# Analytic Waveforms in General Relativity From Scattering Amplitudes

**Giacomo Brunello**

- G.B., S. De Angelis [in progress]
- G.B., S. De Angelis [2403.08009]
- G.B., G. Crisanti, M. Giroux, P. Mastrolia, S. Smith [2311.14432]

**High Precision for Hard Processes**
**Turin, 10th September 2024**

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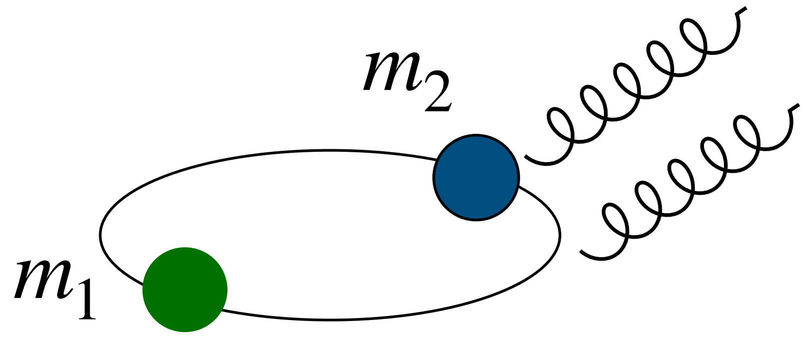
# Motivation: Gravitational Waves



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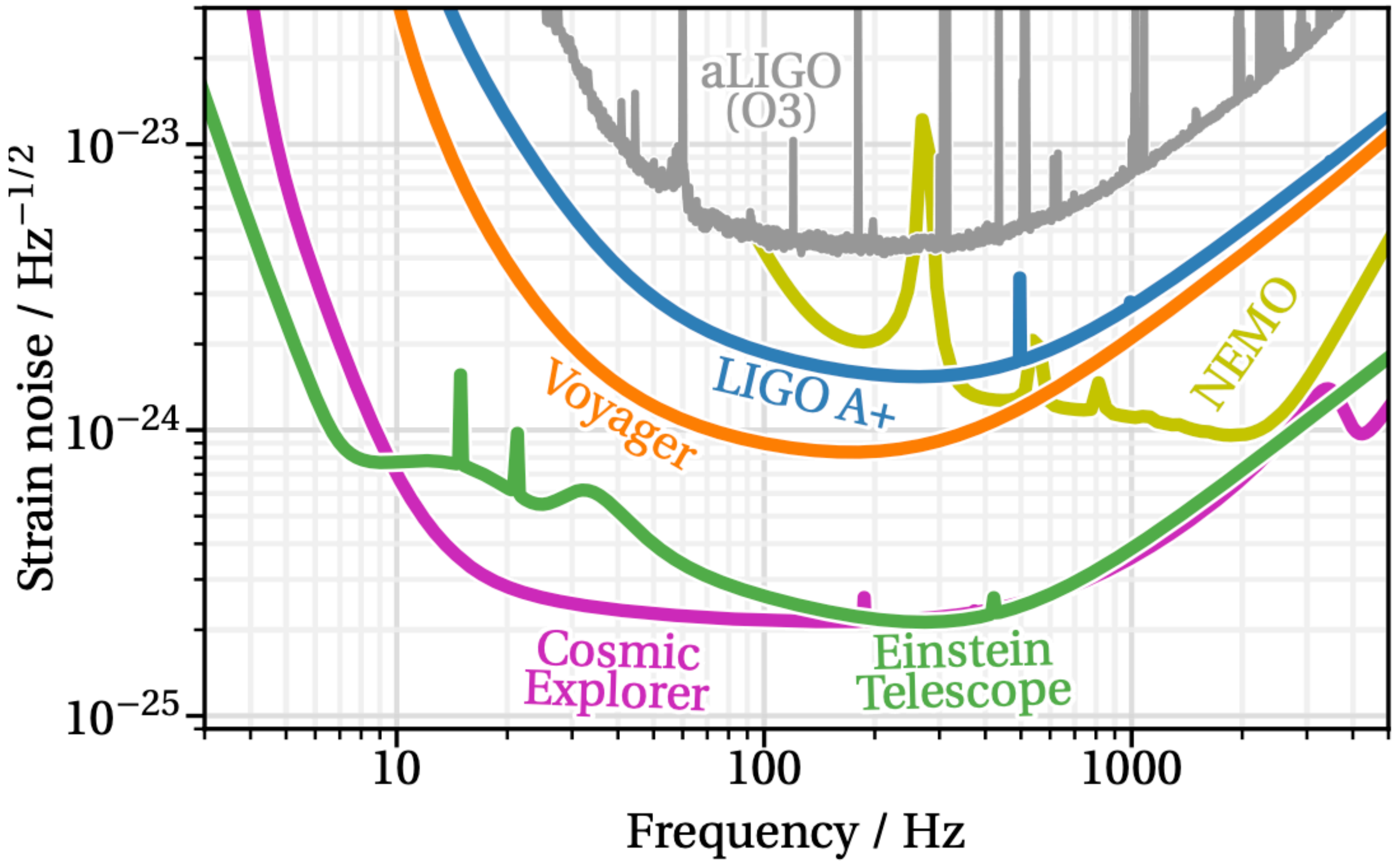

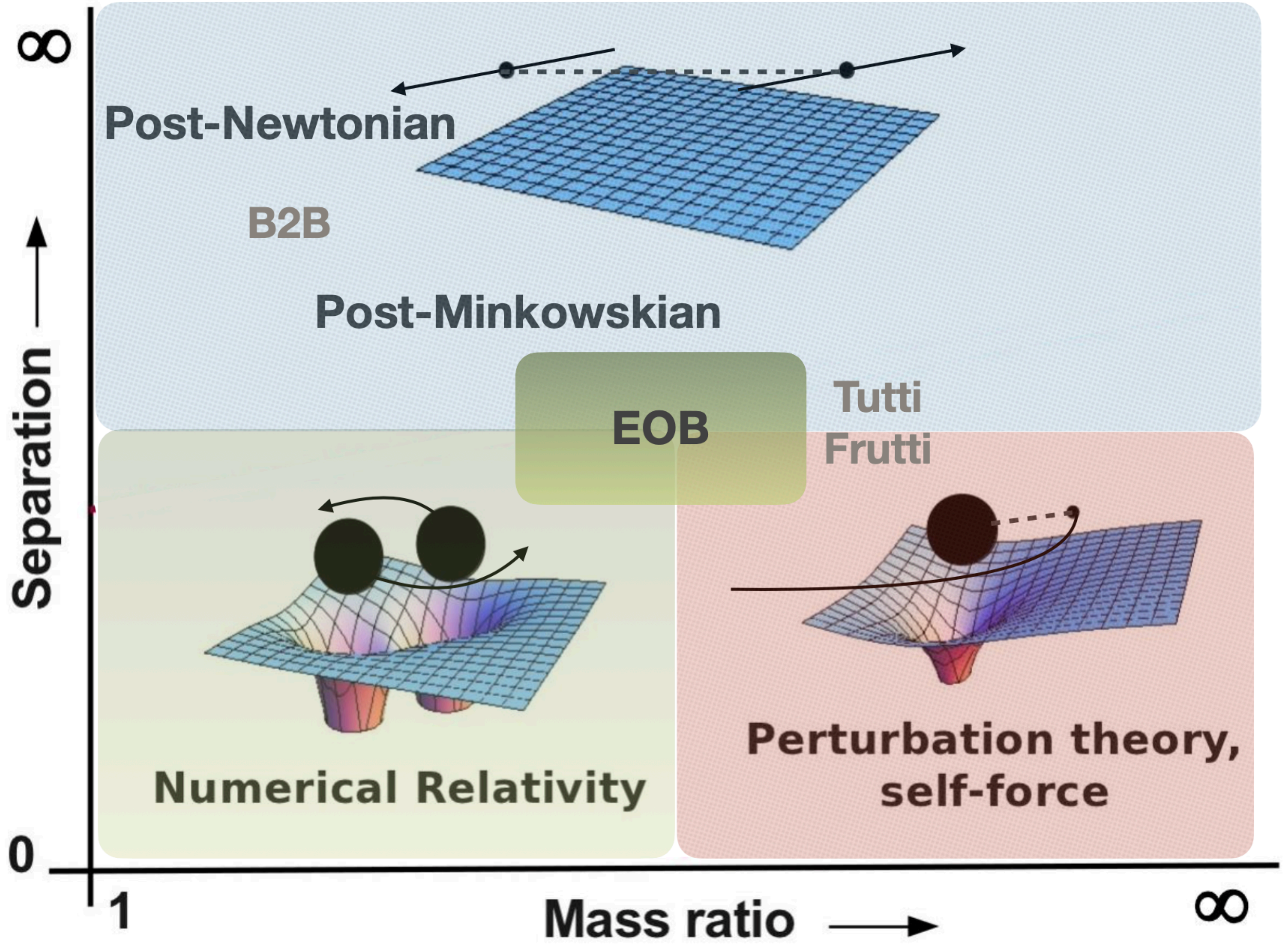

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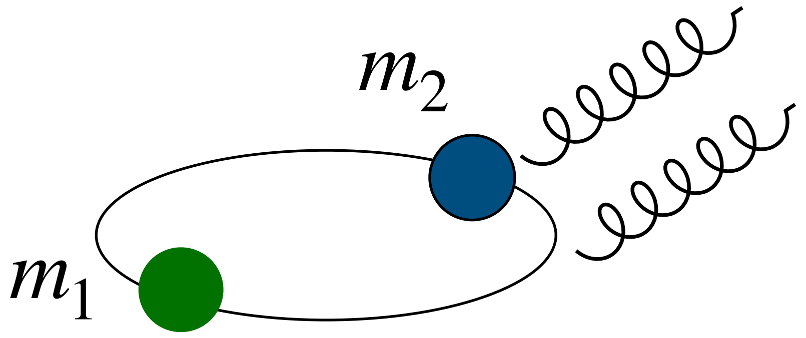

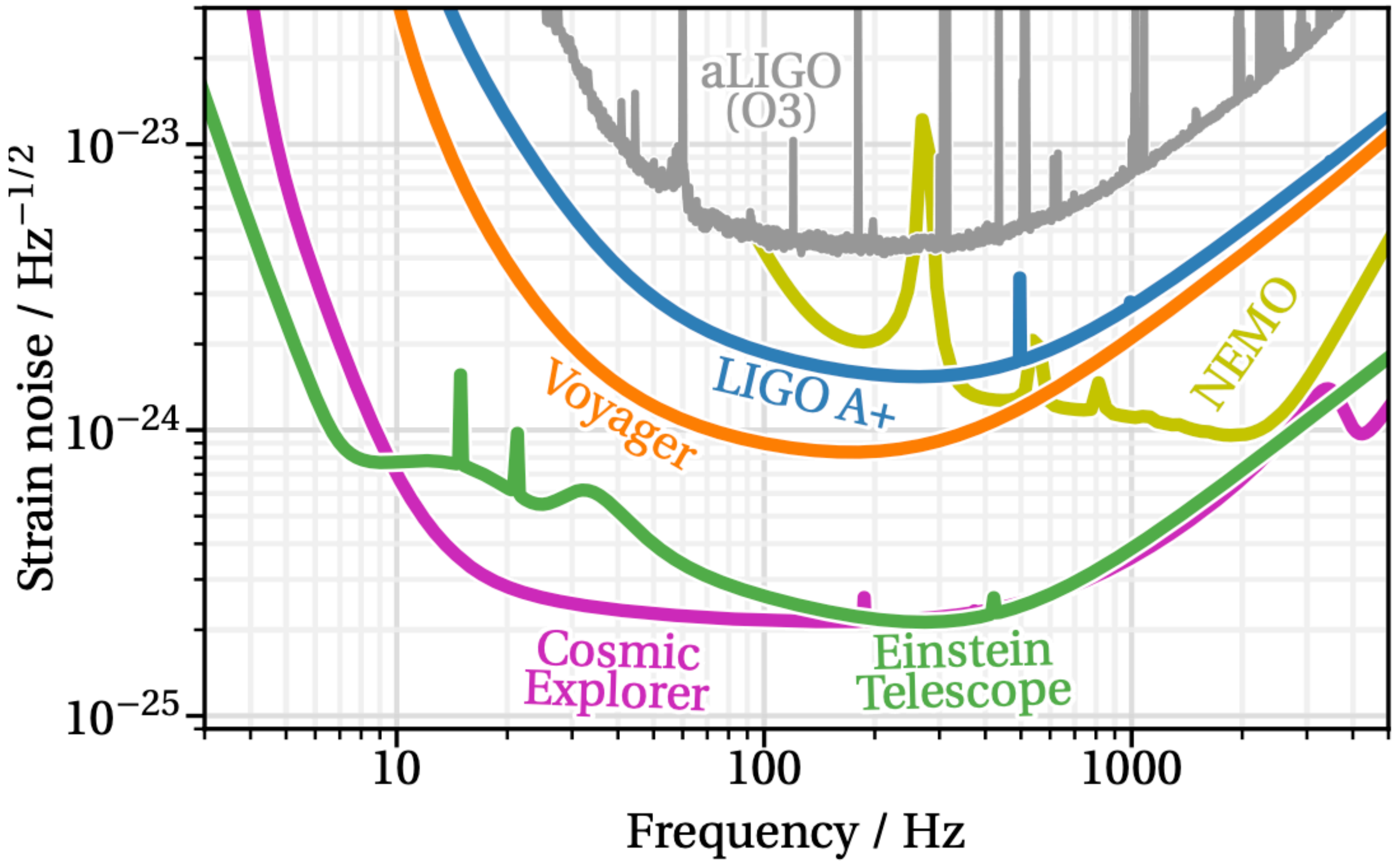

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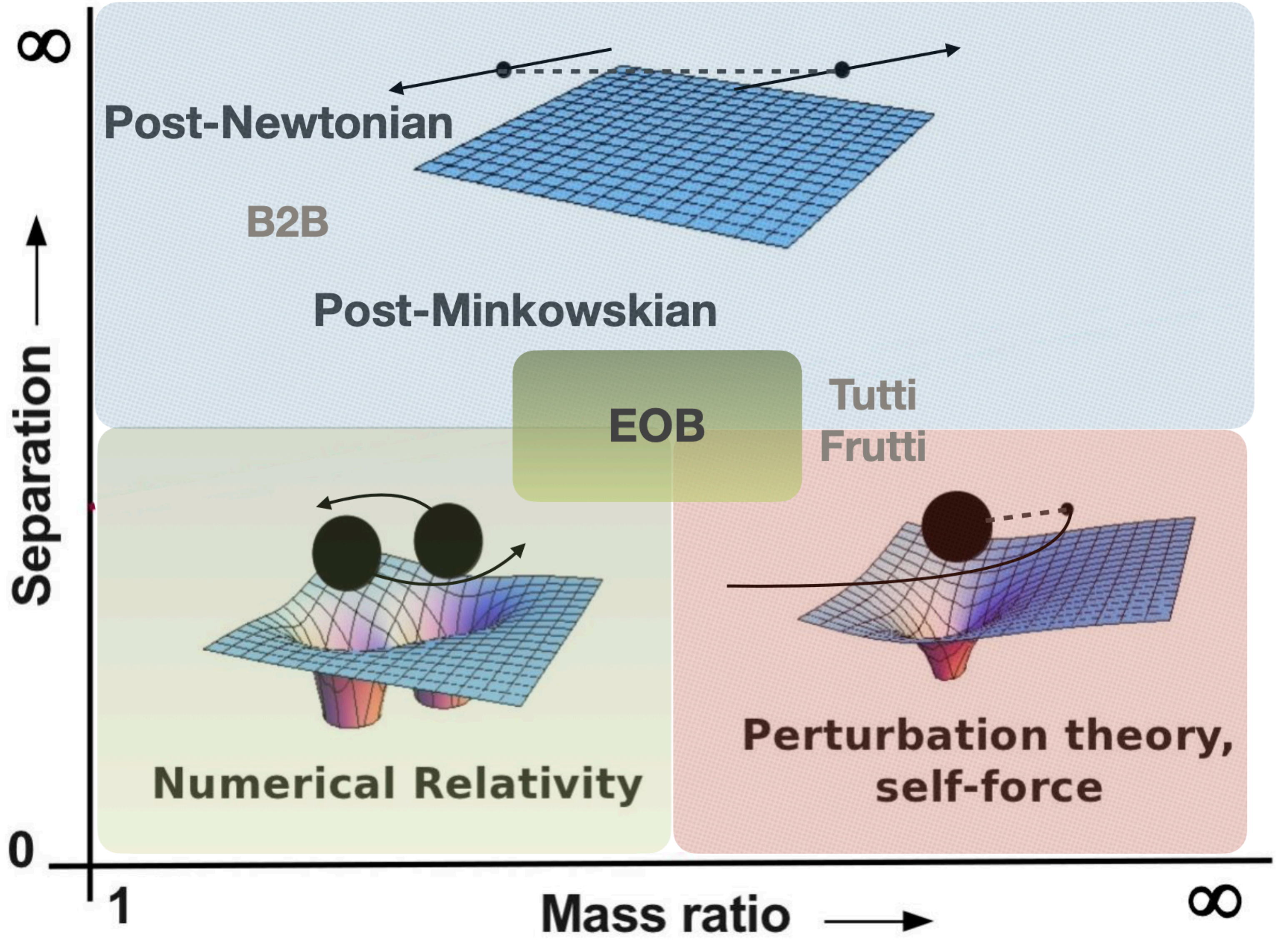

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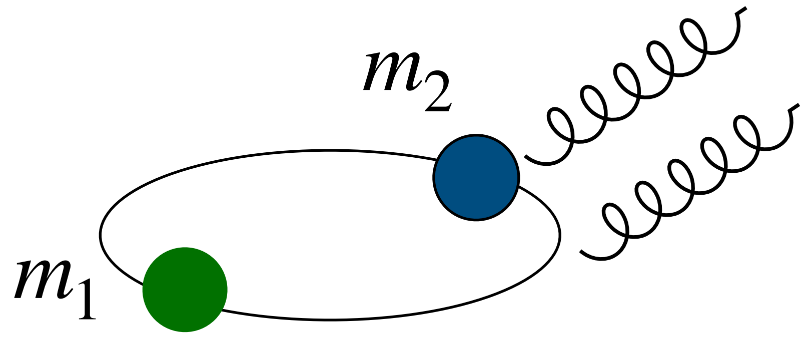

