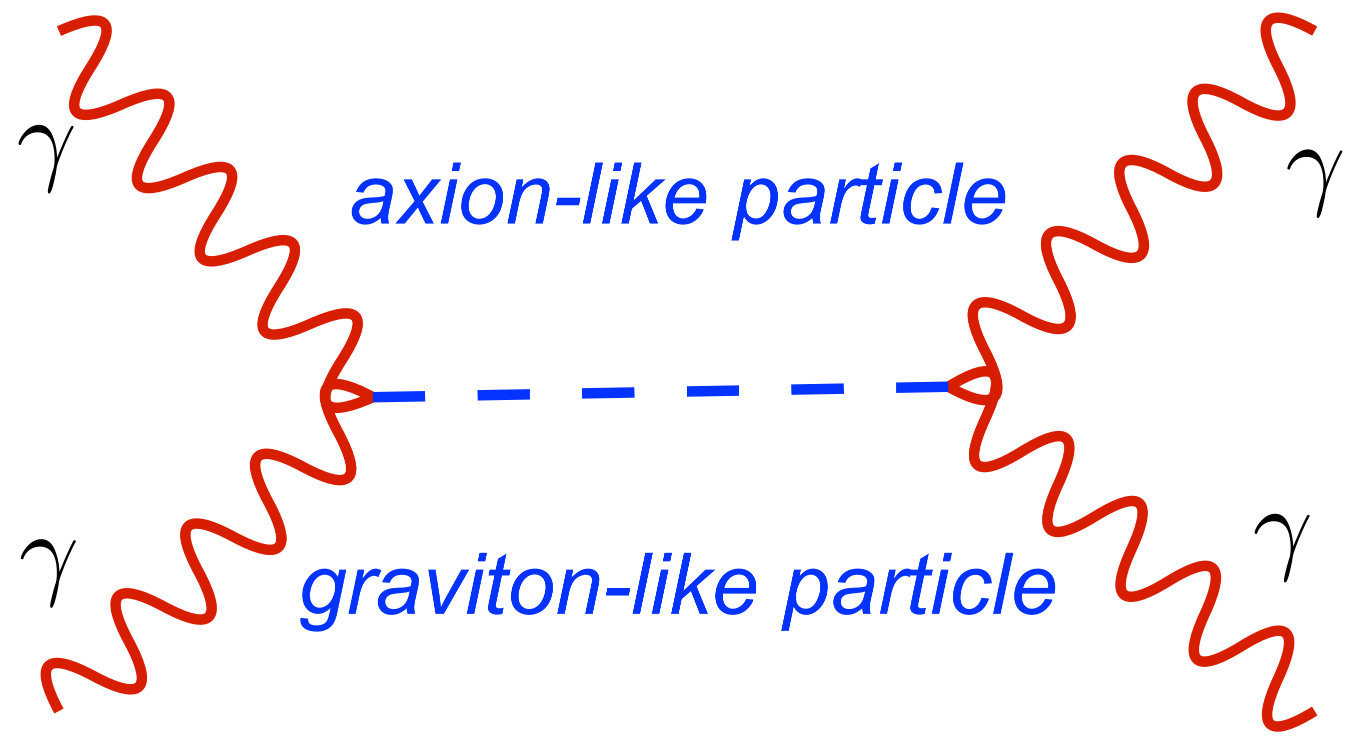
- First complete LO : [Karplus & Neuman PR'51]

- A fundamental process for many interesting questions…

- How light interacts with itself?

Sensitive channel to search for new physics

BSM resonances?



Motivation

- One of the earliest predictions in Dirac's theory : in 30s by Heisenberg & Euler

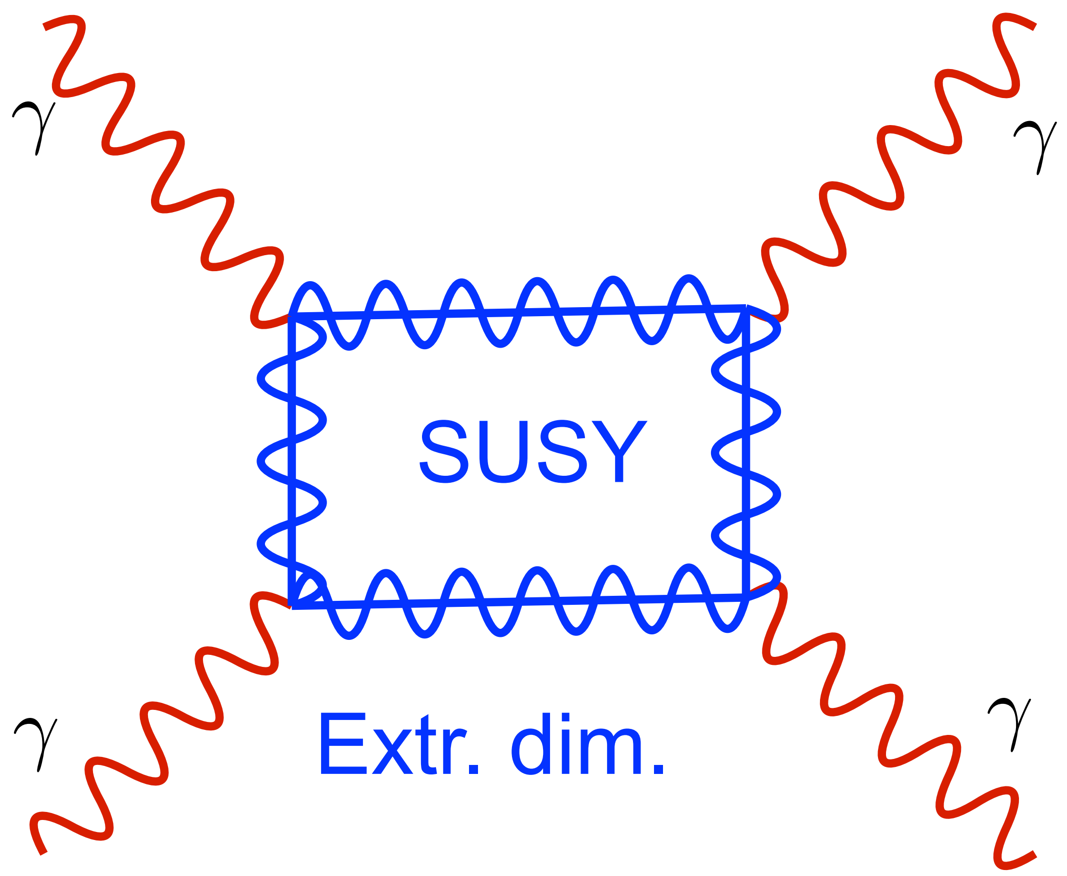
  Euler-Heisenberg Lagrangian for low energy limit

- First complete LO : [Karplus & Neuman PR'51]

- A fundamental process for many interesting questions…

- How light interacts with itself?

