MAX-PLANCK-INSTITUT FÜR PHYSIK

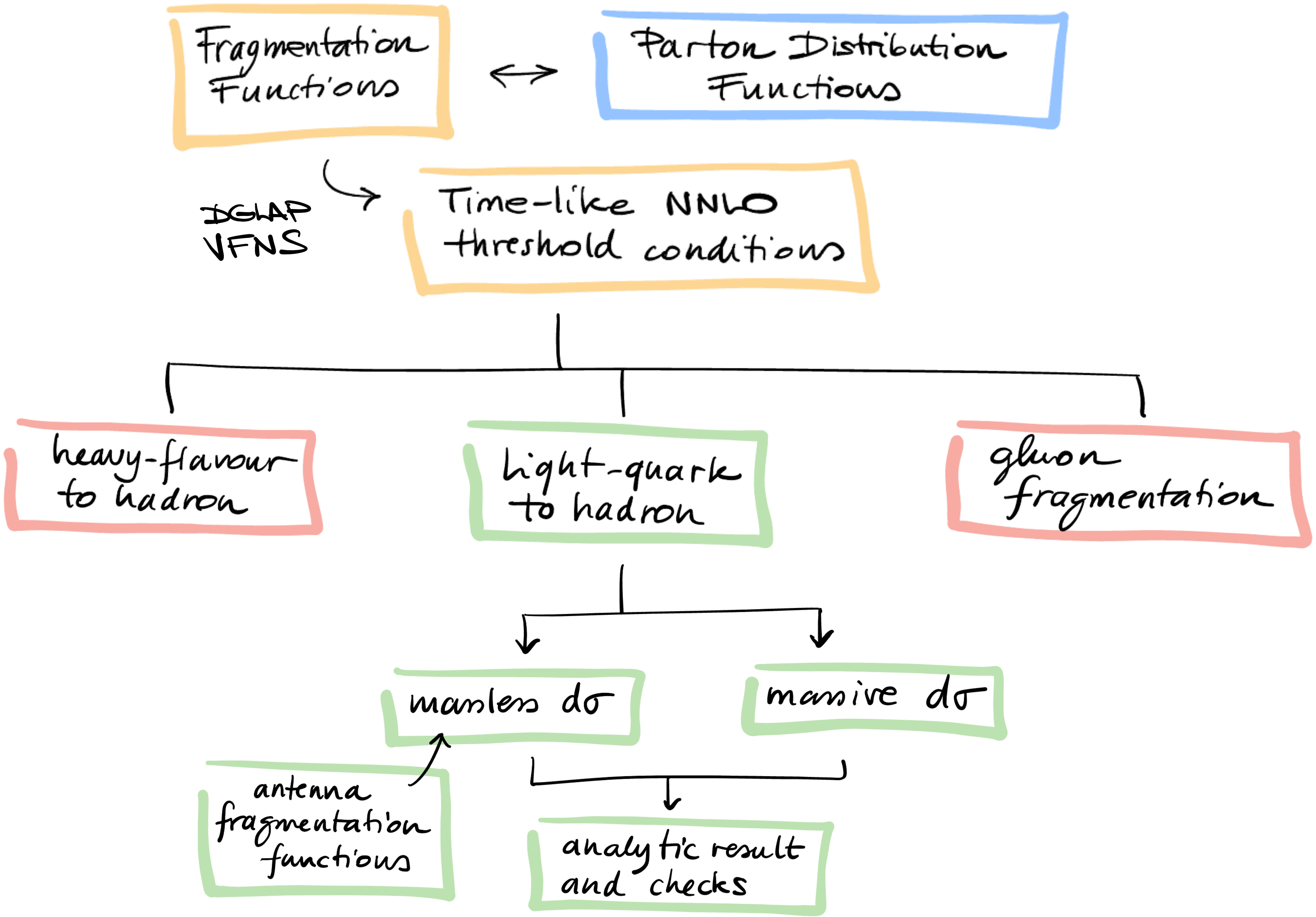
Crossing heavy-flavour thresholds in Fragmentation Functions

Christian Biello (MPP)

in collaboration with Leonardo Bonino (UZH)

based on 2407.07623

9th International Workshop on High Precision for Hard Processes at the LHC
September 12th, 2024
Torino, Italy


Fragmentation Functions
Parton Distribution Functions
DGLAP
VFNS
Time-like NNLO threshold conditions
heavy-flavour to hadron
light-quark to hadron
gluon fragmentation
massless dσ
massive dσ
antenna fragmentation functions
analytic result and checks

Recap on fragmentation functions (FFs)

Growing interest in pheno predictions with identified hadrons
⇝ B and D meson production

Matteo Cacciari's talk
Terry Generet's talk

In processes with identified hadrons, the factorisation formula reads

$$\frac{\mathrm{d}\sigma_{pp\to H}}{\mathrm{d}x} = \int \mathrm{d}\xi_1 \mathrm{d}\xi_2 \mathrm{d}z\, f^i(\xi_1) f^j(\xi_2) D^k(z) \,\frac{\mathrm{d}\sigma_{ijk}}{\mathrm{d}\hat{x}}\, \delta(x - z\hat{x})\mathrm{d}\hat{x}, \quad x = \frac{E_H}{E_{beam}}$$

In Mellin space for e^+e^-,

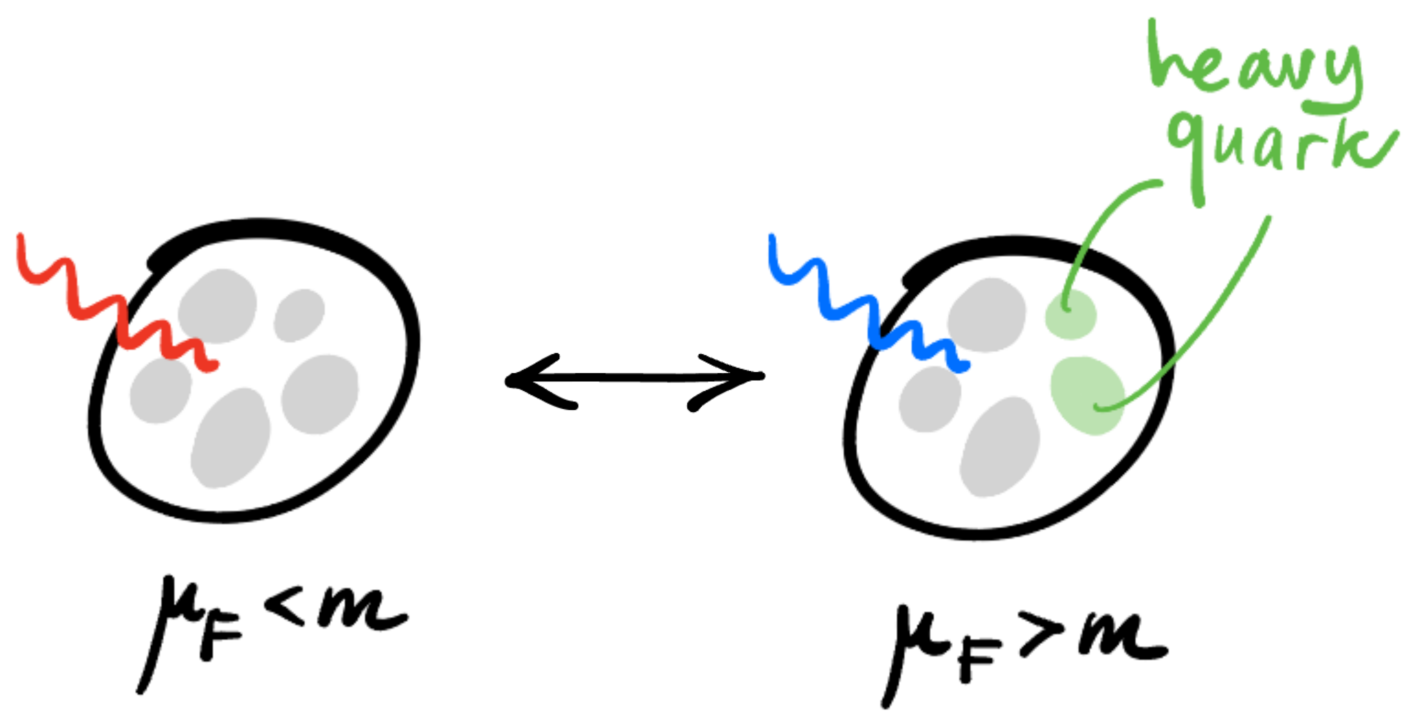
$$\frac{\mathrm{d}\sigma_H}{\mathrm{d}x} = \int \frac{\mathrm{d}z}{z} \sum_k D^k\left(\frac{x}{z}, \mu\right) \frac{\mathrm{d}\sigma_k}{\mathrm{d}z}, \quad k \in \text{partons}$$

$$\Sigma(N, Q^2) \sim \mathcal{M}_{(N,x)}\left[\frac{\mathrm{d}\sigma_H}{\mathrm{d}x}\right] = \sum_k D^k(N,\mu)\, \mathcal{M}_{(N,x)}\left[\frac{\mathrm{d}\sigma_k}{\mathrm{d}z}\right]$$

$$\mathcal{M}_{(N,x)}[D(x)] = \int_0^1 dx\, x^{N-1} D(x)$$

Improved description of Σ ratio in Andrea Ghira's talk

Heavy-flavour thresholds



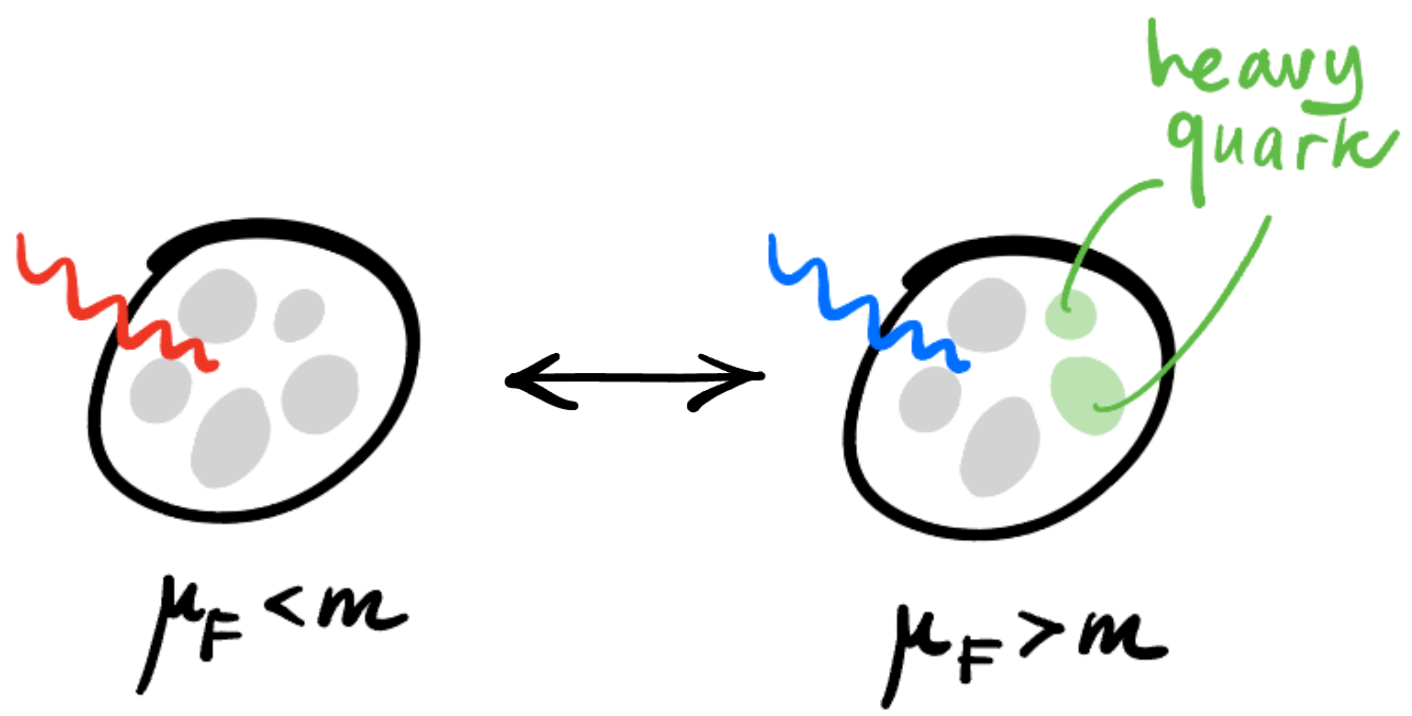
PDFs and FFs change when heavy-flavour thresholds are crossed.

$$\left\{ D_g^{(n_L)}, D_i^{(n_L)} \right\} \xleftrightarrow{\mu_F = m} \left\{ D_g^{(n)}, D_i^{(n)}, D_h^{(n)} \right\}$$

with n_L quarks (i) for $\mu_F < m$

with $n = n_L + 1$ flavours for $\mu_F > m$

Heavy-flavour thresholds



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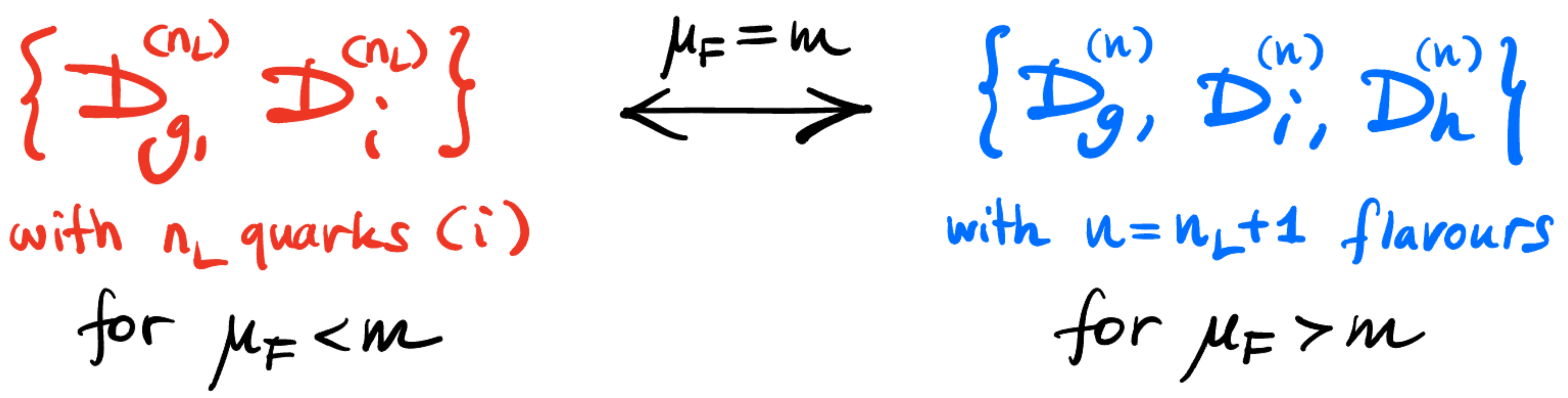
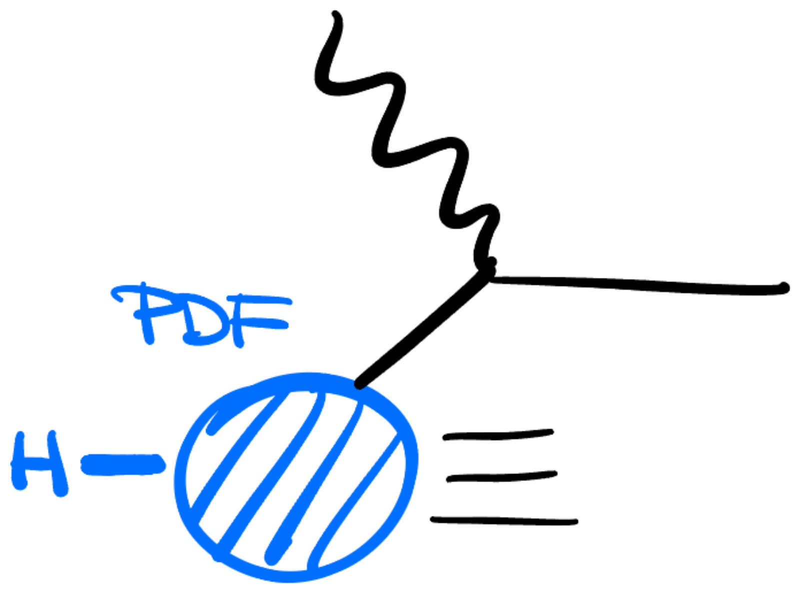


The matching conditions of the two sets around the mass m is an ingredient of the **DGLAP evolution in the VFNS**.

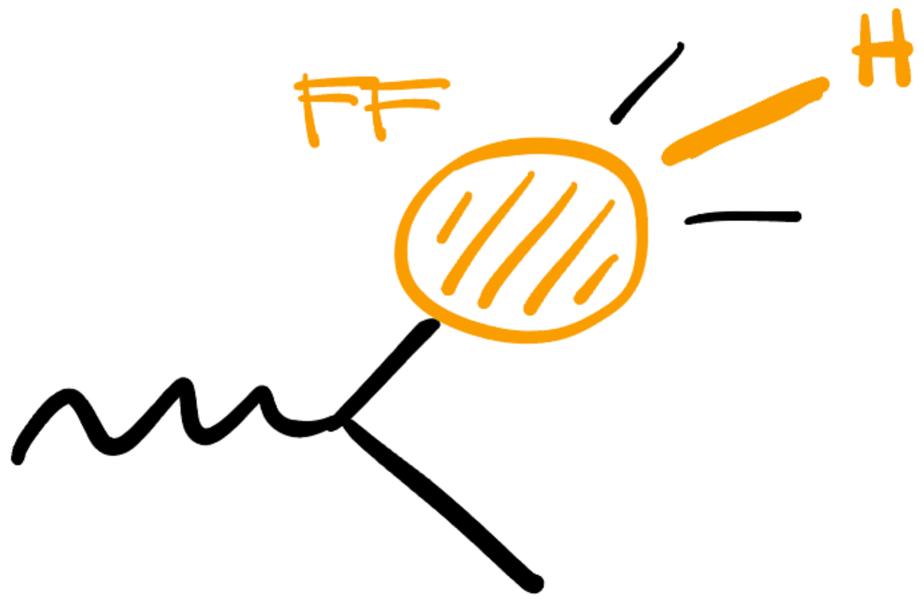
They should be implemented in the **PDFs and FFs fits** for the extraction of light-hadron and heavy-hadron functions at fixed accuracy.

The NNLO space-like conditions played a fundamental role in the evidence of **intrinsic charm** in the proton!

NNPDF Collaboration [2208.08372, 2311.00743]



State-of-the-art



Space-like conditions

NNLO matching conditions have long been known using the formalism of Operator Matrix Elements

Buza et al. [9612398]

All the necessary contributions for the N3LO accuracy are computed.

Blümlein et al. [0904.3563, 1406.4654, 2211.05462, 2311.00644, 2403.00513]

The matching equations are included in the PDF evolution libraries, like `APFEL` and `EKO`.