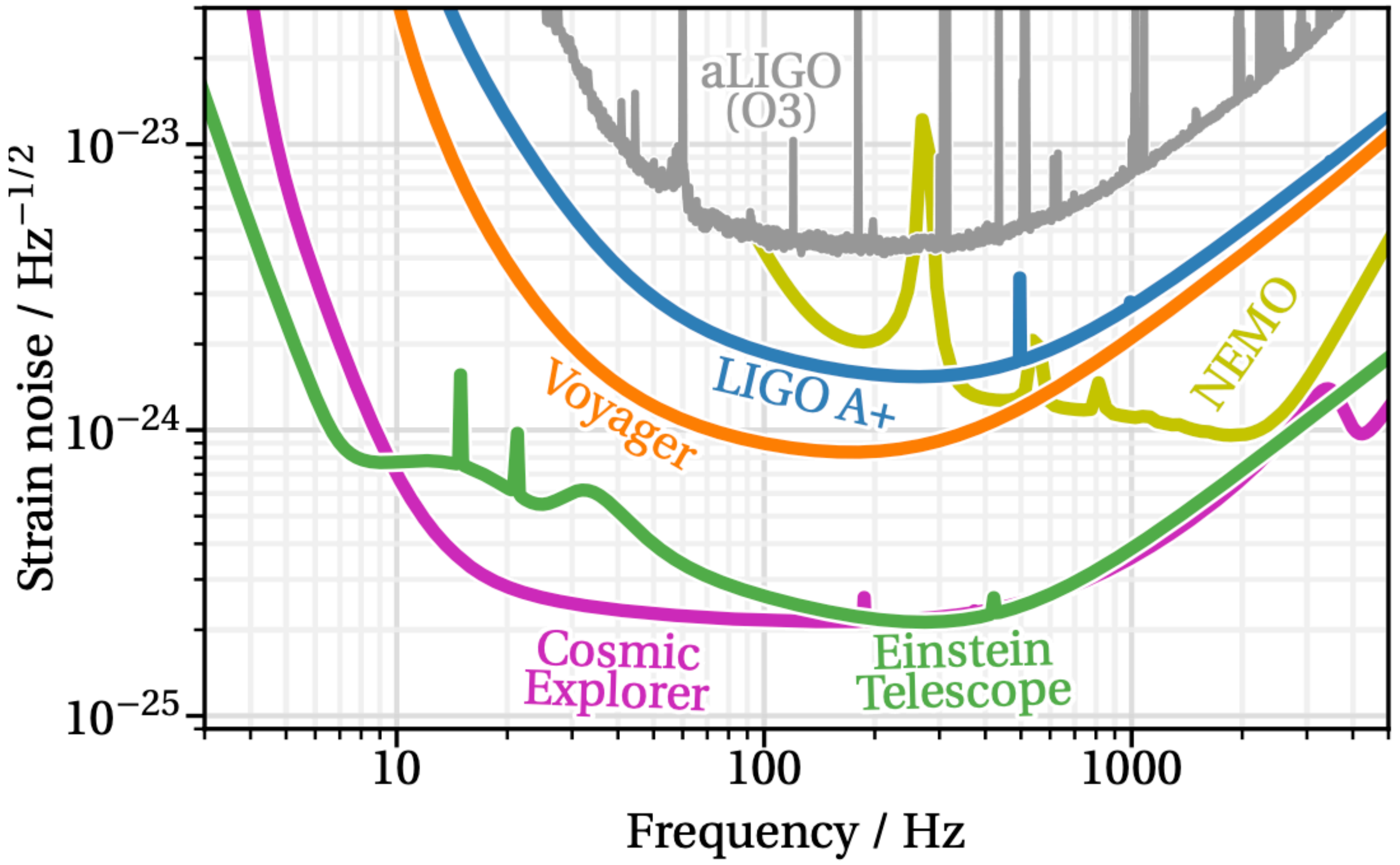

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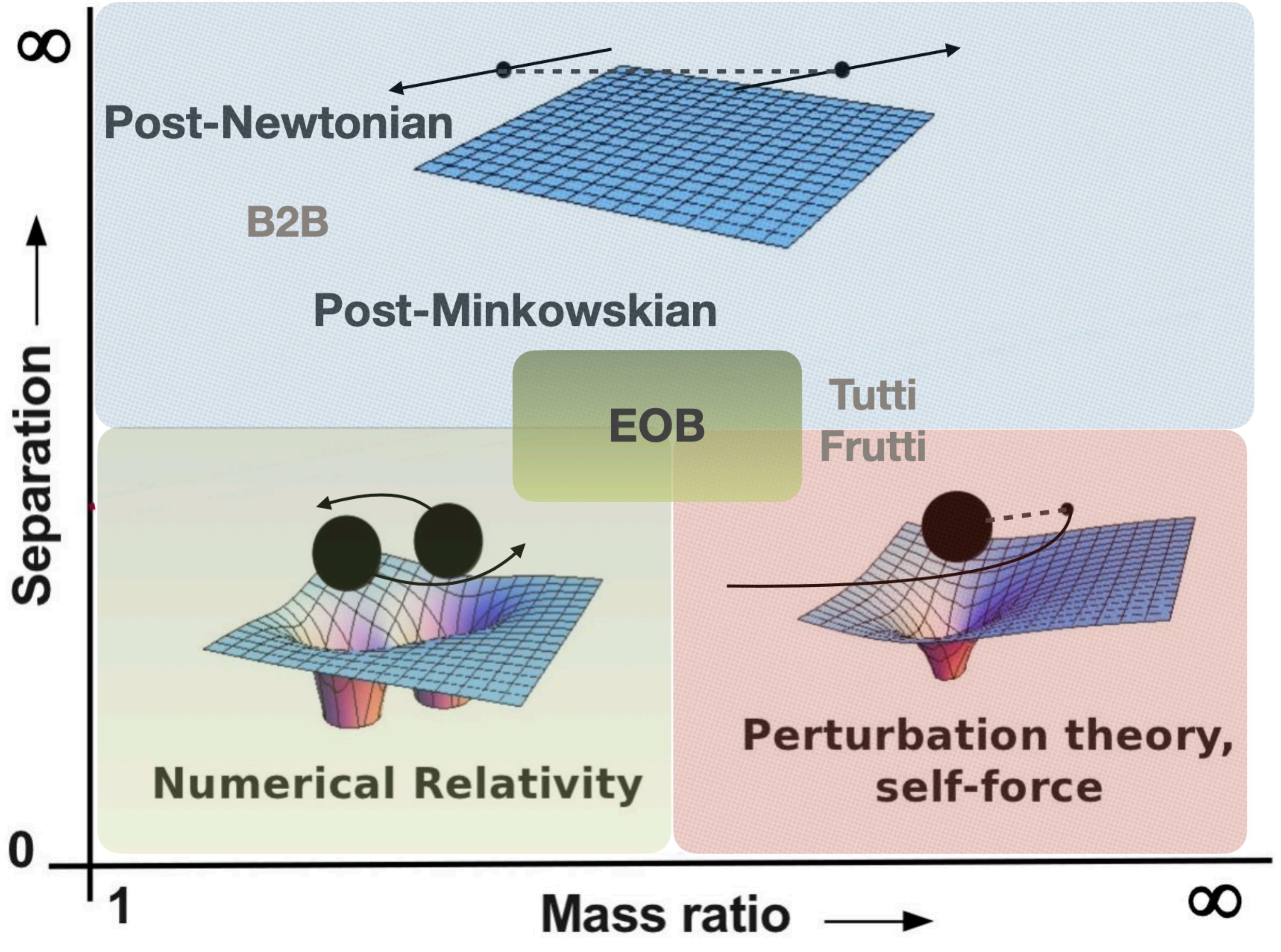

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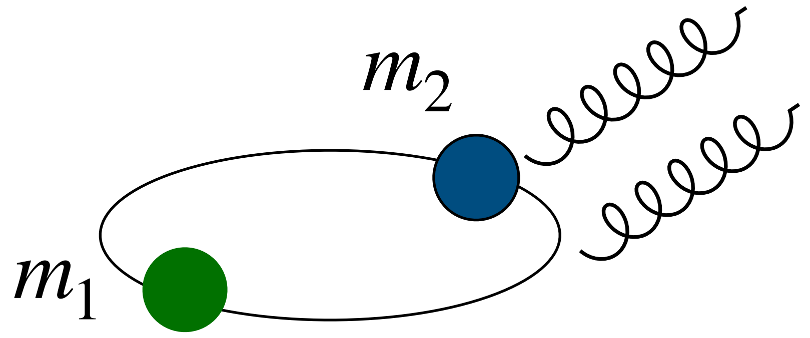

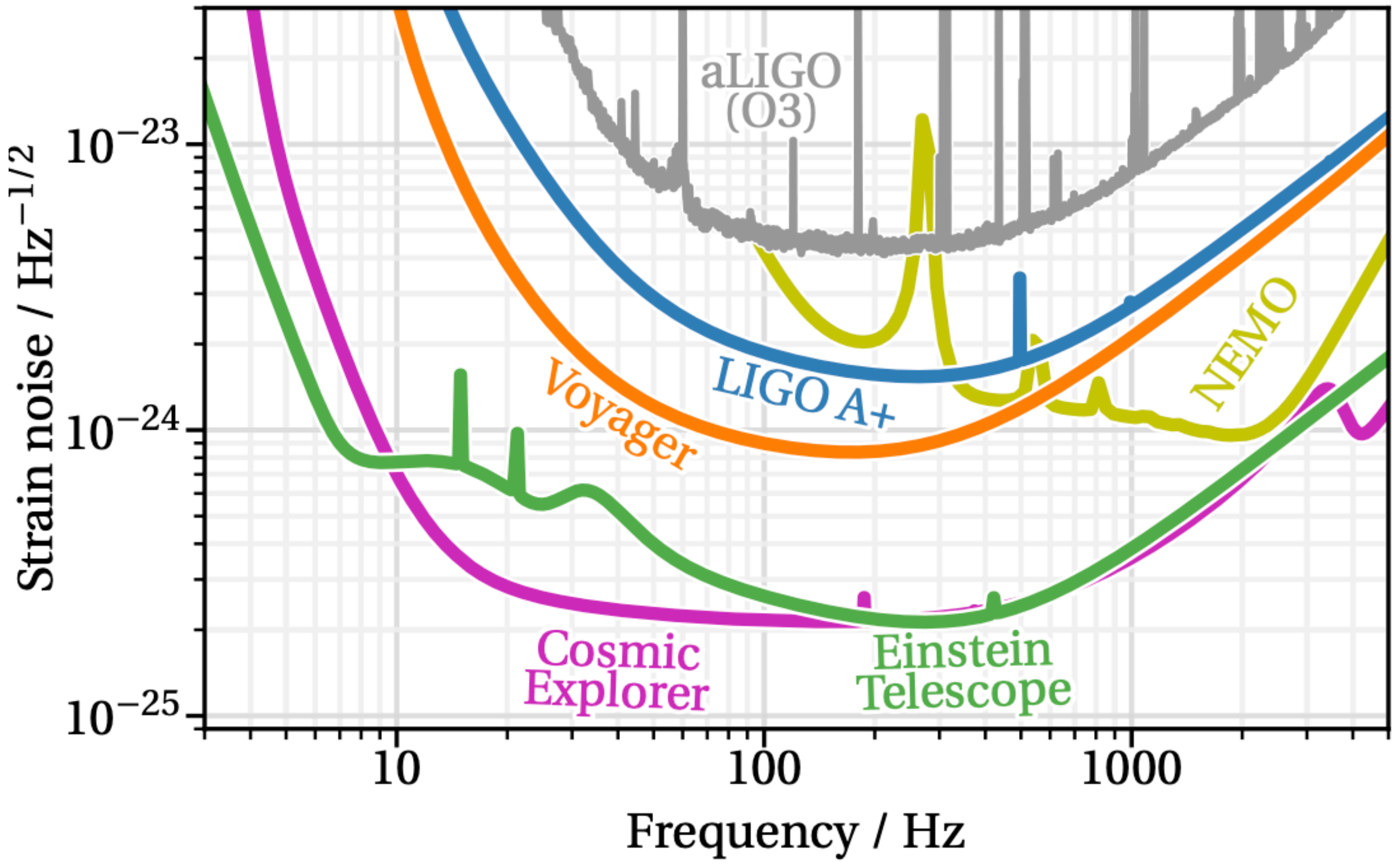

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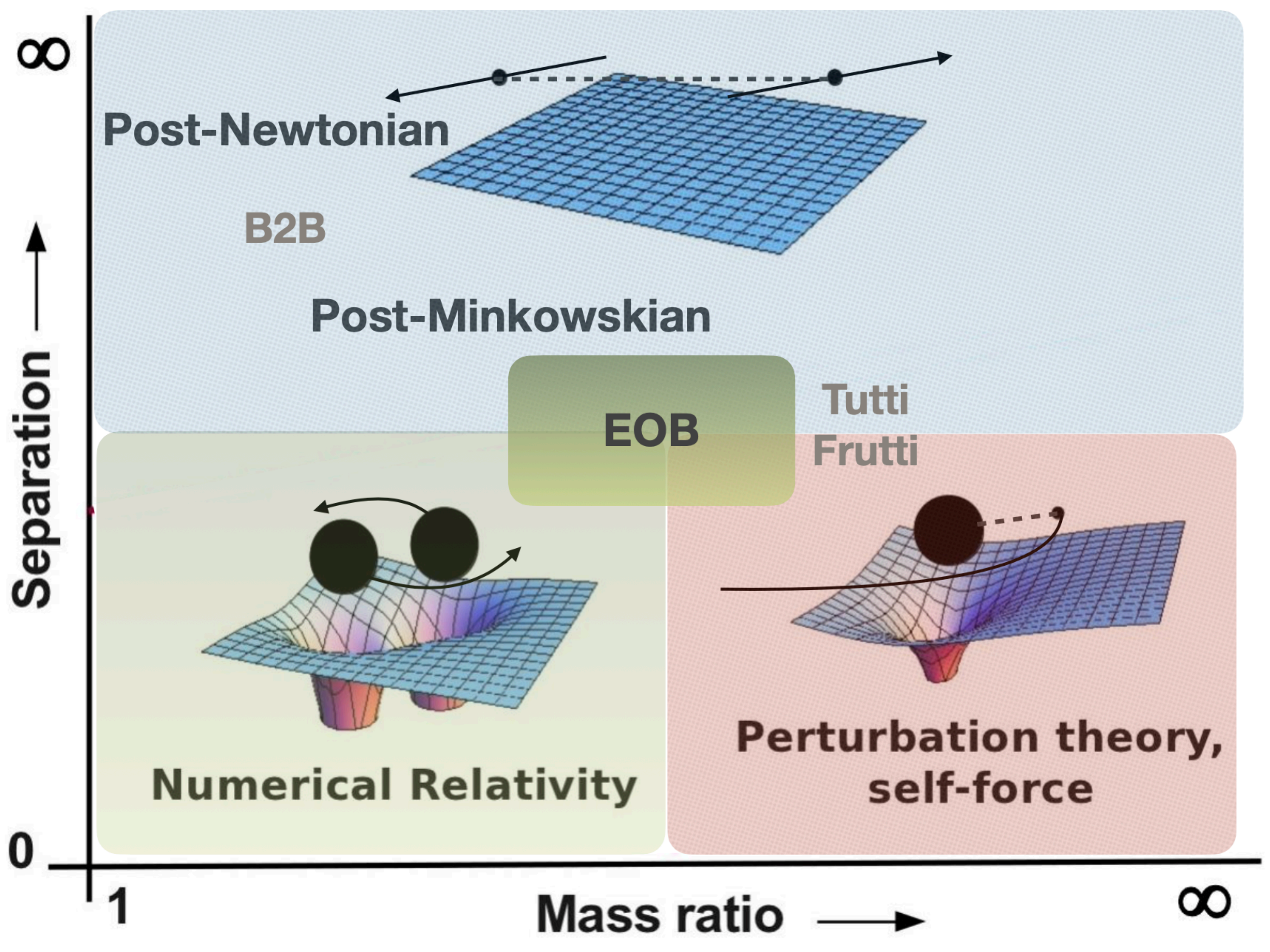

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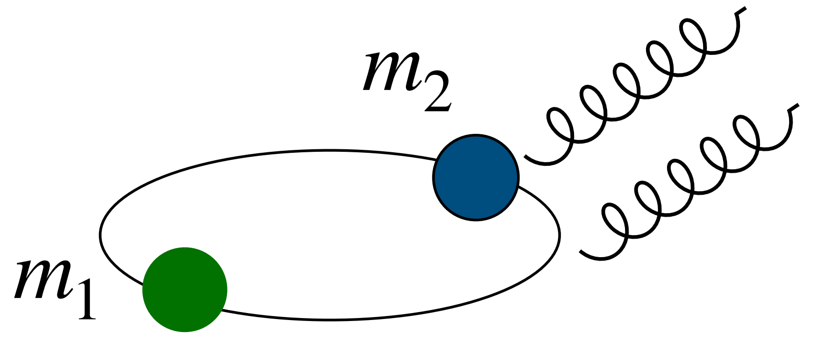

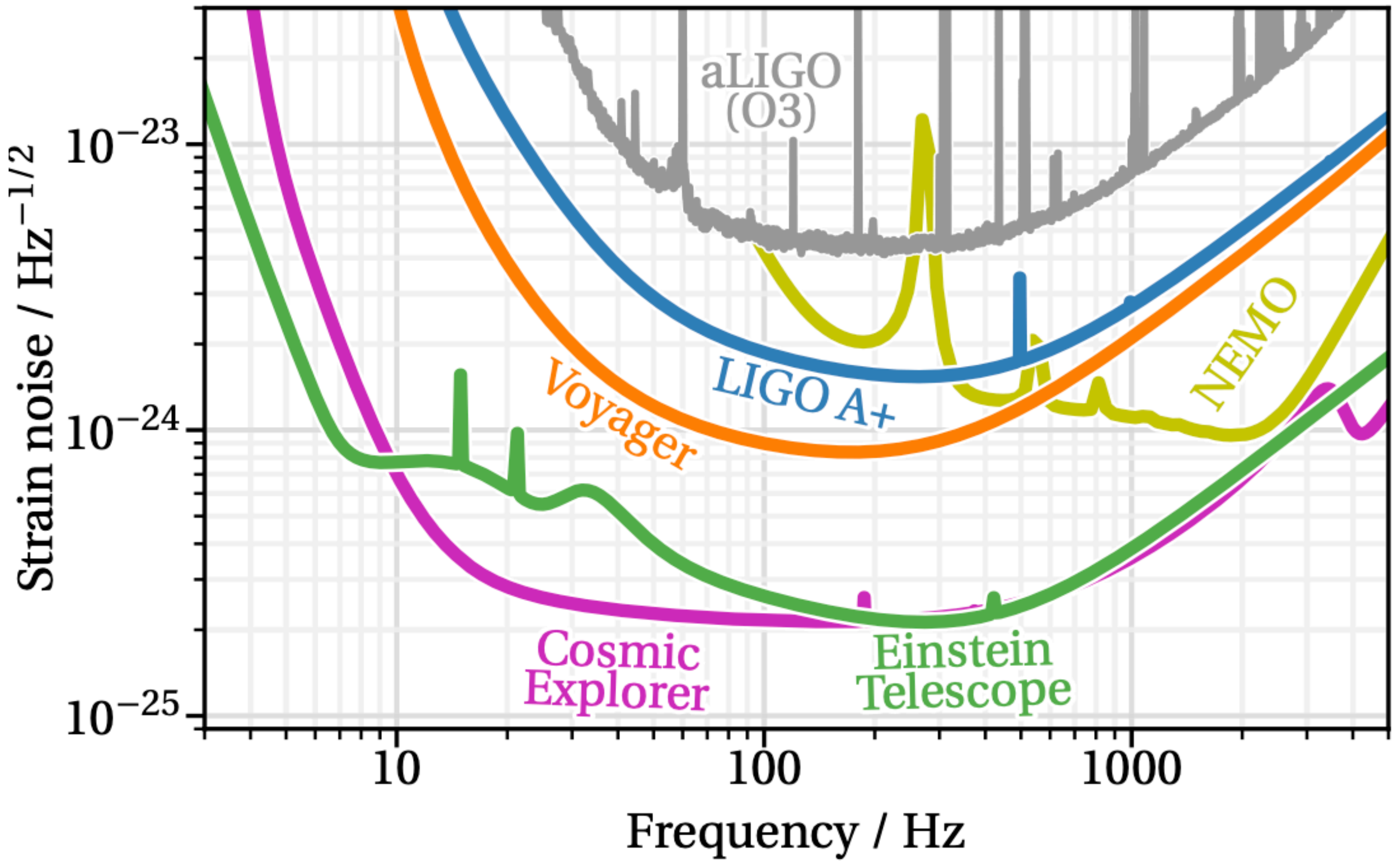

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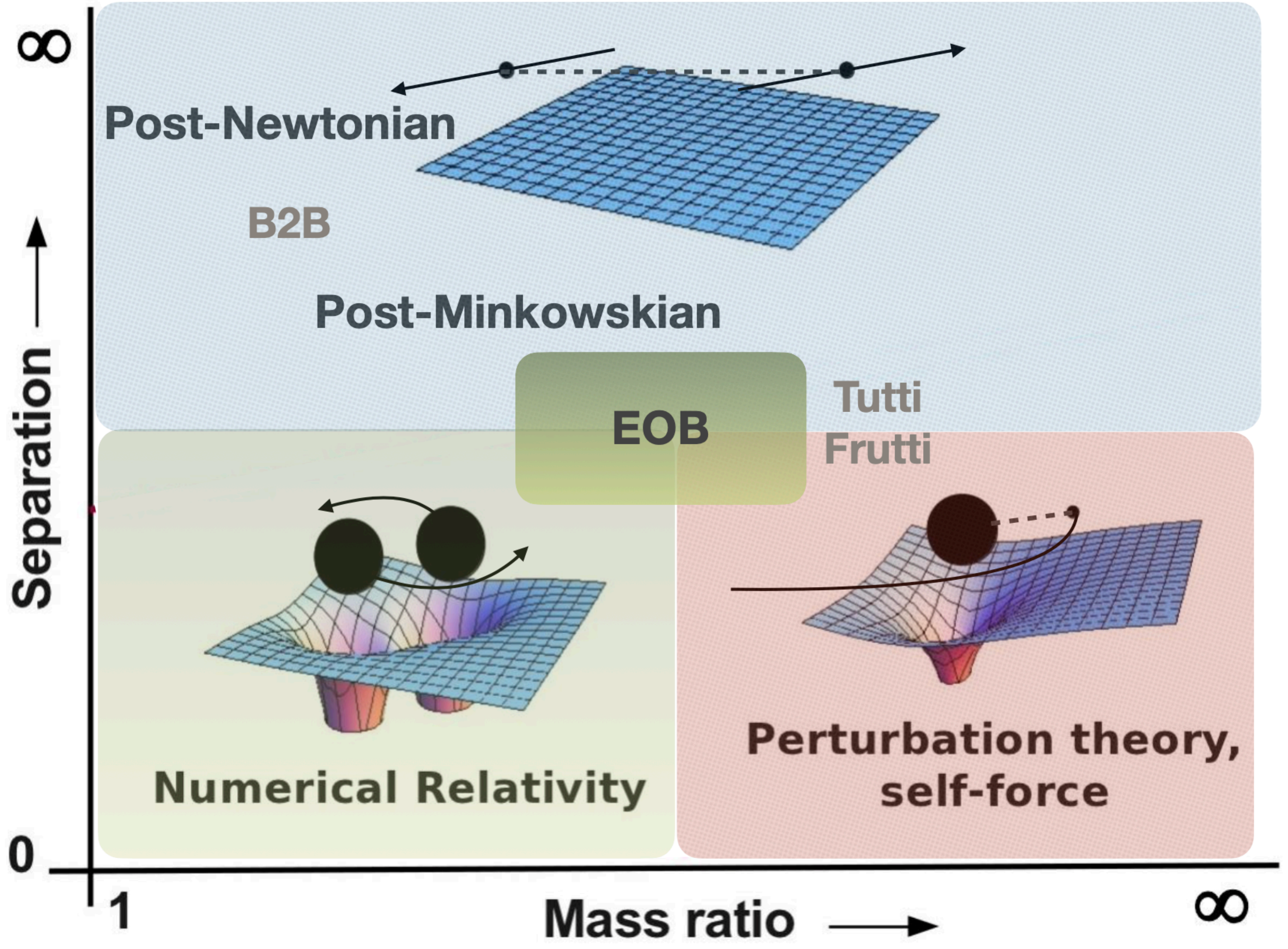
- Extreme need for precise theoretical predictions for signal templates for Matched filtering analyses