Sensitive channel to search for new physics

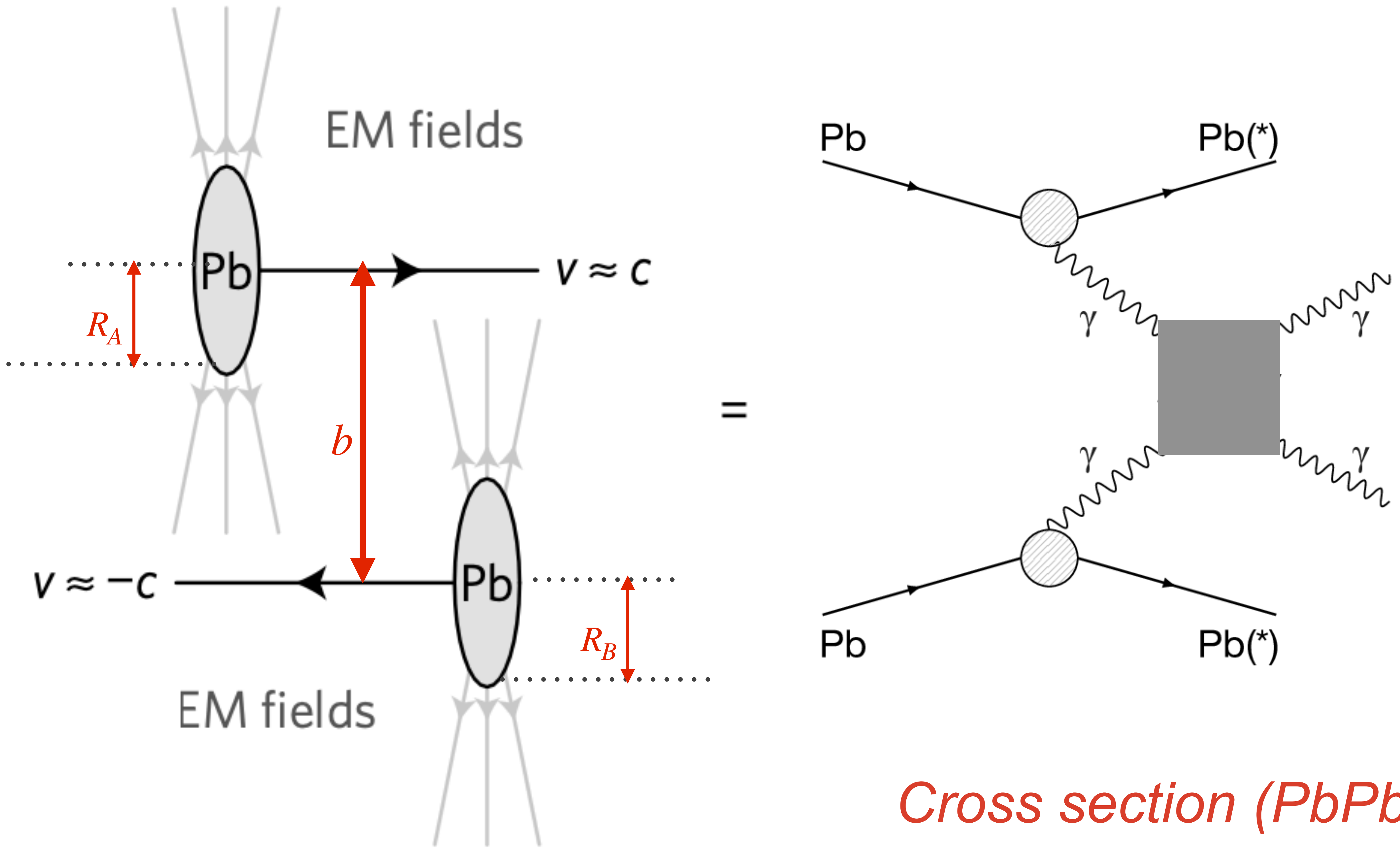


BSM particles in loop?

Measurement of LbL

- First direct detection by ATLAS in 2019 [ATLAS; PRL'19]
- In Ultra-peripheral heavy ion collisions (UPCs)

Earlier evidence [ATLAS; Nature Physics '19] [CMS; PLB'19]



$$b_{min} > R_A + R_B$$

EM field associated with highly relativistic charged particles can be treated as a beam of coherent photons with small virtuality (Equivalent photon approximation)

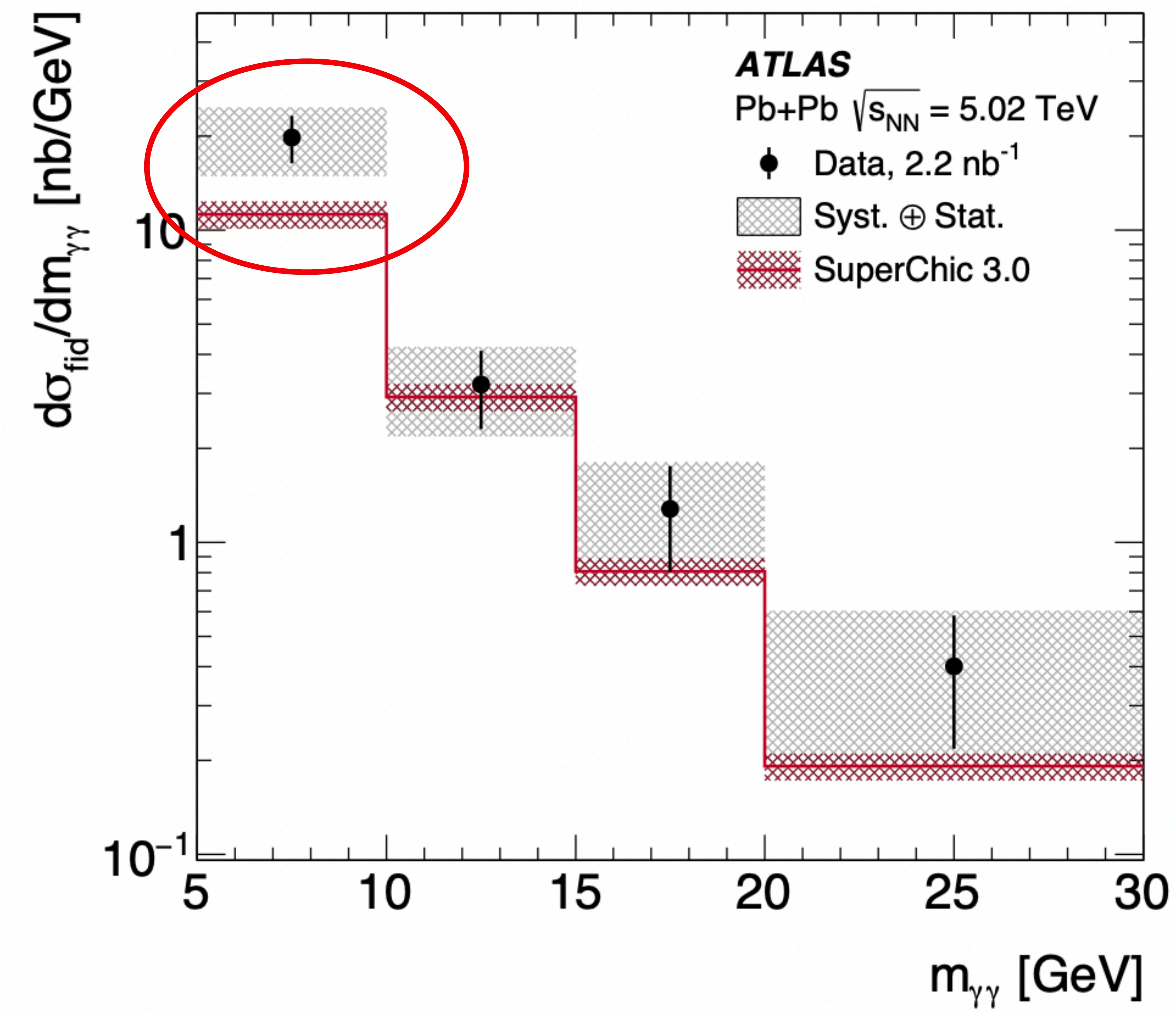
Large photon flux $\sim Z^2, Z = 82$ for Pb