Bertone, Carrazza, Rojo [1310.1394]
Candido, Hekhorn, Magni [2202.02338]

Time-like conditions

The matching conditions were firstly derived at NLO accuracy

Cacciari, Nason, Oleari [0504192]

We have extended the approach at NNLO and computed the light-quark matching,

$$D_i^{(n)} = D_i^{(n_L)} \left(1 + \alpha_s^2 \delta D_i\right) + \mathcal{O}(\alpha_s^3)\,.$$

CB, Bonino [2407.07623]

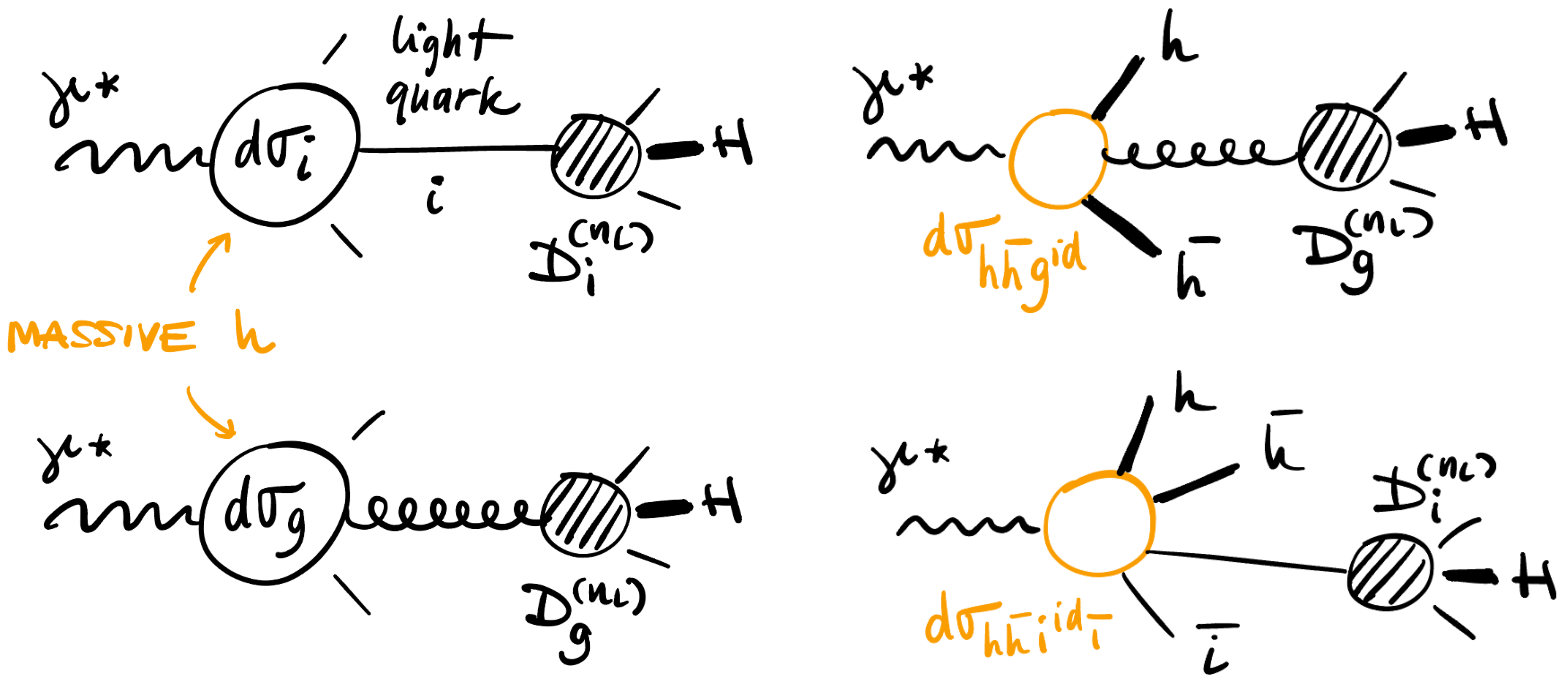
The NLO matching equations are included in the FF evolution library `MELA`.

Bertone, Carrazza, Nocera [1501.00494]

Electron-positron annihilation

Due to their universality, we focus on the simplest process: $e^+e^- \to \gamma^* \to H$.

Below threshold



$$\frac{d\sigma_H^{massive}}{dx} = \int_x^1 \frac{dz}{z} \left\{ \sum_i D_i^{(n_L)} \frac{d\sigma_i}{dz} + D_g^{(n_L)} \frac{d\sigma_g}{dz} + D_g^{(n_L)} \frac{d\sigma_{h\bar{h}g^{id.}}}{dz} + \sum_i D_i^{(n_L)} \frac{d\sigma_{h\bar{h}i^{id.}\bar{i}}}{dz} \right\}$$

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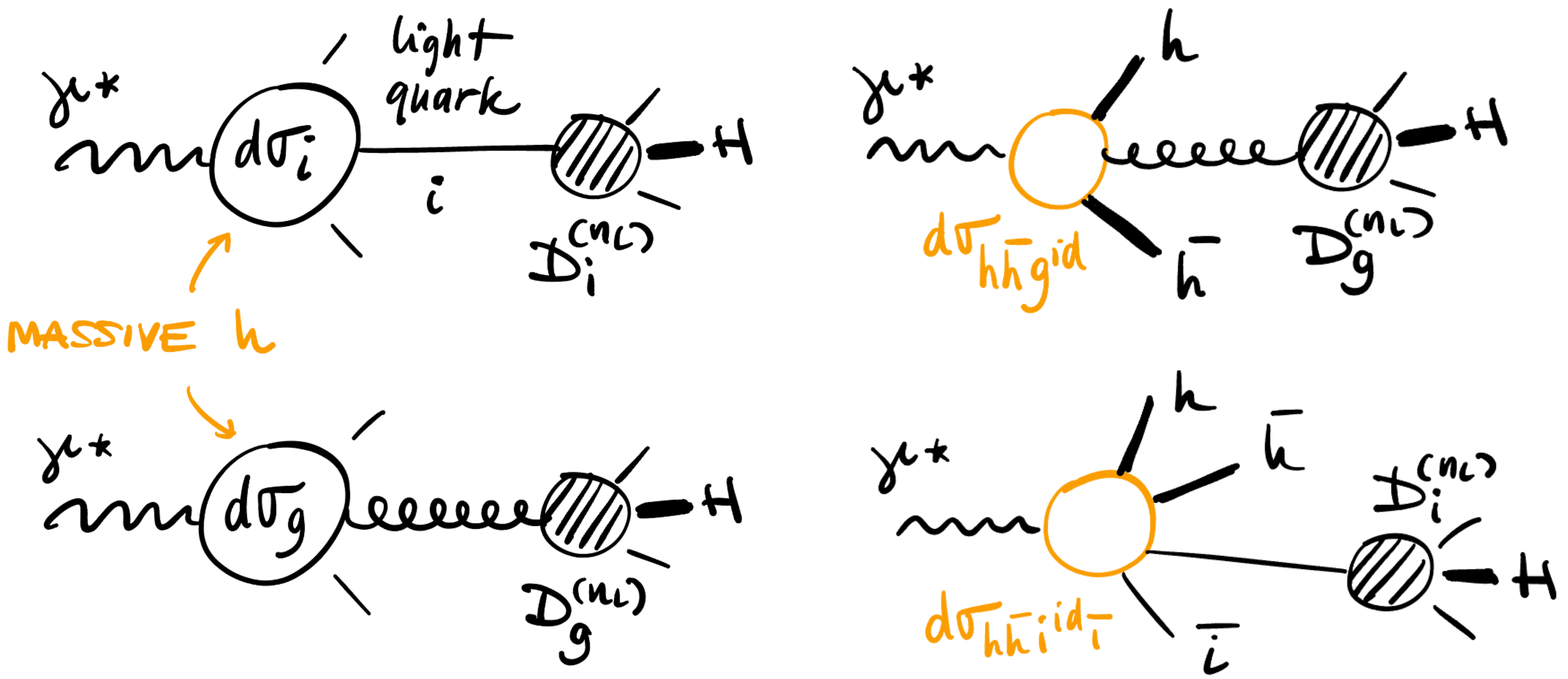
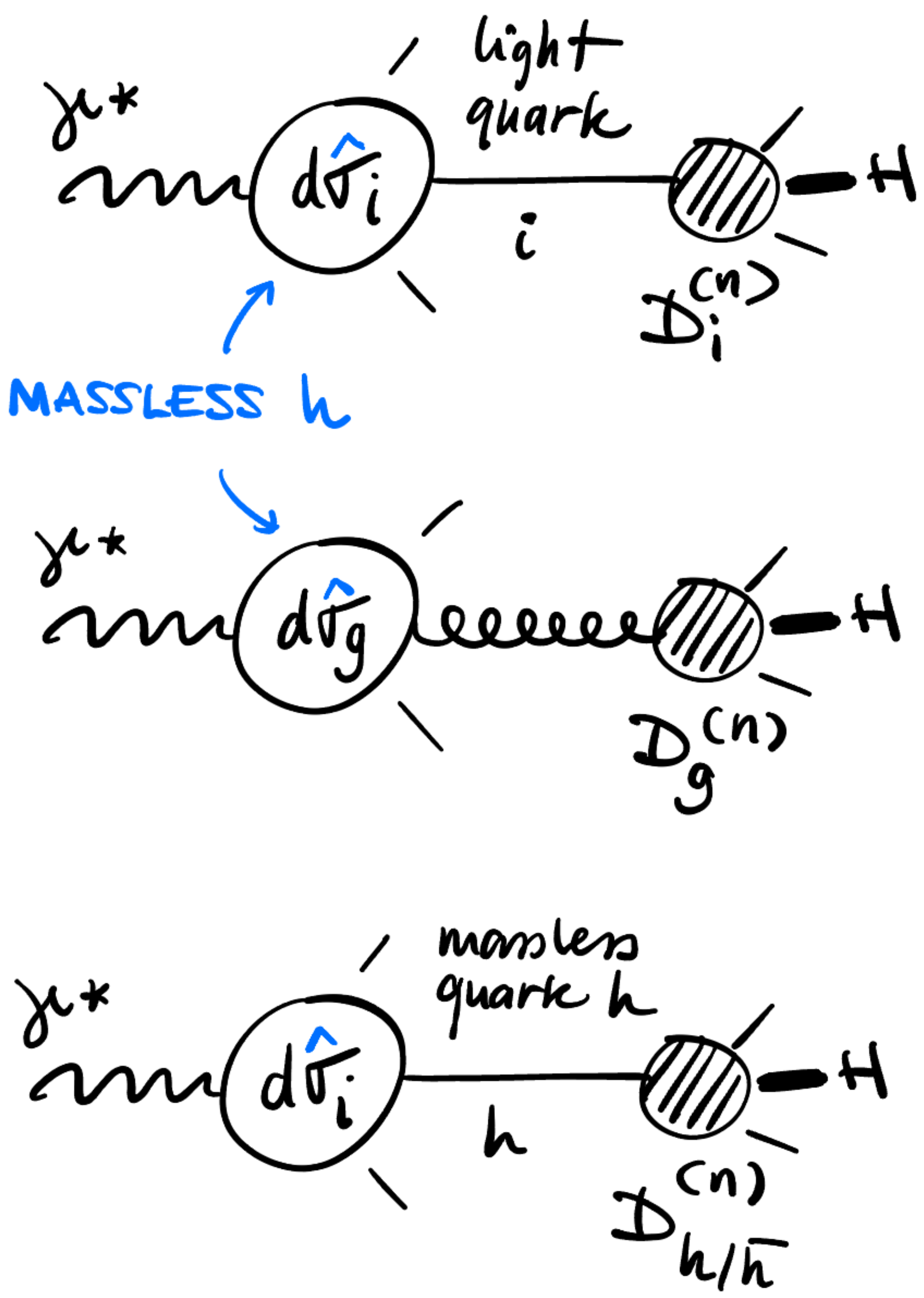
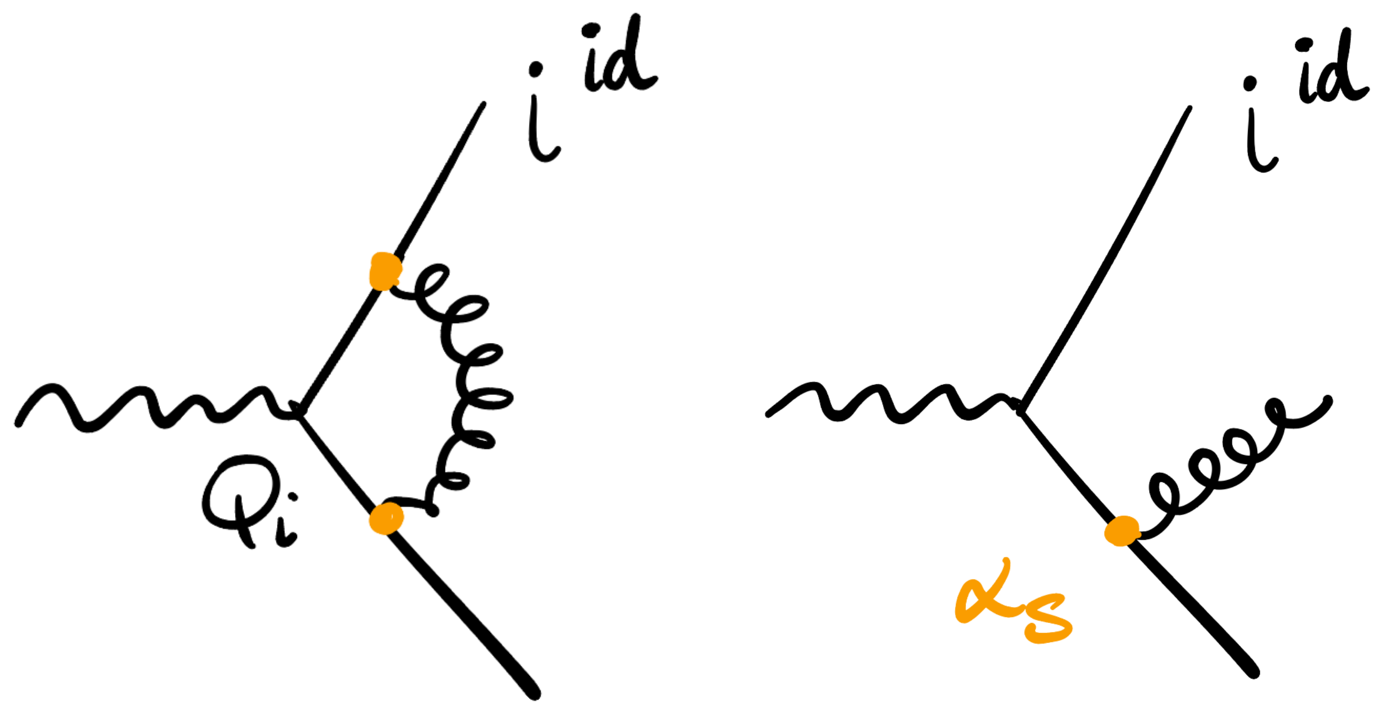


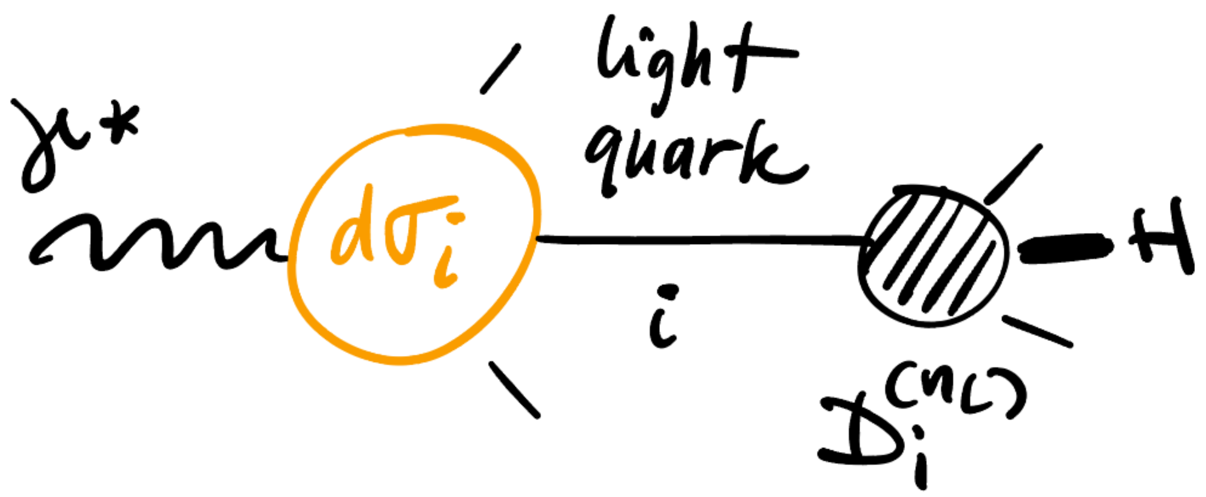
Above threshold

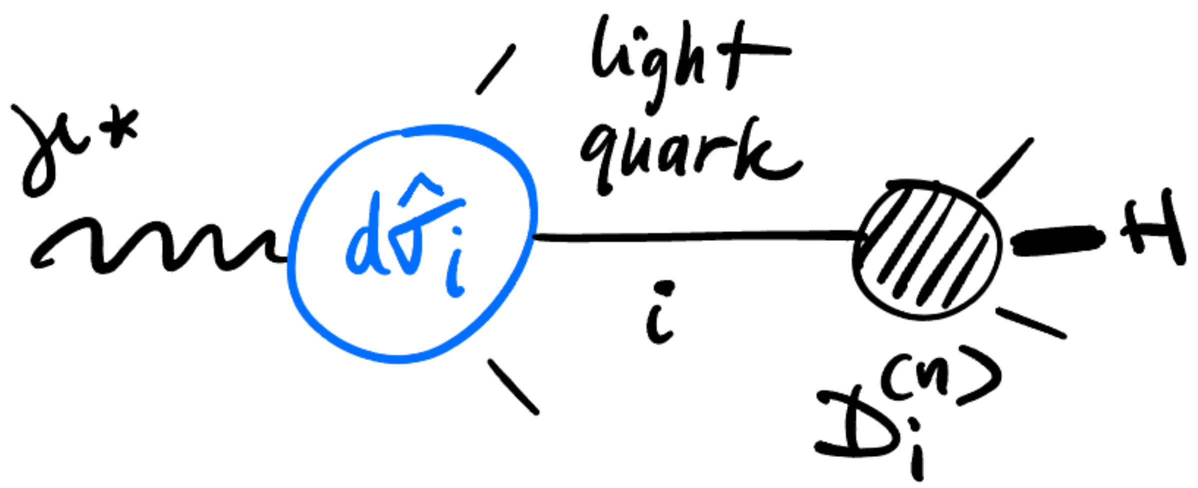


$$\frac{d\sigma_H^{massive}}{dx} = \int_x^1 \frac{dz}{z}\left\{\sum_i D_i^{(n_L)}\frac{d\sigma_i}{dz} + D_g^{(n_L)}\frac{d\sigma_g}{dz} + D_g^{(n_L)}\frac{d\sigma_{h\bar{h}g^{id.}}}{dz} + \sum_i D_i^{(n_L)}\frac{d\sigma_{h\bar{h}i^{id.}\bar{i}}}{dz}\right\}$$

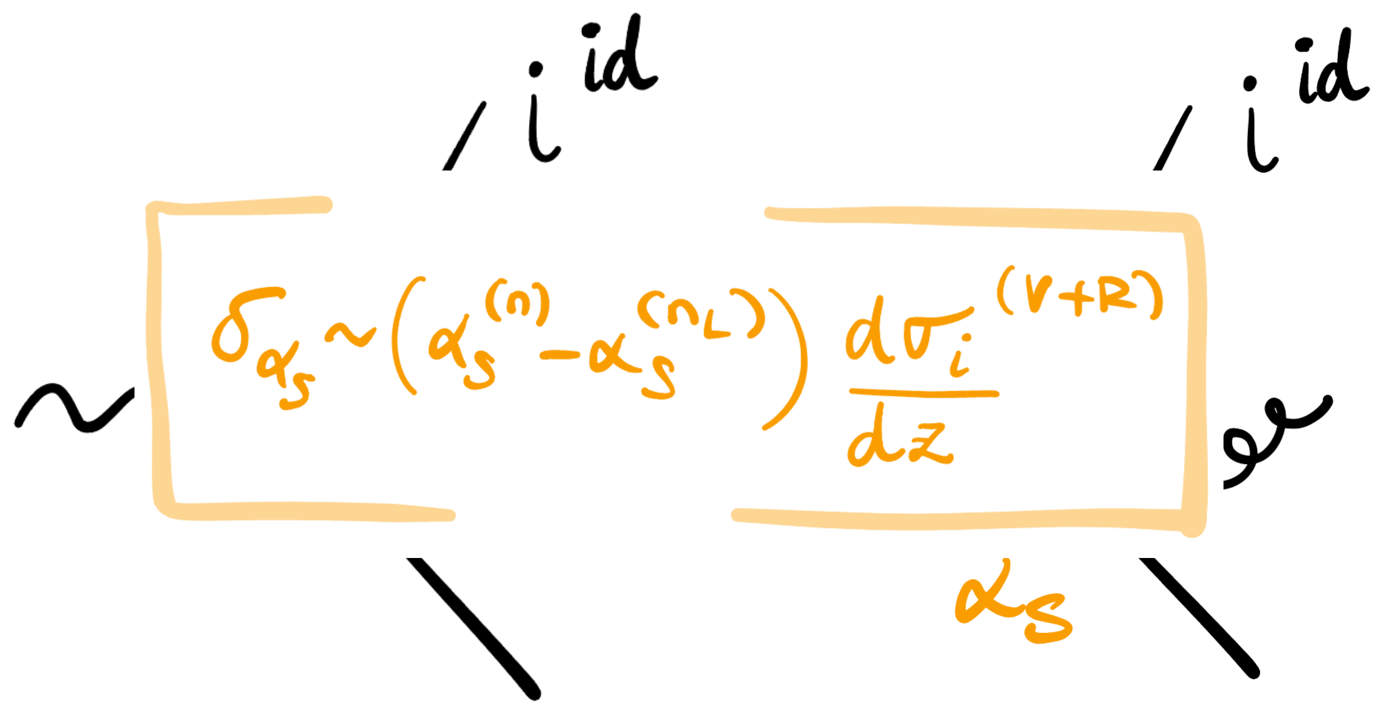
Focus on the hadron production via a light-flavour fragmentation