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$$h_{\mu\nu}\Big|_{|x|\to\infty} = \eta_{\mu\nu} + \frac{\kappa^2 M}{8\pi|x|}\left[\underbrace{\frac{\kappa^2 M}{\sqrt{-b^2}}\hat{h}^{(1)}_{\mu\nu}}_{\text{LO}} + \underbrace{\left(\frac{\kappa^2 M}{\sqrt{-b^2}}\right)^2 \hat{h}^{(2)}_{\mu\nu}}_{\text{NLO}} + \dots\right]$$

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  Jakobsen, Mogull, Plefka,Steinhoff
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- **Spin:**
  De Angelis, Gonzo, Novichkov
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- **NLO:**
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- Why waveforms:
  - Template for matched-filtering analyses
  - Boundary to bound
  - Properties of scattering amplitudes

# Classical Observables From Scattering Amplitudes

Kosower, Maybee, O'Connel

- Consider well defined asymptotic states:

$$|\psi\rangle_{in} = \int d\phi(p_1) d\phi(p_2)\ \phi_1(p_1)\phi_2(p_2) e^{i(b_1\cdot p_1 + b_2\cdot p_2)}\ |p_1, p_2\rangle_{in} \qquad |\psi\rangle_{out} = S|\psi\rangle_{in} \qquad S = 1 + i\ T$$

On-shell phase space integral

wavefunction

Two different quanta

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- Expectation value of a physical observable:

$$\Delta\langle\mathcal{O}\rangle = \langle\mathcal{O}\rangle_{out} - \langle\mathcal{O}\rangle_{in} = {}_{out}\langle\psi|\,\mathcal{O}\,|\psi\rangle_{out} - {}_{in}\langle\psi|\,\mathcal{O}\,|\psi\rangle_{in} = {}_{in}\langle\psi|\,S^\dagger[\mathcal{O}, S]\,|\psi\rangle_{in}$$

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$$1/m \quad \ll \quad G\,m \quad \ll \quad b$$

Compton wavelength

Schwarzschild radius

Impact parameter

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- Waveform observable:

Cristofoli, Gonzo, Kosower, O'Connel

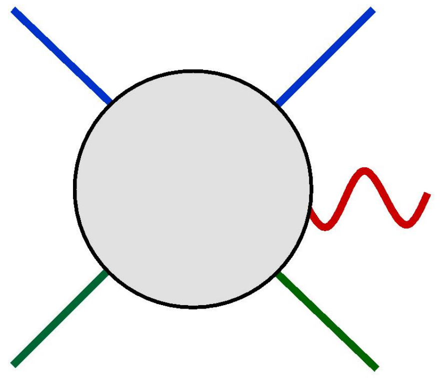
$$\mathcal{O} = \mathcal{W}_{GR} = \epsilon_h^{\mu\nu}h_{\mu\nu} \qquad g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$$

$$\mathcal{O} = \mathcal{W}_{ED} = \epsilon_h^{\mu}A_{\mu} \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

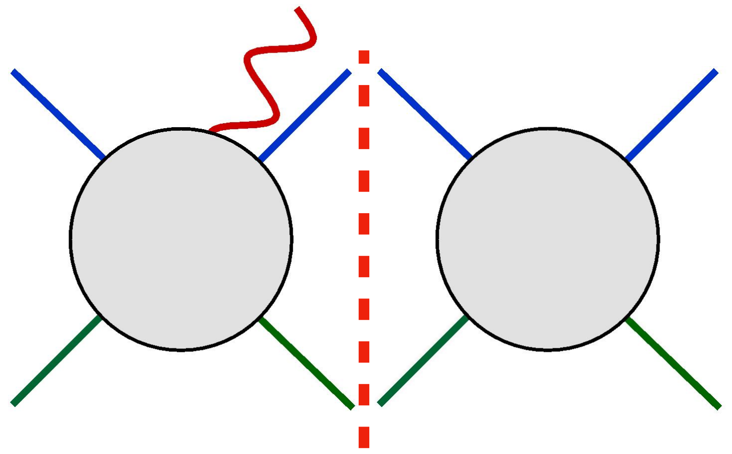
# Waveform From Amplitudes

- Waveform as fourier transform of scattering amplitudes

$$\Delta\langle \mathscr{W}_h\rangle(u,\vec{n})\frac{1}{4\pi r}\int_0^\infty \hat{d}\omega \int d\mu \Bigg\{ \hat{\delta}^D(q_1+q_2-k)e^{-i\omega u}\Bigg[$$



$$-\, i \int d(LIPS)$$



$$\Bigg] + c.c.\Bigg\}$$

Fourier Transform

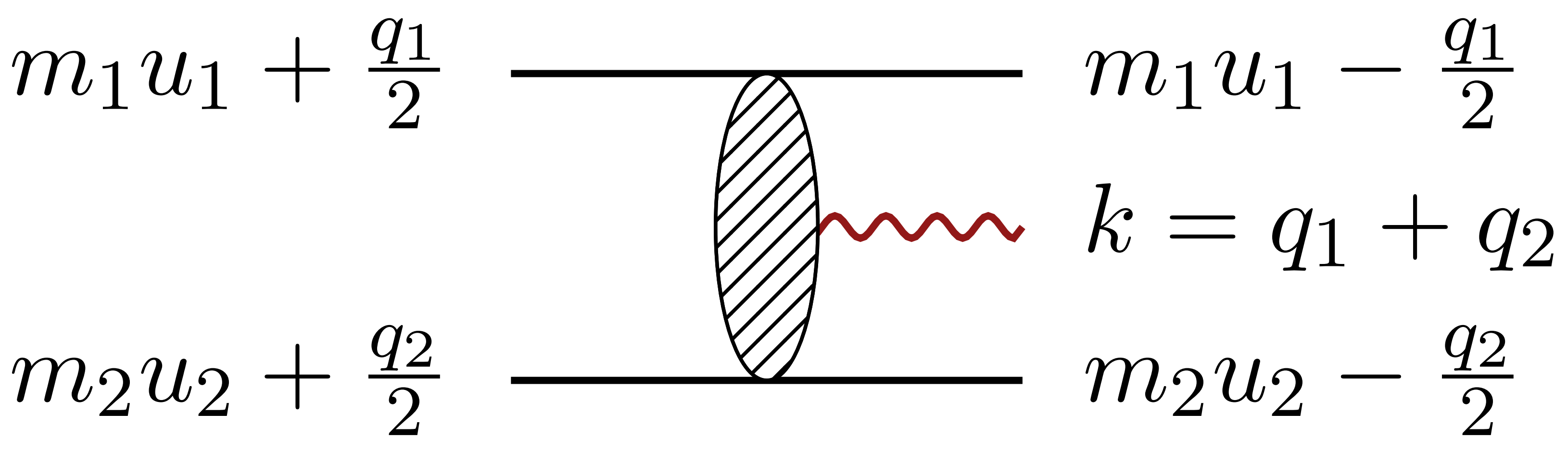
$$d\mu = \prod_{i=1}^{2} \hat{d}^D q_i \delta(2p_i\cdot q_i + q_i^2)e^{ib_i\cdot q_i}$$

Five-points amplitude

$$\mathscr{A}(p_1p_2 \to p_1'p_2'k^h)$$

Iteration terms

$$\mathscr{A}*(\tilde{p}_1'\tilde{p}_2' \to \tilde{X}) \otimes \mathscr{A}(p_1p_2 \to Xk^{-h})$$



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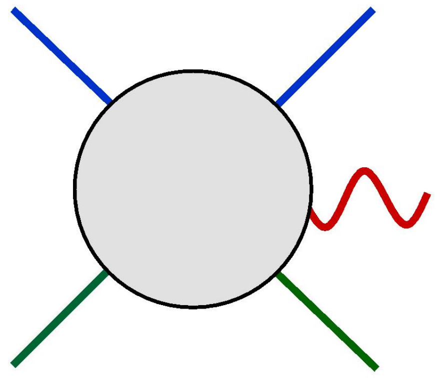

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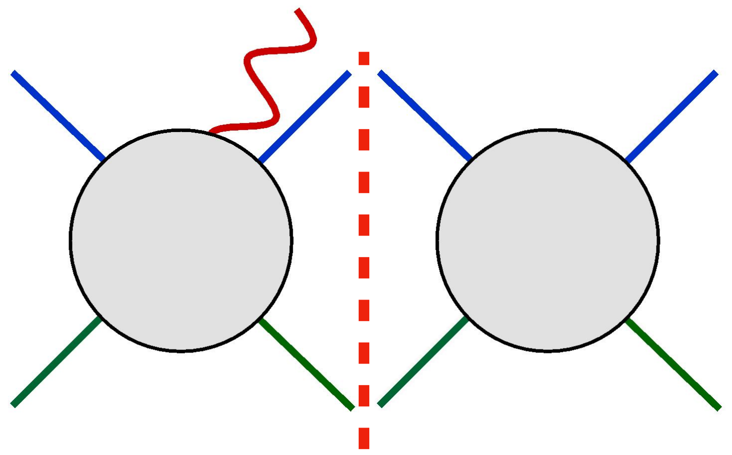

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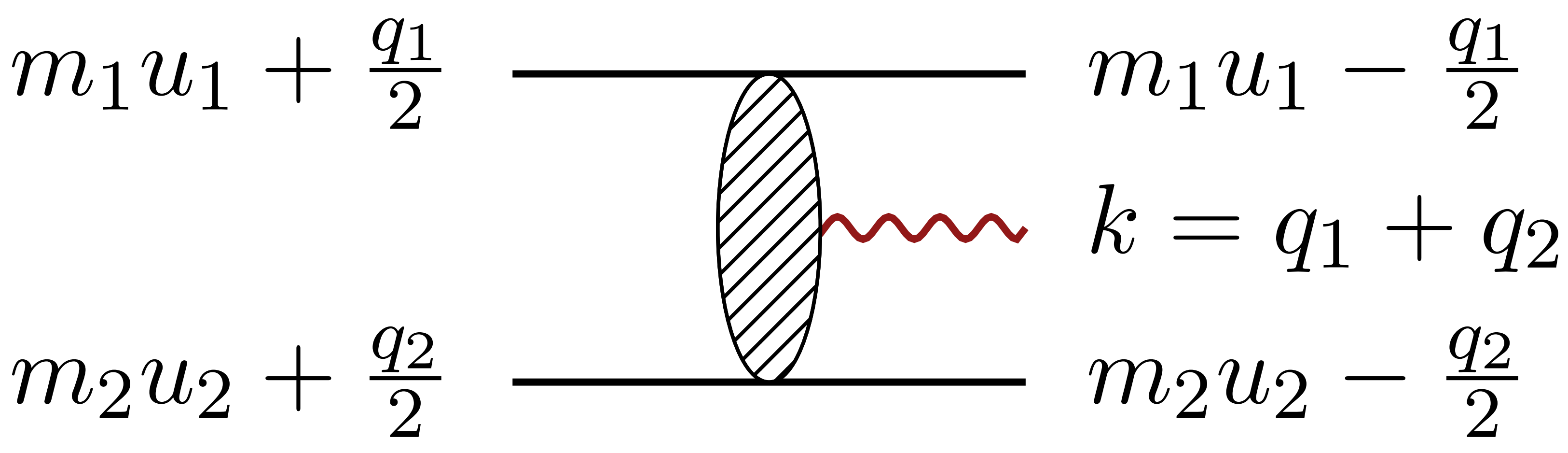
- Iteration terms needed to restore the correct **in-in prescription**

Caron-Huot, Giroux, Hannesdottir, Mizera



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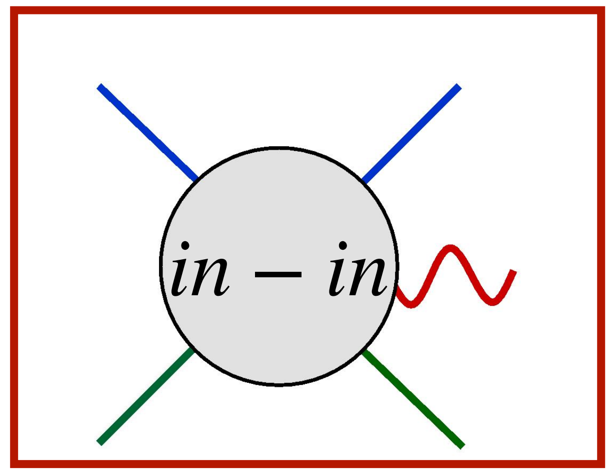

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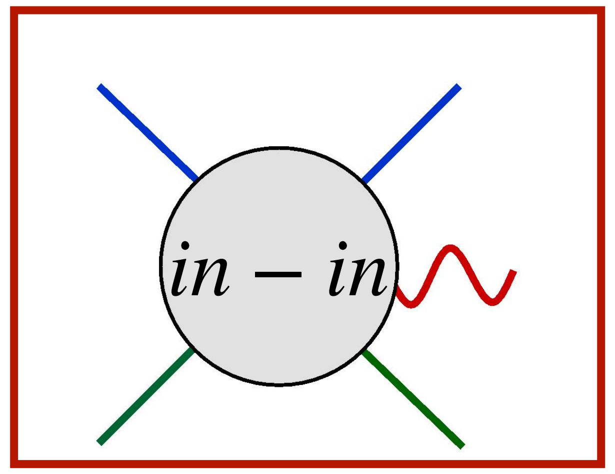

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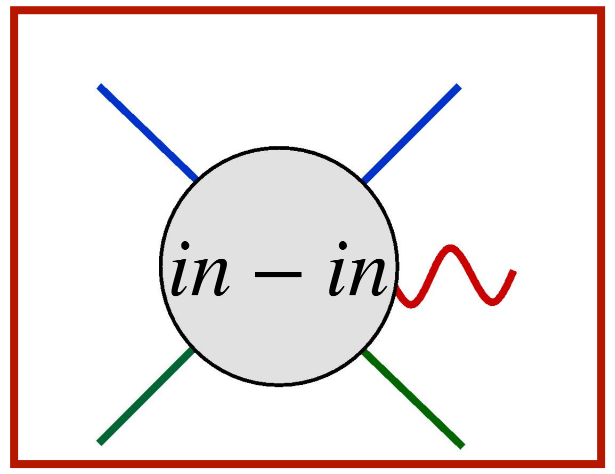

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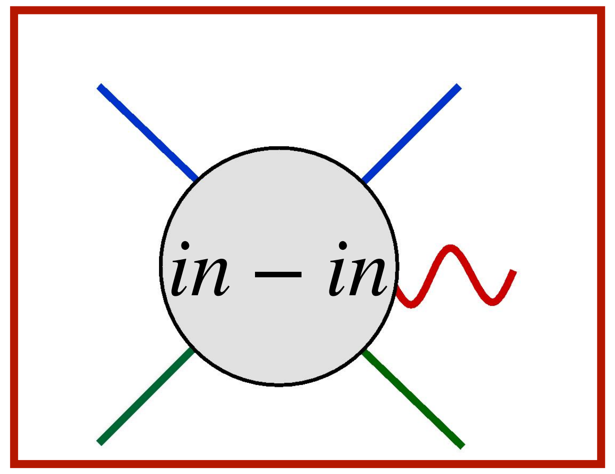

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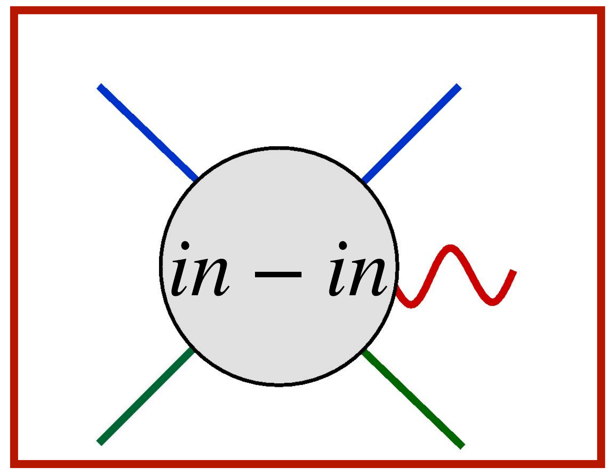

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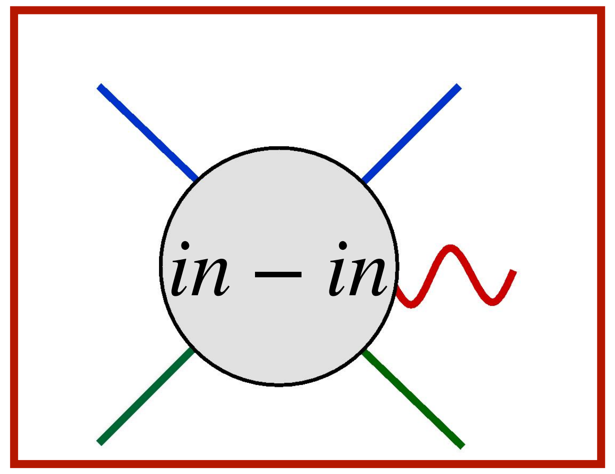

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- Confirmation against multipolar-post-minkowskian formalism (MPM)

Bini, Damour, Geralico
Georgoudis, Heissenberg, Russo
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G.B., De Angelis

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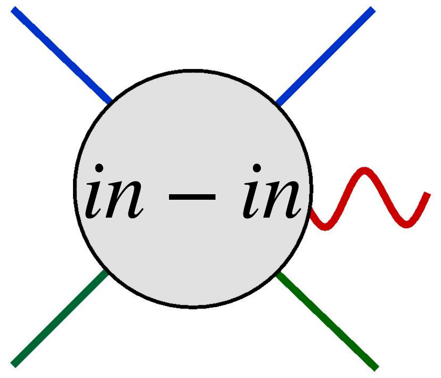

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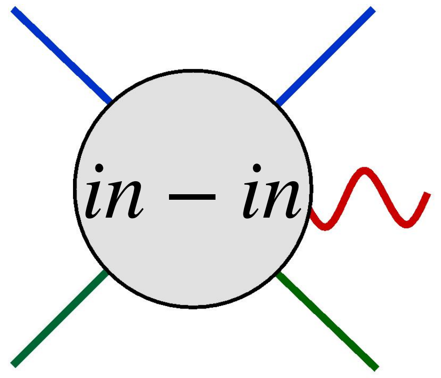
$\Delta\langle \mathscr{W}_h\rangle(\omega,\vec{n})$ Frequency-space waveform
As twisted period integral

# How to compute waveforms?

G.B., De Angelis

- **Generalised approach**:



$$\Delta\langle \mathscr{W}_h\rangle(u,\vec{n}) \sim \boxed{\int d\mu \Big\{ \hat{\delta}^D(q_1+q_2-k) \qquad\qquad} + c\,.\,c. \Big\}$$

$\Delta\langle \mathscr{W}_h\rangle(\omega,\vec{n})$ Frequency-space waveform
As twisted period integral

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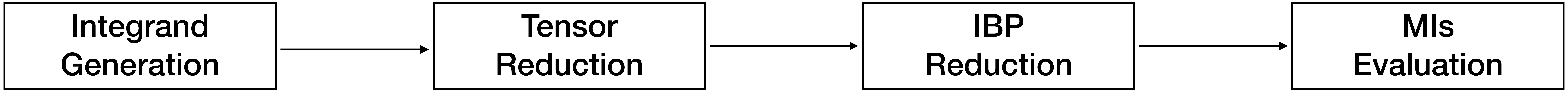

Integrand Generation → Tensor Reduction → IBP Reduction → MIs Evaluation

# How to compute waveforms?

G.B., De Angelis

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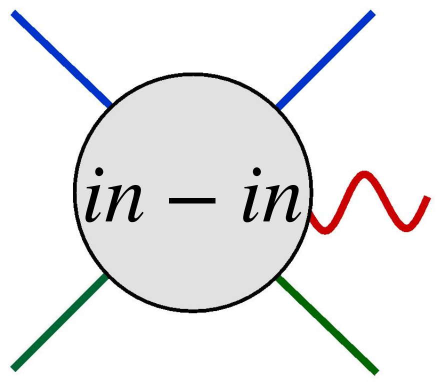