Cross section (PbPb) scales like $Z^4 \sim 5 \cdot 10^7$ larger than pp or $e^\pm$

Data-theory comparison

- Descrepancy between theory (LO) and ATLAS measure

[ATLAS; JHEP'21]



$$\sigma_{\text{ATLAS}} = 120 \pm 22 \text{ nb}$$

$$\sigma_{\text{LO}} = 76 \text{ nb}$$

[L. A. Harland-Lang, V. A. Khoze, and M. G. Ryskin '19]

[H.S Shao, D. d'Enterria '22]

Status - LbL

- Earlier works at two-loop
    - Low energy approx: two loop corrections to Euler-Heisenberg Lagrangian [Martin, Schubert, Sandoval; NPB'03]
    - Massless limit of two-loop amplitudes with internal fermions in QCD and QED

      [Binoth, Glover, Marquard, & van der Bij JHEP'02]

      [Bern, Freitas, Dixon, Ghinculov & Wong; JHEP'01

- Aim : complete QCD & QED corrections at NLO with *massive* fermion loops

Status - LbL

- Earlier works at two-loop
  - Low energy approx: two loop corrections to Euler-Heisenberg Lagrangian [Martin, Schubert, Sandoval; NPB'03]
  - Massless limit of two-loop amplitudes with internal fermions in QCD and QED
    [Binoth, Glover, Marquard, & van der Bij JHEP'02]
    [Bern, Freitas, Dixon, Ghinculov & Wong; JHEP'01
- Aim : complete QCD & QED corrections at NLO with *massive* fermion loops

$$\sigma(\mathrm{A\ B} \xrightarrow{\gamma\gamma} \mathrm{A}\ X\ \mathrm{B}) = \int \frac{dE_{\gamma_1}}{E_{\gamma_1}} \frac{dE_{\gamma_2}}{E_{\gamma_2}} \frac{\mathrm{d}^2 N^{(\mathrm{AB})}_{\gamma_1/\mathrm{Z}_1,\gamma_2/\mathrm{Z}_2}}{\mathrm{d}E_{\gamma_1}\mathrm{d}E_{\gamma_2}} \sigma_{\gamma\gamma\to\gamma\gamma}(W_{\gamma\gamma})$$

Status - LbL

- Earlier works at two-loop
  - Low energy approx: two loop corrections to Euler-Heisenberg Lagrangian [Martin, Schubert, Sandoval; NPB'03]
  - Massless limit of two-loop amplitudes with internal fermions in QCD and QED [Binoth, Glover, Marquard, & van der Bij JHEP'02] [Bern, Freitas, Dixon, Ghinculov & Wong; JHEP'01]
- Aim : complete QCD & QED corrections at NLO with *massive* fermion loops

$$\sigma(\mathrm{A\,B} \xrightarrow{\gamma\gamma} \mathrm{A}\ X\ \mathrm{B}) = \int \frac{dE_{\gamma_1}}{E_{\gamma_1}} \frac{dE_{\gamma_2}}{E_{\gamma_2}} \boxed{\frac{\mathrm{d}^2 N^{(\mathrm{AB})}_{\gamma_1/\mathrm{Z}_1,\gamma_2/\mathrm{Z}_2}}{\mathrm{d}E_{\gamma_1}\mathrm{d}E_{\gamma_2}}} \sigma_{\gamma\gamma\to\gamma\gamma}(W_{\gamma\gamma})$$

Photon-photon luminosity

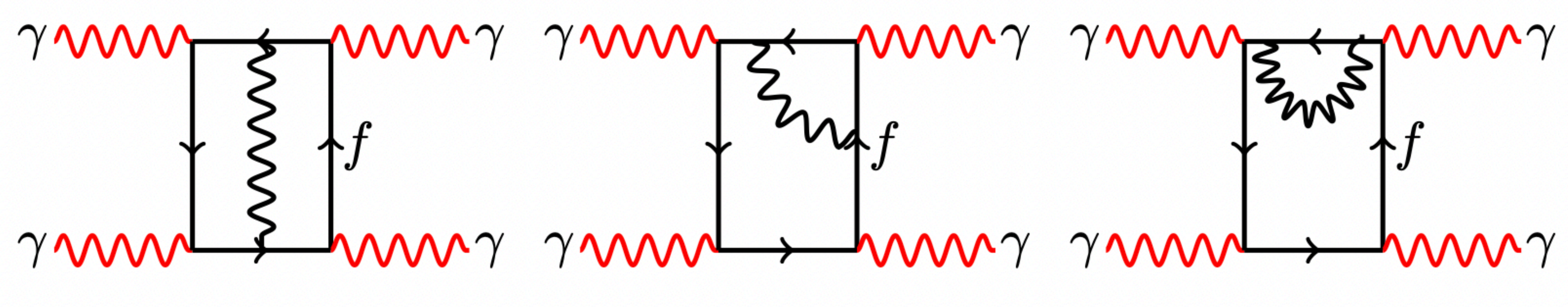
gamma-UPC

[H.S Shao & D. d'Enterria JHEP'22]

Status - LbL

- Earlier works at two-loop
  - Low energy approx: two loop corrections to Euler-Heisenberg Lagrangian [Martin, Schubert, Sandoval; NPB'03]
  - Massless limit of two-loop amplitudes with internal fermions in QCD and QED
    [Binoth, Glover, Marquard, & van der Bij JHEP'02]
    [Bern, Freitas, Dixon, Ghinculov & Wong; JHEP'01
- Aim : complete QCD & QED corrections at NLO with *massive* fermion loops

$$\sigma(\mathrm{A\ B} \xrightarrow{\gamma\gamma} \mathrm{A}\ X\ \mathrm{B}) = \int \frac{dE_{\gamma_1}}{E_{\gamma_1}} \frac{dE_{\gamma_2}}{E_{\gamma_2}} \frac{\mathrm{d}^2 N^{(\mathrm{AB})}_{\gamma_1/\mathrm{Z}_1,\gamma_2/\mathrm{Z}_2}}{\mathrm{d}E_{\gamma_1}\mathrm{d}E_{\gamma_2}} \boxed{\sigma_{\gamma\gamma\to\gamma\gamma}(W_{\gamma\gamma})}$$



$$\overline{\sum}_{\vec{\lambda}} |\mathcal{M}_{\vec{\lambda}}|^2$$

arXiv: 2312.16956, arXiv:2312.16966

Helicity amplitudes

$$\gamma(p_1,\lambda_1)+\gamma(p_2,\lambda_2)+\gamma(p_3,\lambda_3)+\gamma(p_4,\lambda_4)\to 0,$$