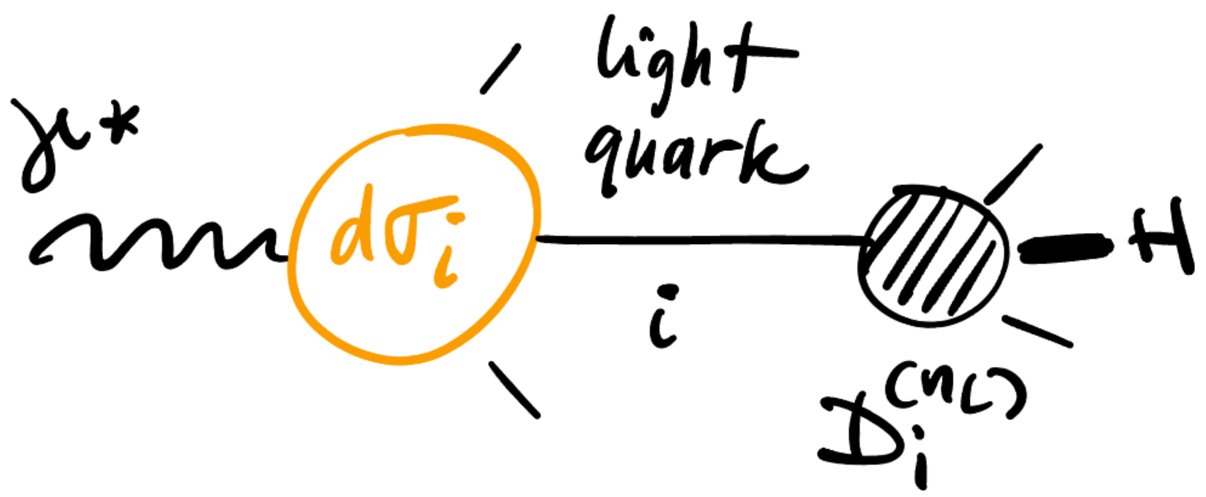


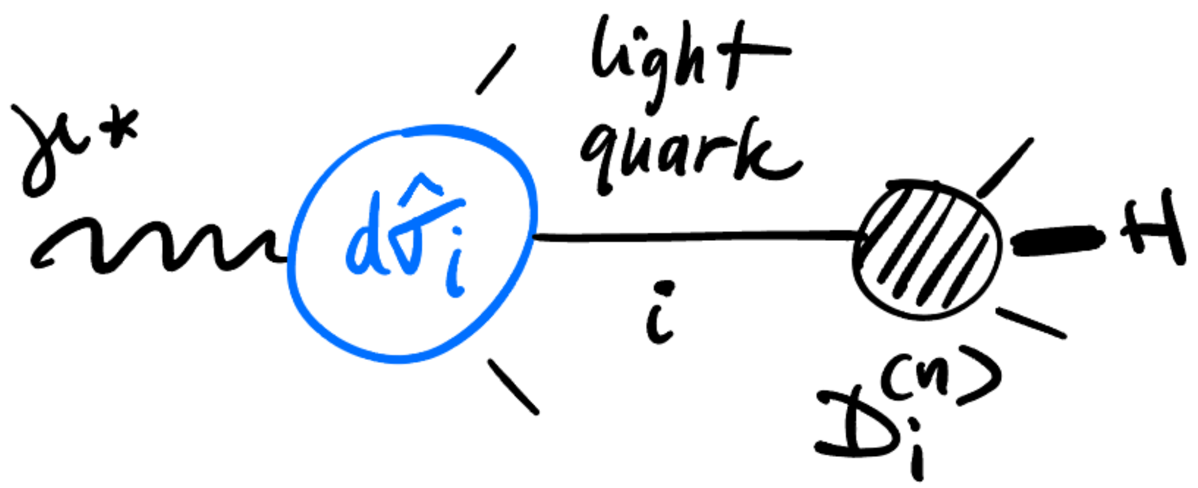




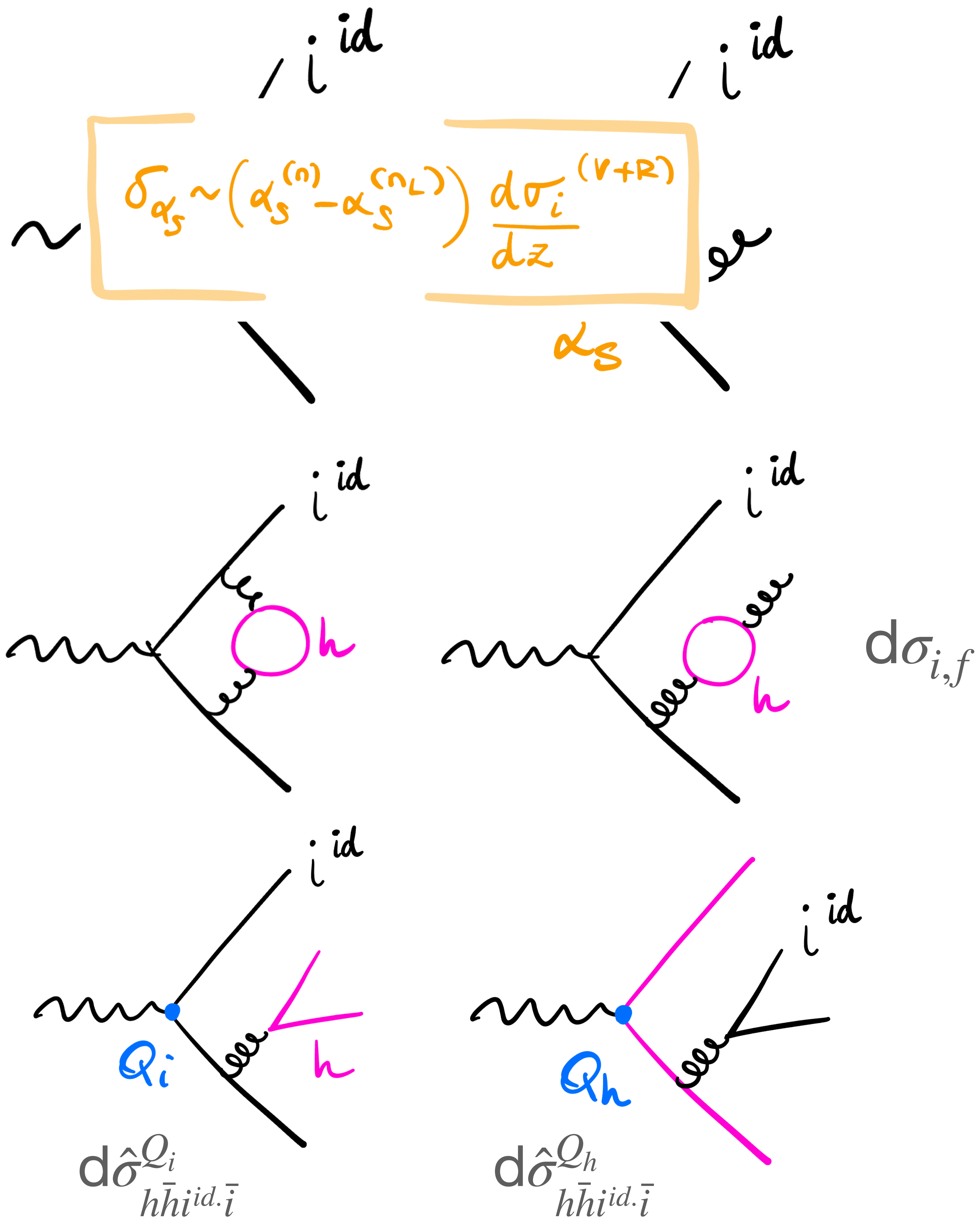
Focus on the hadron production via a light-flavour fragmentation

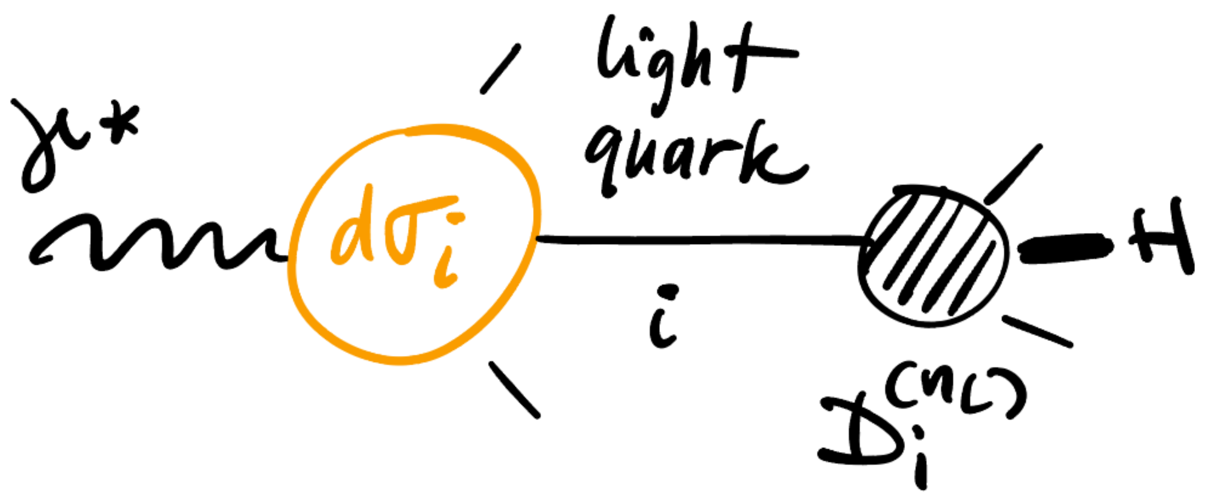


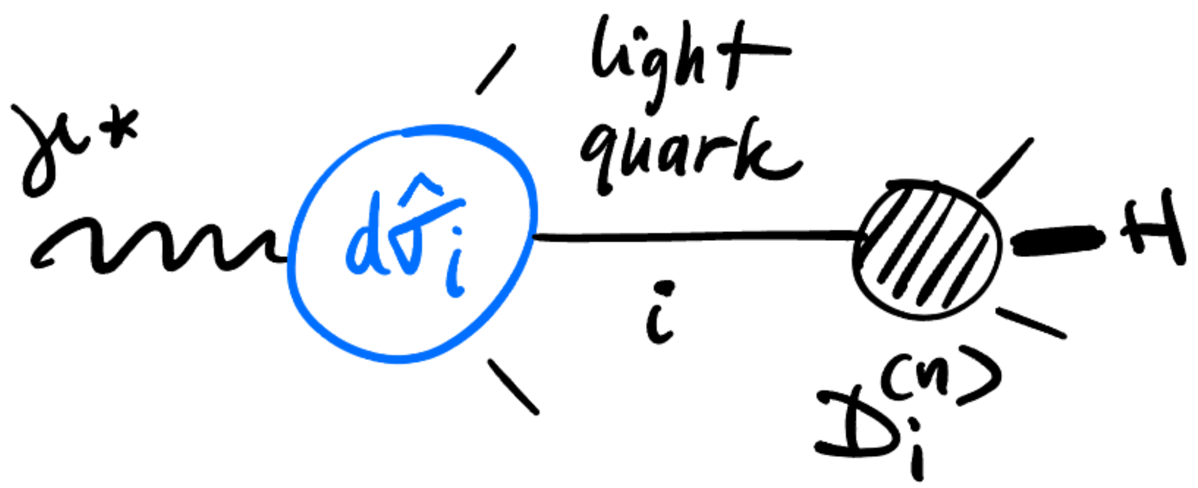




Focus on the hadron production
via a light-flavour fragmentation







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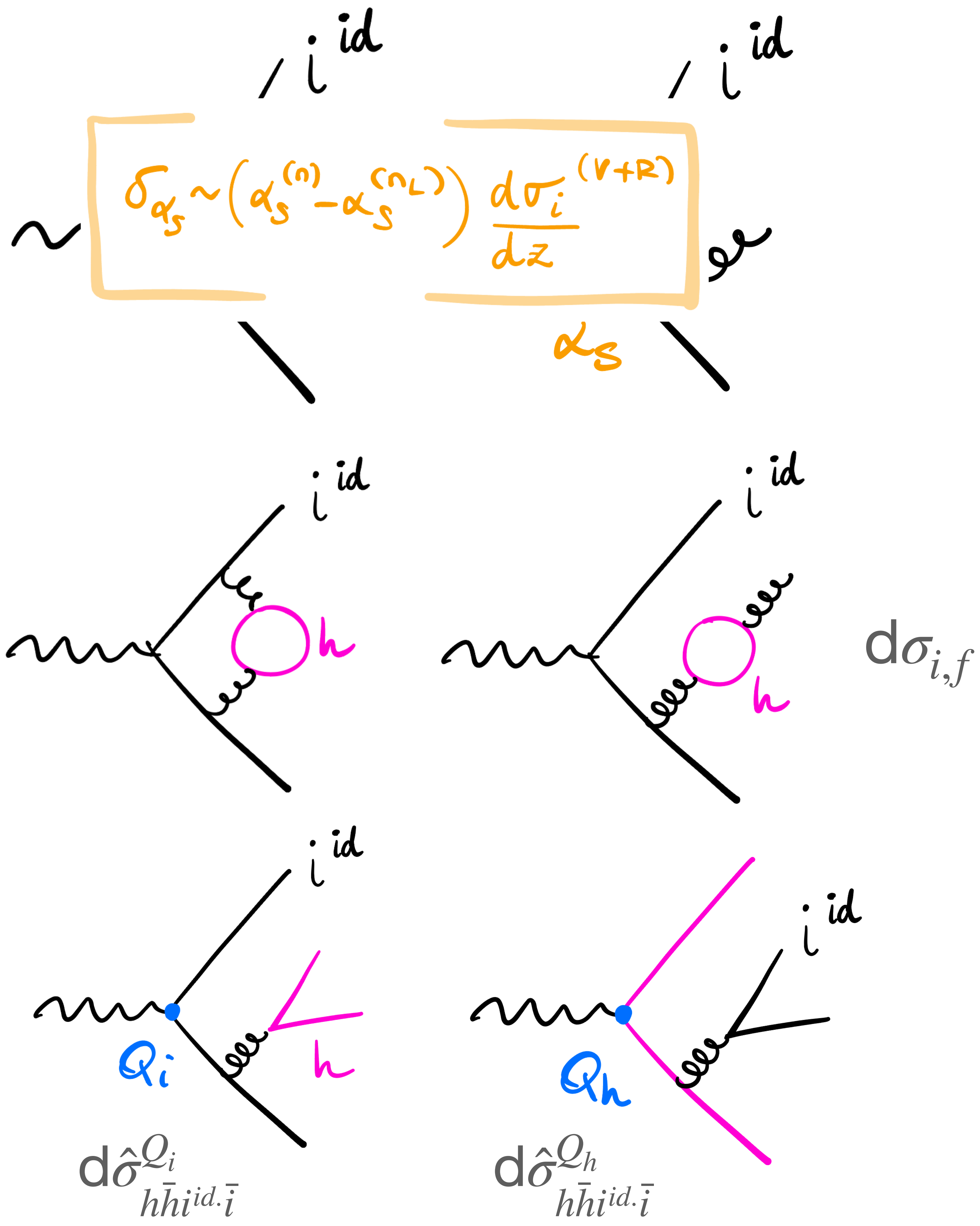

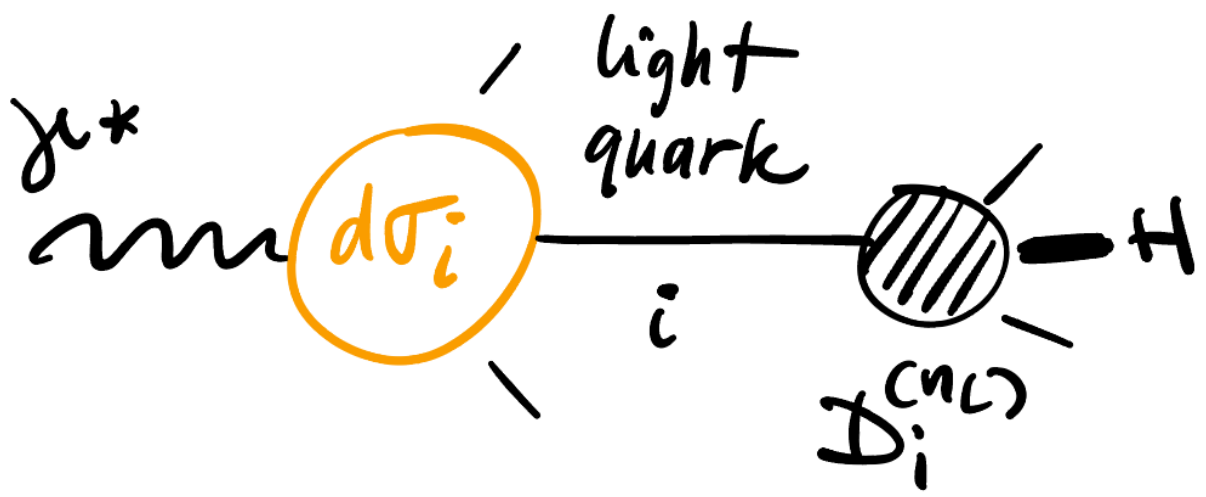

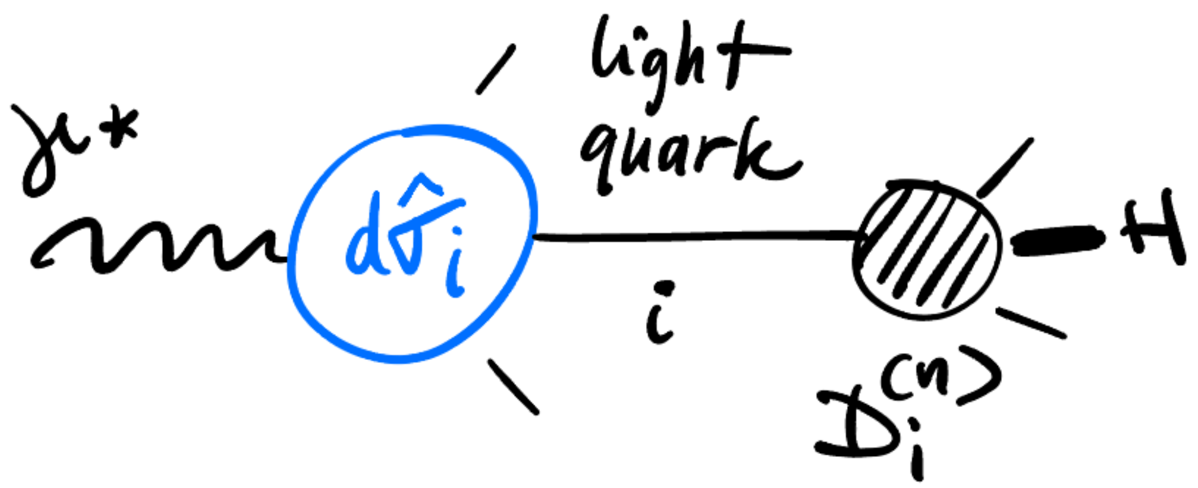

The independent EM charges produce independent matching conditions

$$d\sigma_H^{massive} - d\sigma_H^{massless} \sim \sum_i Q_i^2(\dots) + Q_h^2(\dots)$$

$$(\dots) = 0 \qquad (\dots) = 0$$

D_i can be easily isolated from Q_i^2

$$D_i^{(n)}(x,\mu) - D_i^{(n_L)}(x,\mu) =$$

$$\frac{1}{\sigma_{i\bar i}}\int_x^1 dz\, D_i^{(n_L)}\left(\frac{x}{z},\mu\right)\left[\frac{d\sigma^{Q_i}_{h\bar h i\bar i}}{dz} - \frac{d\hat\sigma^{Q_i}_{h\bar h i\bar i}}{dz} + \frac{d\sigma_{i,f}}{dz} - \frac{d\hat\sigma_{i,f}}{dz} + \delta_{\alpha_s}\right]$$

Partonic cross-sections: massless

The NNLO massless cross-sections are computed for example in antenna subtraction.