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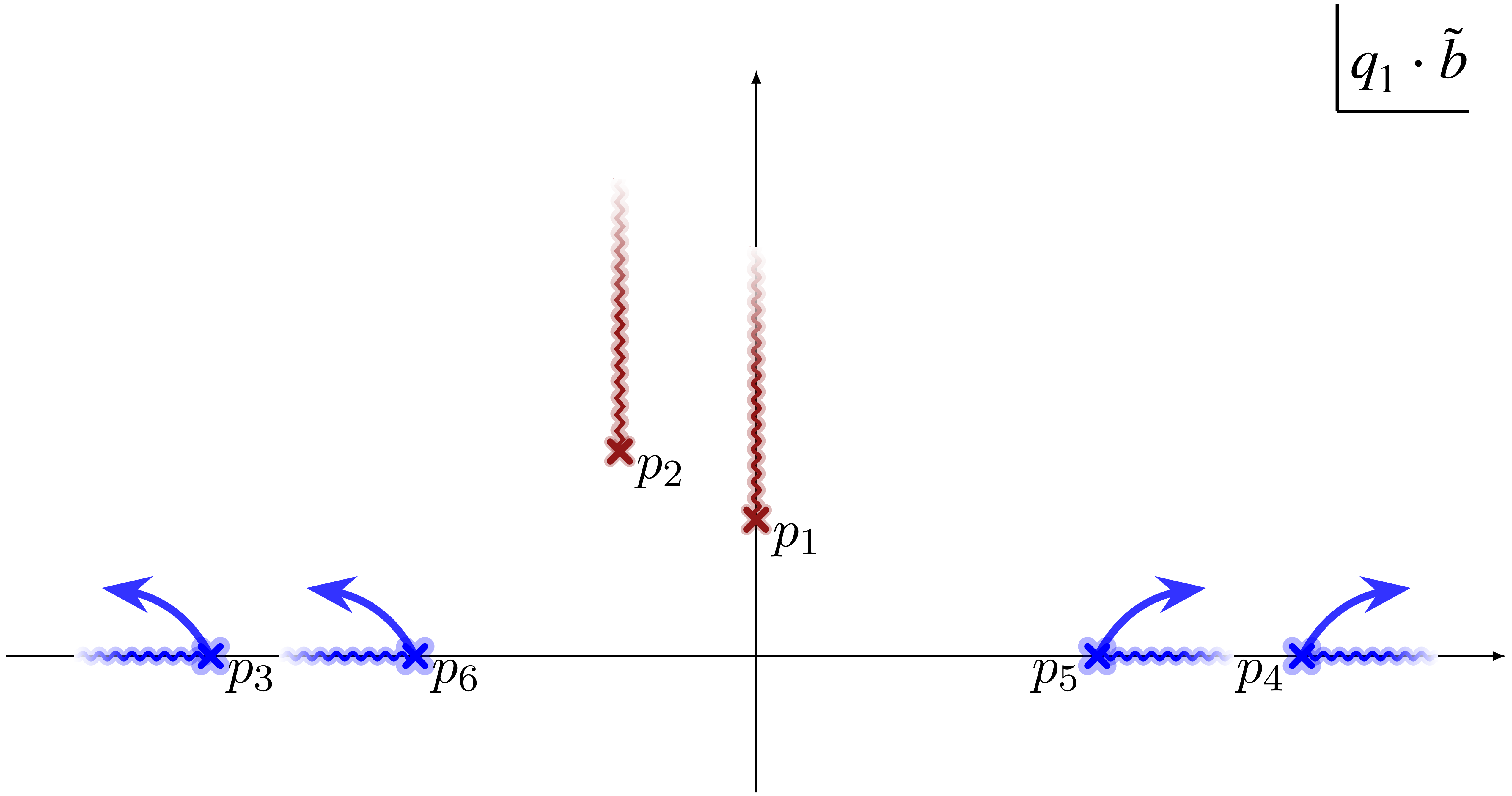
- The **amplitudes factory with Fourier Exponentials**:

Integrand Generation → Tensor Reduction → IBP Reduction → MIs Evaluation

- No spurious poles, analytic results into a finite basis of Master Integrals

# Singularity structure

- 2 types of singularities of the quantum amplitudes:



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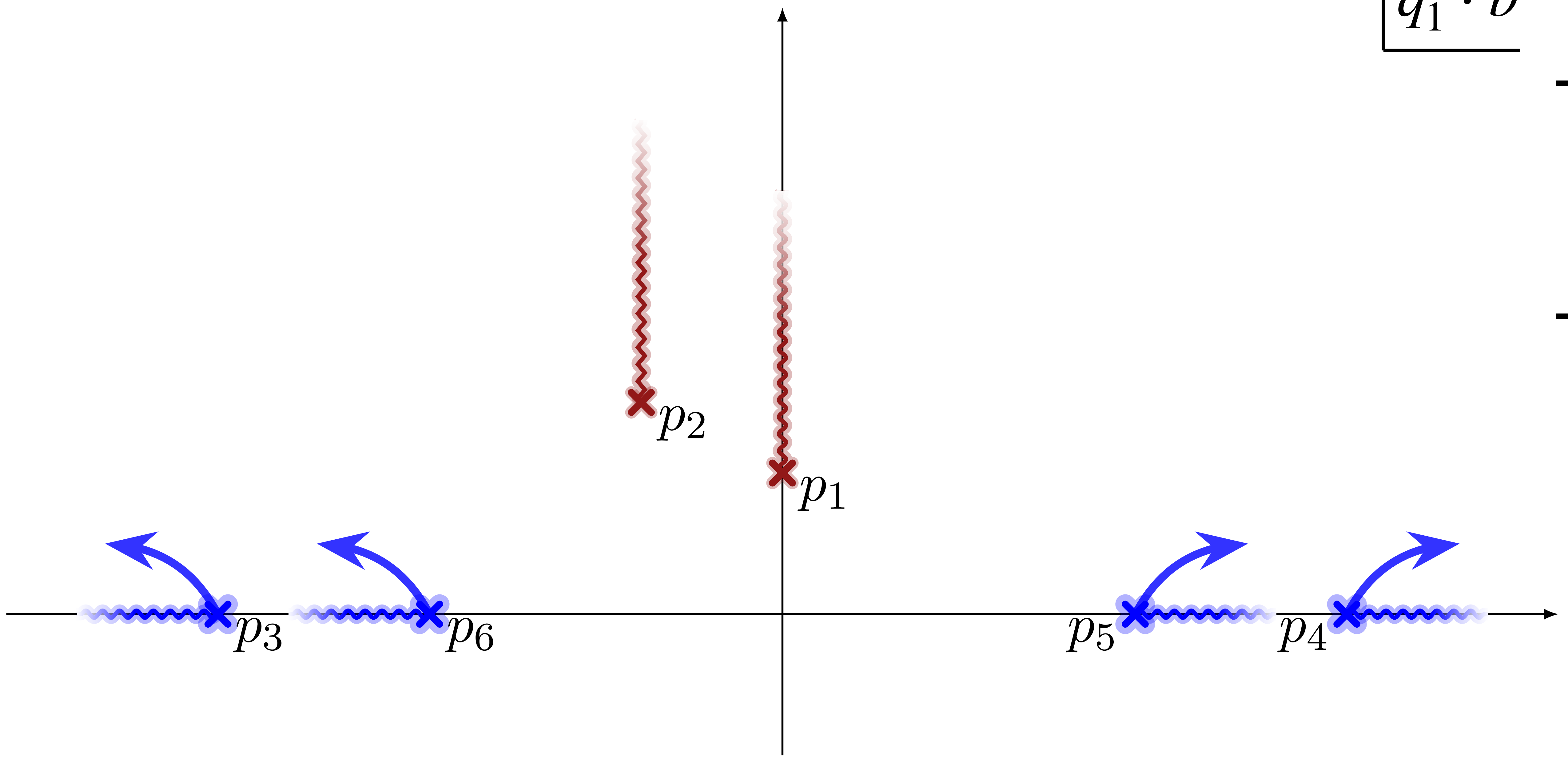

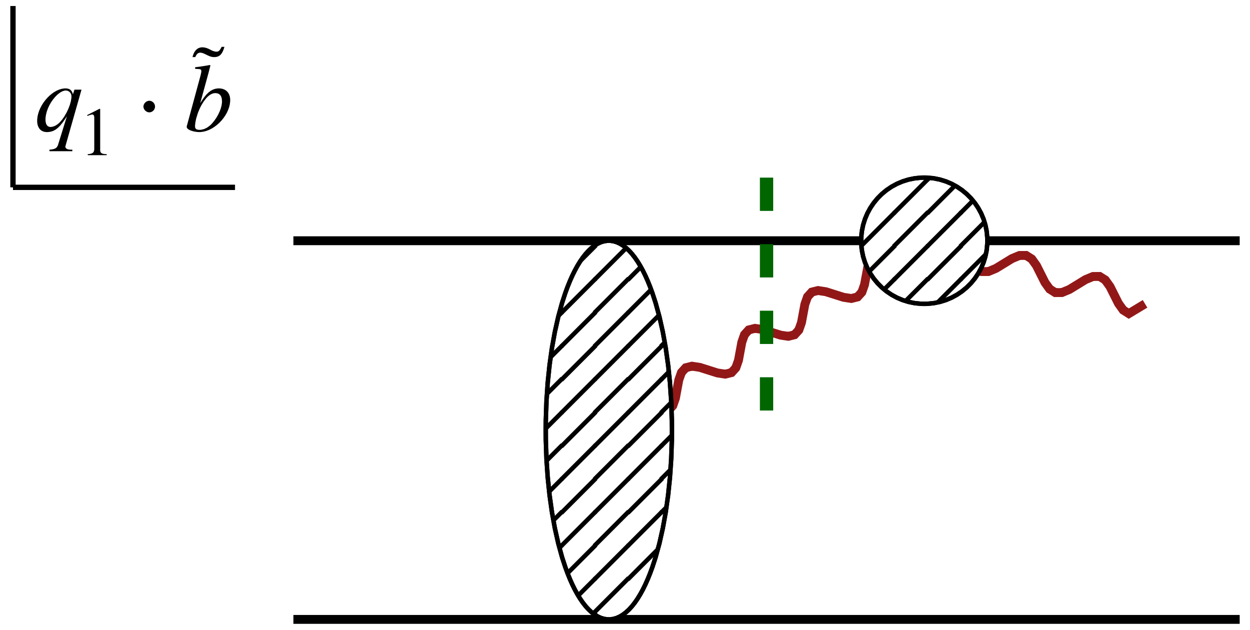

Massive (classical) particles going on-shell

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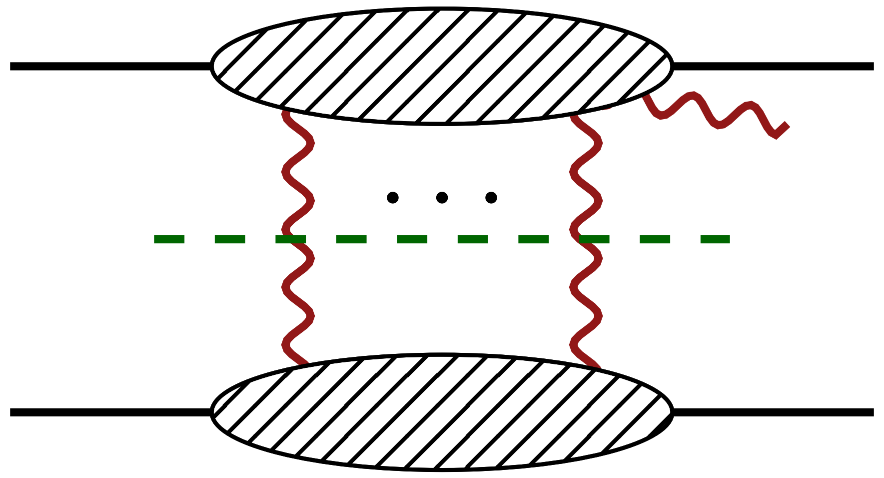
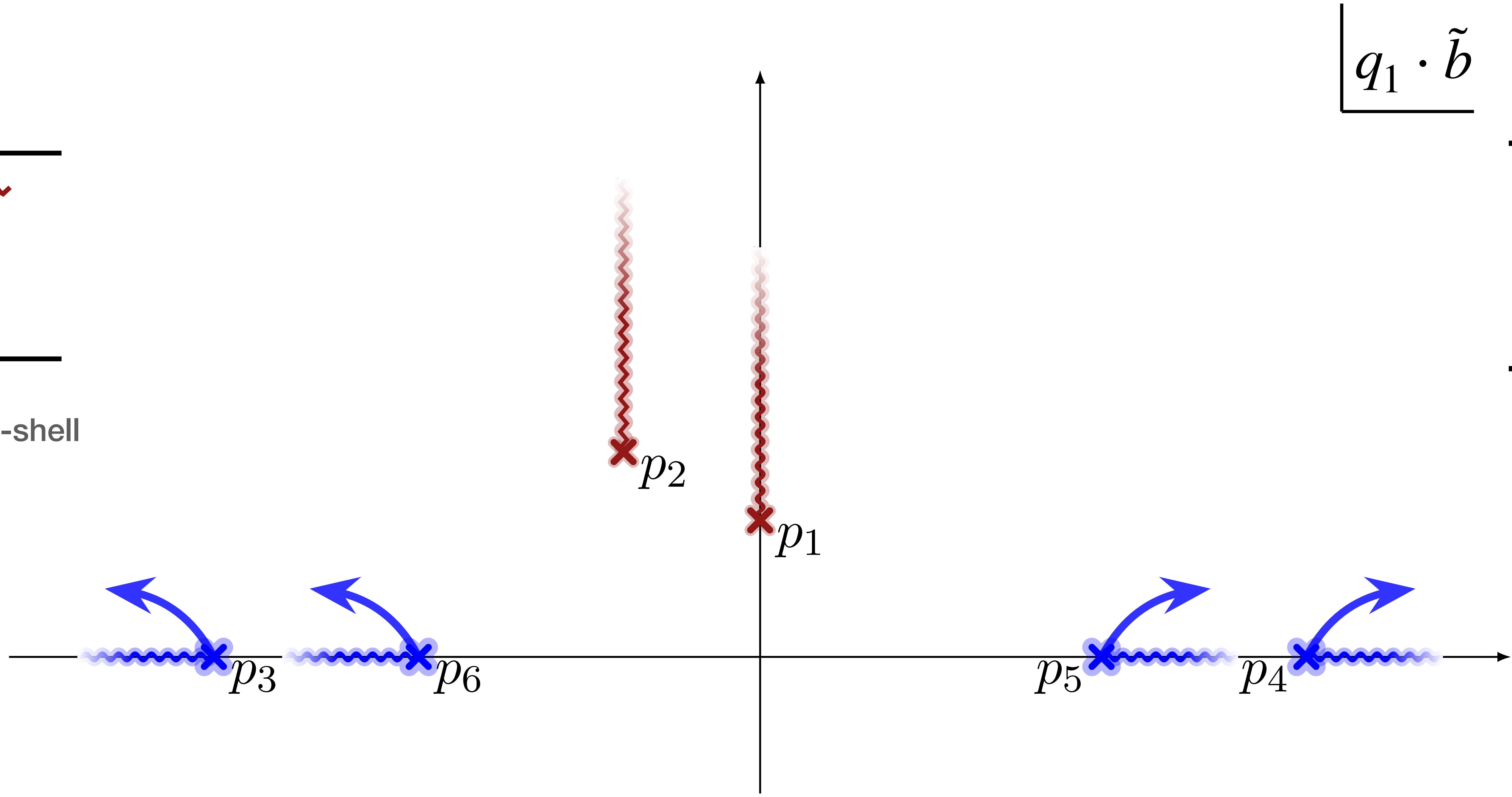
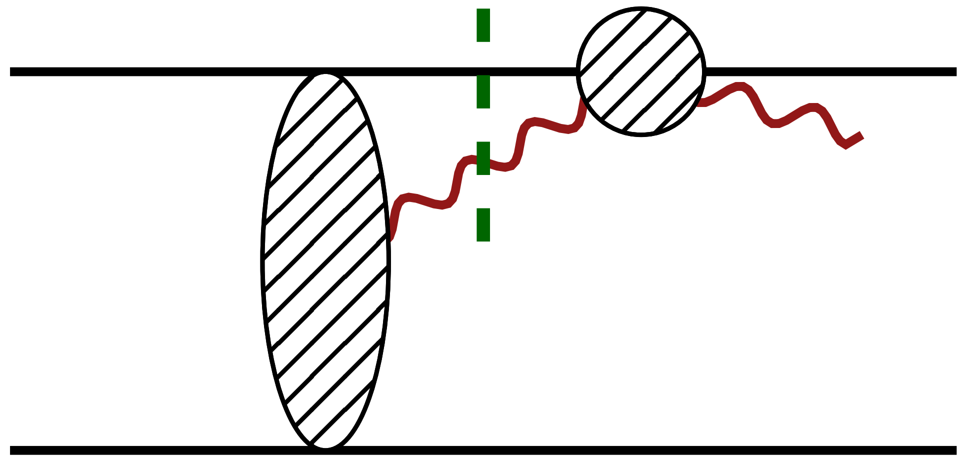


Intermediate gravitons going on-shell

$q_i^2 = 0$





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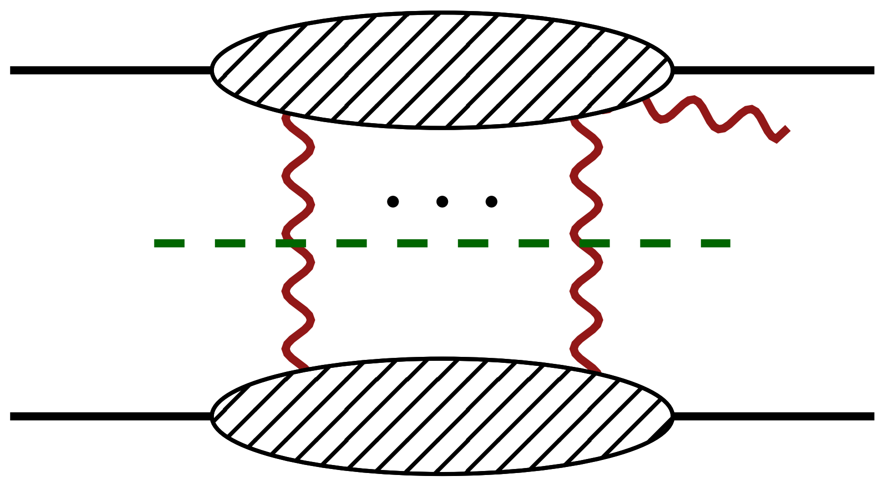

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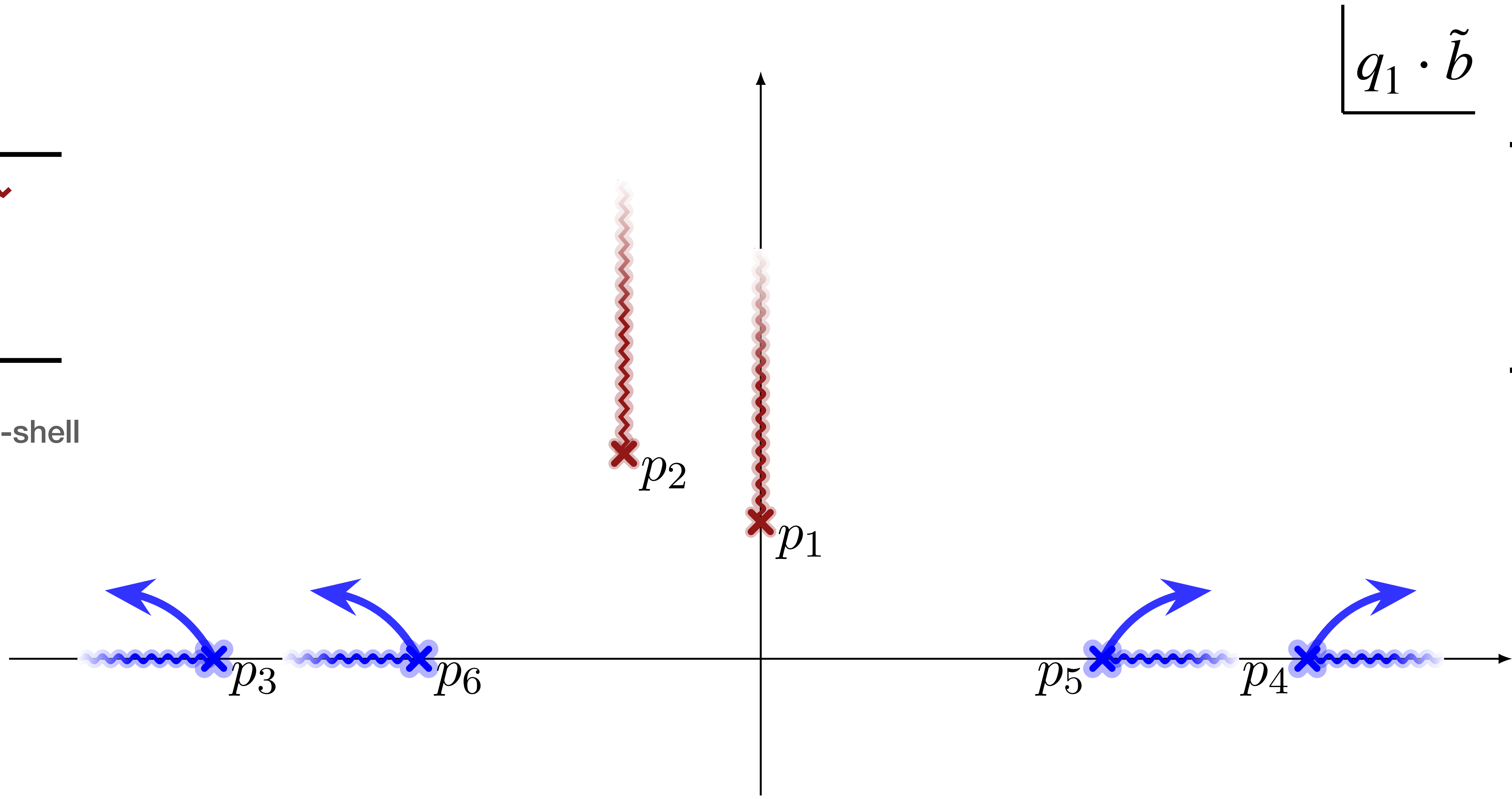
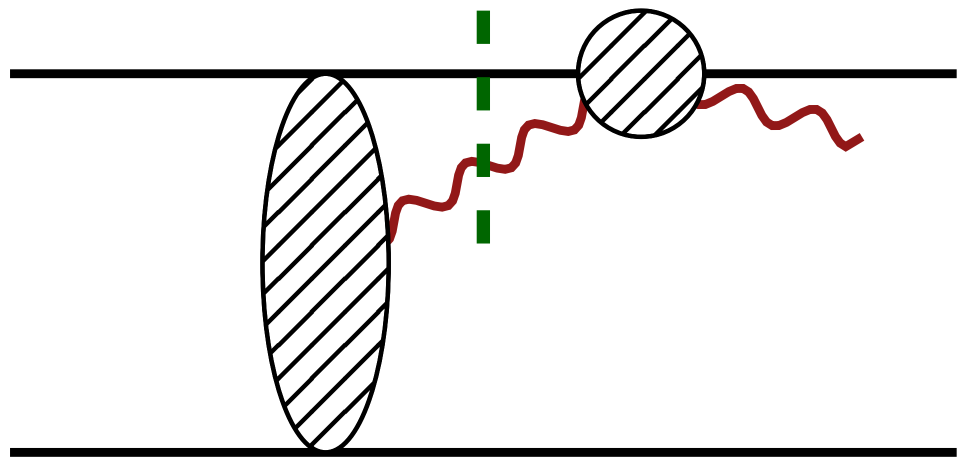