- Lorentz decomposition

$$\mathcal{M}_{\vec{\lambda}} = \left(\prod_{i=1}^{4}\varepsilon_{\lambda_i,\mu_i}(p_i)\right)\mathcal{M}^{\mu_1\mu_2\mu_3\mu_4}(p_1,p_2,p_3,p_4),$$

$$\begin{aligned}\mathcal{M}^{\mu_1\mu_2\mu_3\mu_4} = {} & A_1 g^{\mu_1\mu_2}g^{\mu_3\mu_4} + A_2 g^{\mu_1\mu_3}g^{\mu_2\mu_4} + A_3 g^{\mu_1\mu_4}g^{\mu_2\mu_3} + \sum_{j_1,j_2=1}^{3}\Big(B^1_{j_1 j_2} g^{\mu_1\mu_2}p^{\mu_3}_{j_1}p^{\mu_4}_{j_2} + B^2_{j_1 j_2} g^{\mu_1\mu_3}p^{\mu_2}_{j_1}p^{\mu_4}_{j_2} \\ & + B^3_{j_1 j_2} g^{\mu_1\mu_4}p^{\mu_2}_{j_1}p^{\mu_3}_{j_2} + B^4_{j_1 j_2} g^{\mu_2\mu_3}p^{\mu_1}_{j_1}p^{\mu_4}_{j_2} + B^5_{j_1 j_2} g^{\mu_2\mu_4}p^{\mu_1}_{j_1}p^{\mu_3}_{j_2} + B^6_{j_1 j_2} g^{\mu_3\mu_4}p^{\mu_1}_{j_1}p^{\mu_2}_{j_2}\Big) + \sum_{j_1,j_2,j_3,j_4=1}^{3} C_{j_1 j_2 j_3 j_4}p^{\mu_1}_{j_1}p^{\mu_2}_{j_2}p^{\mu_3}_{j_3}p^{\mu_4}_{j_4},\end{aligned}$$

138 form factors $A_i, B^i_{jk}, C_{ijkl} : (s,t,u;m^2) \sim r(s,t,u;m^2)\, I(s,t,u;m^2)$

$s = (p_1+p_2)^2$

$t = (p_2+p_3)^2$

$u = -s-t$

Tranversality $\varepsilon(p_i)\cdot p_i = 0$

Bose symmetry

Gauge invariance

5 independent linear combinations $\mathcal{M}_{++++}, \mathcal{M}_{-+++}, \mathcal{M}_{--++}, \mathcal{M}_{+-+-}, \mathcal{M}_{+--+}$

Computation

Generate amplitudes
Qgraf/FeynArts
FORM/Mathematica

- 60 Feynman diagrams
- 10k+ integrals before IBP
- 18 top sectors

$$\mathcal{M}_{\vec{\lambda}} = \sum_j r_j\left(s,t,u;m^2,\epsilon\right) I_{j;a_1,\cdots,a_9}(s,t,u;m^2,\epsilon)$$

$$I_{j;a_1,\cdots,a_9}(s,t,u;m^2,\epsilon) = \left(\frac{e^{\epsilon\gamma_E} m_f^{2\epsilon}}{i\pi^{\frac{d}{2}}}\right)^2 \int \mathrm{d}^d\ell_1 \mathrm{d}^d\ell_2 \frac{1}{D_1^{a_1} D_2^{a_2} D_3^{a_3} D_4^{a_4} D_5^{a_5} D_6^{a_6} D_7^{a_7} D_8^{a_8} D_9^{a_9}}$$

$$k_1^2 - m_f^2, (k_1+p_1)^2 - m_f^2, (k_1+p_1+p_2)^2 - m_f^2, (k_1+p_1+p_2+p_3)^2 - m_f^2,$$
$$k_2^2 - m_f^2, (k_2+p_1)^2 - m_f^2, (k_2+p_1+p_2)^2 - m_f^2, (k_2+p_1+p_2+p_3)^2 - m_f^2, (k_2-k_1)^2$$

Computation

Generate amplitudes
Qgraf/FeynArts
FORM/Mathematica

IBP Reduction
Kira/FiniteFlow+LiteRed

$$I_j = \sum_k C_{jk}\, f_k, \quad f_k \to \text{Master Integrals (MIs)}$$

At 2 loop: 30 MIs + crossing

Computation

Generate amplitudes
Qgraf/FeynArts
FORM/Mathematica

IBP Reduction
Kira/FiniteFlow+LiteRed

Solve MIs in terms of special functions

- Canonical form [Caron-huot & Henn; JHEP'14]
- Diff. Eq is used to solve the MIs

$$\partial_s \vec{f} = \epsilon A_s \vec{f}$$

$$\partial_t \vec{f} = \epsilon A_t \vec{f}$$

- Square roots :

$$\sqrt{s(s-4m_f^2)},\ \sqrt{t(t-4m_f^2)},\ \sqrt{st(st-4m_f^2(s+t))},\ \sqrt{s(m_f^4 s-2m_f^2 t(s+2t)+st^2)}$$

- Choice of variables :

$$\frac{s}{m_f^2} = -\frac{4(w-z)^2}{(1-w^2)(1-z^2)} \qquad \frac{t}{m_f^2} = -\frac{(w-z)^2}{wz}$$

Non-rationalisable : $\rho = \sqrt{-2wz + z^2 + w^4 z^2 - 2w^3 z^3 + w^2(1+z^2+z^4)}$

Computation

Generate amplitudes
Qgraf / FeynArts
FORM / Mathematica

IBP Reduction
Kira / FiniteFlow+LiteRed

Solve MIs in terms of special functions

- Solution in terms of Chen's iterated integrals with logarithmic one-forms
- Boundary condition : $f_i(s=0, t=0; m^2) = \delta_{i,1}$
- MIs with only rationalisable square roots
  - in terms of Goncharov's Polylogarithms (GPLs)

$$G(a_1, \ldots a_n; z) = \int_0^z \frac{dt}{t - a_1} G(a_2, \ldots, a_n; t)\,,$$

  - Numerical evaluation of GPLs [Vollinga & Weinzierl; CPC'05]

FastGPL
[Wang, Yang & Zhou '21]

HandyG
[Naterop, Signer, Ulrich; CPC'20]

Computation

Generate amplitudes
Qgraf/FeynArts
FORM/Mathematica

IBP Reduction
Kira/FiniteFlow+LiteRed

Solve MIs in terms of special functions

- 5 MIs with non-rationalisable square roots with weight 3/4

- First two-fold integrals are expressed in terms of logs or classical PolyLogs by matching symbols [Duhr, Gangl, Rhodes, JHEP`12]

$$\int_\gamma I(w_1,\cdots,w_4;\lambda) = \int_0^\lambda d\lambda_1 f_1(\lambda_1) \int_0^{\lambda_1} d\lambda_2 f_1(\lambda_2) \underbrace{\int_0^{\lambda_2} d\lambda_3 f_1(\lambda_3) \int_0^{\lambda_3} d\lambda_4 f_1(\lambda_4)}_{\{\ln^2(z),\, \mathrm{Li}_2(z)\}}$$

- The remaining integrals are converted into one-dimensional integrals, and perform them numerically

[Caron-huot & Henn JHEP'14], [Chicherin, Sotnikov, Zoia JHEP`22]

Computation

Generate amplitudes
Qgraf/FeynArts
FORM/Mathematica

IBP Reduction
Kira/FiniteFlow+LiteRed

Solve MIs in terms of special functions

- Numerical evaluation of integrals
  - 3 regions to consider $s < 0, t < 0, u > 0$
    $s > 0, t < 0, u < 0$
    $s < 0, t > 0, u < 0$
  - We obtain distinct analytic results valid in different region, specific boundary constants for each region
  - Integrals are cross-checked for few phase points using AMFLOW [Liu & Ma, CPC'23]