Leonardo Bonino's talk

Gehrmann, Stagnitto [2208.02650]
Bonino, Gehrmann, Marcoli, Schürmann, Stagnitto [2406.09925]

$$\frac{\mathrm{d}\hat{\sigma}^{Q_i}_{h\bar{h}i^{id}\cdot\bar{i}}}{\mathrm{d}z} + \frac{\mathrm{d}\hat{\sigma}_{i,f}}{\mathrm{d}z} = \sigma_{i\bar{i}} \left(\frac{\alpha_s}{2\pi}\right)^2 2C_F \left(\mathcal{I}^{(2)}_{q\bar{q}}\Big|_{N_f} + F^{(2)}_q\Big|_{N_f} \right)$$

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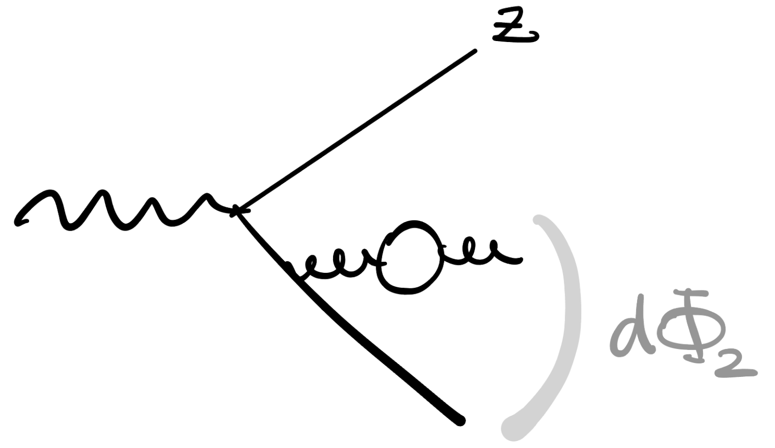
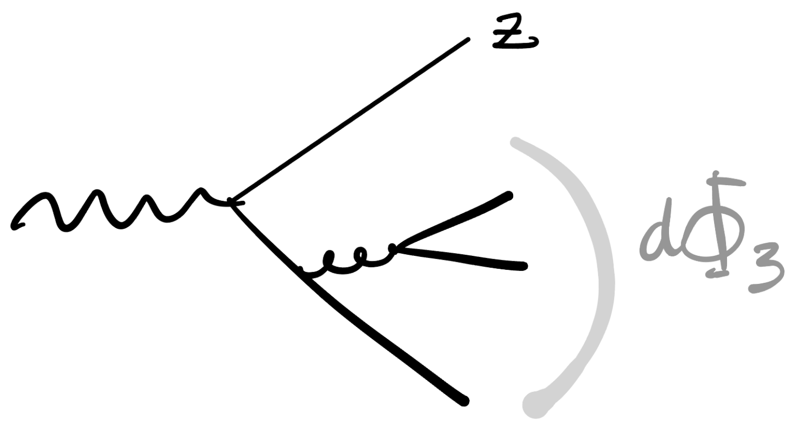
The RR and RV corrections can be captured by the N_f part of the antenna dipole $\mathcal{J}^{(2)}$, which depends on the antennae

$$\mathcal{A}^{1,id.q}_3 \sim \int \mathrm{d}\Phi^{(4-2\epsilon)}_2\, z^{1-2\epsilon} A^1_3$$

$$\mathcal{B}^{0,id.q}_4 \sim \int \mathrm{d}\Phi^{(4-2\epsilon)}_3\, z^{1-2\epsilon} B^0_4$$





Mass factorisation is also performed at dipole level.

The plus-prescription emerges from dimensional regularisation

$$(1-z)^{-1+k\epsilon} = -\frac{1}{k\epsilon}\delta(1-z) + \left(\frac{1}{1-z}\right)_+ - k\epsilon\left(\frac{\log(1-z)}{1-z}\right)_+ + \ldots$$

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→pole cancellation checked

→we reproduced the R-ratio for N=1

Gehrmann-De Ridder, Gehrmann, Glover [0403057]

$$\mathcal{A}^{1,id.q}_3 \sim \int \mathrm{d}\Phi^{(4-2\epsilon)}_2 \, z^{1-2\epsilon} A^1_3$$

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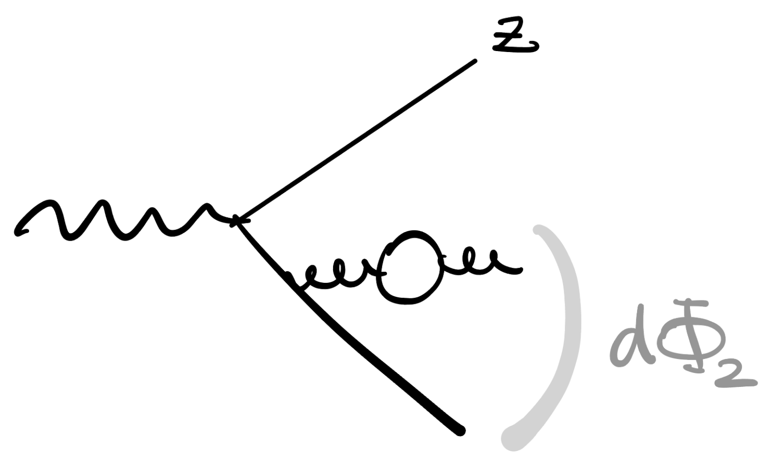

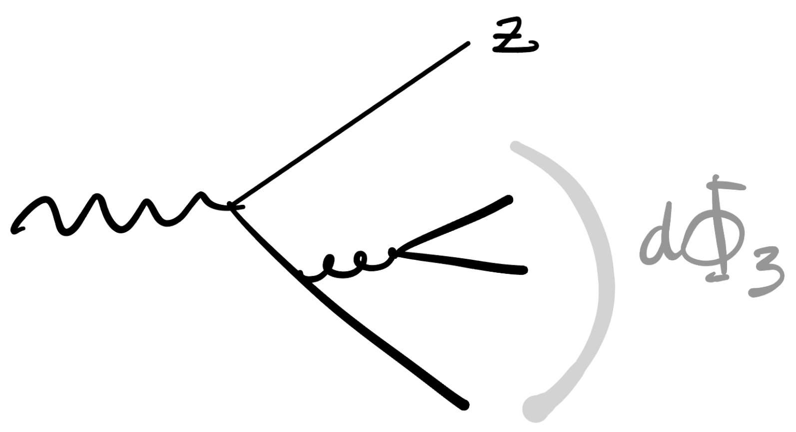

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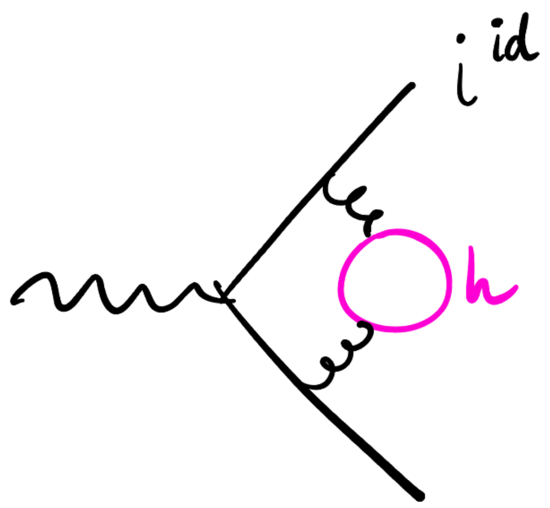
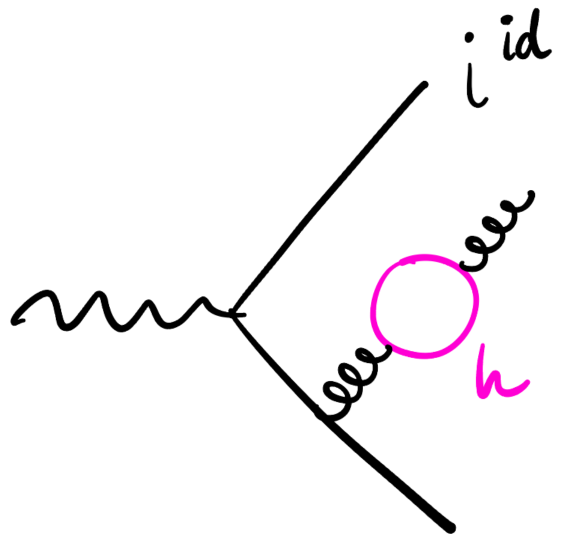
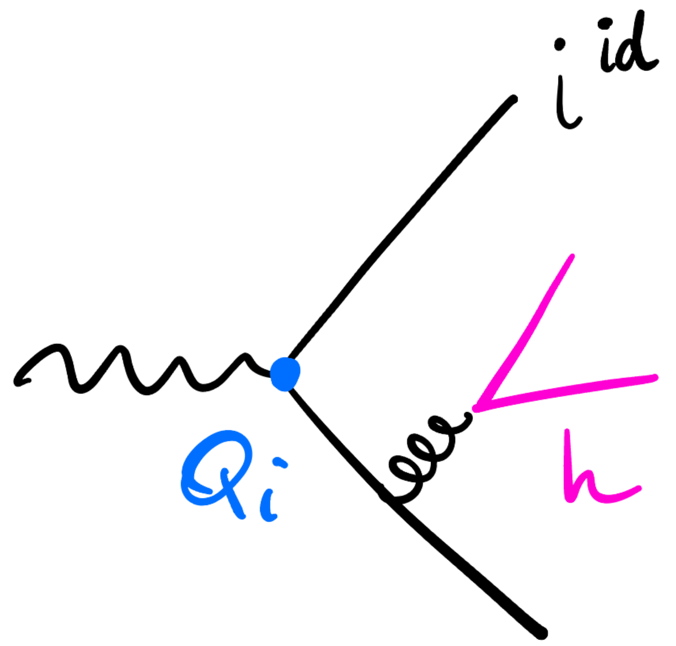
Partonic cross-sections: massive





The virtual corrections are computed for space-like purposes.

Blümlein, Falcioni, De Freitas [1605.05541]



We have computed the RR correction in the fragmentation kinematics

$$\frac{\mathrm{d}\sigma^{Q_i}_{h\bar{h}i^{id}.i}}{\mathrm{d}z} \sim \bar{\mathscr{B}}^{0,id.i}_{ih\bar{h}i}(z,m) \sim \int \mathrm{d}\Phi_3 \; z^{1-2\epsilon}\bar{B}^{0}_{ih\bar{h}i}$$

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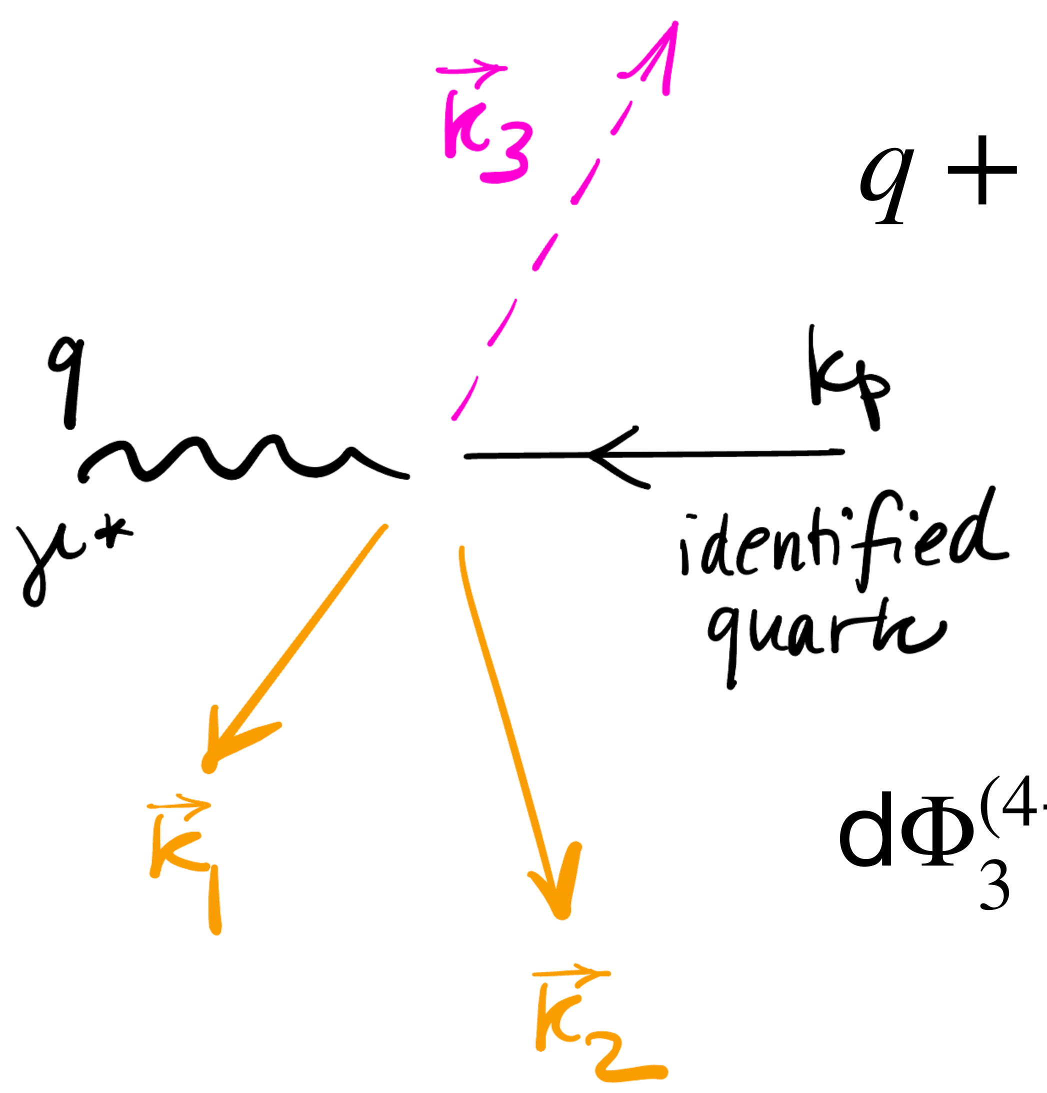
Two different phase-space parametrisation as a check



$$q + (-k_p^{(0)}) \to k_1^{(m)} + k_2^{(m)} + k_3^{(0)}$$

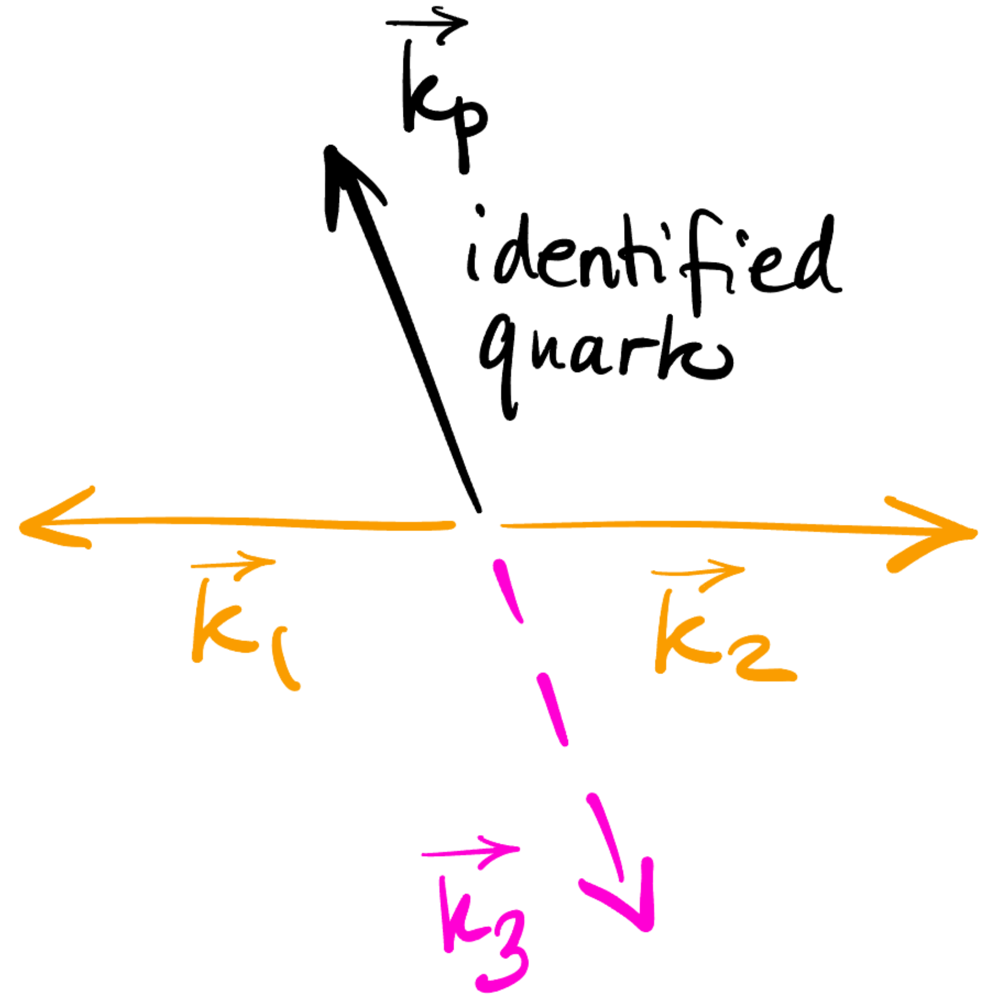
Compute $\mathrm{d}\Phi_3$ with $(1-z)Q^2$ CoM energy

$$\mathrm{d}\Phi_3^{(4-2\epsilon)} \sim \mathrm{d}s_{12}\mathrm{d}s_{23}\mathrm{d}\cos\theta_3\mathrm{d}\phi_3[\ldots]$$

$$q \to k_p^{(0)} + k_1^{(m)} + k_2^{(m)} + k_3^{(0)}$$

Write $\mathrm{d}\Phi_4$ in the off-shell γ frame and do not perform the integration in E_p

$$\mathrm{d}\Phi_4^{(4)} \sim \mathrm{d}z\mathrm{d}\Phi_3^{(4)}$$



Partonic cross-sections: massive

We perform the analytic integration with the two phase-space parametrisations and expand in the **small-mass limit.**

For the Mellin transformation, we regularised the end-point divergence

$$\int_0^{1-\frac{4m^2}{Q^2}} \mathrm{d}z\, \mathscr{B}^{0,id.i}_{ih\bar{h}i}(z,m) g(z) = \int_0^1 \mathrm{d}z \mathscr{B}^{0,id.i}_{ih\bar{h}i,reg}(z,m) g(z) + \mathcal{O}(m^2)$$

$$\mathscr{B}^{0,id.i}_{ih\bar{h}i,reg}(z,m) = \left[\mathscr{B}^{0,id.i}_{ih\bar{h}i}(z,m)\right]_+ + \delta(1-z) \int_0^{1-\frac{4m^2}{Q^2}} \mathrm{d}z \mathscr{B}^{0,id.i}_{ih\bar{h}i}(z,m)$$

Now the Mellin transform can be computed and it can talk with the massless!

Cancellation of $\log^3 \frac{m^2}{Q^2}$ with the virtual

i.e. cancellation of the massified poles ϵ^{-3}

$$\frac{1}{\epsilon} \to \log m^2 + \text{const.}$$

As expected from the universality properties, all the $\log^2 Q$ and $\log Q$ vanish in the difference.