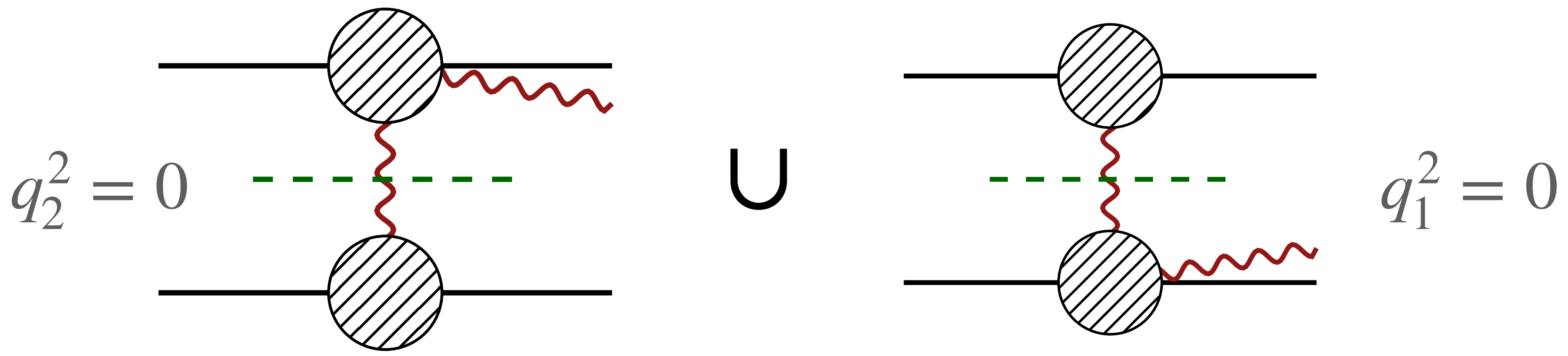
Massive (classical) particles going on-shell

- Classical limit captured by on-shell gravitons, corresponding to long-range interactions

# Playground: Waveform at Leading Order

- Integrand generation from Generalised Unitarity

$\mathcal{I}^{(0)}_{\mathscr{W}} =$



# Playground: Waveform at Leading Order

- Integrand generation from Generalised Unitarity

- Building blocks: heavy mass expansion

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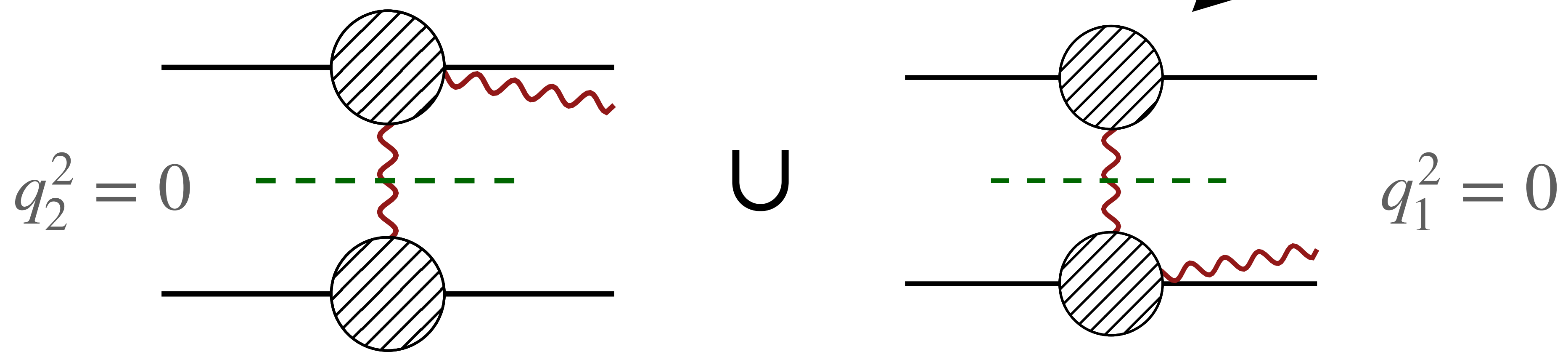

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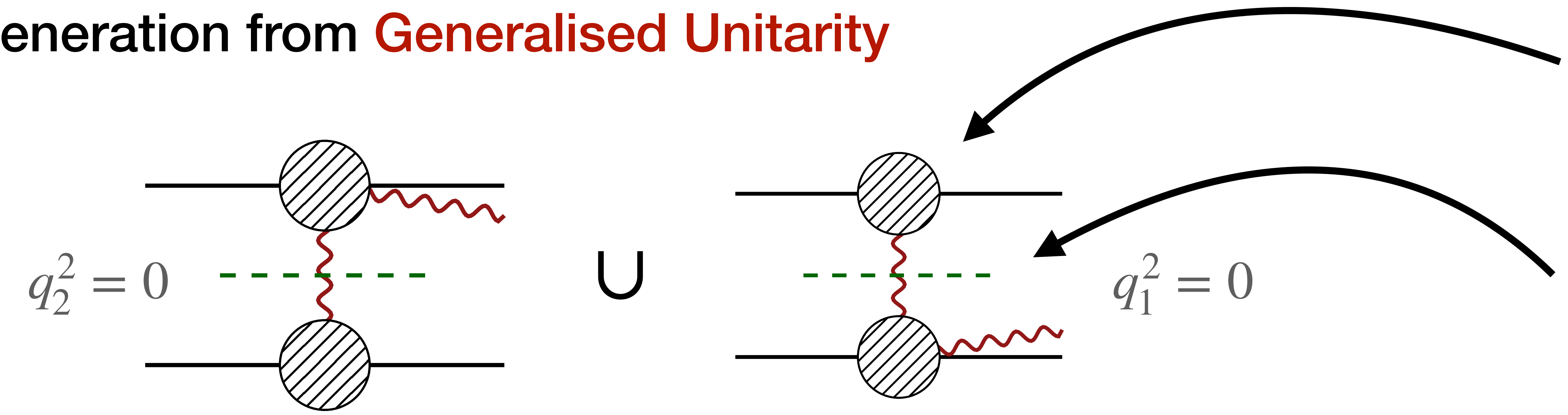

- Building blocks: heavy mass expansion

- Polarisation sum

$$\sum_h \epsilon^{\mu_1\nu_1}_{-k}\epsilon^{\mu_2\nu_2}_{k} = \frac{1}{2}(P^{\mu_1\mu_2}P^{\nu_1\nu_2} + P^{\nu_1\mu_2}P^{\nu_1\mu_2}) - \frac{2}{d-2}P^{\mu_1\nu_1}P^{\mu_2\nu_2}$$

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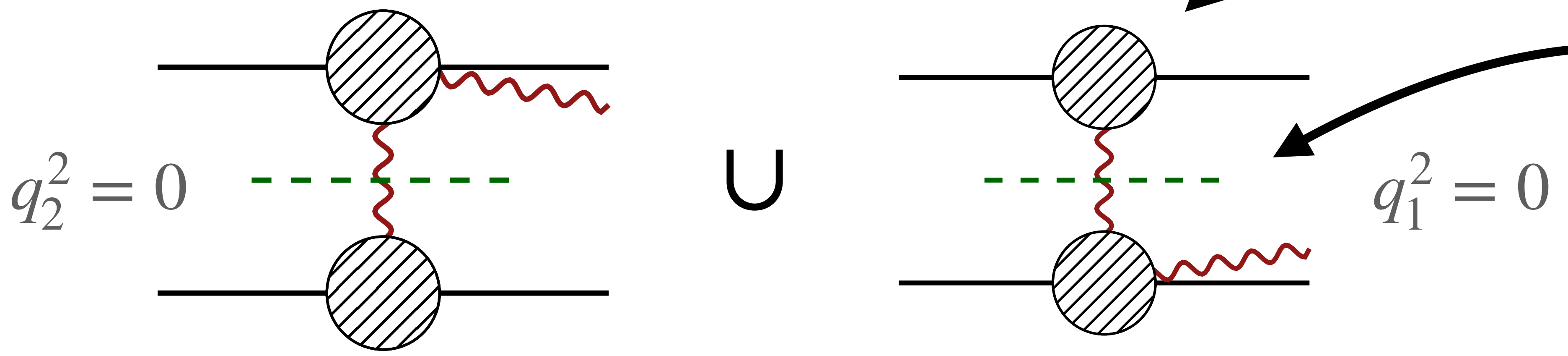

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Tensor structures appearing

$$\epsilon_k \cdot q$$

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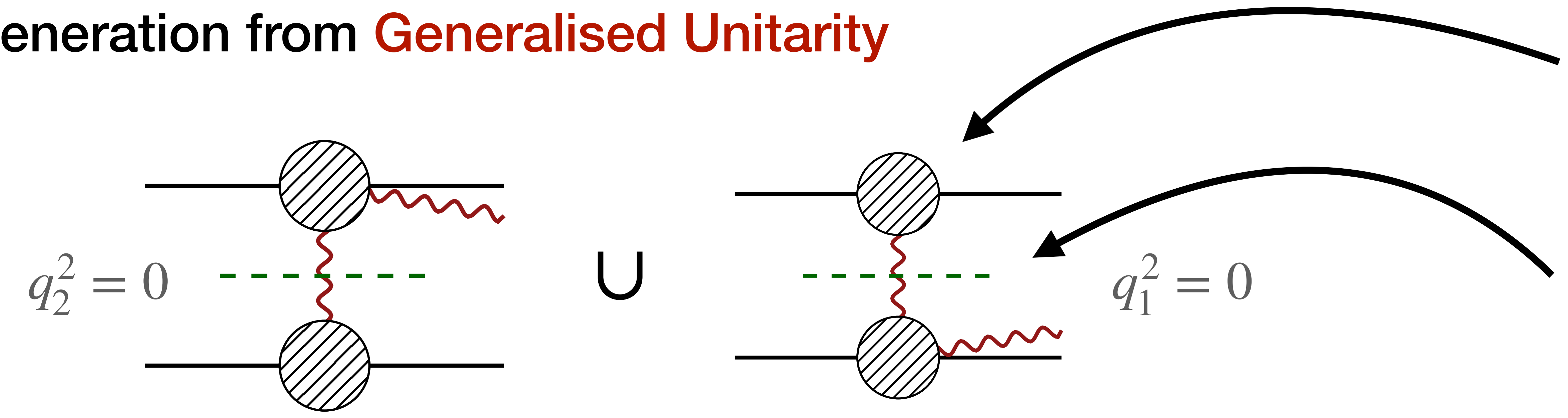

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- Decomposition in scalar form factors:

Anastasiou, Karlen, Vicini

$$\Delta\langle \mathcal{W}^{(0)}_{h} \rangle = \sum_i T_i\ \Delta\langle \mathcal{W}^{(0)}_{h,\,i} \rangle$$

# About Fourier Integrals

G.B., S. De Angelis

G.B., G. Crisanti, M. Giroux, P. Mastrolia, S. Smith

- Form factors in terms of scalar integrals:

$$I_{a_1\,1\,1\,a_4 a_5} = \int_{\hat{q}} e^{i\,b\cdot q}\,\frac{(i\,b\cdot q)^{-a_1}\hat{\delta}(u_1\cdot q)\hat{\delta}(u_2\cdot(q-k))}{(q^2)^{a_4}((q-k)^2)^{a_5}}$$

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Integral family

$D_1 = i\ b\cdot q,$

$D_2 = q^2,$

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LiteRed
FiniteFlow

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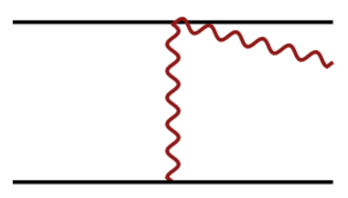
$J_{1+n} \;=\delta_b^{(n)}\,\mathcal{F}\Big[$  $\Big]\,,$

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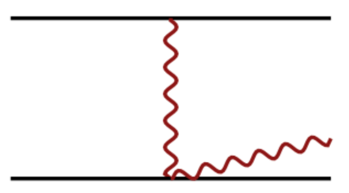
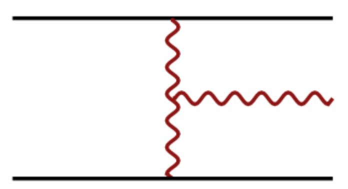
$J_{3+n} \;=\delta_b^{(n)}\,\mathcal{F}\Big[$  $\Big]\,,$

$J_{5+n} \;=\delta_b^{(n)}\,\mathcal{F}\Big[$  $\Big]$

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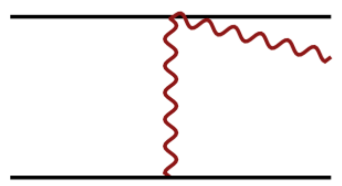
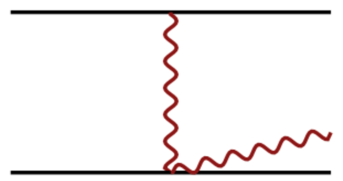
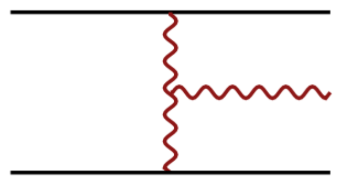
$J_{1+n} = \delta_b^{(n)}\,\mathcal{F}\Big[$  $\Big]$ ,

$J_{3+n} = \delta_b^{(n)}\,\mathcal{F}\Big[$  $\Big]$ ,

$J_{5+n} = \delta_b^{(n)}\,\mathcal{F}\Big[$  $\Big]$

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Complex analysis
Differential Equations

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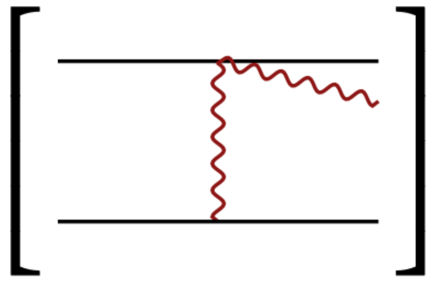
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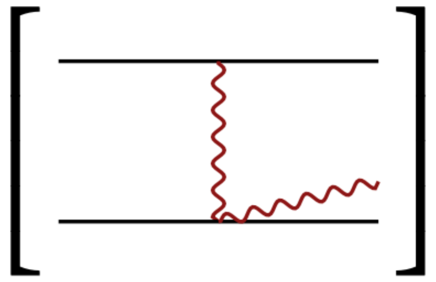
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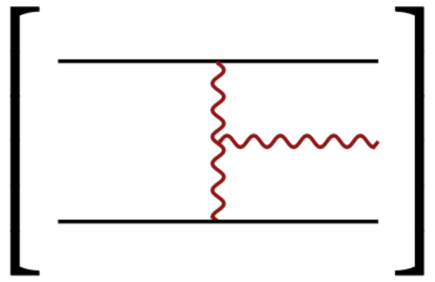
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Complex analysis
Differential Equations

- Alternative: coefficients from **Intersection numbers**

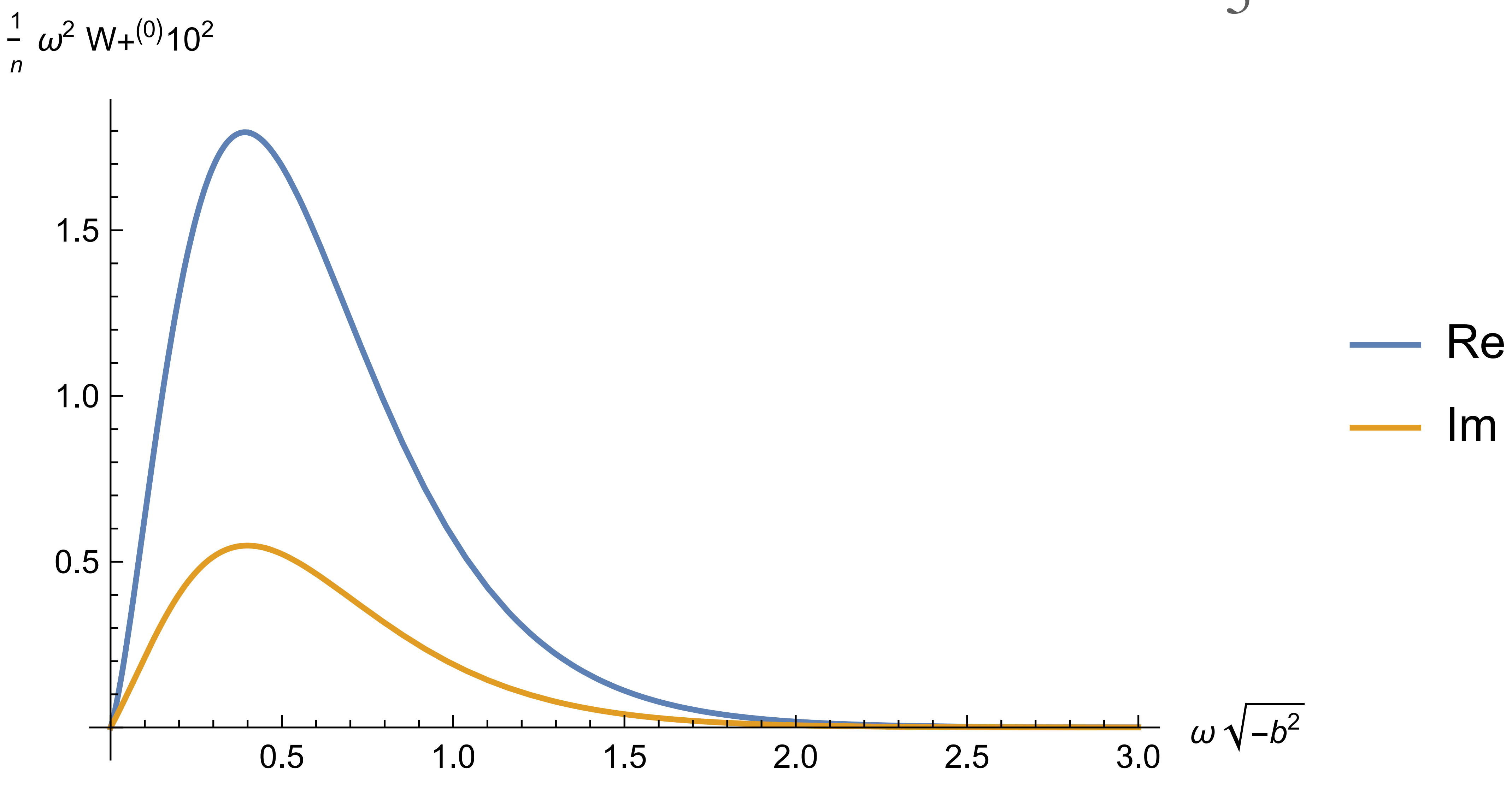
Mastrolia, Mizera, (2018)