Computation

Generate amplitudes
Qgraf/FeynArts
FORM/Mathematica

IBP Reduction
Kira/FiniteFlow+LiteRed

Solve MIs in terms of special functions

Simplifying amplitudes

- Relations among MIs at a given transcendental weight using
  - Shuffle properties of iterated integrals
  - Matching symbols of different integrals

- Finding linear relations among the rational coefficients and express complex ones into simple ones. FiniteFlow MultivariateApart

- This step was necessary for the analytic pole cancellation. This also help to avoid numerical cancellation between different pieces

Computation

Generate amplitudes
Qgraf/FeynArts
FORM/Mathematica

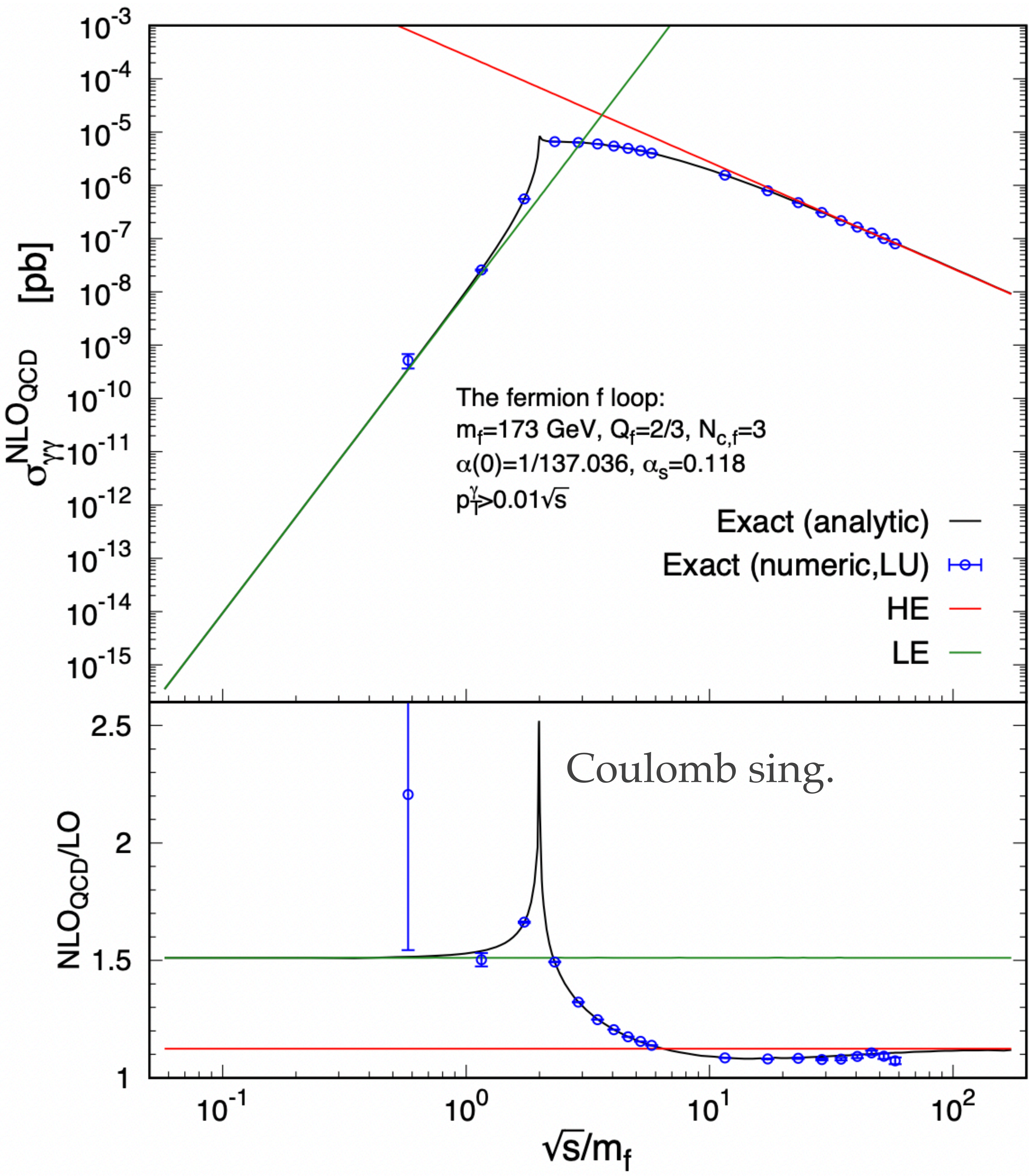
IBP Reduction
Kira/FiniteFlow+LiteRed

Solve MIs in terms of special functions

Simplifying amplitudes

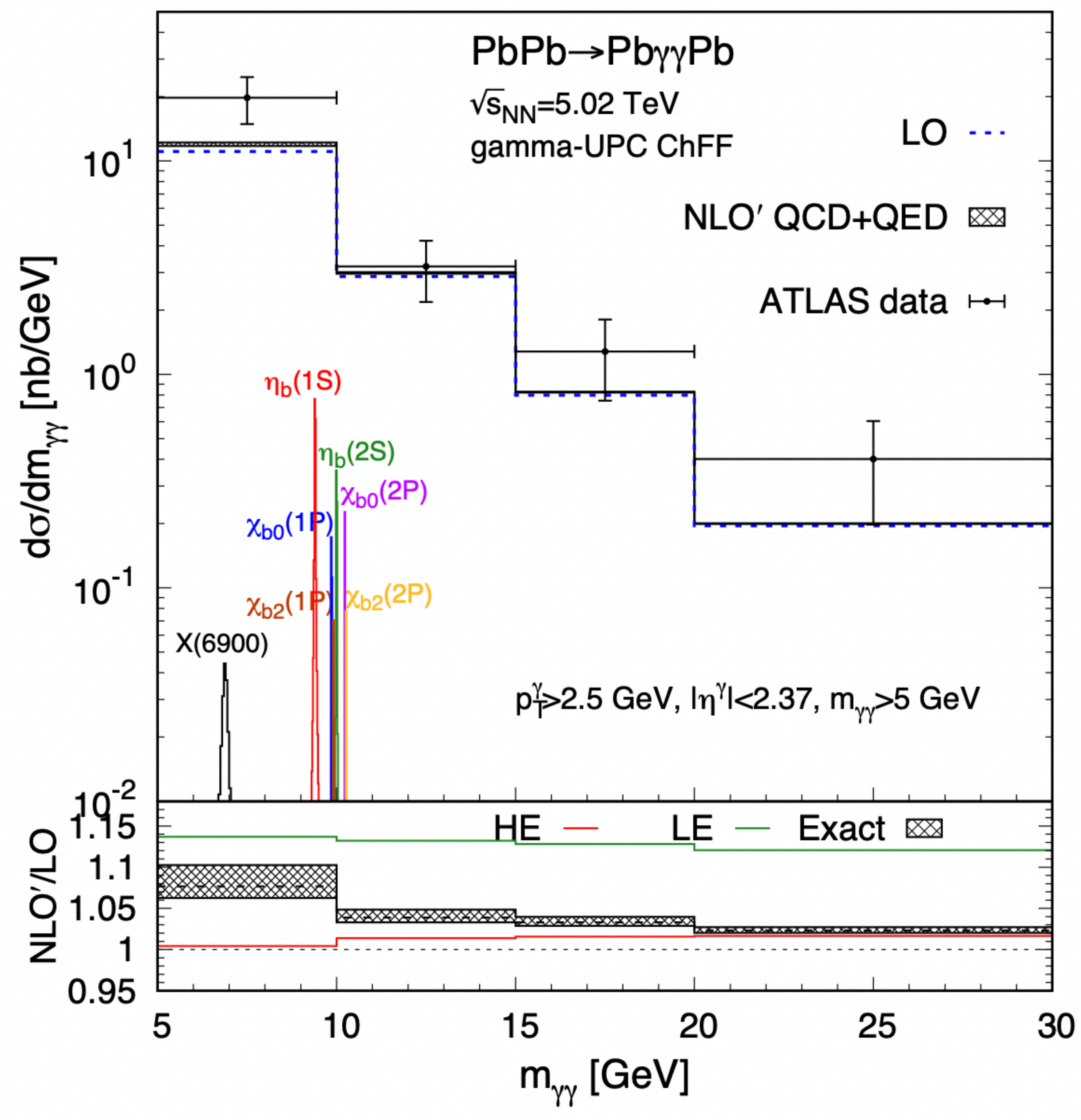
Number of scalar integrals to Master integrals	10k+	30 + crossing
Master integrals in UT basis at different weights	300+	84
Rational coefficients	200+	31+ crossing
Total size	300 Mb	180 Kb