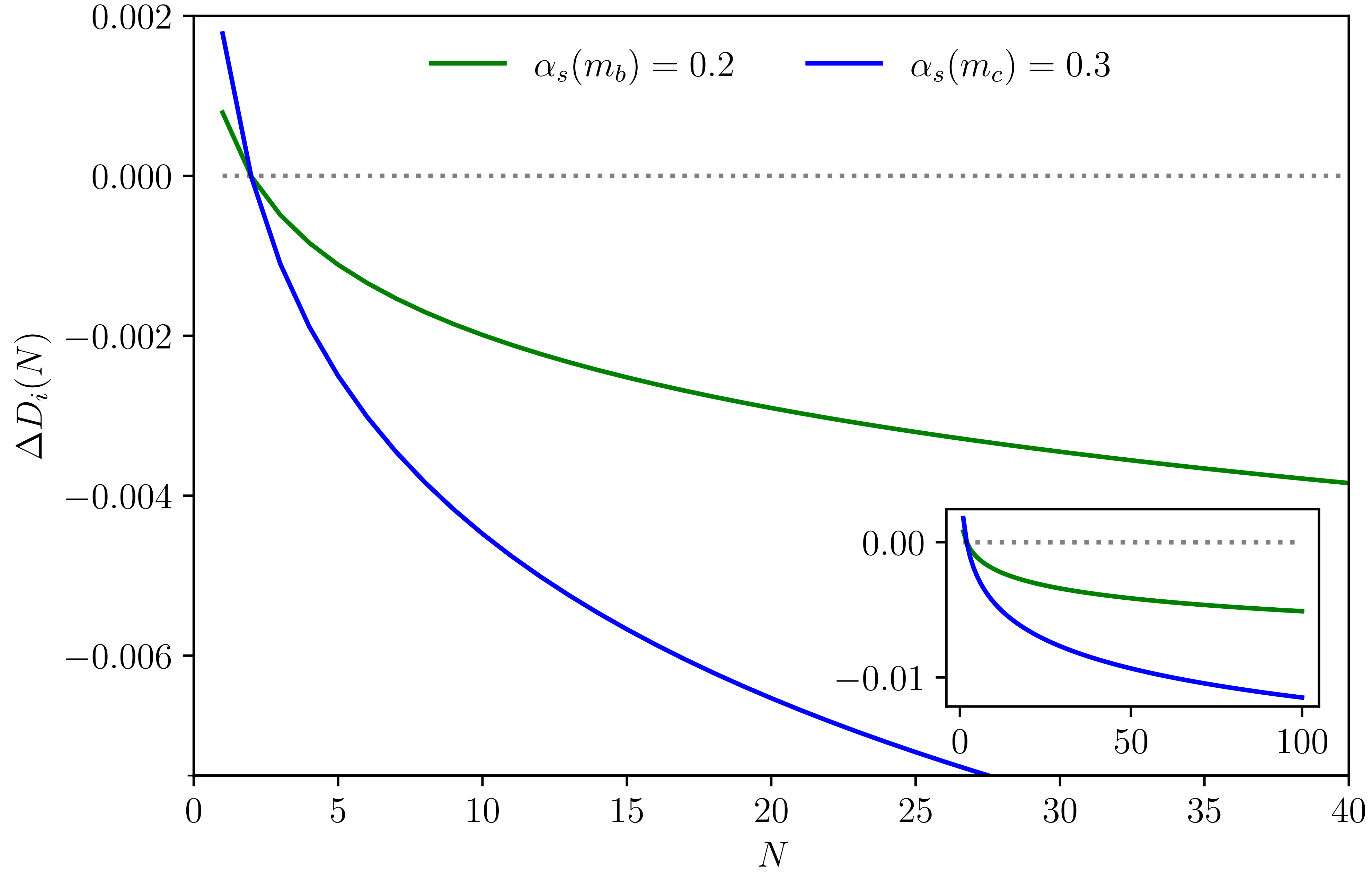
Compact result

Mellin space

$$D_i^{(n)}(N,\mu) = \Bigg\{1 + \left(\frac{\alpha_s}{2\pi}\right)^2 C_F \frac{1}{N^3(N+1)^3}\Bigg[-\frac{2}{3}N^3(N+1)^3 S_{1,2}(N) - \frac{2}{3}N^3(N+1)^3 S_{2,1}(N) + \frac{1}{3}N^3(N+1)^3 S_3(N)$$
$$+\frac{5}{9}N^3(N+1)^3 S_2(N) + S_1(N)\left(\frac{2}{3}N^3(N+1)^3 S_2(N) - \frac{28}{27}N^3(N+1)^3\right) - \frac{4}{3}N^3(N+1)^3\zeta_3$$
$$+\left(\frac{9307}{1296} - \frac{29}{108}\pi^2\right)N^6 + \left(\frac{9307}{432} - \frac{29}{36}\pi^2\right)N^5 + \left(\frac{3281}{144} - \frac{29}{36}\pi^2\right)N^4 + \left(\frac{10939}{1296} - \frac{29}{108}\pi^2\right)N^3 - \frac{5}{54}N^2 - \frac{1}{9}N + \frac{1}{6}$$
$$-\frac{8}{9}N^3(N+1)^3\log^3 2 + \frac{29}{9}N^3(N+1)^3\log^2 2 + \frac{1}{54}\left(12\pi^2 - 359\right)N^3(N+1)^3\log 2$$
$$+\left(\frac{10}{9}N^3(N+1)^3 S_1(N) - \frac{2}{3}N^3(N+1)^3 S_2(N) - \frac{1}{36}N\left(3N^5 + 9N^4 + 53N^3 + 67N^2 + 8N - 12\right)\right)\log\left(\frac{\mu^2}{m^2}\right)$$
$$+\left(\frac{1}{12}N^2(N+1)^2\left(3N^2 + 3N + 2\right) - \frac{1}{3}N^3(N+1)^3 S_1(N)\right)\log^2\left(\frac{\mu^2}{m^2}\right)\Bigg] + \mathcal{O}(\alpha_s^3)\Bigg\}D_i^{(n_L)}(N,\mu)\,.$$

for a straightforward implementation in MELA

$$\Delta D_i(N) = \frac{D_i^{(n)}(N,m) - D_i^{(n_L)}(N,m)}{D_i^{(n_L)}(N,m)}$$



The growing behaviour for larger N is governed by the RR emissions

Compact result

Mellin space

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$$+\frac{5}{9}N^3(N+1)^3 S_2(N) + S_1(N)\left(\frac{2}{3}N^3(N+1)^3 S_2(N) - \frac{28}{27}N^3(N+1)^3\right) - \frac{4}{3}N^3(N+1)^3\zeta_3$$
$$+\left(\frac{9307}{1296}-\frac{29}{108}\pi^2\right)N^6 + \left(\frac{9307}{432}-\frac{29}{36}\pi^2\right)N^5 + \left(\frac{3281}{144}-\frac{29}{36}\pi^2\right)N^4 + \left(\frac{10939}{1296}-\frac{29}{108}\pi^2\right)N^3 - \frac{5}{54}N^2 - \frac{1}{9}N + \frac{1}{6}$$
$$-\frac{8}{9}N^3(N+1)^3\log^3 2 + \frac{29}{9}N^3(N+1)^3\log^2 2 + \frac{1}{54}\left(12\pi^2-359\right)N^3(N+1)^3\log 2$$
$$+\left(\frac{10}{9}N^3(N+1)^3 S_1(N) - \frac{2}{3}N^3(N+1)^3 S_2(N) - \frac{1}{36}N\left(3N^5+9N^4+53N^3+67N^2+8N-12\right)\right)\log\left(\frac{\mu^2}{m^2}\right)$$
$$+\left(\frac{1}{12}N^2(N+1)^2\left(3N^2+3N+2\right) - \frac{1}{3}N^3(N+1)^3 S_1(N)\right)\log^2\left(\frac{\mu^2}{m^2}\right)\Big] + \mathcal{O}(\alpha_s^3)\Big\} D_i^{(n_L)}(N,\mu)\,.$$

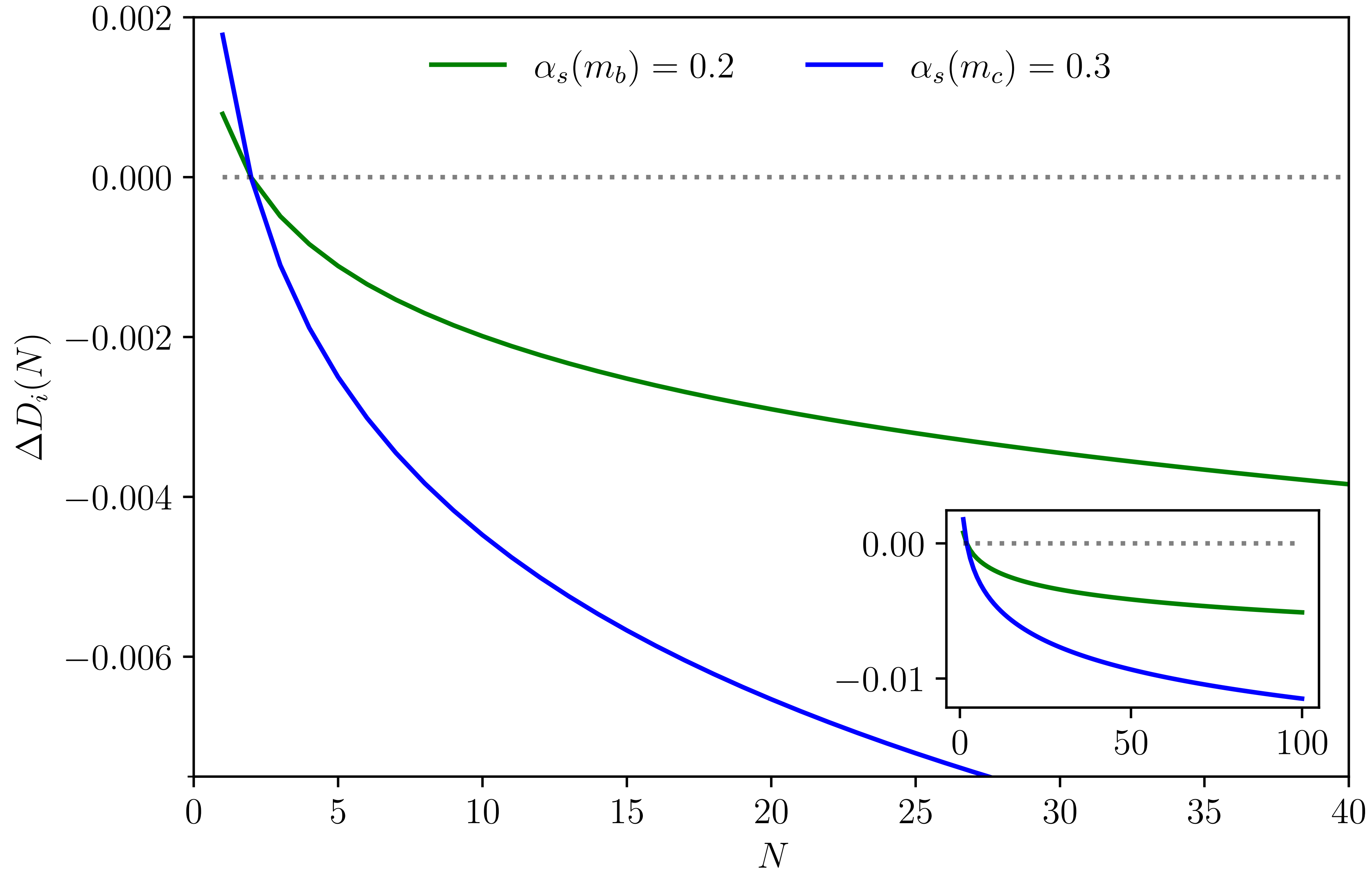
for a straightforward implementation in MELA

Direct space

Same of the non-singlet quark OME

$$D_i^{(n)}(z,\mu) = \Big\{1+\left(\frac{\alpha_s}{2\pi}\right)^2 C_F T_F\Big[\left(-\frac{1}{3}-\frac{1}{3}z+\frac{1}{2}\delta(1-z)+\frac{2}{3}\left(\frac{1}{1-z}\right)_+\right)\log^2\frac{\mu^2}{m^2} +$$
$$\left(-\frac{2}{9}+\frac{22}{9}z-\frac{2}{3}\frac{1+z^2}{1-z}\log z - \left(\frac{1}{6}+\frac{2\pi^2}{9}\right)\delta(1-z) - \frac{20}{9}\left(\frac{1}{1-z}\right)_+\right)\log\frac{\mu^2}{m^2} +$$
$$-\frac{67z}{27} + \left(-\frac{z}{6}+\frac{1}{3(1-z)}-\frac{1}{6}\right)\log^2(z) + \left(-\frac{11z}{9}+\frac{10}{9(1-z)}+\frac{1}{9}\right)\log(z) + \frac{11}{27} + \Big(\frac{9307}{648}-\frac{10}{3}\zeta_3-\frac{19\pi^2}{54}$$
$$-\frac{16\log^3(2)}{9}+\frac{58\log^2(2)}{9}+\left(\frac{4\pi^2}{9}-\frac{359}{27}\right)\log(2)\Big)\delta(1-z) + \frac{56}{27}\left(\frac{1}{1-z}\right)_+\Big] + \mathcal{O}(\alpha_s^3)\Big\} D_i^{(n_L)}(z,\mu).$$

$$\Delta D_i(N) = \frac{D_i^{(n)}(N,m) - D_i^{(n_L)}(N,m)}{D_i^{(n_L)}(N,m)}$$



The growing behaviour for larger N is governed by the RR emissions

The matching equation obeys to a RGE

$$\frac{\partial\,\Delta D}{\partial \log\mu^2} = \beta_0 \log\frac{\mu^2}{m^2} P_{ii,0} + \left(P_{ii,1}^{(n)} - P_{ii,1}^{(n_L)}\right)$$

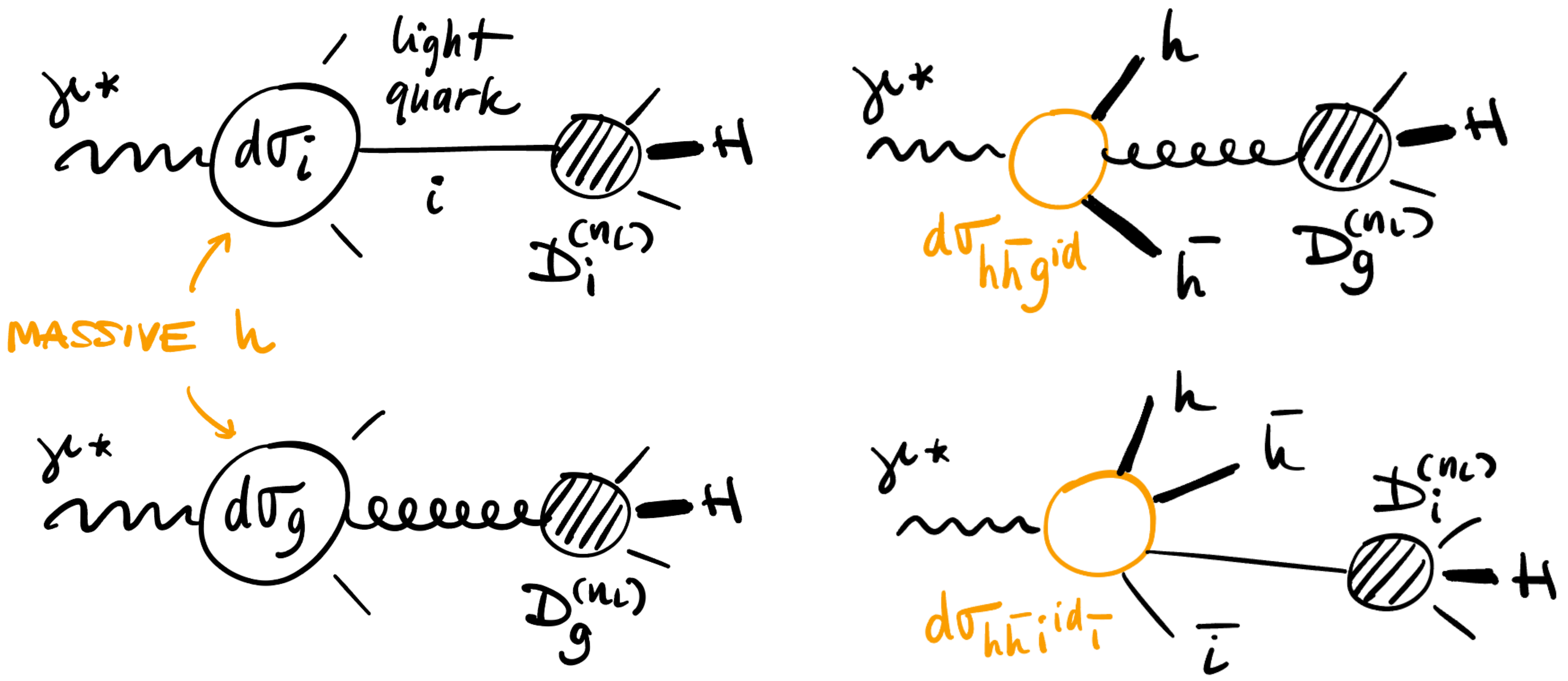
Same space-like and time-like splitting functions at tree-level

Heavy-quark matching equation

$$\mathrm{d}\sigma_H^{massive} - \mathrm{d}\sigma_H^{massless} \sim \sum_i Q_i^2(\ldots) + Q_h^2(\ldots)$$

The D_h condition can be extracted in the same process focusing on the Q_h^2 contributions.