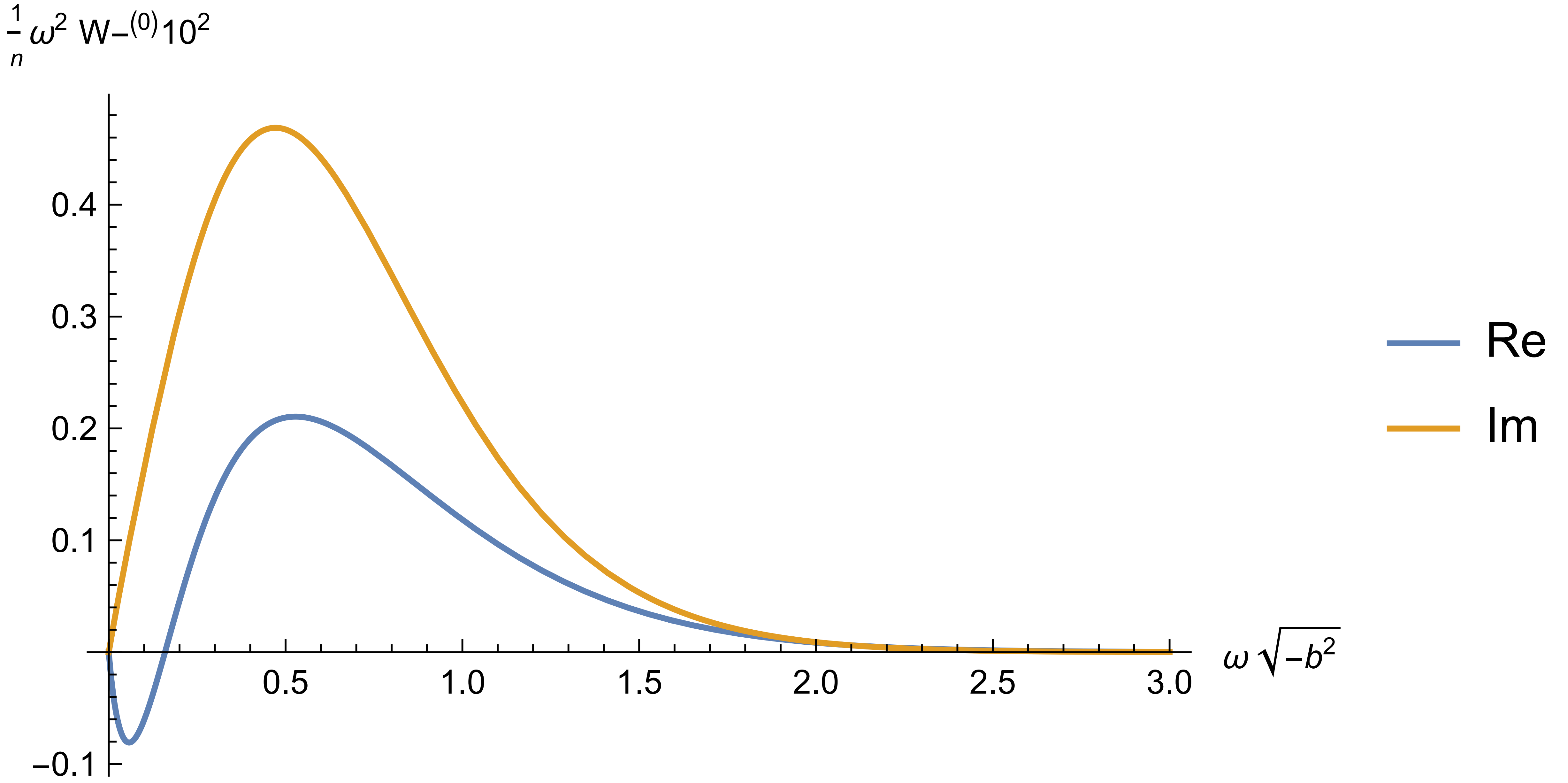
See Pierpaolo Mastrolia,
Vsevolod Chestnov talks

# Results at LO

$$\phi = \frac{7\pi}{10}\,, \quad \theta = \frac{7\pi}{5}\,, \quad m_1 = m_2\,, \quad b = 1$$

- **Frequency- domain waveform (CoM)** $v = \frac{1}{5}$





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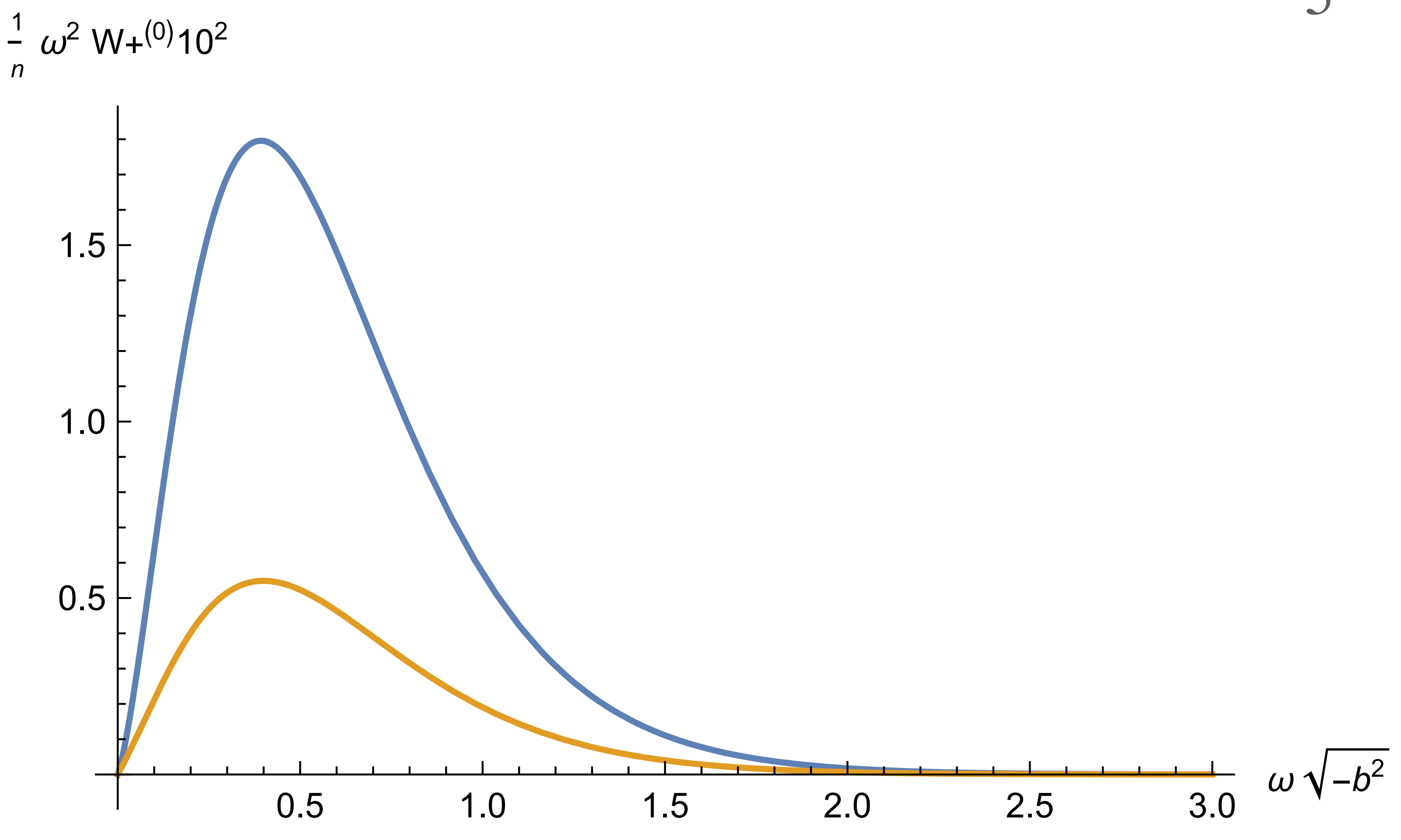
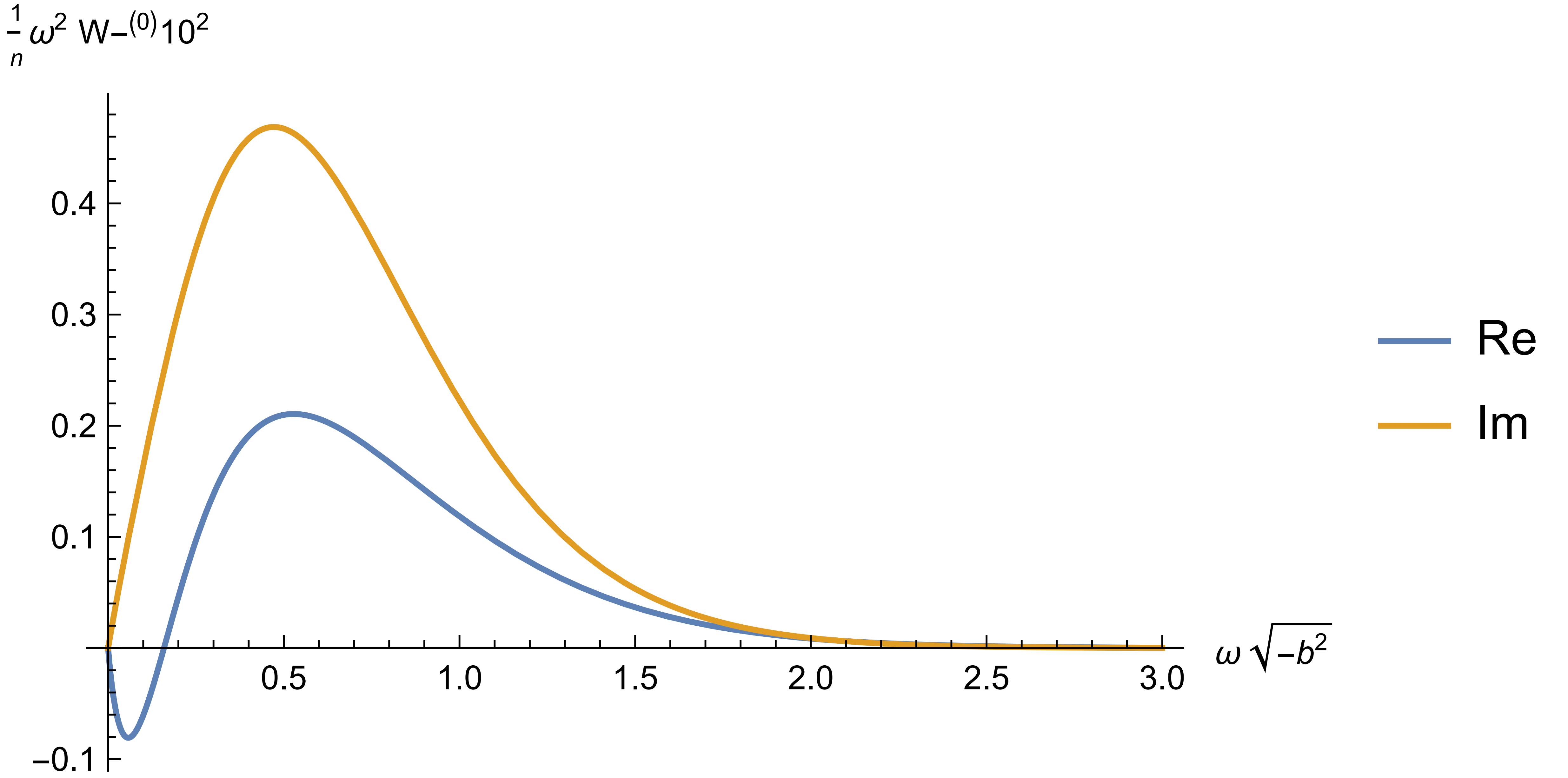
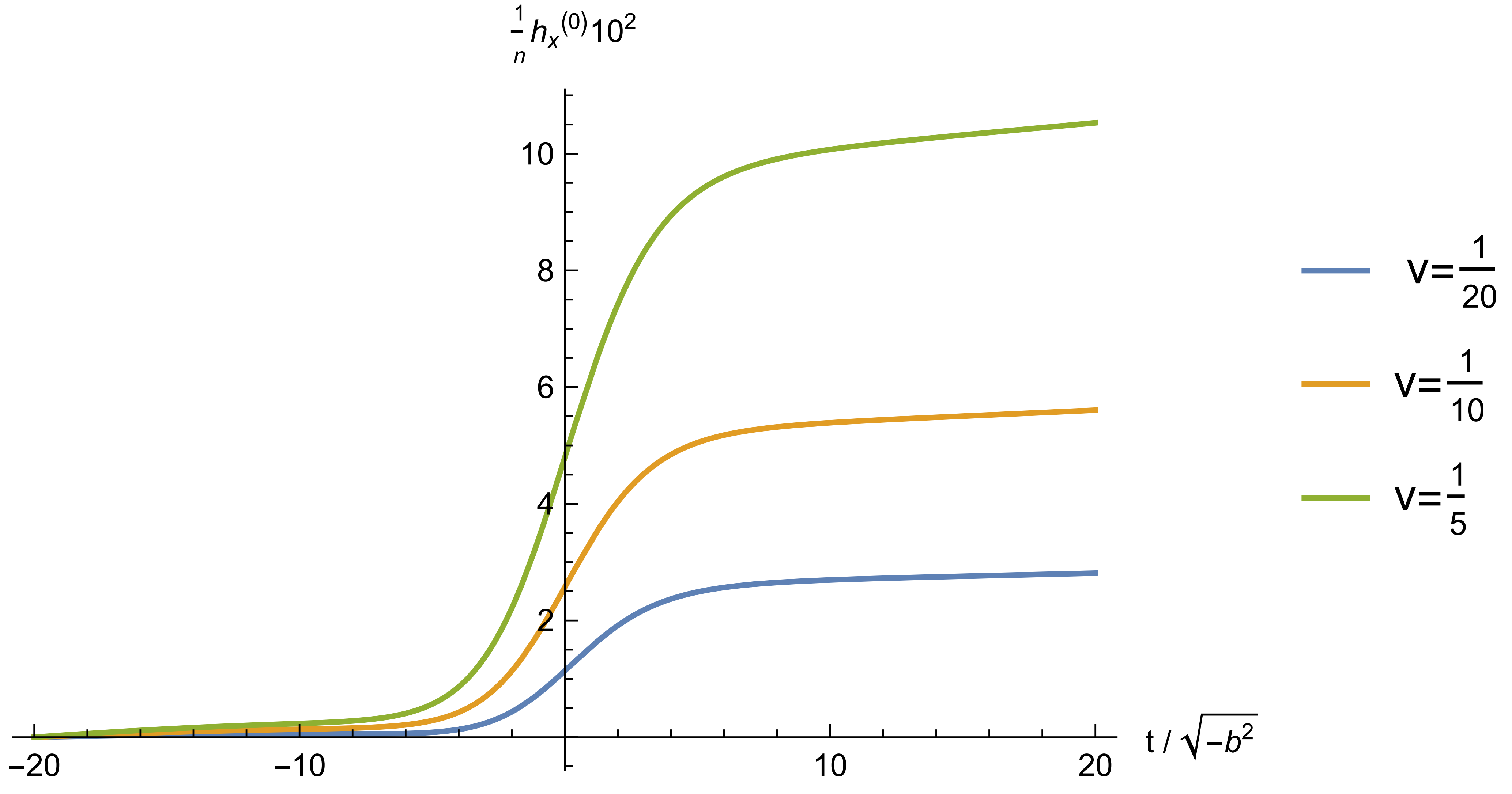
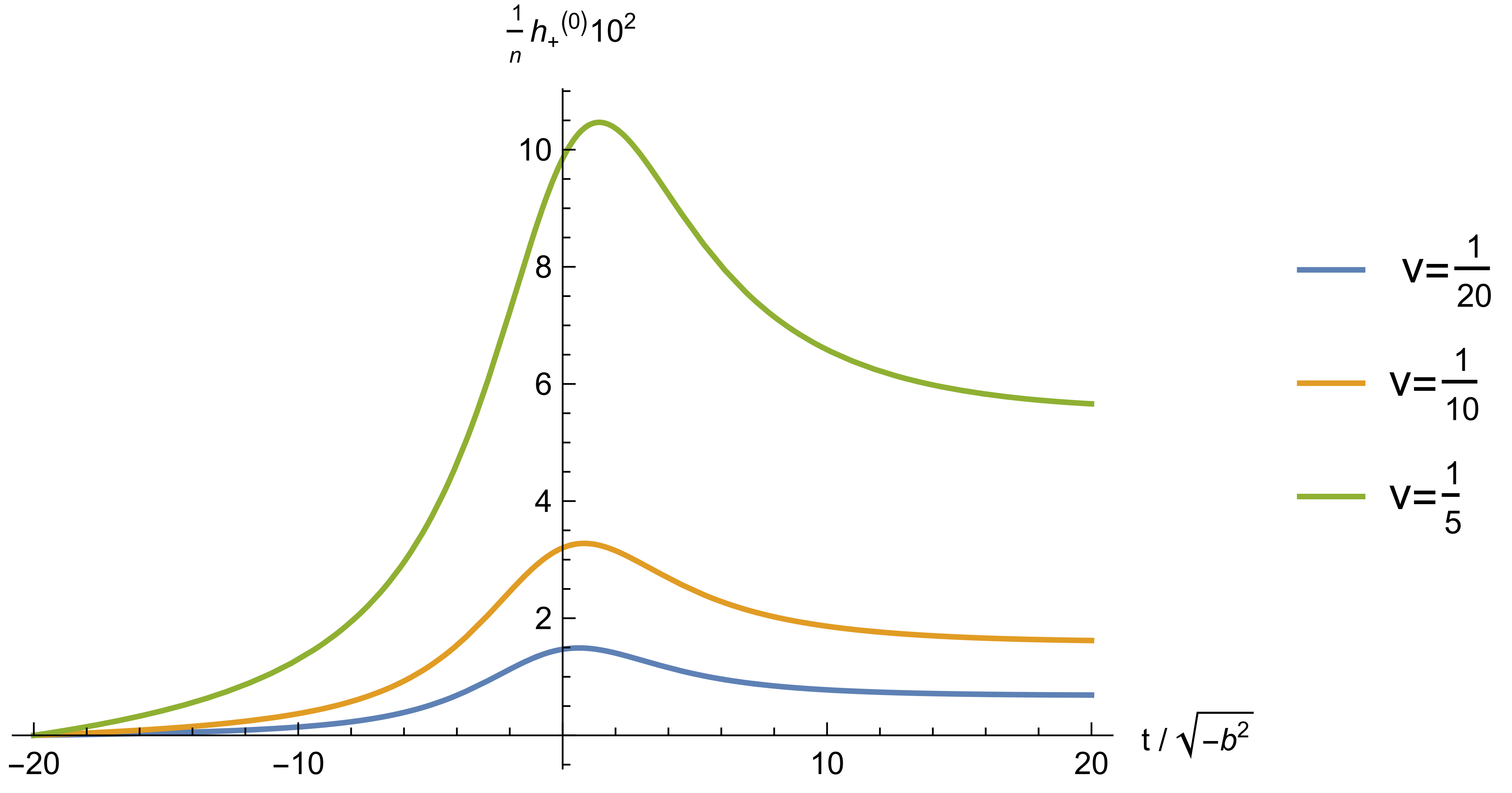




- Time-domain waveform (CoM)





# Waveform at NLO: electrodynamics

- Integrand generation from double and single graviton exchange

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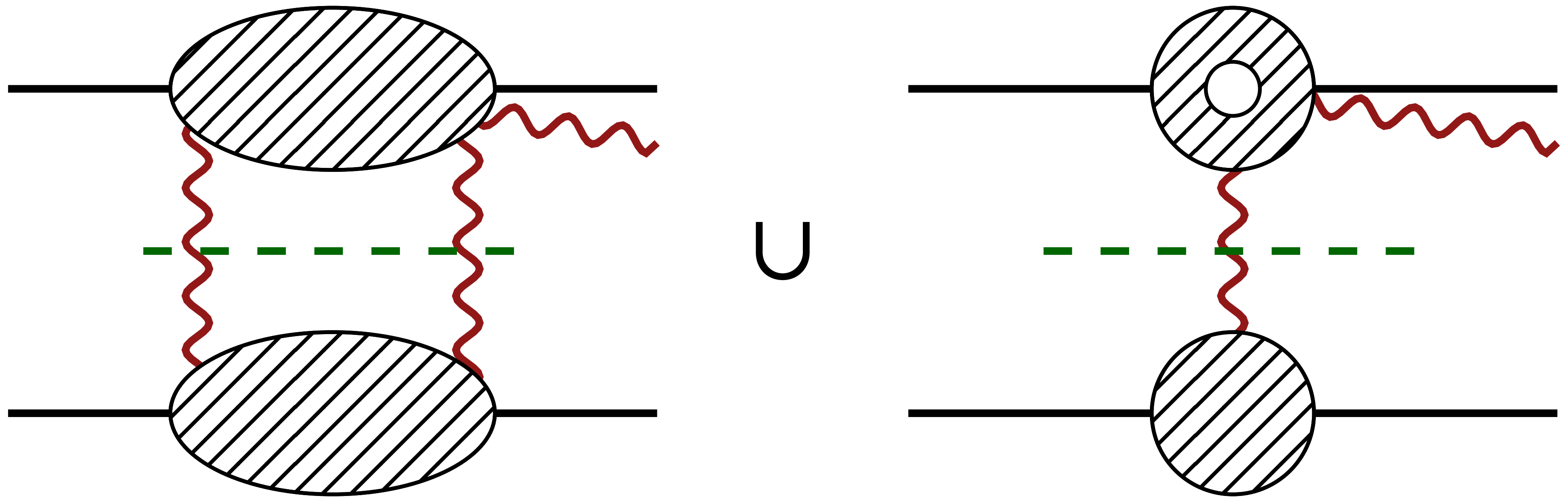