Cross section : Analytic and numeric methods



- Numerical approach is based on local unitarity method [Capatti, Hirschi, Pelloni & Ruijl PRL'21]
- Analytical exact result matches with numerical one
- Exact results matches with the HE/LE approximations in their applicable regimes
- Exact K-factor exhibit more richer structure compared to both LE and HE approximations

Data-Theory comparison



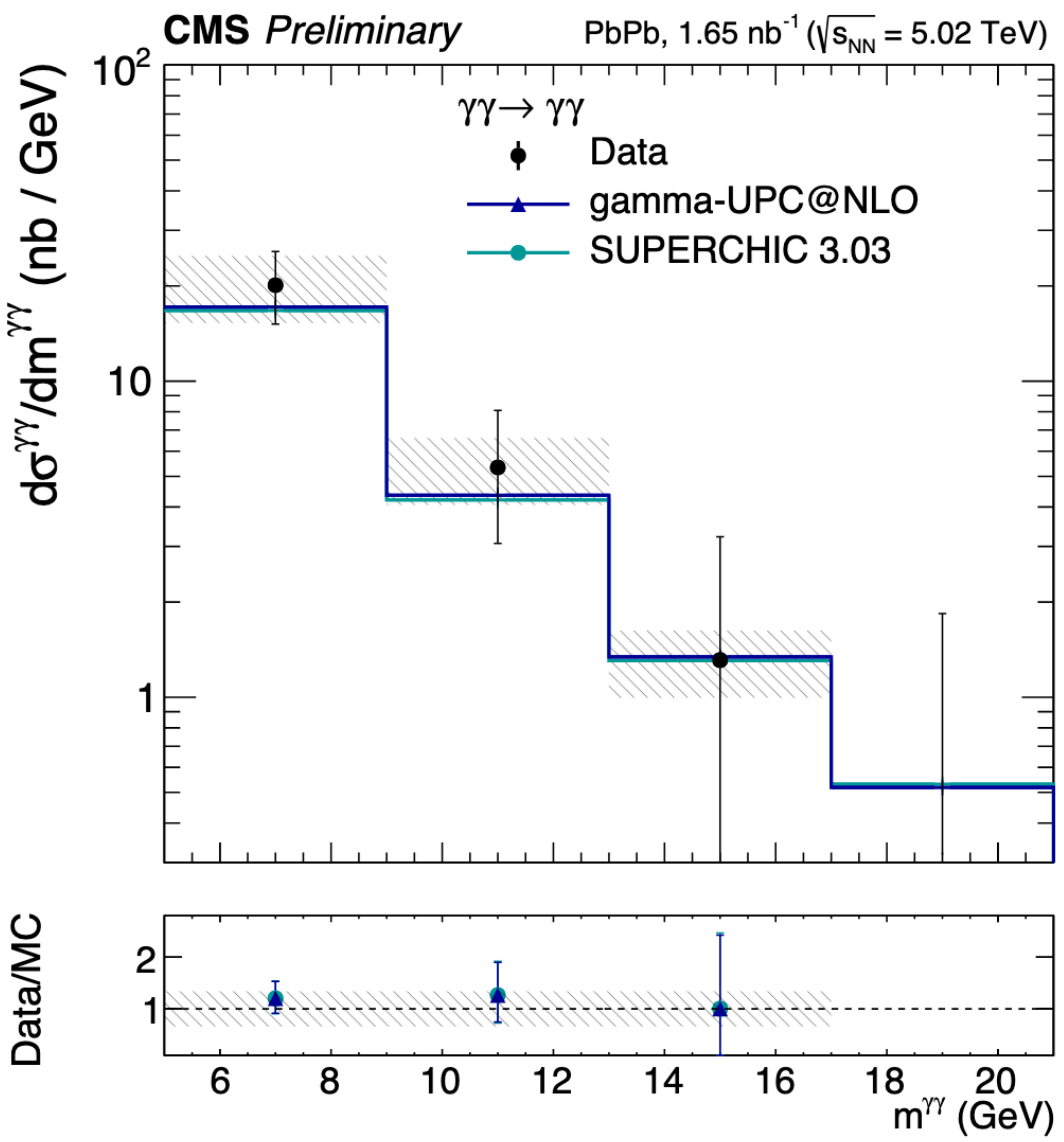
$$\sigma_{\mathrm{ATLAS}} = 120 \pm 22 \text{ nb}$$

$$\sigma_{\mathrm{LO}} = 76 \text{ nb} \qquad \sigma_{\mathrm{NLO}} = 81.2^{+1.6}_{-0.9} \text{ nb}$$

- Data-theory tension persists, though it reduced a bit compared to LO
- HE (LE) under (over) estimates the corrections.
- 6 C-even bottomonia and X(6900) seems cannot explain the discrepancy neither

Improved theory predictions and/or measurements?