Below threshold



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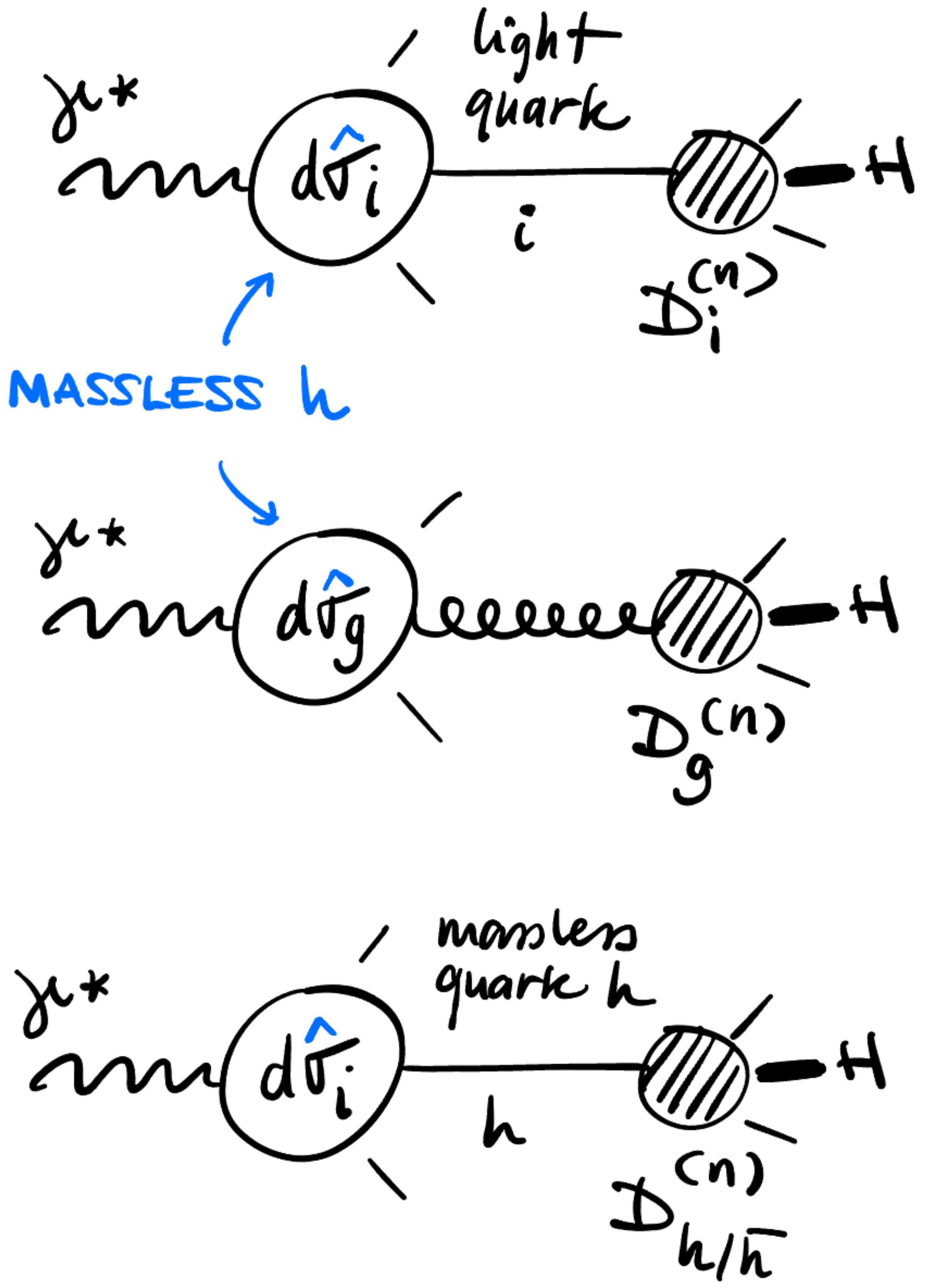

Heavy-quark matching equation

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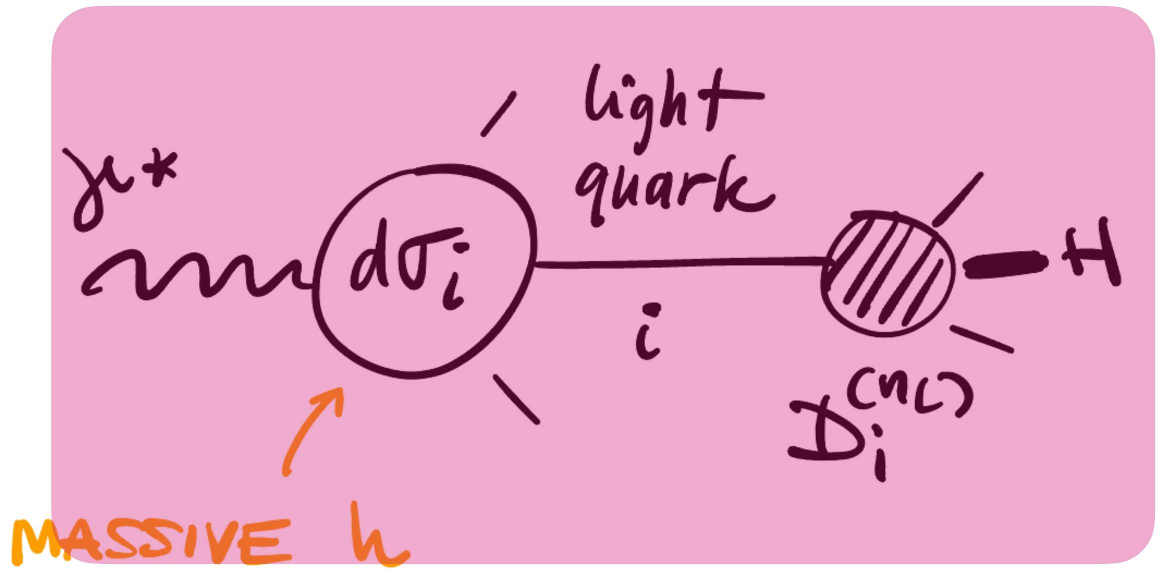
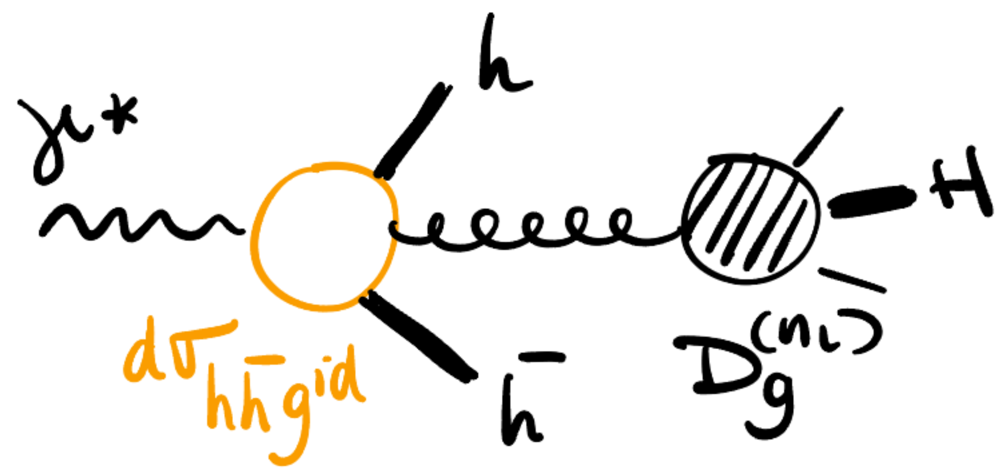
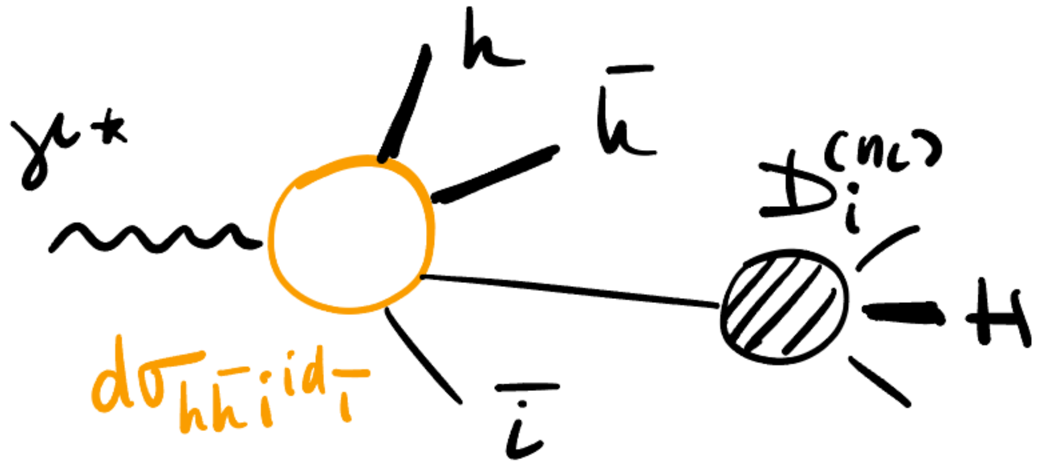
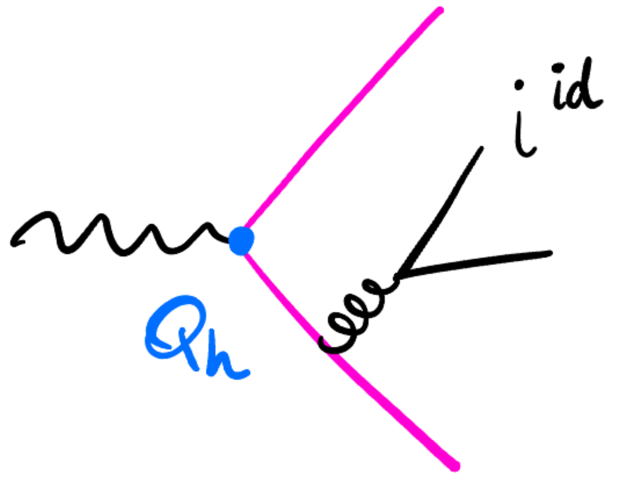


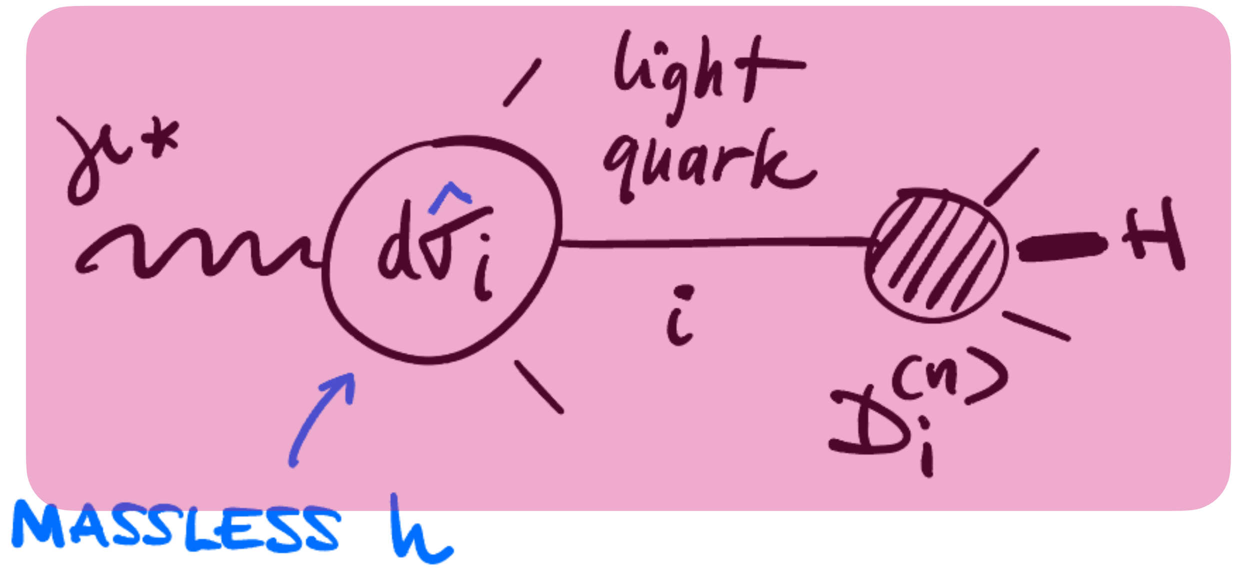


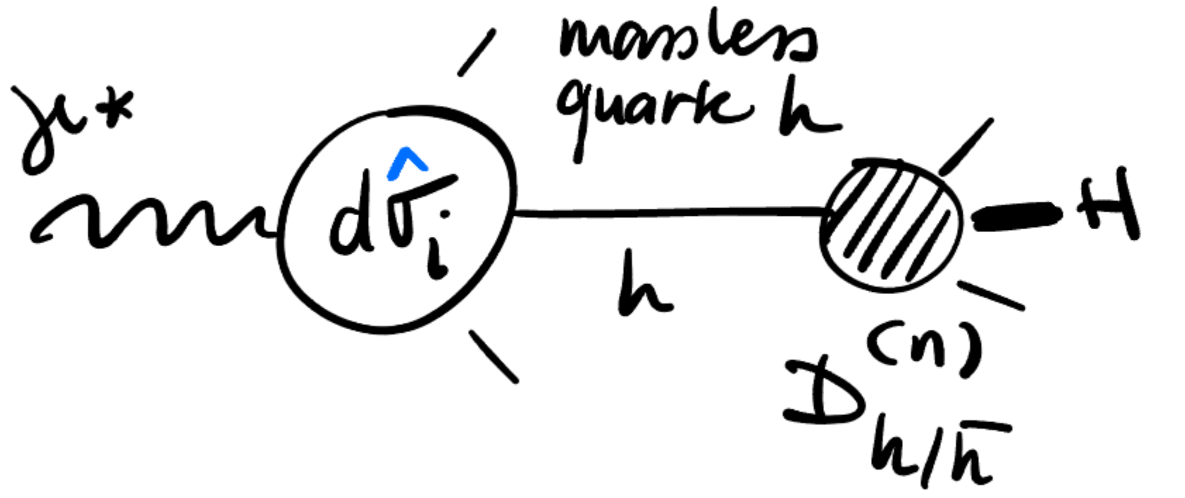




In the difference $d\sigma_i - d\hat{\sigma}_i$,
only $d\hat{\sigma}^{Q_h}_{h i^{id} \bar{i} \bar{h}}$ remains

Above threshold





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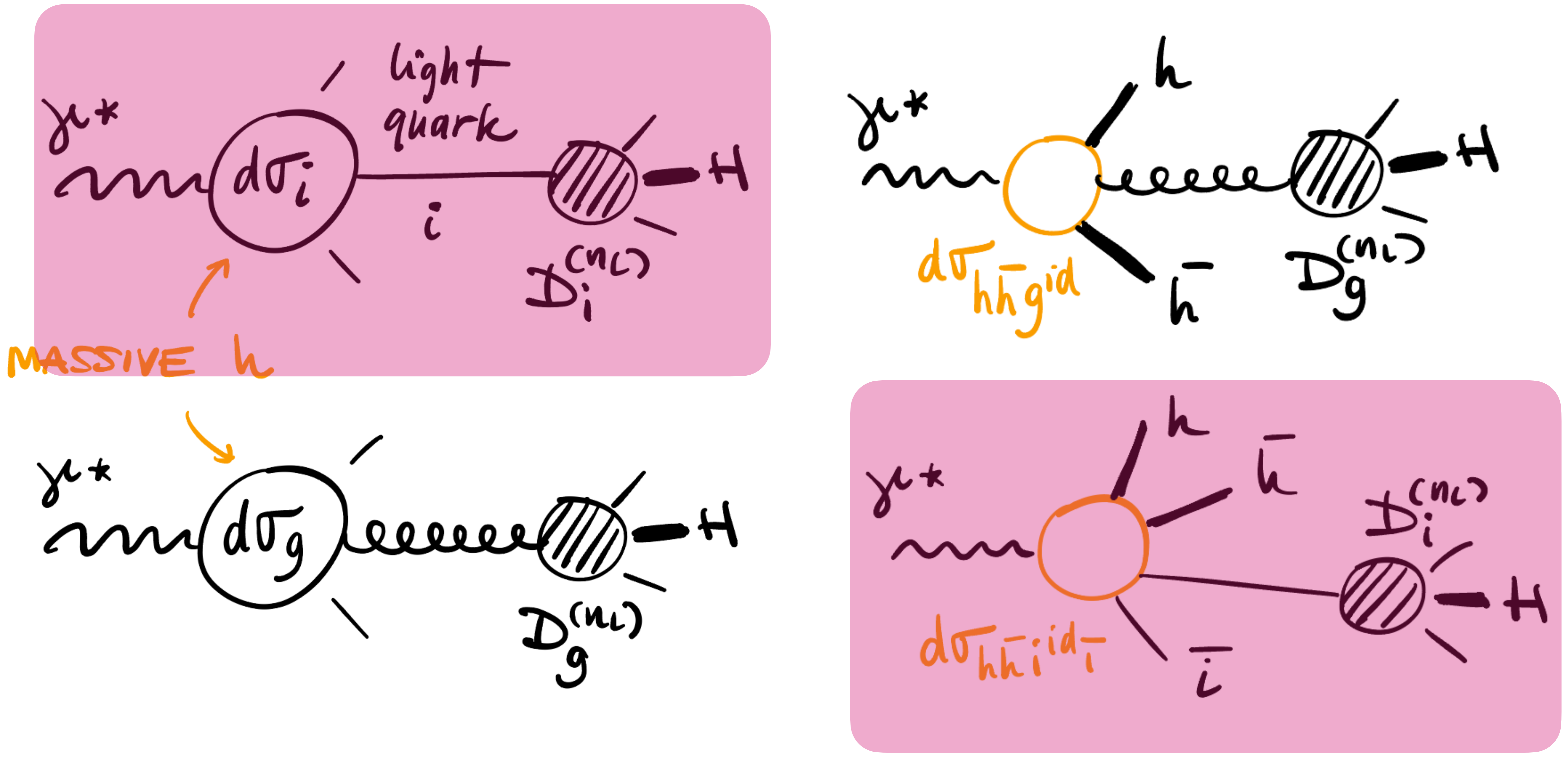

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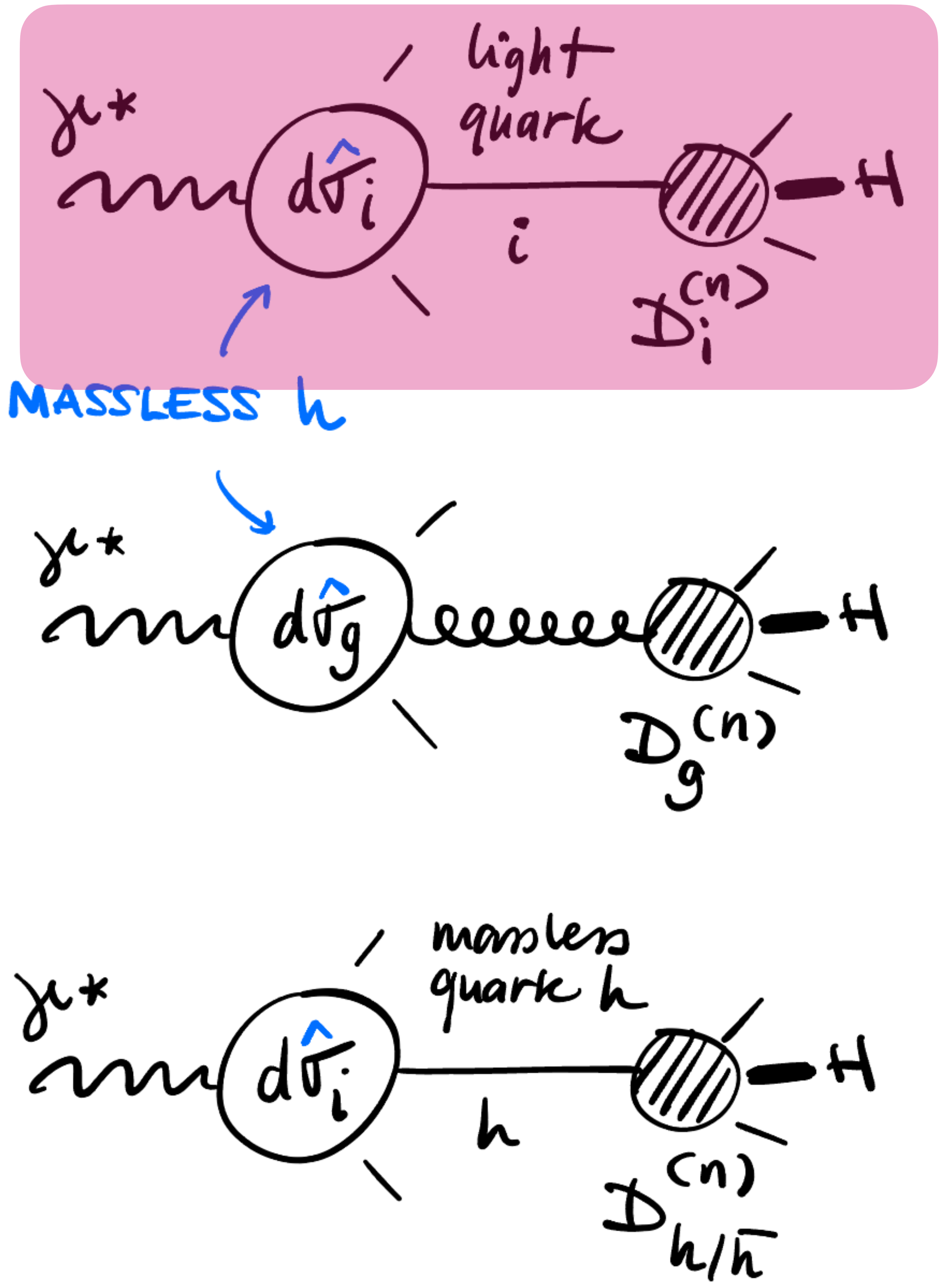
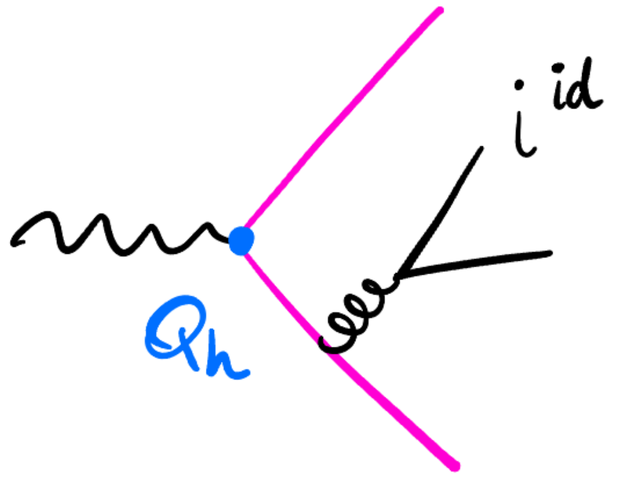