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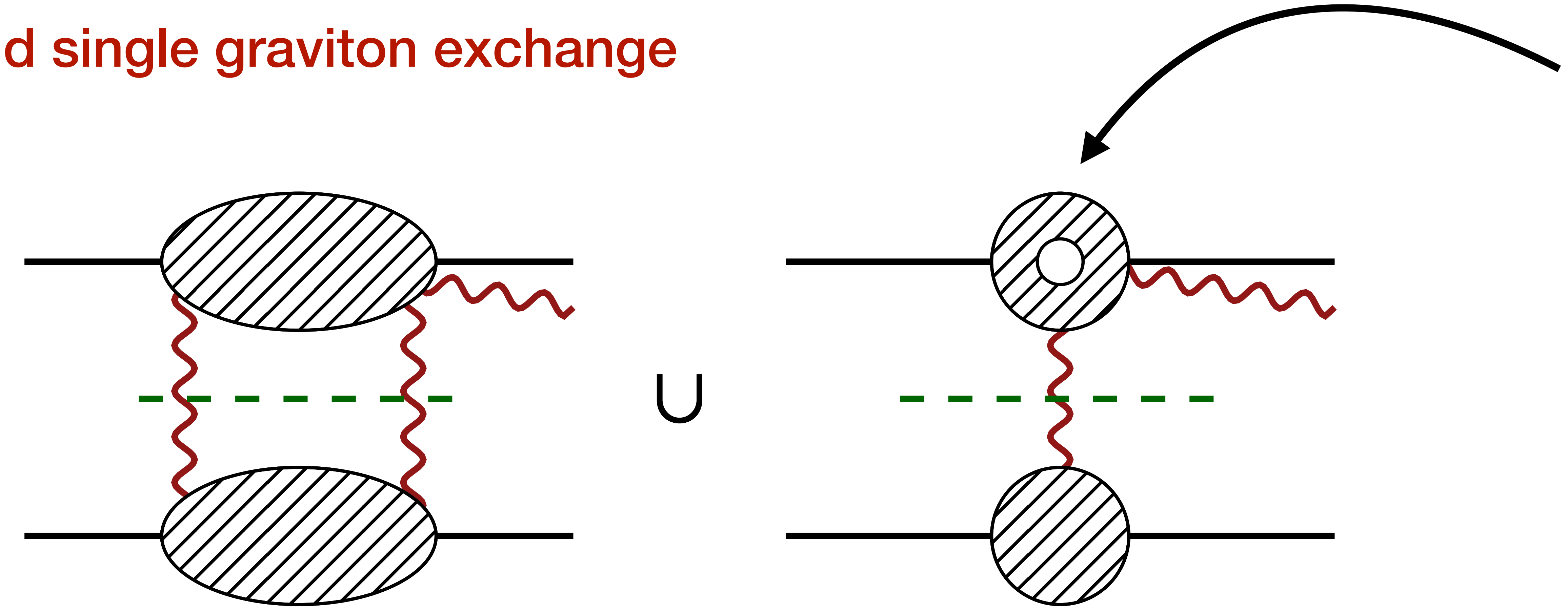
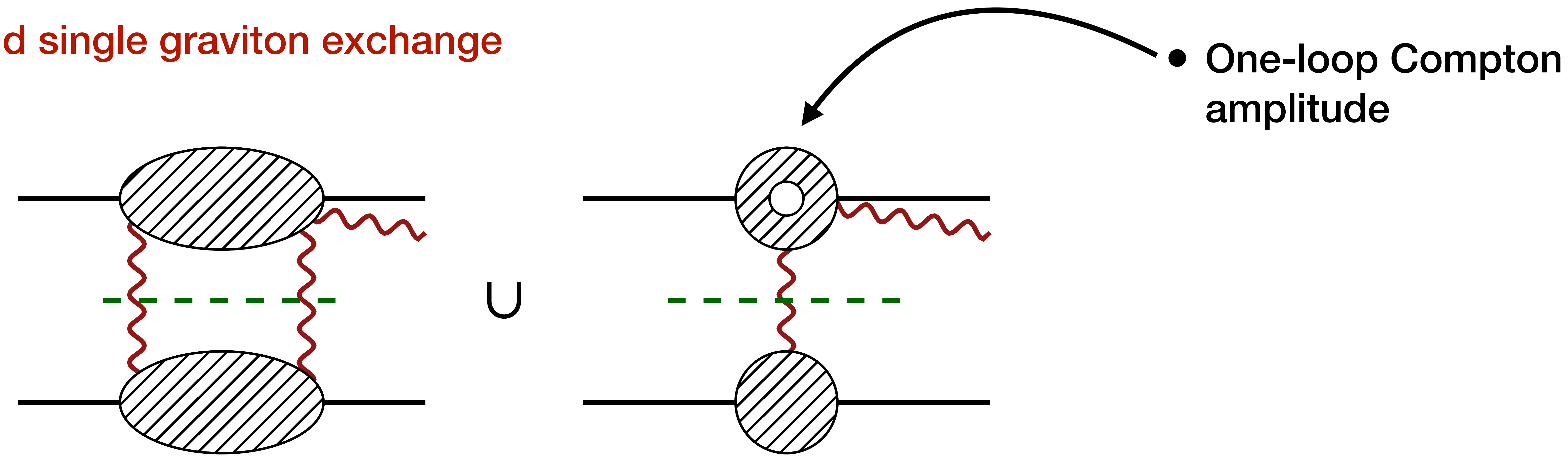


- One-loop Compton amplitude

# Waveform at NLO: electrodynamics

- Integrand generation from double and single graviton exchange



$$\mathcal{I}_{\mathcal{W}}^{(1)} = \quad \cup$$

- Form factors in terms of scalar integrals:

$$I^{u_1}_{a_1a_2a_3a_4a_5a_6a_7a_8a_9a_{10}a_{11}} = \int_{\hat{\ell},\hat{q}} e^{D_1} \frac{1}{\prod_{i=1}^{11} D_i^{a_i}}$$

$D_1 = i\ b \cdot q,\ D_2 = q^2,\ D_3 = (q-k)^2,$

$D_4 = u_1 \cdot q,\ D_5 = u_2 \cdot (k-q)$

$D_6 = u_1 \cdot \ell\ ,\ \ D_7 = u_2 \cdot \ell,\ \ D_8 = \ell^2,$

$D_9 = (\ell - q_2)^2,\ \ D_{10} = (\ell + q_1)^2,\ D_{11} = i\ b \cdot \ell$

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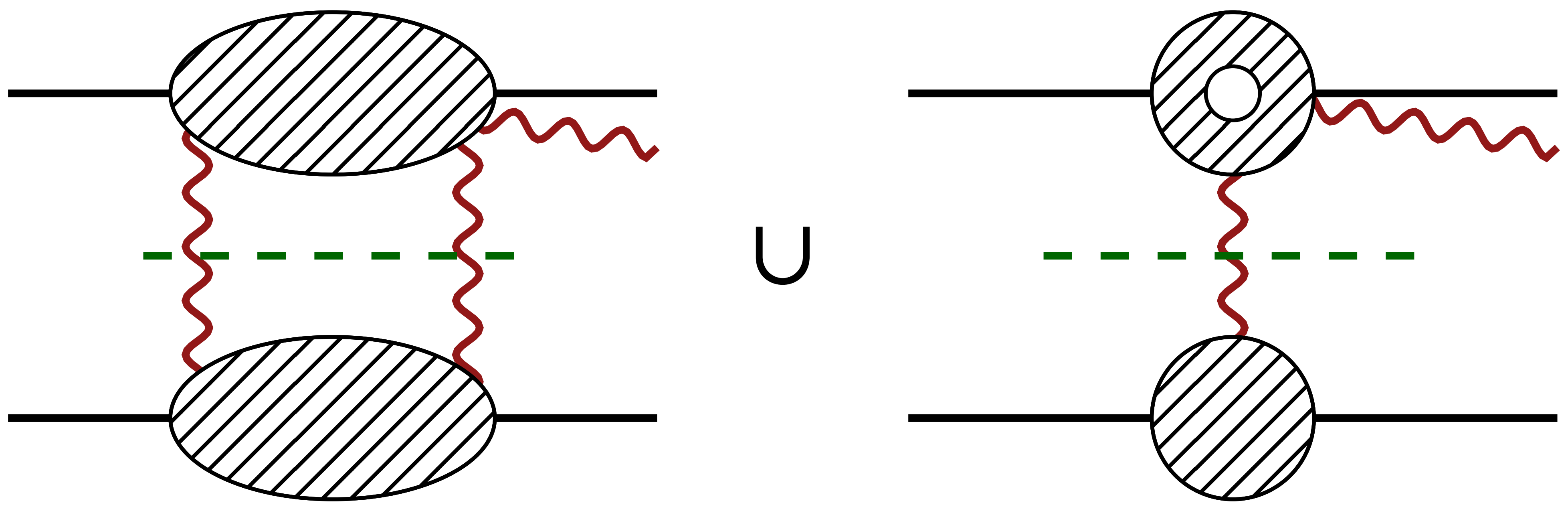

- One-loop Compton amplitude

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- Fourier-Loop IBPs:

$$\int_{\hat{\ell},\hat{q}} \frac{\partial}{\partial_{\{\ell^\mu, q^\mu\}}} \left( e^{D_1} \frac{v^\mu}{\prod_{i=1}^{11} D_i^{a_i}} \right) = 0$$

$$\hat{\delta}(D_4)\hat{\delta}(D_5)\hat{\delta}(D_6) \qquad \hat{\delta}(D_4)\hat{\delta}(D_5)\hat{\delta}(D_7)$$

$$(1 - D_1) IBP[a_1, \ldots, a_{11}] + IBP[a_1 - 1, \ldots, a_{11}] = 0$$

# Results from IBP decomposition

- 2 families of 12 MIs appearing in ED:

$$\Delta\langle \mathscr{W}_h^{(1)}\rangle(\omega,\vec{n}) = \frac{e^{i\omega\, n\cdot b_2}}{4\pi r}\sum_{i=1}^{10}\left(c_i^{\bar{u}_1}J_i^{\bar{u}_1}+c_i^{\bar{u}_2}J_i^{\bar{u}_2}\right)+\frac{e^{i\omega\, n\cdot b_2}}{4\pi r}\sum_{i=1}^{2}\left(c_i^{\bar{u}_1,C}J_i^{\bar{u}_1,C}+c_i^{\bar{u}_2,C}J_i^{\bar{u}_2,C}\right),$$

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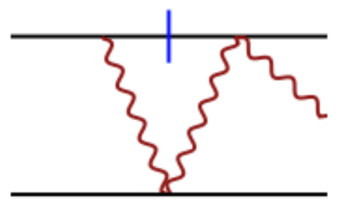
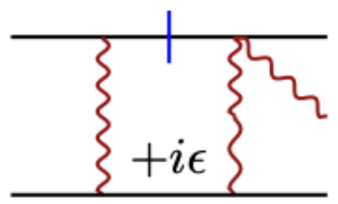
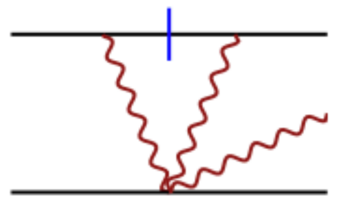
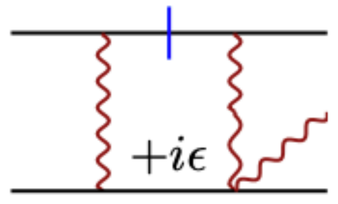
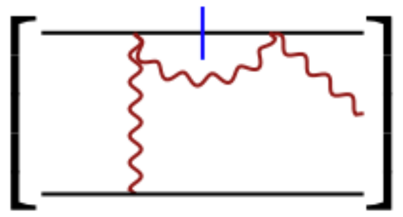
$J^{\bar{u}_1}_{1+n} = \delta_b^{(n)}\mathcal{F}\Bigg[$  $\Bigg]$ ,

$J^{\bar{u}_1}_{5+n} = \delta_b^{(m)}\mathcal{F}\Bigg[$  $\Bigg]$ ,

$J^{\bar{u}_1}_{7+n} = \delta_b^{(m)}\mathcal{F}\Bigg[$  $\Bigg]$ ,

$J^{\bar{u}_1}_{9+n} = \delta_b^{(m)}\mathcal{F}\Bigg[$  $\Bigg]$ ,

$J^{\bar{u}_1\,C}_{1+n} = \delta_b^{(n)}\mathcal{F}\Bigg[$ 

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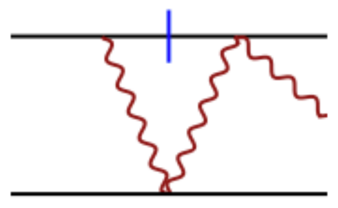
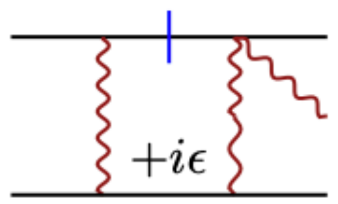
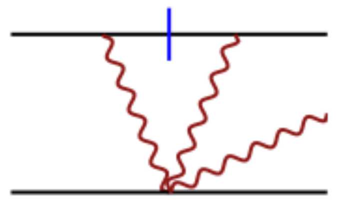
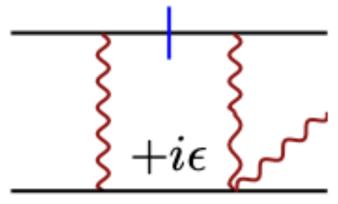
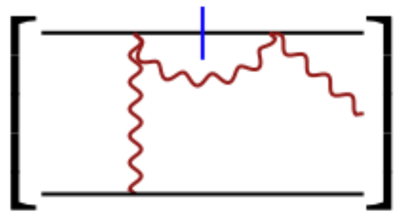
$J^{\bar{u}_1}_{1+n} = \delta_b^{(n)}\mathcal{F}\Big[$  $\Big]$,

$J^{\bar{u}_1}_{5+n} = \delta_b^{(m)}\mathcal{F}\Big[$  $\Big]$,

$J^{\bar{u}_1}_{7+n} = \delta_b^{(m)}\mathcal{F}\Big[$  $\Big]$,

$J^{\bar{u}_1}_{9+n} = \delta_b^{(m)}\mathcal{F}\Big[$  $\Big]$,

$J^{\bar{u}_1\,C}_{1+n} = \delta_b^{(n)}\mathcal{F}\Big[$ 

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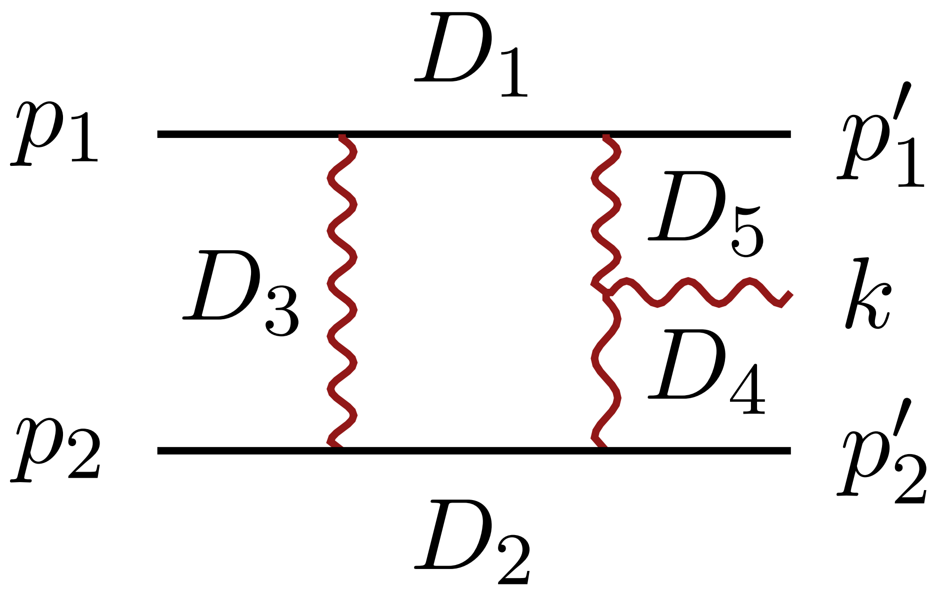

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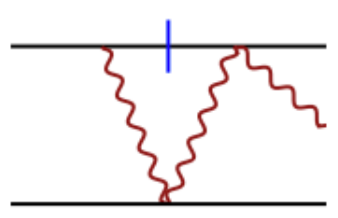
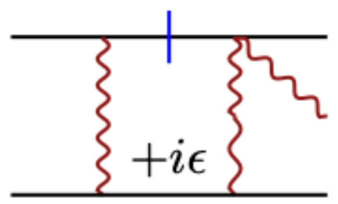
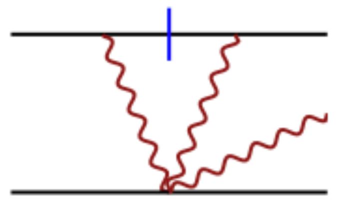
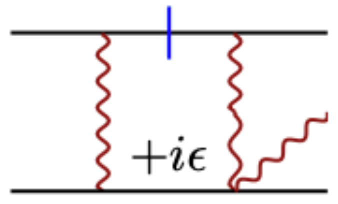
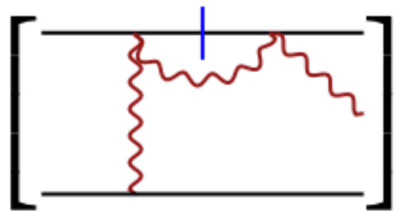
$J^{\bar{u}_1}_{1+n} = \delta_b^{(n)}\mathcal{F}\Bigg[$  $\Bigg]$, $J^{\bar{u}_1}_{5+n} = \delta_b^{(m)}\mathcal{F}\Bigg[$ 

$\Bigg]$, $J^{\bar{u}_1}_{7+n} = \delta_b^{(m)}\mathcal{F}\Bigg[$  $\Bigg]$, $J^{\bar{u}_1}_{9+n} = \delta_b^{(m)}\mathcal{F}\Bigg[$ 

$\Bigg]$, $J^{\bar{u}_1\,C}_{1+n} = \delta_b^{(n)}\mathcal{F}\Bigg[$ 

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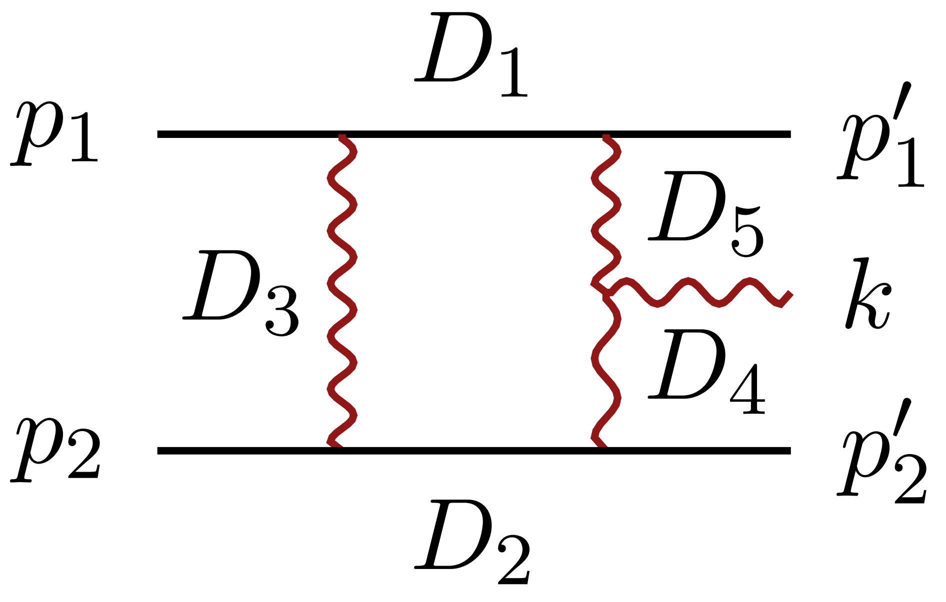

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$$d\mathcal{G} = \epsilon d\mathbb{A}\mathcal{G} \qquad d\mathbb{A} = \sum_{i=1}^{42}\mathbb{M}_i d\log(\eta_i)$$

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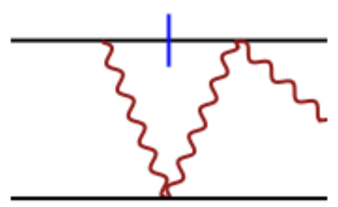
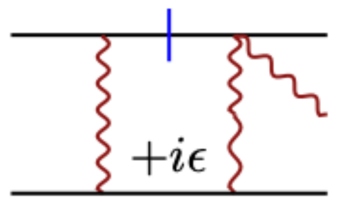
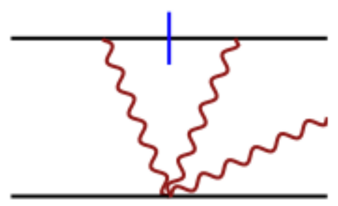
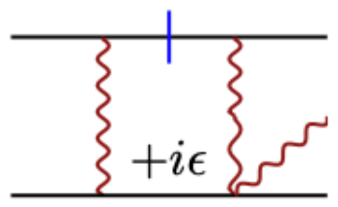
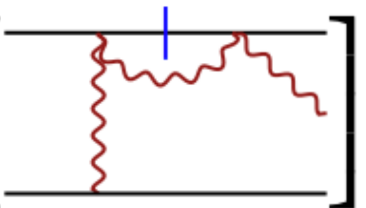
$J^{\bar{u}_1}_{1+n} = \delta_b^{(n)}\mathcal{F}\Big[$  $\Big]$ , $J^{\bar{u}_1}_{5+n} = \delta_b^{(m)}\mathcal{F}\Big[$  $\Big]$ , $J^{\bar{u}_1}_{7+n} = \delta_b^{(m)}\mathcal{F}\Big[$  $\Big]$ , $J^{\bar{u}_1}_{9+n} = \delta_b^{(m)}\mathcal{F}\Big[$  $\Big]$ , $J^{\bar{u}_1\,C}_{1+n} = \delta_b^{(n)}\mathcal{F}\Big[$ 