Data-Theory comparison



[CMS-PAS-HIN-21-015]

$$\sigma_{\mathrm{CMS}} = 107 \pm 33_{\mathrm{stat}} \pm 20_{\mathrm{syst}}\ \mathrm{nb}$$

$$\sigma_{\mathrm{NLO}} = 95.4\,\binom{+2.0}{-1.0}_{\mathrm{scale}}\ \mathrm{nb}$$

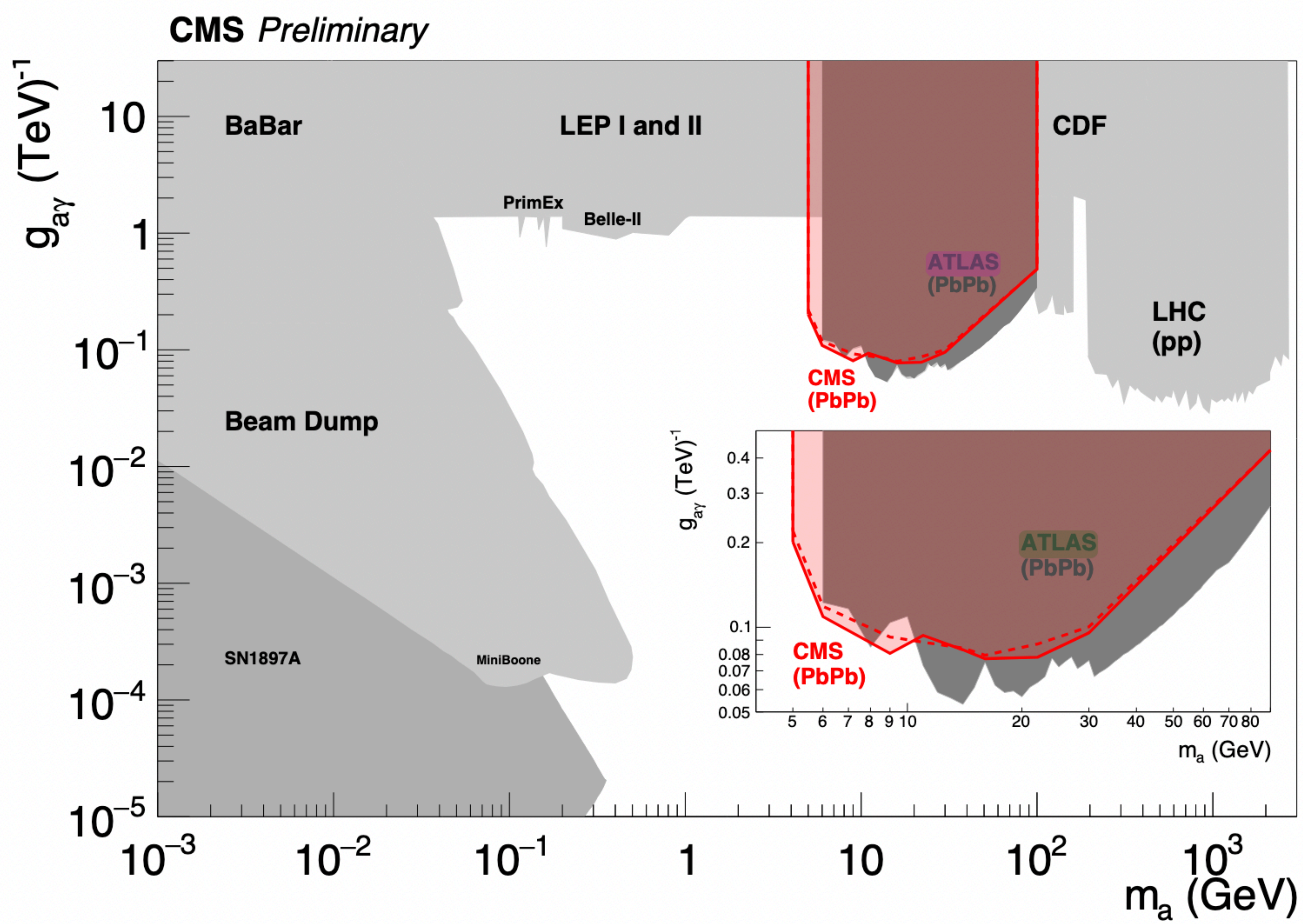
Summary

- NLO computations with exact mass for Light by Light scattering in QCD & QED in ultra-peripheral heavy ion collisions
- Two completely different approach have been compared and serve as a strong check for our calculations
- Compact analytic results at two loop, overcoming the difficulties arising due to internal mass scales.
- The theory tension with ATLAs data reduces, but does not eliminates.
- If data-theory agrees as in CMS, that gives the strongest constraints on axion or graviton like particles

Outlook

Limits on axion like particle

[CMS-PAS-HIN-21-015]



Strongest limit on the axion-like particles over mass above 5 GeV

Outlook

- Intriguing to look at the EW corrections at NLO or even go to the NNLO in QCD and QED
- Phenomenological relevance will show up in HL-LHC where the precision is aim to be within 3%

Thank you for the attention!

Backup Slides

Local unitarity technique

- Direct Monte-Carlo integration in momentum space
- Consider all relevant forward scattering graphs relevant to a given cross section and collect all the cutkosky cuts

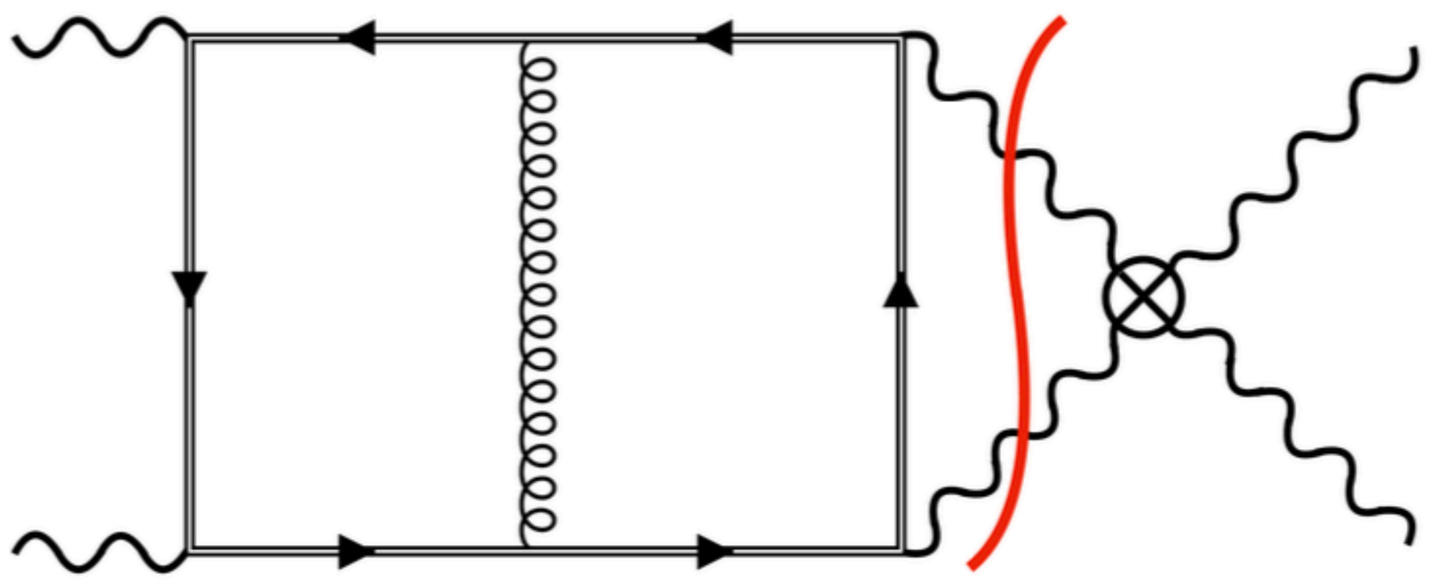


Figure 1: Example of one of the 16 distinct 3-loop FSG contributing to the NLO correction of the LbL cross-section. The single Cutkosky cut contributing is shown in red. The effective four-photon vertex is denoted with a cross and is implemented with the exact 1-loop amplitude. The double line corresponds to a massive fermion.