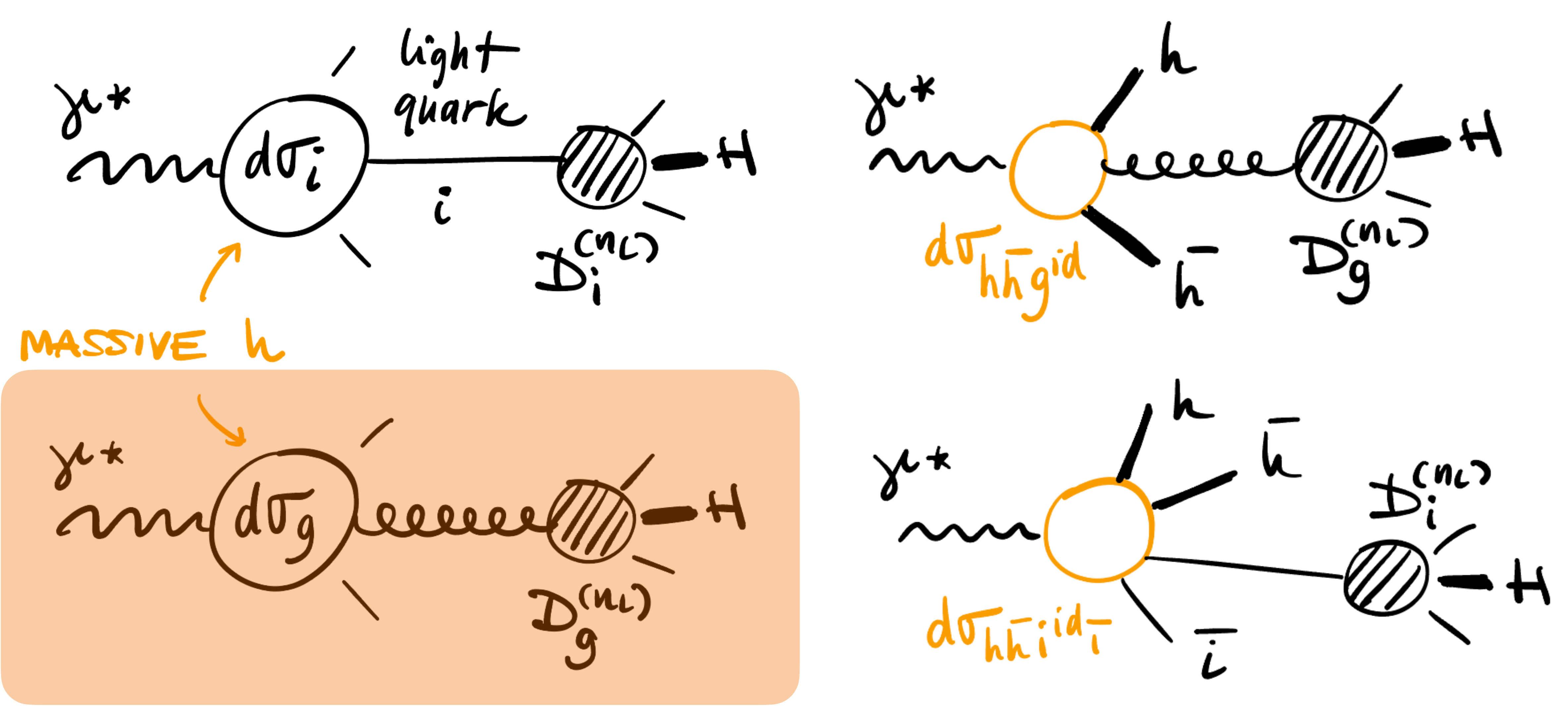
In the difference $d\sigma_i - d\hat{\sigma}_i$,
only $d\hat{\sigma}^{Q_h}_{hi^{id}.\bar{i}\bar{h}}$ remains

⇝ combine it with the massive counterpart

Heavy-quark matching equation

The D_h condition can be extracted in the same process focusing on the Q_h^2 contributions.

Below threshold



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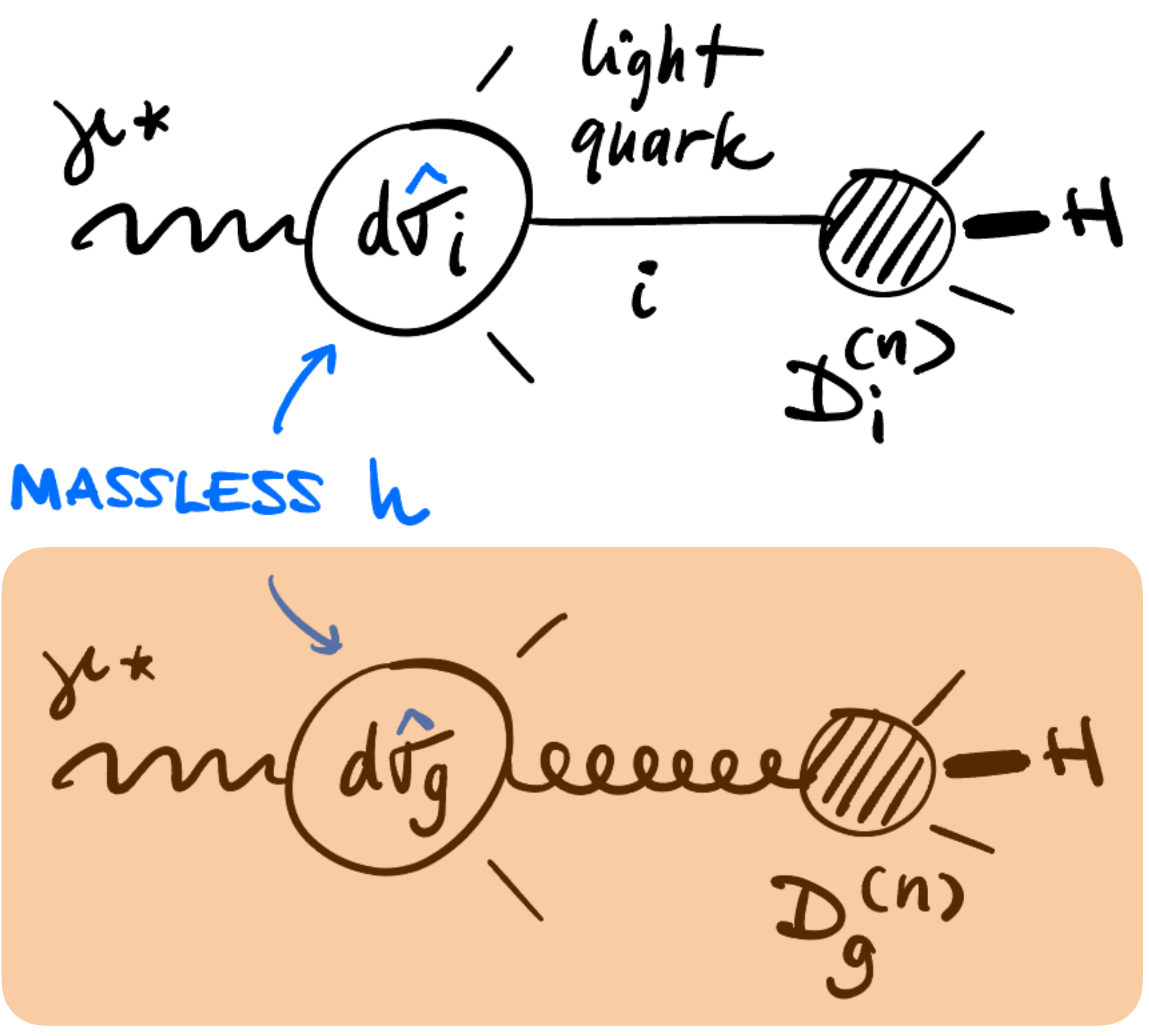
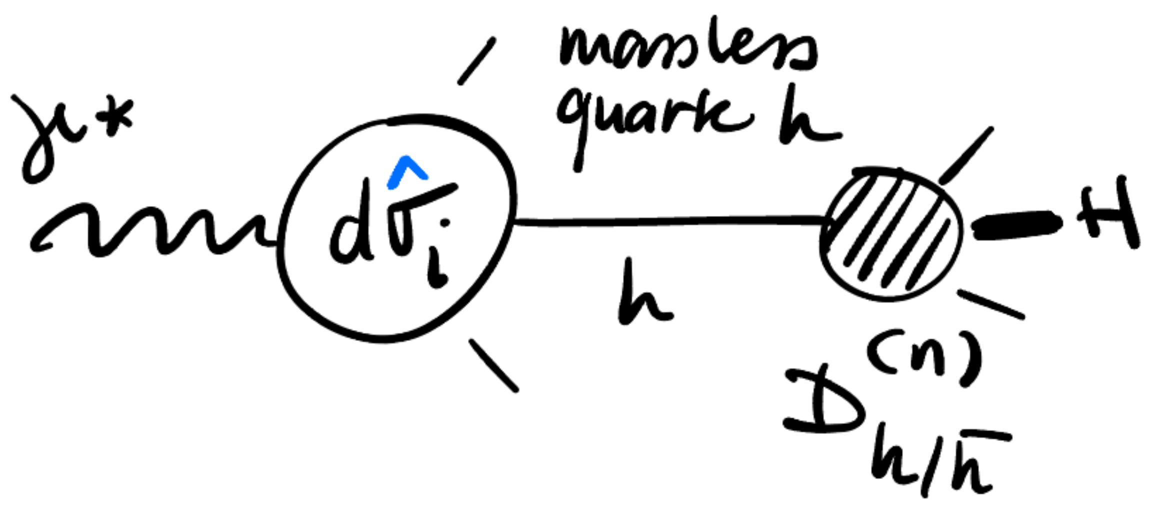


In the difference $\mathrm{d}\sigma_g - \mathrm{d}\hat{\sigma}_g$ only the NLO correction to $\mathrm{d}\hat{\sigma}_{h\bar{h}g}$ survives



Heavy-quark matching equation

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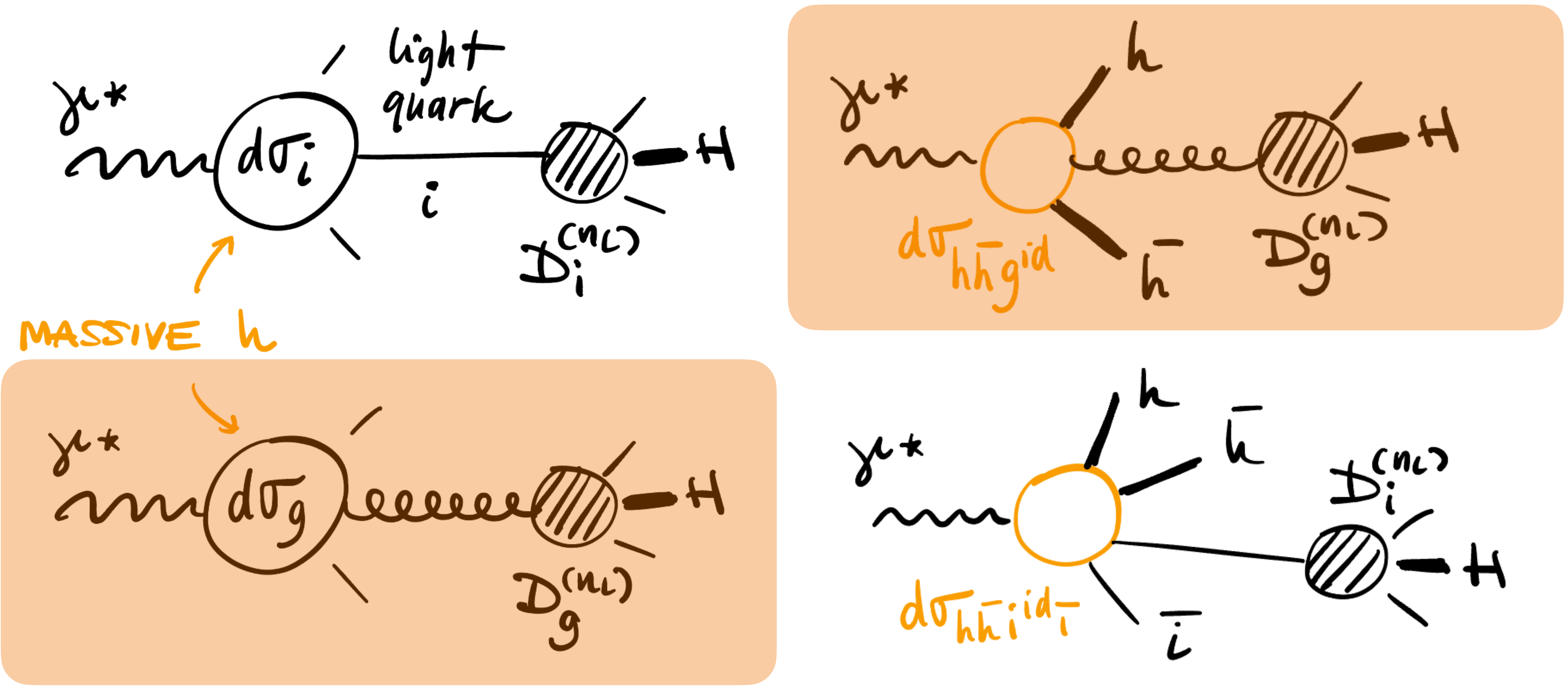
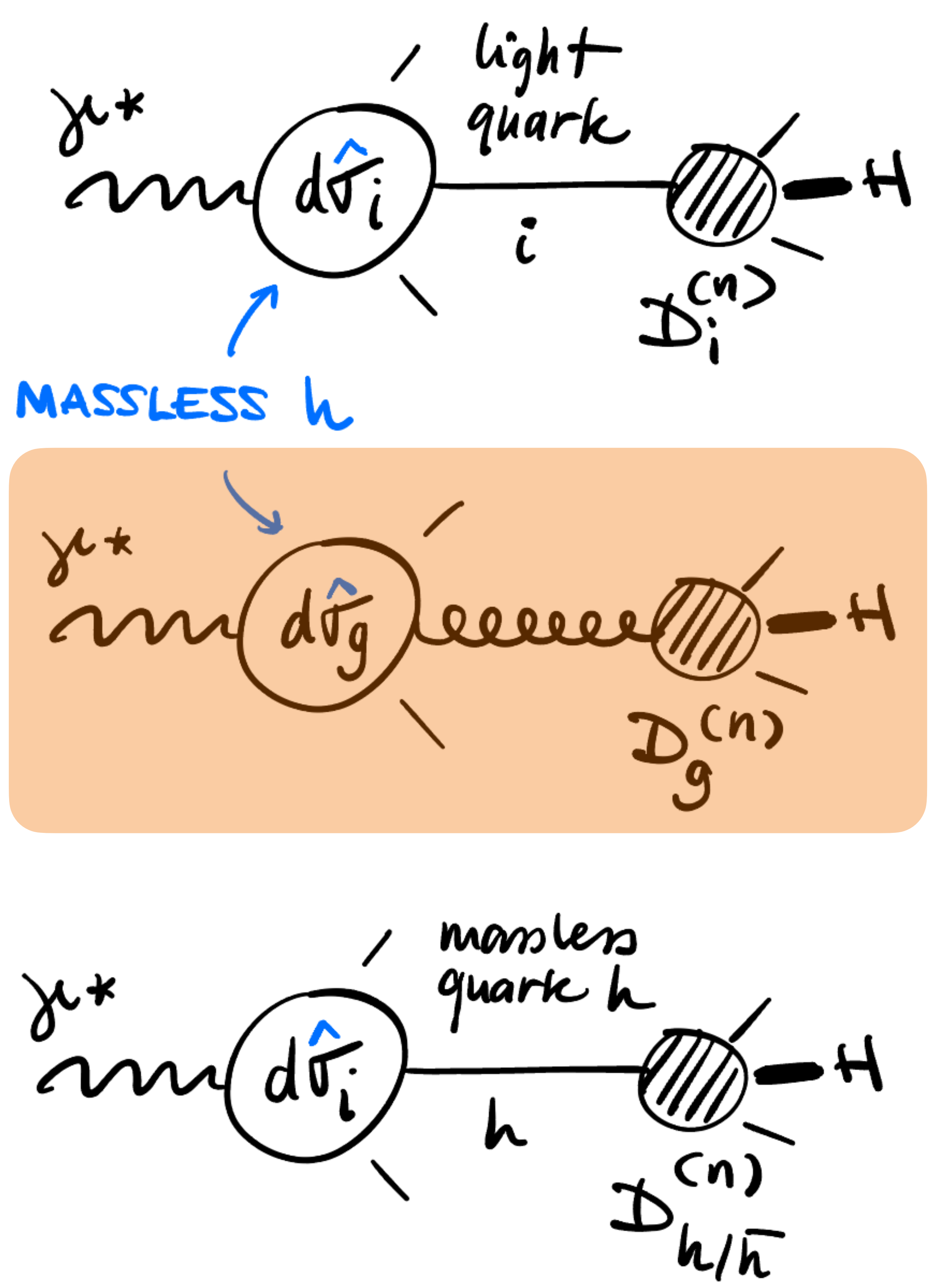
Below threshold



Above threshold



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⇝ combine it with the massive counterpart

Heavy-quark matching equation

The D_h condition can be extracted in the same process focusing on the Q_h^2 contributions.

$$D_h^{(n)}(N,\mu) = \left(\mathcal{M}_{(N,z)}\left[\frac{\mathrm{d}\hat{\sigma}_h}{\mathrm{d}z}\right]\right)^{-1}\left\{D_g^{(n_L)}(N,\mu)\ \mathcal{M}_{(N,z)}\left[\frac{\mathrm{d}\sigma_{h\bar{h}g^{id.}}}{\mathrm{d}z} - \frac{\mathrm{d}\hat{\sigma}_{h\bar{h}g^{id.}}}{\mathrm{d}z}\right] + \sum_{i\in\mathbb{I}_{n_L}-g} D_i^{(n_L)}(N,\mu)\ \mathcal{M}_{(N,z)}\left[\frac{\mathrm{d}\sigma^{Q_h}_{h\bar{h}i^{id.}\bar{i}}}{\mathrm{d}z} - \frac{\mathrm{d}\hat{\sigma}^{Q_h}_{h\bar{h}i^{id.}\bar{i}}}{\mathrm{d}z}\right] + \mathcal{O}(\alpha_s^3)\right\}$$

- The $\mathcal{O}(\alpha_s)$ term reproduces Cacciari-Nason-Oleari matching equation

$$D_h^{(n)}(N,\mu) = \frac{1}{\sigma_0}\left\{D_g^{(n_L)}(N,\mu)\ \mathcal{M}_{(N,z)}\left[\frac{\mathrm{d}\sigma_{h\bar{h}g^{id.}}}{\mathrm{d}z} - \frac{\mathrm{d}\hat{\sigma}_{h\bar{h}g^{id.}}}{\mathrm{d}z}\right] + \mathcal{O}(\alpha_s^2)\right\}$$

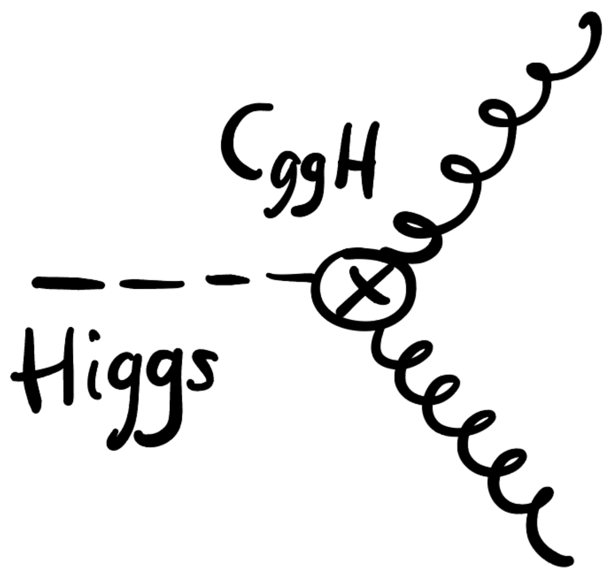
- The RR massive correction proportional to D_i must be integrated in $4-2\epsilon$ dimensions
- The RV massive correction is still missing at differential level, although the fully-integrated antenna is known

Bernreuther, (Bogner,) Dekkers [1309.6887,1409.3124]

Gluon matching equation

The NNLO calculation in $e^+e^- \to \gamma^* \to H$ is not sensible to the gluon NNLO matching condition.