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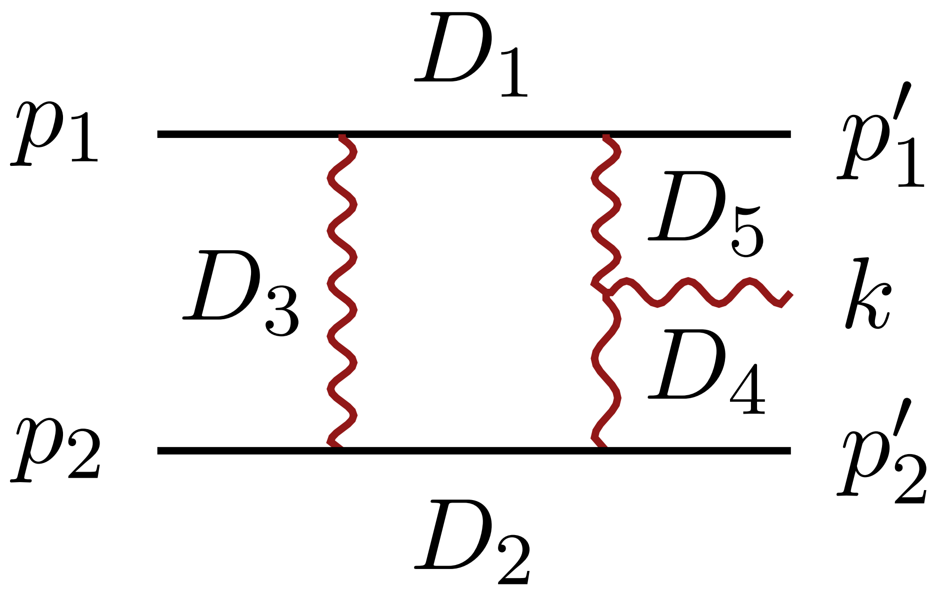

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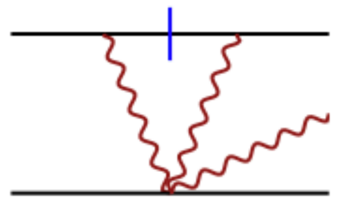
- Retarded boundary conditions as prescribed by in-in, using AMFlow + PSLQ

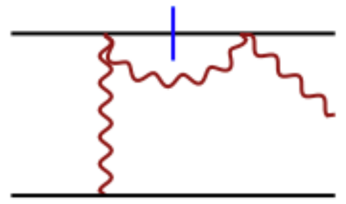
# Integral Evaluation - Fourier integral

G.B., S. De Angelis

- **Waveform integrals appearing in ED:**



$$= \frac{1}{8\sqrt{(-q_1^2)}} + \mathcal{O}\left(\epsilon^1\right),$$



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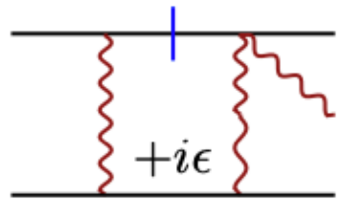
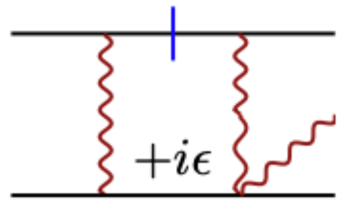


$$= \frac{i}{8\pi(-q_2^2)\sqrt{\gamma^2-1}}\left(\frac{-q_2^2}{w_1\mu_{\rm IR}}\right)^{-2\epsilon}\left[\frac{1}{\tilde{\epsilon}} - \log\left(\gamma^2-1\right) - 2\log\left(\gamma+\sqrt{\gamma^2-1}\right) + i\pi\right] + \mathcal{O}\left(\epsilon^1\right)$$



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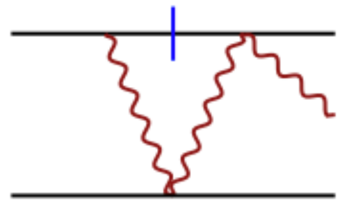

$$= \frac{\pi + 2i\log\frac{w_1+\sqrt{w_1^2-q_2^2}}{\sqrt{-q_2^2}}}{8\pi\sqrt{w_1^2-q_2^2}} + \mathcal{O}\left(\epsilon^1\right),$$

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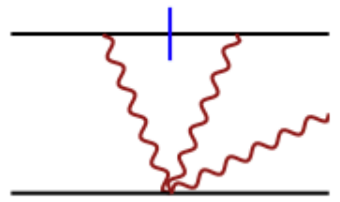

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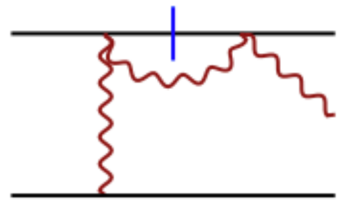

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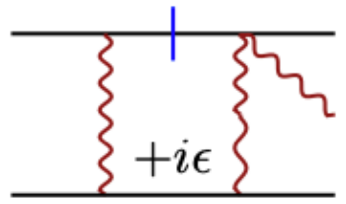

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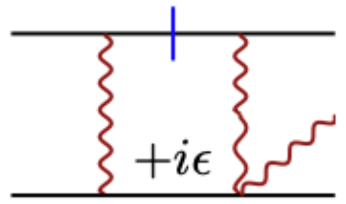

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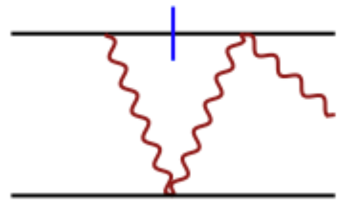

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$$\mathscr{F}[(-q^2)^\alpha] \propto K_{\alpha+\frac{D}{2}-1}(\sqrt{-b^2}\hat{w}_2)$$

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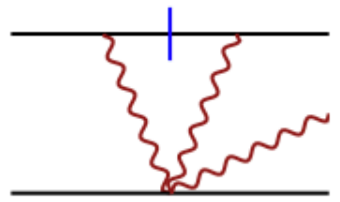

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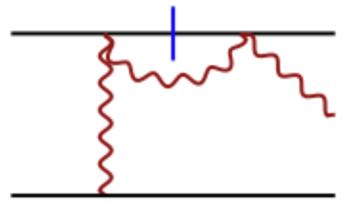

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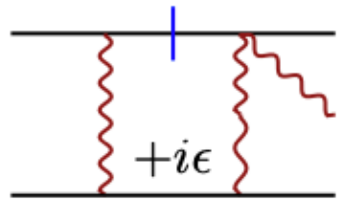
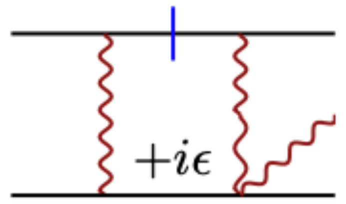


$$= \frac{i}{8\pi(-q_2^2)\sqrt{\gamma^2-1}}\left(\frac{-q_2^2}{w_1\mu_{\text{IR}}}\right)^{-2\epsilon}\left[\frac{1}{\tilde{\epsilon}} - \log\left(\gamma^2-1\right) - 2\log\left(\gamma+\sqrt{\gamma^2-1}\right) + i\pi\right] + \mathcal{O}\left(\epsilon^1\right)$$



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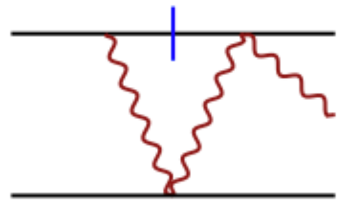


$$= \frac{\pi + 2i\log\frac{w_1+\sqrt{w_1^2-q_2^2}}{\sqrt{-q_2^2}}}{8\pi\sqrt{w_1^2-q_2^2}} + \mathcal{O}\left(\epsilon^1\right),$$

$$\mathscr{F}[(-q^2)^\alpha] \propto K_{\alpha+\frac{D}{2}-1}(\sqrt{-b^2}\hat{w}_2)$$

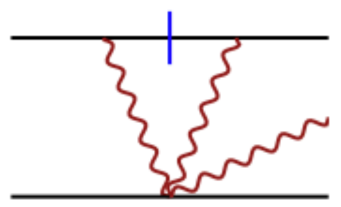
$$\frac{i}{16\pi\sqrt{(-b^2)p_\infty}}\left\{z\int_0^\infty dx\left[e^{-z\ cosh\ x}\ \mathbf{H}_{-1}\left(z\sqrt{p_\infty}sinh\ x\right)\right] - i\frac{e^{-z\sqrt{1+p_\infty}}}{\sqrt{p_\infty}}\right\}$$

Struve H-function

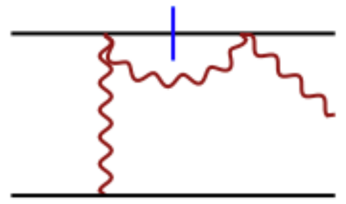
# Integral Evaluation - Fourier integral

G.B., S. De Angelis

- Waveform integrals appearing in ED:



$$= \frac{1}{8\sqrt{(-q_1^2)}} + \mathcal{O}\left(\epsilon^1\right),$$



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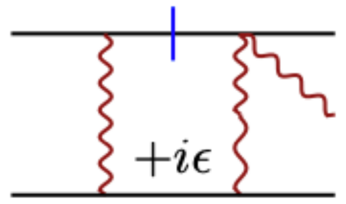

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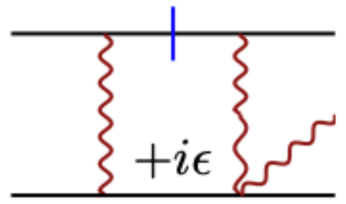

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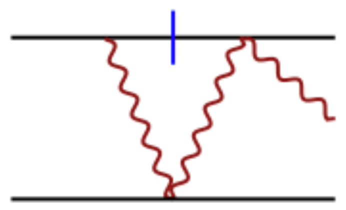

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Struve H-function

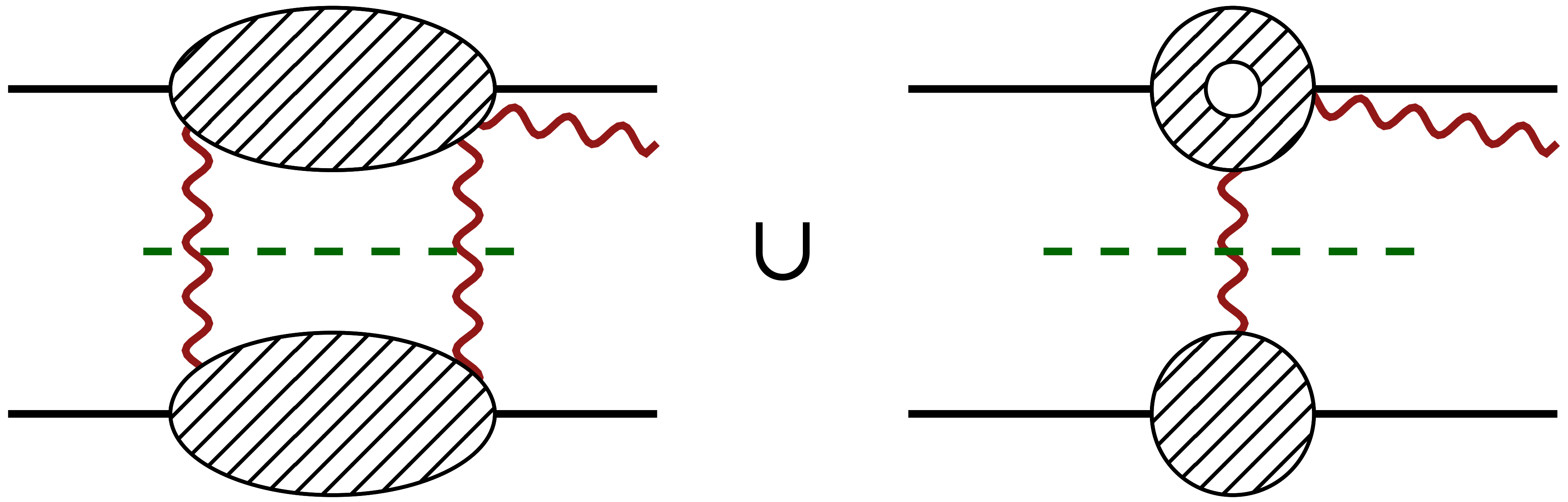
- Fast numerical convergence

# Waveform at NLO: General Relativity

G.B., S. De Angelis,
In progress

- Integrand generation from double and single graviton exchange

$$\mathscr{I}^{(1)}_{\mathscr{W}} =$$



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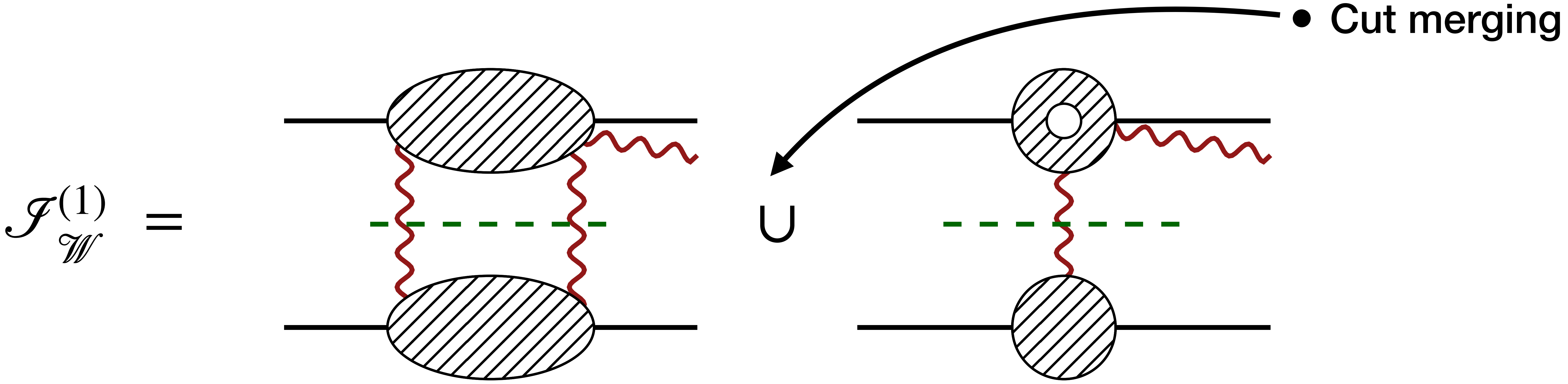

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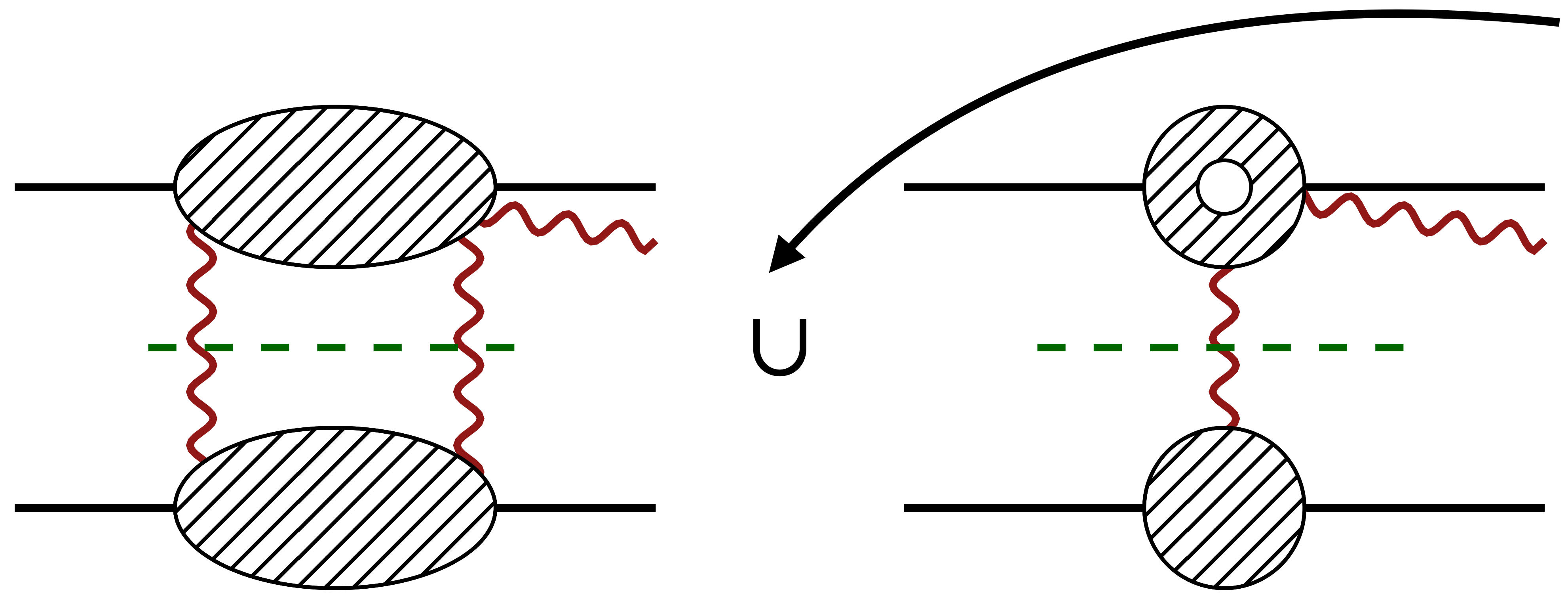

- Cut merging

- 2 families of 28 MIs appearing in General Relativity:

$$\Delta\langle \mathscr{W}_h^{(1)}\rangle(\omega,\vec{n}) = \frac{e^{i\omega\, n\cdot b_2}}{4\pi r}\sum_{i=1}^{28}\left(c_i^{\bar{u}_1}J_i^{\bar{u}_1} + c_i^{\bar{u}_2}J_i^{\bar{u}_2}\right)$$

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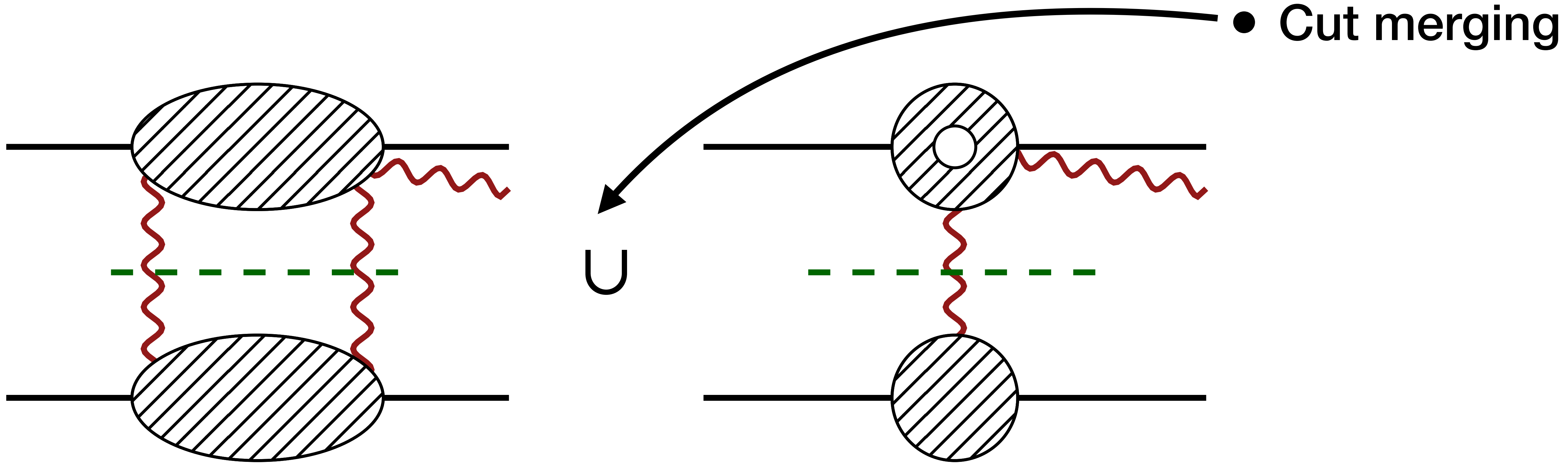

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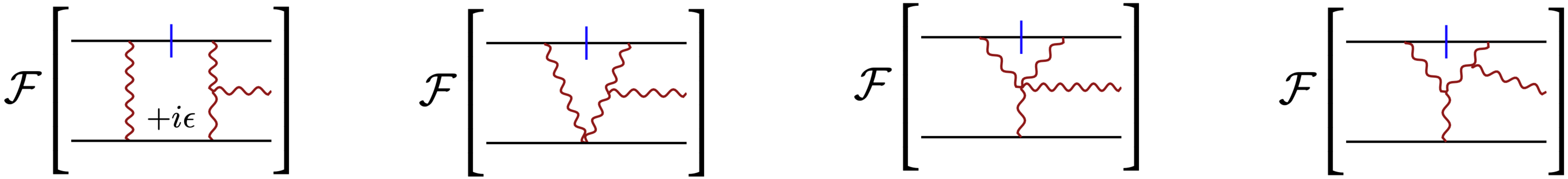
- New master integrals need to be computed:



# Outlooks

- Gravitational Waveforms can be evaluated from Scattering Amplitudes

- Multi-loop technology applied at the level of the frequency-space waveform

- Fourier-like integrals can be understood as twisted period integrals G.B., G. Crisanti, M. Giroux, P. Mastrolia, Sid Smith [2311.14432]

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- On one-loop corrections to the Bunch-Davies wavefunction of the Universe P. Benincasa, G.B., M.K. Mandal, P. Mastrolia, F. Vazão [2408.16386]

# Thank you!