Iterated integrals

- Shuffle properties : For example

$$\int_\gamma \omega_1 \int_\gamma \omega_2\omega_3 = \int_\gamma \omega_1\omega_2\omega_3 + \int_\gamma \omega_2\omega_1\omega_3 + \int_\gamma \omega_2\omega_3\omega_1$$

$$\begin{aligned}
f_{24}^{(2,4)}(x_{ij},x_{jk},x_{ik}) &= \frac{1}{4}\left(f_3^{(1,1)}(x_{jk},x_{ij},x_{ik})\right)^4 ,\\
f_{16}^{(2,2)}(x_{ij},x_{jk},x_{ik}) &= -\frac{1}{4}\left(f_3^{(1,1)}(x_{ij},x_{jk},x_{ik})\right)^2 ,\\
f_{5}^{(2,3)}(x_{ij},x_{jk},x_{ik}) &= -\frac{1}{2}\left(f_3^{(1,1)}(x_{jk},x_{ij},x_{ik})\right)^3 ,\\
f_{3}^{(2,3)}(x_{ij},x_{jk},x_{ik}) &= -2f_3^{(1,1)}(x_{jk},x_{ij},x_{ik})f_3^{(1,2)}(x_{jk},x_{ij},x_{ik}) ,\\
f_{5}^{(2,4)}(x_{ij},x_{jk},x_{ik}) &= -f_2^{(1,2)}(x_{ij},x_{jk},x_{ik})f_4^{(1,2)}(x_{ij},x_{jk},x_{ik}) - \\
&\qquad f_2^{(1,1)}(x_{ij},x_{jk},x_{ik})f_4^{(1,3)}(x_{ij},x_{jk},x_{ik}) ,\\
f_{10}^{(2,3)}(x_{ij},x_{jk},x_{ik}) &= \frac{1}{3}f_{14}^{(2,3)}(x_{jk},x_{ij},x_{ik}) - 2f_5^{(1,3)}(x_{jk},x_{ij},x_{ik}) + \\
&\quad \frac{1}{6}f_3^{(1,1)}(x_{jk},x_{ij},x_{ik})\left(-6f_3^{(1,2)}(x_{jk},x_{ij},x_{ik}) + f_{15}^{(2,2)}(x_{jk},x_{ij},x_{ik})\right) ,\\
f_{16}^{(2,3)}(x_{ij},x_{jk},x_{ik}) &= f_5^{(1,3)}(x_{ij},x_{jk},x_{ik}) - \frac{5}{12}f_{14}^{(2,3)}(x_{ij},x_{jk},x_{ik}) + \\
&\quad \frac{1}{6}f_3^{(1,1)}(x_{ij},x_{jk},x_{ik})\left(-6f_3^{(1,2)}(x_{ij},x_{jk},x_{ik}) + f_{15}^{(2,2)}(x_{ij},x_{jk},x_{ik})\right) ,\\
f_{23}^{(2,3)}(x_{ij},x_{jk},x_{ik}) &= -\frac{1}{4}f_{10}^{(2,3)}(x_{ij},x_{jk},x_{ik}) - \frac{1}{2}f_{10}^{(2,3)}(x_{jk},x_{ij},x_{ik}) - f_{20}^{(2,3)}(x_{ij},x_{jk},x_{ik})\\
&\quad + \frac{3}{2}f_{11}^{(2,2)}(x_{ij},x_{jk},x_{ik})f_2^{(2,1)}(x_{ij},x_{jk},x_{ik})\\
&\quad + f_{11}^{(2,2)}(x_{jk},x_{ij},x_{ik})f_2^{(2,1)}(x_{jk},x_{ij},x_{ik}) . \qquad (3.14)
\end{aligned}$$

Photon luminosity

- **Cross section:**

$$\sigma(\mathrm{A\ B} \xrightarrow{\gamma\gamma} \mathrm{A}\ X\ \mathrm{B}) = \int \frac{dE_{\gamma_1}}{E_{\gamma_1}} \frac{dE_{\gamma_2}}{E_{\gamma_2}} \frac{\mathrm{d}^2 N^{(\mathrm{AB})}_{\gamma_1/\mathrm{Z}_1,\gamma_2/\mathrm{Z}_2}}{\mathrm{d}E_{\gamma_1}\mathrm{d}E_{\gamma_2}} \sigma_{\gamma\gamma\to X}(W_{\gamma\gamma})$$

- **Effective two-photon luminosity:**

$$\frac{\mathrm{d}^2 N^{(\mathrm{AB})}_{\gamma_1/\mathrm{Z}_1,\gamma_2/\mathrm{Z}_2}}{\mathrm{d}E_{\gamma_1}\mathrm{d}E_{\gamma_2}} = \int \mathrm{d}^2\boldsymbol{b}_1 \mathrm{d}^2\boldsymbol{b}_2 \boxed{P_{\text{no inel}}\left(|\boldsymbol{b}_1 - \boldsymbol{b}_2|\right)} N_{\gamma_1/\mathrm{Z}_1}(E_{\gamma_1}, \boldsymbol{b}_1) N_{\gamma_2/\mathrm{Z}_2}(E_{\gamma_2}, \boldsymbol{b}_2)$$
$$\times\theta(b_1 - \epsilon R_\mathrm{A})\theta(b_2 - \epsilon R_\mathrm{B})$$

- **No hadronic/inelastic interaction probability density:**

$$P_{\text{no inel}}(b) = \begin{cases} e^{-\sigma^{\mathrm{NN}}_{\text{inel}}\cdot T_{\mathrm{AB}}(b)}, & \text{nucleus-nucleus} \\ e^{-\sigma^{\mathrm{NN}}_{\text{inel}}\cdot T_{\mathrm{A}}(b)}, & \text{proton-nucleus} \\ |1-\Gamma(s_{\mathrm{NN}}, b)|^2, \quad \text{with } \Gamma(s_{\mathrm{NN}}, b) \propto e^{-b^2/(2b_0)} & \text{p-p} \end{cases}$$

Photon number density calculated using EdFF (Electron dipole form factor or ChFF (charge form factor)

Low energy approx.

$$
\begin{aligned}
-i\mathcal{M}^{(0,0,f)}_{++++,\mathrm{LE}} &= -\frac{1}{15} N_{c,f} Q_f^4 \alpha^2 \left(x_s^2 + x_t^2 + x_u^2\right), \\
-i\mathcal{M}^{(0,0,f)}_{-+++,\mathrm{LE}} &= 0, \\
-i\mathcal{M}^{(0,0,f)}_{--++,\mathrm{LE}} &= \frac{11}{45} N_{c,f} Q_f^4 \alpha^2 x_s^2.
\end{aligned}
$$

The two-loop QCD amplitudes in the same limit are

$$
\begin{aligned}
\mathcal{M}^{(1,0,f)}_{++++,\mathrm{LE}} &= \frac{25}{4}\frac{\alpha_s}{\pi} C_{F,f} \mathcal{M}^{(0,0,f)}_{++++,\mathrm{LE}}, \\
\mathcal{M}^{(1,0,f)}_{-+++,\mathrm{LE}} &= 0, \\
\mathcal{M}^{(1,0,f)}_{--++,\mathrm{LE}} &= \frac{1955}{396}\frac{\alpha_s}{\pi} C_{F,f} \mathcal{M}^{(0,0,f)}_{--++,\mathrm{LE}}.
\end{aligned}
$$

Background for exclusive LbL