Alternative process: $e^+e^- \to h^0 \to H$ in the effective theory



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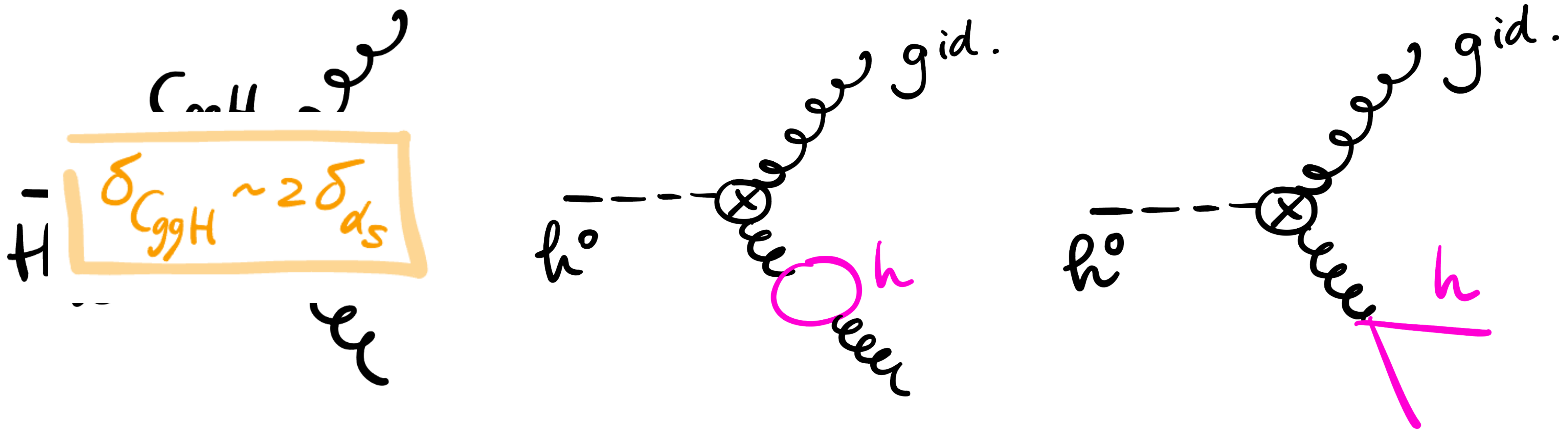

$$D_g^{(n)}(x,\mu) = D_g^{(n_L)}(x,\mu) + \frac{1}{\sigma_0}\int_x^1 \frac{\mathrm{d}z}{z} D_g^{(n_L)}\left(\frac{x}{z},\mu\right)\left(\frac{\mathrm{d}\sigma_{g,f}}{\mathrm{d}z} - \frac{\mathrm{d}\hat{\sigma}_{g,f}}{\mathrm{d}z} + \frac{\mathrm{d}\sigma_{g^{id.}h\bar{h}}}{\mathrm{d}z} - \frac{\mathrm{d}\hat{\sigma}_{g^{id.}h\bar{h}}}{\mathrm{d}z} - \delta_{C_{h^0gg}}\right)$$

$$= D_g^{(n_L)}(x,\mu)\left(1 - \frac{T_F\alpha_s}{3\pi}\log\frac{\mu^2}{m^2} + \mathcal{O}(\alpha_s^2)\right)$$

Gluon matching equation

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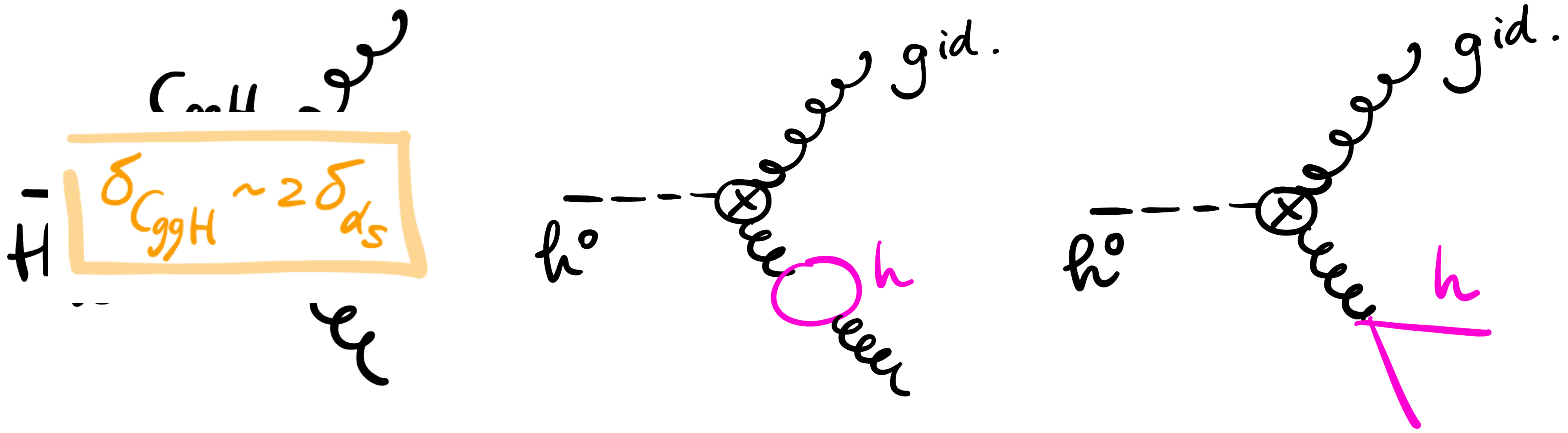
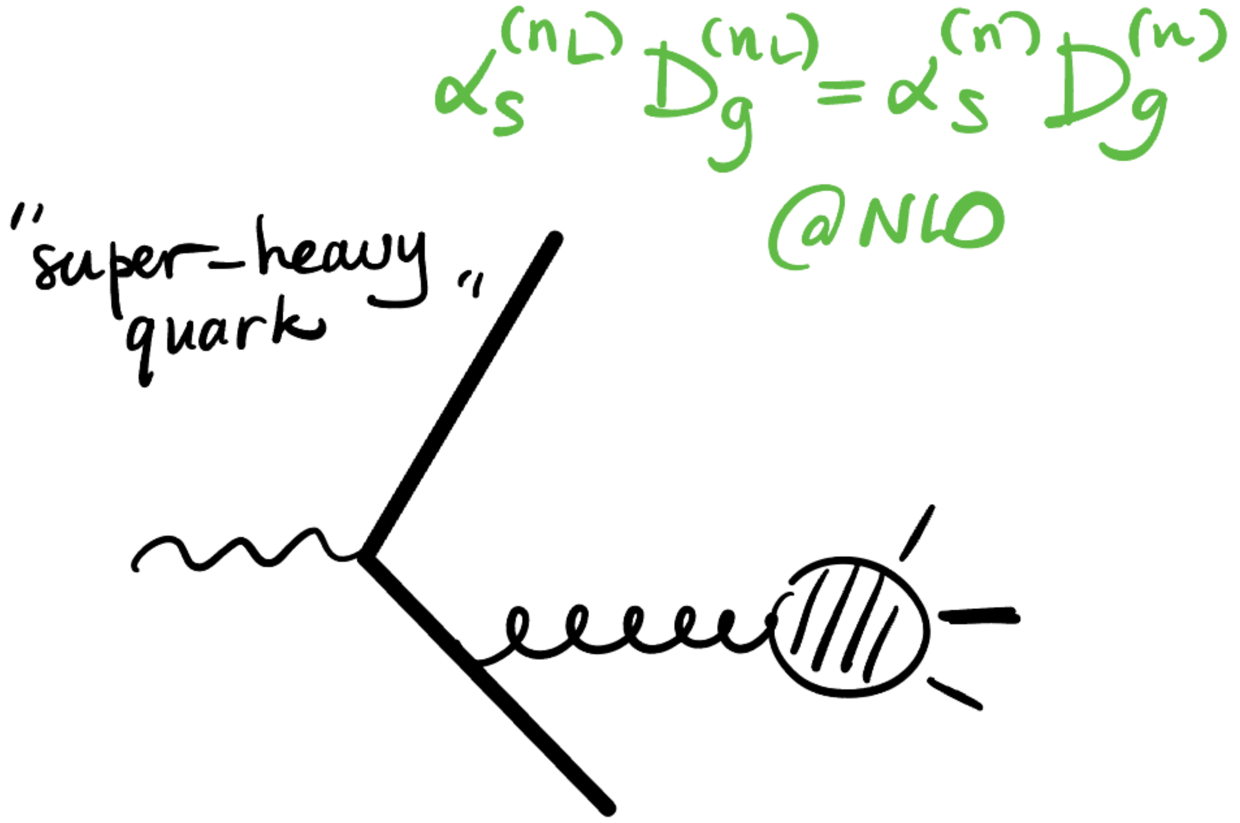




$$D_g^{(n)}(x,\mu) = D_g^{(n_L)}(x,\mu) + \frac{1}{\sigma_0}\int_x^1 \frac{\mathrm{d}z}{z} D_g^{(n_L)}\left(\frac{x}{z},\mu\right)\left(\frac{\mathrm{d}\sigma_{g,f}}{\mathrm{d}z} - \frac{\mathrm{d}\hat{\sigma}_{g,f}}{\mathrm{d}z} + \frac{\mathrm{d}\sigma_{g^{id.}h\bar{h}}}{\mathrm{d}z} - \frac{\mathrm{d}\hat{\sigma}_{g^{id.}h\bar{h}}}{\mathrm{d}z} - \delta_{C_{h^0gg}}\right)$$

$$= D_g^{(n_L)}(x,\mu)\left(1 - \frac{T_F\alpha_s}{3\pi}\log\frac{\mu^2}{m^2} + \mathcal{O}(\alpha_s^2)\right)$$

After the NLO check, we have extended the master formula at NNLO in the h^0 channel.

The gluon-gluon massless antennae are useful but some massive ingredients are still missing.

Summary and outlook

Matching conditions are the last missing ingredients for the **NNLO consistency of FF fits**.

We presented the **first time-like NNLO matching** equation D_i

- Extension of the NLO master formulae focusing on physical processes
- Explicit analytic **result for light-quark FF**
    - Several checks including cancellation of $\log Q$ and RGE scaling
    - D_g and D_h conditions still missing
    - Massive fragmentation antennae can solve the problem

Thank you for your attention!

Backup slides

RGE of the matching equation

$$D_i^{(n)} = \left(1 + \alpha_s^2 \delta D\right) \otimes D_i^{(n_L)}$$

The matching equation obeys to a RGE

$$\frac{\partial}{\partial \log \mu} D_i^{(n)} = \left(1 + \alpha_s^2 \frac{\partial\, \delta D}{\partial \log \mu}\right) D_i^{(n_L)} + \frac{\partial}{\partial \log \mu} D_i^{(n_L)} + \mathcal{O}(\alpha_s^3)$$

$$\gamma_{ii} \otimes D_i^{(n)} + \gamma_{gi} \otimes D_g^{(n)} = \left(1 + \alpha_s^2 \frac{\partial\, \delta D}{\partial \log \mu}\right) D_i^{(n_L)} + \gamma_{ii} \otimes D_i^{(n_L)} + \gamma_{gi} \otimes D_g^{(n_L)}$$

The RG dependence on D_g cancels at $\mathcal{O}(\alpha_s^2)$

$$\left\{\alpha_s^{(n)} \gamma_{gi,0} + (\alpha_s^{(n)})^2 \gamma_{gi,1}\right\} \otimes D_i^{(n)} = \left\{\alpha_s^{(n_L)} \gamma_{gi,0} + (\alpha_s^{(n_L)})^2 \gamma_{gi,1}\right\} \otimes D_i^{(n_L)}$$

thanks to the NLO matching $\alpha_s^{(n)} D_g^{(n)} = \alpha_s^{(n_L)} D_g^{(n_L)}$

RGE of the matching equation

$$D_i^{(n)} = \left(1 + \alpha_s^2 \delta D\right) \otimes D_i^{(n_L)}$$

The matching equation obeys to a RGE

$$\frac{\partial}{\partial \log \mu} D_i^{(n)} = \left(1 + \alpha_s^2 \frac{\partial\, \delta D}{\partial \log \mu}\right) D_i^{(n_L)} + \frac{\partial}{\partial \log \mu} D_i^{(n_L)} + \mathcal{O}(\alpha_s^3)$$

$$\gamma_{ii} \otimes D_i^{(n)} + \gamma_{gi} \otimes D_g^{(n)} = \left(1 + \alpha_s^2 \frac{\partial\, \delta D}{\partial \log \mu}\right) D_i^{(n_L)} + \gamma_{ii} \otimes D_i^{(n_L)} + \gamma_{gi} \otimes D_g^{(n_L)}$$

By expanding the quark anomalous coupling,

$$\gamma_{ii} \otimes D_i^{(n)} = \left\{ \alpha_s^{(n_L)} \left(1 + \alpha_s \beta_0 \log \frac{\mu^2}{m^2}\right) \gamma_{ii,0} + \alpha_s^2 \gamma_{ii,1}^{(n_L)} \right\} \otimes D_i^{(n)}$$

we deduce the RGE

$$\frac{\partial\, \delta D}{\partial \log \mu} = \beta_0 \log \frac{\mu^2}{m^2} P_{ii,0} + \left(P_{ii,1}^{(n)} - P_{ii,1}^{(n_L)}\right)$$

The non-singlet OME for quark PDFs $(f_i - f_j)$ is controlled by the same splittings in the space-like kinematics

⇝ Gribov-Lipatov reciprocity implies the same $\log^2 \frac{\mu^2}{m^2}$

Antenna dipole

The integrated dipole allows for a natural organisation of IR singularities.

- The needed identity-preserving dipole reproduces the structure of the virtual correction.

$$\mathscr{J}^{(2)}_{q\bar{q}}\Big|_{N_f} = \hat{J}_2^{(2)} - \frac{b_{0,F}}{\epsilon} J_2^{(1)}$$

Real

$$J_2^{(1)}(z) = \left(\frac{Q^2}{\mu_R^2}\right)^{-\epsilon} \mathscr{A}_3^{0,\mathrm{id}.q}(z) - \left(\frac{\mu_F^2}{\mu_R^2}\right)^{-\epsilon} \Gamma_{qq}^{(1)}(z),$$

Real-Real, Real-Virtual

$$\hat{J}_2^{(2)}(z) = \left(\frac{Q^2}{\mu_R^2}\right)^{-2\epsilon} \mathscr{B}_4^{0,\mathrm{id}.q}(z) + \left(\frac{Q^2}{\mu_R^2}\right)^{-2\epsilon} \hat{\mathscr{A}}_3^{1,\mathrm{id}.q}(z)$$

$$+ \frac{b_{0,F}}{\epsilon}\left(\frac{Q^2}{\mu_R^2}\right)^{-2\epsilon} \mathscr{A}_3^{0,\mathrm{id}.q}(z) - \left(\frac{\mu_F^2}{\mu_R^2}\right)^{-2\epsilon} \widehat{\overline{\Gamma}}_{qq}^{(2)}(z).$$

Nf piece of the beta function

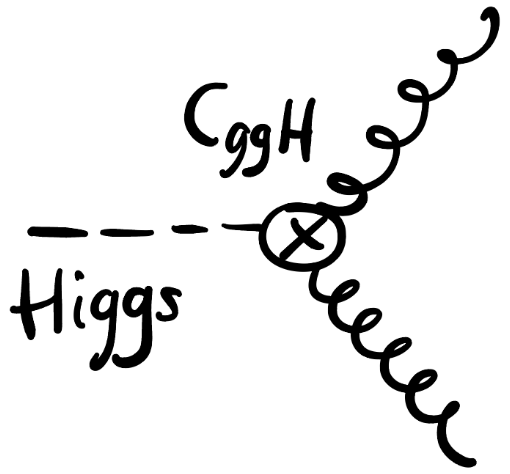
Real-Virtual $\quad \hat{\mathscr{A}}_3^{1,\mathrm{id}.q}(z) = \frac{1}{C(\epsilon)} \int \mathrm{d}\Phi_2 \frac{Q^2}{2\pi} z^{1-2\epsilon} \hat{A}_3^1,$

Real-Real $\quad \mathscr{B}_4^{0,\mathrm{id}.q}(z) = \frac{1}{[C(\epsilon)]^2} \int \mathrm{d}\Phi_3 \frac{Q^2}{2\pi} z^{1-2\epsilon} B_4^0.$

Mass factorisation kernels absorb the poles of the integrated antenna functions coming from the FF collinear singularities

Bonino, Gehrmann, Marcoli, Schürmann, Stagnitto [2406.09925]

NNLO gluon matching equation: details



Let's study the NNLO corrections to $e^+e^- \to h^0 \to H$ in the effective theory.

$D_g^{(n)}$ cannot depend on $D_i^{(n_L)}$ around the threshold, since $$\int_x^1 \frac{\mathrm{d}z}{z}\left(D_i^{(n_L)}\frac{\mathrm{d}\hat{\sigma}_i}{\mathrm{d}z} - D_i^{(n)}\frac{\mathrm{d}\sigma_i}{\mathrm{d}z}\right) = \mathcal{O}(\alpha_s^3)\,.$$

⇝ only the gluon fragmentation function is present in the gluon NNLO matching

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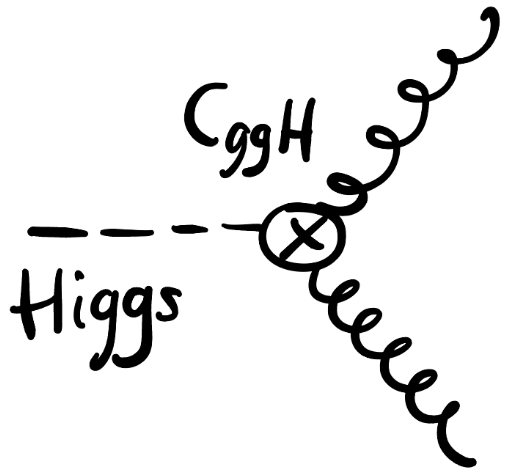

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⇝ only the gluon fragmentation function is present in the gluon NNLO matching

$$D_g^{(n)}(x,\mu) = D_g^{(n_L)}(x,\mu)\left(1 - \frac{T_F\alpha_s^{(n_L)}}{3\pi}\log\frac{\mu^2}{m^2}\right) + \int_x^1 \frac{\mathrm{d}z}{z} D_g^{(n_L)}\left(\frac{x}{z},\mu\right)\left\{\frac{\mathrm{d}\sigma_{g,f}}{\mathrm{d}z} - \left(\frac{\mathrm{d}\hat{\sigma}_{g,f}}{\mathrm{d}z} - \frac{\mathrm{d}\hat{\sigma}_{g^{id.}h\bar{h},f}}{\mathrm{d}z}\right) + \frac{\mathrm{d}\sigma_{g^{id.}h\bar{h}}}{\mathrm{d}z} - \frac{\mathrm{d}\hat{\sigma}_{g^{id.}h\bar{h}}}{\mathrm{d}z} - \delta_{C_{h^0gg},\alpha_s} + \frac{T_F\alpha_s^{(n_L)}}{3\pi}\log\frac{\mu^2}{m^2}\frac{\mathrm{d}\hat{\sigma}_g}{\mathrm{d}z}\right\} - \int_x^1 \frac{\mathrm{d}z}{z}\left(D_h^{(n)} + D_{\bar{h}}^{(n)}\right)\frac{\mathrm{d}\hat{\sigma}_{gh^{id.}\bar{h}}}{\mathrm{d}z}\,.$$

NLO matching

Genuine NNLO fermionic loop contributions

NLO $g^{id.}h\bar{h}$ cross-section, $g^{id.}gh\bar{h}$ included

Decoupling effects

These contributions are w.r.t $D_g^{(n_L)}$ using the NLO heavy-flavour matching